the aid of Lagrange's analysis; but it is an obvious consequence of the Hamiltonian form of his equations, as stated in words in § 319 above. In the precisely equivalent case, of a liquid extending infinitely in all directions, and given at rest; and two equal balls projected through it with equal velocities perpendicular to the line joining their centres—the result that the two balls will seem to attract one another is most remarkable, and very suggestive.

Seeming attraction between two ships moving side by side in the same direction.

Hydro-dynamical examples continued.

Example (2).—*A solid symmetrical round an axis, moving through a liquid so as to keep its axis always in one plane.* Let ω be the angular velocity of the body at any instant about any axis perpendicular to the fixed plane, and let u and q be the component velocities along and perpendicular to the axis of figure, of any chosen point, C, of the body in this line. By the general principle stated in § 320 (since changing the sign of u cannot alter the kinetic energy), we have

$$T = \tfrac{1}{2}(Au^2 + Bq^2 + \mu'\omega^2 + 2E\omega q) \text{ .....} (a),$$

where A, B, μ', and E are constants depending on the figure of the body, its mass, and the density of the liquid. Now let v denote the velocity, perpendicular to the axis, of a point which we shall call the *centre of reaction*, being a point in the axis and at a distance $\frac{E}{B}$ from C, so that (§ 87) $q = v - \frac{E}{B}\omega$. Then, denoting $\mu' - \frac{E^2}{B}$ by μ, we have $T = \frac{1}{2}(Au^2 + Bv^2 + \mu\omega^2)$.....$(a')$.

"Centre of reaction" defined.

Let x and y be the co-ordinates of the centre of reaction relatively to any fixed rectangular axes in the plane of motion of the axis of figure, and let θ be the angle between this line and OX, at any instant, so that

$$\omega = \dot\theta,\ u = \dot x\cos\theta + \dot y\sin\theta,\ v = -\dot x\sin\theta + \dot y\cos\theta \text{ .....} (b).$$

Substituting in T, differentiating, and retaining the notation u, v where convenient for brevity, we have

$$\left.\begin{aligned} &\frac{dT}{d\dot\theta} = \mu\dot\theta,\ \frac{dT}{d\dot x} = Au\cos\theta - Bv\sin\theta,\ \frac{dT}{d\dot y} = Au\sin\theta + Bv\cos\theta,\\ &\frac{dT}{d\theta} = (A-B)\,uv,\ \frac{dT}{dx} = 0,\ \frac{dT}{dy} = 0,\end{aligned}\right\} \quad (c)$$

Hydro-dynamical examples continued.

Hence the equations of motion are

$$\left.\begin{aligned} &\mu\ddot\theta - (A-B)\,uv = L,\\ &\frac{d(Au\cos\theta - Bv\sin\theta)}{dt} = X,\quad \frac{d(Au\sin\theta + Bv\cos\theta)}{dt} = Y\end{aligned}\right\} \quad (d),$$

where X, Y are the component forces in lines through C parallel to OX and OY, and L the couple, applied to the body.

Denoting by λ, ξ, η the impulsive couple, and the components of impulsive force through C, required to produce the motion at any instant, we have of course [§ 313 (c)],

$$\lambda = \frac{dT}{d\dot\theta} = \mu\dot\theta,\ \xi = \frac{dT}{d\dot x},\ \eta = \frac{dT}{d\dot y} \text{ .....} (e),$$

and therefore by (c), and (b),

$$u = \frac{1}{A}(\xi\cos\theta + \eta\sin\theta),\ v = \frac{1}{B}(-\xi\sin\theta + \eta\cos\theta),\ \dot\theta = \frac{\lambda}{\mu} \text{.....} (f),$$

$$\left.\begin{aligned} \dot x &= \left(\frac{\cos^2\theta}{A} + \frac{\sin^2\theta}{B}\right)\xi + \left(\frac{1}{A} - \frac{1}{B}\right)\sin\theta\cos\theta\eta,\\ \dot y &= \left(\frac{1}{A} - \frac{1}{B}\right)\sin\theta\cos\theta\xi + \left(\frac{\sin^2\theta}{A} + \frac{\cos^2\theta}{B}\right)\eta\end{aligned}\right\} \text{.....} (g),$$

and the equations of motion become

$$\mu\frac{d^2\theta}{dt^2} - \frac{A-B}{2AB}\{(-\xi^2+\eta^2)\sin 2\theta + 2\xi\eta\cos 2\theta\} = L,\ \frac{d\xi}{dt} = X,\ \frac{d\eta}{dt} = Y, \quad (h).$$

The simple case of $X=0$, $Y=0$, $L=0$, is particularly interesting. In it ξ and η are each constant; and we may therefore choose the axes OX, OY, so that η shall vanish. Thus we have, in (g), two first integrals of the equations of motion; and they become

$$\dot x = \xi\left(\frac{\cos^2\theta}{A} + \frac{\sin^2\theta}{B}\right),\quad \dot y = -\frac{A-B}{2AB}\,\xi\sin 2\theta \text{ .....} (k):$$

and the first of equations (h) becomes

$$\mu\frac{d^2\theta}{dt^2} + \frac{A-B}{2AB}\,\xi^2\sin 2\theta = 0 \text{.....} (l).$$

In this let, for a moment, $2\theta = \phi$, and $\frac{A-B}{AB}\xi^2 = ghW$. It becomes

$$\mu\frac{d^2\phi}{dt^2} + ghW\sin\phi = 0,$$

which is the equation of motion of a common pendulum, of mass W, moment of inertia μ round its fixed axis, and length

Hydro-dynamical examples continued.

h from axis to centre of gravity; if ϕ be the angle from the position of equilibrium to the position at time t. As we shall see, under kinetics, the final integral of this equation expresses ϕ in terms of t by means of an elliptic function. By using the value thus found for θ or $\frac{1}{2}\phi$, in (k), we have equations giving x and y in terms of t by common integration; and thus the full solution of our present problem is reduced to quadratures. The detailed working out to exhibit both the actual curve described by the centre of reaction, and the position of the axis of the body at any instant, is highly interesting. It is very easily done approximately for the case of very small angular vibrations; that is to say, when either $A-B$ is positive, and ϕ always very small, or $A-B$ negative, and ϕ very nearly equal to $\frac{1}{2}\pi$. But without attending at present to the final integrals, rigorous or approximate, we see from (k) and (l) that

322. If a solid of revolution in an infinite liquid, be set in motion round any axis perpendicular to its axis of figure, or simply projected in any direction without rotation, it will move with its axis always in one plane, and every point of it moving only parallel to this plane; and the strange evolutions which it will, in general, perform, are perfectly defined by comparison with the common pendulum thus. First, for brevity, we shall call by the name of *quadrantal pendulum* (which will be further exemplified in various cases described later, under electricity and magnetism; for instance, an elongated mass of soft iron pivoted on a vertical axis, in a "uniform field of magnetic force"), a body moving about an axis, according to the same law with reference to a quadrant on each side of its position of equilibrium, as the common pendulum with reference to a half circle on each side.

Quadrantal pendulum defined.

Let now the body in question be set in motion by an impulse, ξ, in any line through the centre of reaction, and an impulsive couple λ in the plane of that line and the axis. This will (as will be proved later in the theory of statical couples) have the same effect as a simple impulse ξ (applied to a point, if not of the real body, connected with it by an imaginary infinitely light framework) in a certain fixed line, which we shall call the line of resultant impulse, or of resultant momentum,

Motion of a solid of revolution through a liquid.

being parallel to the former line, and at a distance from it equal to $\frac{\lambda}{\xi}$. The whole momentum of the motion generated is of course (§ 295) equal to ξ. The body will move ever afterwards according to the following conditions:—(1.) The angular velocity follows the law of the quadrantal pendulum. (2.) The distance of the centre of reaction from the line of resultant impulse varies simply as the angular velocity. (3.) The velocity of the centre of reaction parallel to the line of impulse is found by dividing the excess of the whole constant energy of the motion above the part of it due to the angular velocity round the centre of reaction, by half the momentum. (4.) If A, B, and μ denote constants, depending on the mass of the solid and its distribution, the density of the liquid, and the form and dimensions of the solid, such that $\frac{\xi}{A}$, $\frac{\xi}{B}$, $\frac{\lambda}{\mu}$ are the linear velocities, and the angular velocity, respectively produced by an impulse ξ along the axis, an impulse ξ in a line through the centre of reaction perpendicular to the axis, and an impulsive couple λ in a plane through the axis; the length of the simple gravitation pendulum, whose motion would keep time with the periodic motion in question, is $\frac{g\mu AB}{\xi^2(A-B)}$, and, when the angular motion is vibratory, the vibrations will, according as $A > B$, or $A < B$, be of the axis, or of a line perpendicular to the axis, vibrating on each side of the line of impulse. The angular motion will in fact be vibratory if the distance of the line of resultant impulse from the centre of reaction is anything less than $\sqrt{\frac{(A-B)\mu\cos 2\alpha}{AB}}$ where α denotes the inclination of the impulse to the initial position of the axis. In this case the path of the centre of reaction will be a sinuous curve symmetrical on the two sides of the line of impulse; every time it cuts this line, the angular motion will reverse, and the maximum inclination will be attained; and every time the centre of reaction is at its greatest distance on either side, the angular velocity will be at its greatest, positive or negative, value, and the linear velocity of

Motion of a solid of revolution through a liquid.

the centre of reaction will be at its least. If, on the other hand, the line of the resultant impulse be at a greater distance than $\sqrt{\frac{(A\sim B)\mu\cos 2\alpha}{AB}}$ from the centre of reaction, the angular motion will be always in one direction, but will increase and diminish periodically, and the centre of reaction will describe a sinuous curve on one side of that line; being at its greatest and least deviations when the angular velocity is greatest and least. At the same points the curvature of the path will be greatest and least respectively, and the linear velocity of the describing point will be least and greatest.

323. At any instant the component linear velocities along and perpendicular to the axis of the solid will be $\frac{\xi\cos\theta}{A}$ and $-\frac{\xi\sin\theta}{B}$ respectively, if θ be its inclination to the line of resultant impulse; and the angular velocity will be $\frac{\xi y}{\mu}$ if y be the distance of the centre of reaction from that line. The whole kinetic energy of the motion will be

$$\frac{\xi^2\cos^2\theta}{2A}+\frac{\xi^2\sin^2\theta}{2B}+\frac{\xi^2 y^2}{2\mu},$$

and the last term is what we have referred to above as the part due to rotation round the centre of reaction (defined in § 321). To stop the whole motion at any instant, a simple impulse equal and opposite to ξ in the fixed "line of resultant impulse" will suffice (or an equal and parallel impulse in any line through the body, with the proper impulsive couple, according to the principle already referred to).

324. From Lagrange's equations applied as above to the case of a solid of revolution moving through a liquid, the couple which must be kept applied to it to prevent it from turning is immediately found to be

$$uv\,(A-B),$$

Motion of a solid of revolution through a liquid.

if u and v be the component velocities along and perpendicular to the axis, or [§ 321 (f)]

$$\xi^2\frac{(A-B)\sin 2\theta}{2AB},$$

if, as before, ξ be the generating impulse, and θ the angle between its line and the axis. The direction of this couple must be such as to prevent θ from diminishing or from increasing, according as A or B is the greater. The former will clearly be the case of a flat disc, or oblate spheroid; the latter that of an elongated, or oval-shaped body. The actual values of A and B we shall learn how to calculate (hydrodynamics) for several cases, including a body bounded by two spherical surfaces cutting one another at any angle a submultiple of two right angles; two complete spheres rigidly connected; and an oblate or a prolate spheroid.

Observed phenomena.

325. The tendency of a body to turn its flat side, or its length (as the case may be), across the direction of its motion through a liquid, to which the accelerations and retardations of rotatory motion described in § 322 are due, and of which we have now obtained the statical measure, is a remarkable illustration of the statement of § 319; and is closely connected with the dynamical explanation of many curious observations well known in practical mechanics, among which may be mentioned:—

(1) That the course of a symmetrical square-rigged ship sailing in the direction of the wind with rudder amidships is unstable, and can only be *kept* by manipulating the rudder to check infinitesimal deviations;—and that a child's toy-boat, whether "square-rigged" or "fore-and-aft rigged*," cannot be

* "Fore-and-aft" rig is any rig in which (as in "cutters" and "schooners") the chief sails come into the plane of mast or masts and keel, by the action of the wind upon the sails when the vessel's head is to wind. This position of the sails is unstable when the wind is right astern. Accordingly, in "wearing" a fore-and-aft rigged vessel (that is to say turning her round stern to wind, from sailing with the wind on one side to sailing with the wind on the other side) the mainsail must be hauled in as closely as may be towards the middle position before the wind is allowed to get on the other side of the sail from that on which it had been pressing, so that when the wind

Applications to nautical dynamics

got to sail permanently before the wind by any permanent adjustment of rudder and sails, and that (without a wind vane, or a weighted tiller, acting on the rudder to do the part of steersman) it always, after running a few yards before the wind, turns round till nearly in a direction perpendicular to the wind (either "gibing" first, or "luffing" without gibing if it is a cutter or schooner):—

(2) That the towing rope of a canal boat, when the rudder is left straight, takes a position in a vertical plane cutting the axis *before* its middle point:—

(3) That a boat sculled rapidly across the direction of the wind, always (unless it is extraordinarily unsymmetrical in its draught of water, and in the amounts of surface exposed to the wind, towards its two ends) requires the weather oar to be worked hardest to prevent it from running up on the wind, and that for the same reason a sailing vessel generally "carries a weather helm*" or "gripes;" and that still more does so a steamer with sail even if only in the forward half of her length—griping so badly with any after canvass† that it is often impossible to steer:—

(4) That in a heavy gale it is exceedingly difficult, and often found impossible, to get a ship out of "the trough of the sea," and that it cannot be done at all without rapid motion ahead, whether by steam or sails:—

(5) That in a smooth sea with moderate wind blowing parallel to the shore, a sailing vessel heading towards the shore with not enough of sail set can only be saved from creeping ashore by setting more sail, and sailing rapidly towards the shore, or the danger that is to be avoided, so as to allow her to be steered away from it. The risk of going ashore in fulfilment

does get on the other side, and when therefore the sail dashes across through the mid-ship position to the other side, carrying massive boom and gaff with it, the range of this sudden motion, which is called "gibing," shall be as small as may be.

* The weather side of any object is the side of it towards the wind. A ship is said to "carry a weather helm" when it is necessary to hold the "helm" or "tiller" permanently on the weather side of its middle position (by which the rudder is held towards the lee side) to keep the ship on her course.

† Hence mizen masts are altogether condemned in modern war-ships by many competent nautical authorities.

of Lagrange's equations is a frequent incident of "getting under way" while lifting anchor, or even after slipping from moorings:—

and gunnery.

(6) That an elongated rifle-bullet requires rapid rotation about its axis to keep its point foremost.

(7) The curious motions of a flat disc, oyster-shell, or the like, when dropped obliquely into water, resemble, no doubt, to some extent those described in § 322. But it must be remembered that the real circumstances differ greatly, because of fluid friction, from those of the abstract problem, of which we take leave for the present.

Least action.

326. Maupertuis' celebrated principle of *Least Action* has been, even up to the present time, regarded rather as a curious and somewhat perplexing property of motion, than as a useful guide in kinetic investigations. We are strongly impressed with the conviction that a much more profound importance will be attached to it, not only in abstract dynamics, but in the theory of the several branches of physical science now beginning to receive dynamic explanations. As an extension of it, Sir W. R. Hamilton* has evolved his method of *Varying Action*, which undoubtedly must become a most valuable aid in future generalizations.

Action.

Time average of energy.

What is meant by "Action" in these expressions is, unfortunately, something very different from the *Actio Agentis* defined by Newton†, and, it must be admitted, is a much less judiciously chosen word. Taking it, however, as we find it, now universally used by writers on dynamics, we define the *Action of a Moving System* as proportional to the average kinetic energy, which the system has possessed during the time from any convenient epoch of reckoning, multiplied by the time. According to the unit generally adopted, the action of a system which has not varied in its kinetic energy, is twice the amount of the energy multiplied by the time from the epoch. Or if the energy has been sometimes greater and sometimes less,

* *Phil. Trans.* 1834—1835.

† Which, however (§ 263), we have translated "activity" to avoid confusion.

Time average of energy doubled.

the action at time t is the double of what we may call the *time-integral* of the energy, that is to say, it is what is denoted in the integral calculus by

$$2\int_0^t T d\tau,$$

where T denotes the kinetic energy at any time τ, between the epoch and t.

Let m be the mass, and v the velocity at time τ, of any one of the material points of which the system is composed. We have

$$T = \Sigma \tfrac{1}{2} m v^2 \quad \ldots\ldots\ldots\ldots (1),$$

and therefore, if A denote the action at time t,

$$A = \int_0^t \Sigma m v^2 d\tau \quad \ldots\ldots\ldots\ldots (2).$$

This may be put otherwise by taking ds to denote the space described by a particle in time $d\tau$, so that $v d\tau = ds$, and therefore

$$A = \int \Sigma m v ds \quad \ldots\ldots\ldots\ldots (3),$$

or, if x, y, z be the rectangular co-ordinates of m at any time,

$$A = \int \Sigma m (\dot{x}dx + \dot{y}dy + \dot{z}dz) \quad \ldots\ldots\ldots\ldots (4).$$

Hence we might, as many writers in fact have virtually done, define action thus :—

Space average of momentums.

The action of a system is equal to the sum of the *average momentums for the spaces* described by the particles from any era each multiplied by the length of its path.

Least action.

327. The principle of Least Action is this :—Of all the different sets of paths along which a conservative system may be guided to move from one configuration to another, with the sum of its potential and kinetic energies equal to a given constant, that one for which the action is the least is such that the system will require only to be started with the proper velocities, to move along it unguided. Consider the Problem :—

General Problem of *aim*.

Given the whole initial kinetic energy; find the initial velocities through one given configuration, which shall send the system unguided to another specified configuration. This problem is essentially determinate, but generally has multiple solutions (§ 363 below); (or only imaginary solutions.)

High aim and low aim for same target. Target too far off to be reached.

Least action.

If there are any real solutions, there is one of them for which the action is less than for any other real solution, and less than for any constrainedly guided motion with proper sum of potential and kinetic energies. Compare §§ 346—366 below.

Let x, y, z be the co-ordinates of a particle, m, of the system, at time τ, and V the potential energy of the system in its particular configuration at this instant; and let it be required to find the way to pass from one given configuration to another with velocities at each instant satisfying the condition

$$\Sigma \tfrac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + V = E, \text{ a constant} \quad \ldots\ldots\ldots\ldots (5),$$

so that A, or

$$\int \Sigma m (\dot{x}dx + \dot{y}dy + \dot{z}dz)$$

may be the least possible.

By the method of variations we must have $\delta A = 0$, where

$$\delta A = \int \Sigma m (\dot{x}d\delta x + \dot{y}d\delta y + \dot{z}d\delta z + \delta\dot{x}dx + \delta\dot{y}dy + \delta\dot{z}dz) \quad \ldots\ldots (6).$$

Taking in this $dx = \dot{x}d\tau$, $dy = \dot{y}d\tau$, $dz = \dot{z}d\tau$, and remarking that

$$\Sigma m (\dot{x}\delta\dot{x} + \dot{y}\delta\dot{y} + \dot{z}\delta\dot{z}) = \delta T \quad \ldots\ldots\ldots\ldots (7),$$

we have

$$\int \Sigma m (\delta\dot{x}dx + \delta\dot{y}dy + \delta\dot{z}dz) = \int_0^t \delta T d\tau \quad \ldots\ldots\ldots\ldots (8).$$

Also by integration by parts,

$$\int \Sigma m (\dot{x}d\delta x + \ldots) = \{\Sigma m (\dot{x}\delta x + \ldots)\} - [\Sigma m (\dot{x}\delta x + \ldots)] - \int \Sigma m (\ddot{x}\delta x + \ldots) d\tau,$$

where $[\ldots]$ and $\{\ldots\}$ denote the values of the quantities enclosed, at the beginning and end of the motion considered, and where, further, it must be remembered that $d\dot{x} = \ddot{x}d\tau$, etc. Hence, from above,

$$\delta A = \{\Sigma m (\dot{x}\delta x + \dot{y}\delta y + \dot{z}\delta z)\} - [\Sigma m (\dot{x}\delta x + \dot{y}\delta y + \dot{z}\delta z)]$$
$$+ \int_0^t d\tau \, [\delta T - \Sigma m (\ddot{x}\delta x + \ddot{y}\delta y + \ddot{z}\delta z)] \quad \ldots\ldots\ldots\ldots (9).$$

This, it may be observed, is a perfectly general kinematical expression, unrestricted by any terminal or kinetic conditions. Now in the present problem we suppose the initial and final positions to be invariable. Hence the terminal variations, δx, etc., must all vanish, and therefore the integrated expressions $\{\ldots\}$, $[\ldots]$ disappear. Also, in the present problem $\delta T = -\delta V$, by the equation of energy (5). Hence, to make $\delta A = 0$, since the intermediate variations, δx, etc., are quite arbitrary, subject only to the con-

Least action.

ditions of the system, we must have

$$\Sigma m\,(\ddot{x}\delta x + \ddot{y}\delta y + \ddot{z}\delta z) + \delta V = 0 \quad\text{.....}\quad (10),$$

which [(4), § 293 above] is the general variational equation of motion of a conservative system. This proves the proposition.

Principle of Least Action applied to find Lagrange's generalized equations of motion.

It is interesting and instructive as an illustration of the principle of least action, to derive directly from it, without any use of Cartesian co-ordinates, Lagrange's equations in generalized co-ordinates, of the motion of a conservative system [§ 318 (24)]. We have

$$A = \int 2T dt,$$

where T denotes the formula of § 313 (2). If now we put

$$T = \tfrac{1}{2}\frac{ds^2}{dt^2},$$

so that $\quad ds^2 = (\psi, \psi)\, d\psi^2 + 2\,(\psi, \phi)\, d\psi d\phi + \text{etc.},$

we have $\quad A = \int \frac{ds}{dt}\, ds.$

Hence

$$\delta A = \int\left(\delta\frac{ds}{dt}\,ds + \frac{ds\delta ds}{dt}\right) = \int dt\frac{ds}{dt}\,\delta\,\frac{ds}{dt} + \int\frac{\frac{1}{2}\delta\,(ds^2)}{dt}$$

$$= \int dt\delta T + \int\frac{(\psi,\psi)\,d\psi + (\psi,\phi)\,d\phi + \text{etc.}}{dt}\,\delta d\psi$$

$$+ \int\frac{(\psi,\phi)\,d\psi + (\phi,\phi)\,d\phi + \text{etc.}}{dt}\,\delta d\phi + \text{etc.} + \int dt\delta_{(\psi,\phi,\text{etc.})}\,T,$$

where $\delta_{(\psi,\phi,\text{etc.})}$ denotes variation dependent on the explicit appearance of ψ, ϕ, etc. in the coefficients of the quadratic function T. The second chief term in the formula for δA is clearly equal to $\int\frac{dT}{d\dot{\psi}}\,d\delta\psi$, and this, integrated by parts, becomes

$$\frac{dT}{d\dot{\psi}}\,\delta\psi - \int d\,\frac{dT}{d\dot{\psi}}\,\delta\psi, \quad\text{or}\quad \left[\frac{dT}{d\dot{\psi}}\,\delta\psi\right] - \int dt\,\frac{d}{dt}\frac{dT}{d\dot{\psi}}\,\delta\psi,$$

where [ ] denotes the difference of the values of the bracketed expression, at the beginning and end of the time $\int dt$. Thus we have finally

$$\delta A = \left[\frac{dT}{d\dot{\psi}}\,\delta\psi + \frac{dT}{d\dot{\phi}}\,\delta\phi + \text{etc.}\right]$$

$$+ \int dt\left\{-\left(\frac{d}{dt}\frac{dT}{d\dot{\psi}}\,\delta\psi + \frac{d}{dt}\frac{dT}{d\dot{\phi}}\,\delta\phi + \text{etc.}\right) + \delta T + \delta_{(\psi,\phi,\text{etc.})}\,T\right\}\text{....}(10)'.$$

Principle of Least Action applied to find Lagrange's generalized equations of motion.

So far we have a purely kinematical formula. Now introduce the dynamical condition [§ 293 (7)]

$$T = C - V \quad\text{.....}\quad (10)''.$$

From it we find

$$\delta T = -\left(\frac{dV}{d\psi}\,\delta\psi + \frac{dV}{d\phi}\,\delta\phi + \text{etc.}\right) \quad\text{.....}\quad (10)'''.$$

Again, we have

$$\delta_{(\psi,\phi,\text{etc.})}\,T = \frac{dT}{d\psi}\,\delta\psi + \frac{dT}{d\phi}\,\delta\phi + \text{etc.}\quad\text{.....}\quad (10)^{\text{iv}}.$$

Hence (10)′ becomes

$$\delta A = \left[\frac{dT}{d\dot{\psi}}\,\delta\psi + \frac{dT}{d\dot{\phi}}\,\delta\phi + \text{etc.}\right]$$

$$+ \int dt\left\{\left(-\frac{d}{dt}\frac{dT}{d\dot{\psi}} + \frac{dT}{d\psi} + \frac{dV}{d\psi}\right)\delta\psi + (\text{etc.})\,\delta\phi + \text{etc.}\right\}\text{...}(10)^{\text{v}}.$$

To make this a minimum we have

$$-\frac{d}{dt}\frac{dT}{d\dot{\psi}} + \frac{dT}{d\psi} + \frac{dV}{d\psi} = 0, \text{ etc.} \quad\text{.....}\quad (10)^{\text{vi}},$$

which are the required equations [§ 318 (24)].

From the proposition that $\delta A = 0$ implies the equations of motion, it follows that

Why called "stationary action" by Hamilton.

328. In any unguided motion whatever, of a conservative system, the Action from any one stated position to any other, though not necessarily a minimum, fulfils the *stationary condition*, that is to say, the condition that the variation vanishes, which secures either a minimum or maximum, or maximum-minimum.

Stationary action.

This can scarcely be made intelligible without mathematical language. Let (x_1, y_1, z_1), (x_2, y_2, z_2), etc., be the co-ordinates of particles, m_1, m_2, etc., composing the system; at any time τ of the actual motion. Let V be the potential energy of the system, in this configuration; and let E denote the given value of the sum of the potential and kinetic energies. The equation of energy is—

$$\tfrac{1}{2}\{m_1(\dot{x}_1^2 + \dot{y}_1^2 + \dot{z}_1^2) + m_2(\dot{x}_2^2 + \dot{y}_2^2 + \dot{z}_2^2) + \text{etc.}\} + V = E\text{...}(5)\text{ bis}.$$

Choosing any part of the motion, for instance that from time 0 to time t, we have, for the action during it,

$$A = \int_0^t (E - V)\,d\tau = Et - \int_0^t V d\tau \quad\text{.....}\quad (11).$$

Stationary action.

Let now the system be guided to move in any other way possible for it, with any other velocities, from the same initial to the same final configuration as in the given motion, subject only to the condition, that the sum of the kinetic and potential energies shall still be E. Let (x_1', y_1', z_1'), etc., be the co-ordinates, and V' the corresponding potential energy; and let $(\dot{x}_1', \dot{y}_1', \dot{z}_1')$, etc., be the component velocities, at time τ in this arbitrary motion; equation (2) still holding, for the accented letters, with only E unchanged. For the action we shall have

$$A' = Et' - \int_0^{t'} V' d\tau \ldots\ldots\ldots\ldots\ldots\ldots (12),$$

where t' is the time occupied by this supposed motion. Let now θ denote a small numerical quantity, and let ξ_1, η_1, etc., be finite lines such that

$$\frac{x_1' - x_1}{\xi_1} = \frac{y_1' - y_1}{\eta_1} = \frac{z_1' - z_1}{\zeta} = \frac{x_2' - x_2}{\xi_2} = \text{etc.} = \theta.$$

The "principle of stationary action" is, that $\dfrac{V' - V}{\theta}$ vanishes when θ is made infinitely small, for every possible deviation ($\xi_1\theta$, $\eta_1\theta$, etc.) from the natural way and velocities, subject only to the equation of energy and to the condition of passing through the stated initial and final configurations: and conversely, that if $\dfrac{V' - V}{\theta}$ vanishes with θ for every possible such deviation from a certain way and velocities, specified by (x_1, y_1, z_1), etc., as the co-ordinates at t, *this* way and *these* velocities are such that the system unguided will move accordingly if only started with proper velocities from the initial configuration.

Varying action.

329. From this principle of stationary action, founded, as we have seen, on a comparison between a natural motion, and any other motion, arbitrarily guided and subject only to the law of energy, the initial and final configurations of the system being the same in each case, Hamilton passes to the consideration of the variation of the action in a natural or unguided motion of the system produced by varying the initial and final configurations, and the sum of the potential and kinetic energies. The result is, that

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330. The rate of *decrease* of the action per unit of increase of any one of the free (generalized) co-ordinates (§ 204) specifying the initial configuration, is equal to the corresponding (generalized) component momentum [§ 313, (*c*)] of the actual motion from that configuration: the rate of *increase* of the action per unit increase of any one of the free co-ordinates specifying the final configuration, is equal to the corresponding component momentum of the actual motion towards this second configuration: and the rate of increase of the action per unit increase of the constant sum of the potential and kinetic energies, is equal to the time occupied by the motion of which the action is reckoned.

Action expressed as a function of initial and final co-ordinates and the energy;

To prove this we must, in our previous expression (9) for δA, now suppose the terminal co-ordinates to vary; δT to become $\delta E - \delta V$, in which δE is a constant during the motion; and each set of paths and velocities to belong to an unguided motion of the system, which requires (10) to hold. Hence

$$\delta A = \{\Sigma m (\dot{x}\delta x + \dot{y}\delta y + \dot{z}\delta z)\} - [\Sigma m (\dot{x}\delta x + \dot{y}\delta y + \dot{z}\delta z)] + t\delta E \ldots (13).$$

If, now, in the first place, we suppose the particles constituting the system to be all free from constraint, and therefore (x, y, z) for each to be three independent variables, and if, for distinctness, we denote by (x_1', y_1', z_1') and (x_1, y_1, z_1) the co-ordinates of m_1 in its initial and final positions, and by $(\dot{x}_1', \dot{y}_1', \dot{z}_1')$, $(\dot{x}_1, \dot{y}_1, \dot{z}_1)$ the components of the velocity it has at those points, we have, from the preceding, according to the ordinary notation of partial differential coefficients,

its differential coefficients equal respectively to initial and final momentums, and to the time from beginning to end.

$$\left.\begin{array}{lll} \dfrac{dA}{dx_1'} = -m_1\dot{x}_1', & \dfrac{dA}{dy_1'} = -m_1\dot{y}_1', & \dfrac{dA}{dz_1'} = -m_1\dot{z}_1', \text{ etc.} \\ \dfrac{dA}{dx_1} = m_1\dot{x}_1, & \dfrac{dA}{dy_1} = m_1\dot{y}_1, & \dfrac{dA}{dz_1} = m_1\dot{z}_1, \text{ etc.} \\ \text{and} & \dfrac{dA}{dE} = t. & \end{array}\right\} \ldots (14).$$

In these equations we must suppose A to be expressed as a function of the initial and final co-ordinates, in all six times as many independent variables as there are of particles; and E, one more variable, the sum of the potential and kinetic energies.

If the system consist not of free particles, but of particles connected in any way forming either one rigid body or any number

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of rigid bodies connected with one another or not, we might, it is true, be contented to regard it still as a system of free particles, by taking into account among the impressed forces, the forces necessary to compel the satisfaction of the conditions of connexion. But although this method of dealing with a system of connected particles is very simple, so far as the law of energy merely is concerned, Lagrange's methods, whether that of "equations of condition," or, what for our present purposes is much more convenient, his "generalized co-ordinates," relieve us from very troublesome interpretations when we have to consider the displacements of particles due to arbitrary variations in the configuration of a system.

Let us suppose then, for any particular configuration (x_1, y_1, z_1) $(x_2, y_2, z_2)\ldots$, the expression

$$m_1(\dot{x}_1\delta x_1 + \dot{y}_1\delta y_1 + \dot{z}_1\delta z_1) + \text{etc., to become } \xi\delta\psi + \eta\delta\phi + \zeta\delta\theta + \text{etc.} \quad (15),$$

Same propositions for generalized co-ordinates.

when transformed into terms of ψ, ϕ, $\theta\ldots$, generalized co-ordinates, as many in number as there are of degrees of freedom for the system to move [§ 313, (c)].

The same transformation applied to the kinetic energy of the system would obviously give

$$\tfrac{1}{2}m_1(\dot{x}_1^2 + \dot{y}_1^2 + \dot{z}_1^2) + \text{etc.} = \tfrac{1}{2}(\xi\dot{\psi} + \eta\dot{\phi} + \zeta\dot{\theta} + \text{etc.}) \ldots\ldots (16),$$

and hence ξ, η, ζ, etc., are those linear functions of the generalized velocities which, in § 313 (e), we have designated as "generalized components of momentum;" and which, when T, the kinetic energy, is expressed as a quadratic function of the velocities (of course with, in general, functions of the co-ordinates ψ, ϕ, θ, etc., for the coefficients) are derivable from it thus:

$$\xi = \frac{dT}{d\dot{\psi}}, \quad \eta = \frac{dT}{d\dot{\phi}}, \quad \zeta = \frac{dT}{d\dot{\theta}}, \text{ etc.} \ldots\ldots\ldots\ldots\ldots (17).$$

Hence, taking as before non-accented letters for the second, and accented letters for the initial, configurations of the system respectively, we have

$$\left.\begin{array}{lll} \dfrac{dA}{d\psi'} = -\xi', & \dfrac{dA}{d\phi'} = -\eta', & \dfrac{dA}{d\theta'} = -\zeta', \text{ etc.} \\ \dfrac{dA}{d\psi} = \xi, & \dfrac{dA}{d\phi} = \eta, & \dfrac{dA}{d\theta} = \zeta, \text{ etc.} \\ \text{and, as before,} & \dfrac{dA}{dE} = t, & \end{array}\right\} \ldots\ldots\ldots\ldots (18).$$

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These equations (18), including of course (14) as a particular case, express in mathematical terms the proposition stated in words above, as the *Principle of Varying Action.*

The values of the momentums, thus, (14) and (18), expressed in terms of differential coefficients of A, must of course satisfy the equation of energy. Hence, for the case of free particles,

Hamilton's "characteristic equation" of motion in Cartesian co-ordinates.

$$\Sigma\frac{1}{m}\left(\frac{dA^2}{dx^2} + \frac{dA^2}{dy^2} + \frac{dA^2}{dz^2}\right) = 2(E - V) \ldots\ldots\ldots\ldots (19),$$

$$\Sigma\frac{1}{m}\left(\frac{dA^2}{dx'^2} + \frac{dA^2}{dy'^2} + \frac{dA^2}{dz'^2}\right) = 2(E - V') \ldots\ldots\ldots\ldots (20).$$

Or, in general, for a system of particles or rigid bodies connected in any way, we have, (16) and (18),

Hamilton's characteristic equation of motion in generalized co-ordinates.

$$\dot{\psi}\frac{dA}{d\psi} + \dot{\phi}\frac{dA}{d\phi} + \dot{\theta}\frac{dA}{d\theta} + \text{etc.} = 2(E - V) \ldots\ldots\ldots (21),$$

$$-\left(\dot{\psi}'\frac{dA}{d\psi'} + \dot{\phi}'\frac{dA}{d\phi'} + \dot{\theta}'\frac{dA}{d\theta'} + \text{etc.}\right) = 2(E - V') \ldots\ldots (22),$$

where $\dot{\psi}$, $\dot{\phi}$, etc., are expressible as linear functions of $\dfrac{dA}{d\psi}$, $\dfrac{dA}{d\phi}$, etc., by the solution of the equations

$$\left.\begin{array}{l} (\psi, \psi)\dot{\psi} + (\psi, \phi)\dot{\phi} + (\psi, \theta)\dot{\theta} + \text{etc.} = \xi = \dfrac{dA}{d\psi} \\ (\phi, \psi)\dot{\psi} + (\phi, \phi)\dot{\phi} + (\phi, \theta)\dot{\theta} + \text{etc.} = \eta = \dfrac{dA}{d\phi} \\ \qquad\text{etc.} \qquad\qquad\qquad\qquad \text{etc.} \end{array}\right\} \ldots\ldots (23),$$

and $\dot{\psi}'$, $\dot{\phi}'$, etc., as similar functions of $-\dfrac{dA}{d\psi'}$, $-\dfrac{dA}{d\phi'}$, etc., by

$$\left.\begin{array}{l} (\psi', \psi')\dot{\psi}' + (\psi', \phi')\dot{\phi}' + (\psi', \theta')\dot{\theta}' + \text{etc.} = \xi' = -\dfrac{dA}{d\psi'} \\ (\phi', \psi')\dot{\psi}' + (\phi', \phi')\dot{\phi}' + (\phi', \theta')\dot{\theta}' + \text{etc.} = \eta' = -\dfrac{dA}{d\phi'} \\ \qquad\text{etc.} \qquad\qquad\qquad\qquad \text{etc.} \end{array}\right\} \ldots (24),$$

where it must be remembered that (ψ, ψ), (ψ, ϕ), etc., are functions of the specifying elements, ψ, ϕ, θ, etc., depending on the kinematical nature of the co-ordinate system alone, and quite independent of the dynamical problem with which we are now concerned; being the coefficients of the half squares and the products of the generalized velocities in the expression for the

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kinetic energy of any motion of the system; and that (ψ', ψ'), (ψ', ϕ'), etc., are the same functions with ψ', ϕ', etc., written for ψ, ϕ, θ, etc.; but, on the other hand, that A is a function of all the elements ψ, ϕ, etc., ψ', ϕ', etc. Thus the first member of (21) is a quadratic function of $\frac{dA}{d\psi}$, $\frac{dA}{d\phi}$, etc., with coefficients, known functions of ψ, ϕ, etc., depending merely on the kinematical relations of the system, and the masses of its parts, but not at all on the actual forces or motions; while the second member is a function of the co-ordinates ψ, ϕ, etc., depending on the forces in the dynamical problem, and a constant expressing the particular value given to the sum of the potential and kinetic energies in the actual motion; and so for (22), and ψ', ϕ', etc.

Proof that the characteristic equation defines the motion, for free particles.

It is remarkable that the single linear partial differential equation (19) of the first order and second degree, for the case of free particles, or its equivalent (21), is sufficient to determine a function A, such that the equations (14) or (18) express the momentums in an actual motion of the system, subject to the given forces. For, taking the case of free particles first, and differentiating (19) still on the Hamiltonian understanding that A is expressed merely as a function of initial and final co-ordinates, and of E, the sum of the potential and kinetic energies, we have

$$2\Sigma \frac{1}{m}\left(\frac{dA}{dx}\frac{d^2A}{dx_1dx} + \frac{dA}{dy}\frac{d^2A}{dx_1dy} + \frac{dA}{dz}\frac{d^2A}{dx_1dz}\right) = -2\frac{dV}{dx_1}.$$

But, by (14),

$$\frac{1}{m_1}\frac{dA}{dx_1} = \dot{x}_1, \quad \frac{1}{m_1}\frac{dA}{dy_1} = \dot{y}_1, \text{ etc.},$$

and therefore

$$\frac{d^2A}{dx_1^2} = m_1\frac{d\dot{x}_1}{dx_1}, \quad \frac{d^2A}{dx_1dy_1} = m_1\frac{d\dot{y}_1}{dx_1} = m_1\frac{d\dot{x}_1}{dy_1}, \quad \frac{d^2A}{dx_1dz_1} = m_1\frac{d\dot{z}_1}{dx_1} = m_1\frac{d\dot{x}_1}{dz_1},$$

$$\frac{d^2A}{dx_1dx_2} = m_2\frac{d\dot{x}_2}{dx_1} = m_1\frac{d\dot{x}_1}{dx_2}, \text{ etc.}$$

Using these properly in the preceding and taking half; and writing out for two particles to avoid confusion as to the meaning of Σ, we have

$$m_1\left(\dot{x}_1\frac{d\dot{x}_1}{dx_1} + \dot{y}_1\frac{d\dot{x}_1}{dy_1} + \dot{z}_1\frac{d\dot{x}_1}{dz_1} + \dot{x}_2\frac{d\dot{x}_1}{dx_2} + \dot{y}_2\frac{d\dot{x}_1}{dy_2} + \dot{z}_2\frac{d\dot{x}_1}{dz_2} + \text{etc.}\right) = -\frac{dV}{dx_1} \quad (25).$$

Now if we multiply the first member by dt, we have clearly the change of the value of $m_1\dot{x}_1$ due to varying, still on the Hamil-

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Proof that the characteristic equation defines the motion, for free particles.

tonian supposition, the co-ordinates of all the points, that is to say, the configuration of the system, from what it is at any moment to what it becomes at a time dt later; and it is therefore the actual change in the value of $m\dot{x}_1$, in the natural motion, from the time, t, when the configuration is $(x_1, y_1, z_1, x_2, \ldots, E)$, to the time $t+dt$. It is therefore equal to $m_1\ddot{x}_1dt$, and hence (25) becomes simply $m_1\ddot{x}_1 = -\frac{dV}{dx_1}$. Similarly we find

$$m_1\ddot{y}_1 = -\frac{dV}{dy_1}, \quad m_1\ddot{z}_1 = -\frac{dV}{dz_1}, \quad m_2\ddot{x}_2 = -\frac{dV}{dx_2}, \text{ etc.}$$

But these are [§ 293, (4)] the elementary differential equations of the motions of a conservative system composed of free mutually influencing particles.

If next we regard x_1, y_1, z_1, x_2, etc., as constant, and go through precisely the same process with reference to x_1', y_1', z_1', x_2', etc., we have exactly the same equations among the accented letters, with only the difference that $-A$ appears in place of A; and end with $m_1\ddot{x}_1' = \frac{dV'}{dx_1'}$, from which we infer that, if (20) is satisfied, the motion represented by (14) is a natural motion through the configuration $(x_1', y_1', z_1', x_2'$, etc.).

Hence if both (19) and (20) are satisfied, and if when $x_1 = x_1'$, $y_1 = y_1'$, $z_1 = z_1'$, $x_2 = x_2'$, etc., we have $\frac{dA}{dx_1} = -\frac{dA}{dx_1'}$, etc., the motion represented by (14) is a natural motion through the two configurations $(x_1', y_1', z_1', x_2'$, etc.), and $(x_1, y_1, z_1, x_2$, etc.). Although the signs in the preceding expressions have been fixed on the supposition that the motion is *from* the former, to the latter configuration, it may clearly be from either towards the other, since whichever way it is, the reverse is also a natural motion (§ 271), according to the general property of a conservative system.

Same proposition for a connected system, and generalized co-ordinates.

To prove the same thing for a conservative system of particles or rigid bodies connected in any way, we have, in the first place, from (18)

$$\frac{d\eta}{d\psi} = \frac{d\xi}{d\phi}, \quad \frac{d\zeta}{d\psi} = \frac{d\xi}{d\theta}, \text{ etc.} \ldots\ldots\ldots\ldots\ldots\ldots (26),$$

where, on the Hamiltonian principle, we suppose $\dot{\psi}$, $\dot{\phi}$, etc., and ξ, η, etc., to be expressed as functions of ψ, ϕ, etc., ψ', ϕ', etc.,

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and the sum of the potential and kinetic energies. On the same supposition, differentiating (21), we have

$$\dot{\xi}\frac{d\xi}{d\psi}+\dot{\phi}\frac{d\eta}{d\psi}+\dot{\theta}\frac{d\zeta}{d\psi}+\text{etc.}+\xi\frac{d\dot{\psi}}{d\psi}+\eta\frac{d\dot{\phi}}{d\psi}+\zeta\frac{d\dot{\theta}}{d\psi}+\text{etc.}=-2\frac{dV}{d\psi}\quad\ldots(27).$$

But, by (26), and by the considerations above, we have

$$\dot{\psi}\frac{d\xi}{d\psi}+\dot{\phi}\frac{d\eta}{d\psi}+\dot{\theta}\frac{d\zeta}{d\psi}+\text{etc.}=\dot{\psi}\frac{d\xi}{d\psi}+\dot{\phi}\frac{d\xi}{d\phi}+\dot{\theta}\frac{d\xi}{d\theta}+\text{etc.}=\dot{\xi}\quad\ldots(28),$$

where $\dot{\xi}$ denotes the rate of variation of ξ per unit of time in the actual motion.

Again, we have

$$\left.\begin{aligned}\frac{d\dot{\psi}}{d\psi}&=\frac{\partial\dot{\psi}}{d\xi}\frac{d\xi}{d\psi}+\frac{\partial\dot{\psi}}{d\eta}\frac{d\eta}{d\psi}+\text{etc.}+\frac{\partial\dot{\psi}}{d\psi}\\ \frac{d\dot{\phi}}{d\psi}&=\frac{\partial\dot{\phi}}{d\xi}\frac{d\xi}{d\psi}+\frac{\partial\dot{\phi}}{d\eta}\frac{d\eta}{d\psi}+\text{etc.}+\frac{\partial\dot{\phi}}{d\psi}\\ &\quad\text{etc.}\qquad\qquad\text{etc.}\end{aligned}\right\}\quad\ldots\ldots\ldots\ldots(29),$$

if, as in Hamilton's system of canonical equations of motion, we suppose $\dot{\psi}$, $\dot{\phi}$, etc., to be expressed as linear functions of ξ, η, etc., with coefficients involving ψ, ϕ, θ, etc., and if we take ∂ to denote the partial differentiation of these functions with reference to the system ξ, η,...ψ, ϕ,..., regarded as independent variables. Let the coefficients be denoted by $[\psi, \psi]$, etc., according to the plan followed above; so that, if the formula for the kinetic energy be

$$T=\tfrac{1}{2}\{[\psi,\psi]\,\xi^2+[\phi,\phi]\,\eta^2+\ldots+2\,[\psi,\phi]\,\xi\eta+\text{etc.}\}\ldots\ldots(30),$$

we have

$$\left.\begin{aligned}\dot{\psi}&=\frac{\partial T}{d\xi}=[\psi,\psi]\,\xi+[\psi,\phi]\,\eta+[\psi,\theta]\,\zeta+\text{etc.}\\ \dot{\phi}&=\frac{\partial T}{d\eta}=[\phi,\psi]\,\xi+[\phi,\phi]\,\eta+[\phi,\theta]\,\zeta+\text{etc.}\\ &\quad\text{etc.}\qquad\qquad\text{etc.}\end{aligned}\right\}\ldots\ldots(31),$$

where of course $[\psi, \phi]$, and $[\phi, \psi]$, mean the same.

Hence $\qquad \dfrac{\partial\dot{\psi}}{d\xi}=[\psi,\psi],\ \dfrac{\partial\dot{\phi}}{d\xi}=[\phi,\psi],\ldots;$

$$\frac{\partial\dot{\psi}}{d\psi}=\frac{d\,[\psi,\psi]}{d\psi}\xi+\frac{d\,[\psi,\phi]}{d\psi}\eta+\text{etc.};\quad\frac{\partial\dot{\phi}}{d\psi}=\frac{d\,[\phi,\psi]}{d\psi}\xi+\text{etc. etc.},$$

and therefore, by (29),

$$\begin{aligned}\xi\frac{d\dot{\psi}}{d\psi}+\eta\frac{d\dot{\phi}}{d\psi}+\zeta\frac{d\dot{\theta}}{d\psi}+\text{etc.}&=\{[\psi,\psi]\xi+[\phi,\psi]\eta+\text{etc.}\}\frac{d\xi}{d\psi}+\{[\psi,\phi]\xi+[\phi,\phi]\eta+\text{etc.}\}\frac{d\eta}{d\psi}\\ &\quad+\text{etc.}+\frac{d[\psi,\psi]}{d\psi}\xi^2+\frac{d[\phi,\phi]}{d\psi}\eta^2+\text{etc.}+2\frac{d[\psi,\phi]}{d\psi}\xi\eta+\text{etc.}\\ &=\dot{\psi}\frac{d\xi}{d\psi}+\dot{\phi}\frac{d\eta}{d\psi}+\text{etc.}+2\frac{\partial T}{d\psi};\end{aligned}$$

Hamiltonian form of Lagrange's generalized equations deduced from characteristic equation.

whence, by (28), we see that

$$\xi\frac{d\dot{\psi}}{d\psi}+\eta\frac{d\dot{\phi}}{d\psi}+\zeta\frac{d\dot{\theta}}{d\psi}+\text{etc.}=\dot{\xi}+2\frac{\partial T}{d\psi}\ldots\ldots\ldots\ldots(32).$$

This, and (28), reduce the first member of (27) to $2\dot{\xi}+2\dfrac{\partial T}{d\psi}$, and therefore, halving, we conclude

$$\dot{\xi}+\frac{\partial T}{d\psi}=-\frac{dV}{d\psi},\ \text{and similarly,}\ \dot{\eta}+\frac{\partial T}{d\phi}=-\frac{dV}{d\phi},\ \text{etc...}(33).$$

These, in all as many differential equations as there are of variables, ψ, ϕ, etc., suffice for determining them in terms of t and twice as many arbitrary constants. But every solution of the dynamical problem, as has been demonstrated above, satisfies (21) and (23); and therefore it must satisfy these (33), which we have derived from them. These (33) are therefore *the* equations of motion, of the system referred to generalized co-ordinates, as many in number as it has of degrees of freedom. They are the Hamiltonian explicit equations of motion, of which a direct demonstration was given in § 318 above. Just as above, it appears therefore, that if (21) and (22) are satisfied, (18) expresses a natural motion of the system from one to another of the two configurations $(\psi, \phi, \theta,\ldots)(\psi', \phi', \theta',\ldots)$. Hence

Hence proof concluded.

331. The determination of the motion of any conservative system from one to another of any two configurations, when the sum of its potential and kinetic energies is given, depends on the determination of a single function of the co-ordinates of those configurations by solution of two quadratic partial differential equations of the first order, with reference to those two sets of co-ordinates respectively, with the condition that the corresponding terms of the two differential equations become separately equal when the values of the two sets of co-ordinates agree. The function thus determined and employed to express the solution of the kinetic problem was called the *Characteristic Function* by Sir W. R. Hamilton, to whom the method is due. It is, as we have seen, the "action" from one of the configurations to the other; but its peculiarity in Hamilton's system is, that it is to be expressed as a function of the co-ordinates and a constant, the whole energy, as explained above. It is evi-

Characteristic function.

dently symmetrical with respect to the two configurations, changing only in sign if their co-ordinates are interchanged.

Characteristic equation of motion.

Since not only the complete solution of the problem of motion gives a solution, A, of the partial differential equation (19) or (21), but, as we have just seen [§ 330 (33), etc.], every solution of this equation corresponds to an actual problem relative to the motion, it becomes an object of mathematical analysis, which could not be satisfactorily avoided, to find what character of completeness a solution or integral of the differential equation must have in order that a complete integral of the dynamical equations may be derivable from it—a question which seems to have been first noticed by Jacobi. What is called a "complete integral" of the differential equation; that is to say, an expression,

Complete integral of characteristic equation.

$$A = A_0 + F(\psi, \phi, \theta, \ldots \alpha, \beta, \ldots) \ldots\ldots\ldots\ldots\ldots\ldots (34),$$

for A satisfying it and involving the same number i, let us suppose, of independent arbitrary constants, A_0, α, β,...as there are of the independent variables, ψ, ϕ, etc.; leads, as he found, to a complete final integral of the equations of motion, expressed as follows:—

$$\frac{dF}{d\alpha} = \mathfrak{A}, \quad \frac{dF}{d\beta} = \mathfrak{B} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (35),$$

and, as above, $$\frac{dF}{dE} = t + \epsilon \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (36),$$

where ϵ is the constant depending on the epoch, or era of reckoning, chosen, and $\mathfrak{A}$, $\mathfrak{B}$,... are $i-1$ other arbitrary constants, constituting in all, with E, α, β,..., the proper number, $2i$, of arbitrary constants. This is proved by remarking that (35) are the equations of the "course" (or *paths* in the case of a system of free particles), which is obvious. For they give

$$\left.\begin{aligned} 0 &= \frac{d}{d\psi}\frac{dF}{d\alpha}d\psi + \frac{d}{d\phi}\frac{dF}{d\alpha}d\phi + \frac{d}{d\theta}\frac{dF}{d\alpha}d\theta + \ldots \\ 0 &= \frac{d}{d\psi}\frac{dF}{d\beta}d\psi + \frac{d}{d\phi}\frac{dF}{d\beta}d\phi + \frac{d}{d\theta}\frac{dF}{d\beta}d\theta + \ldots \\ &\text{etc.} \qquad\qquad\qquad \text{etc.} \end{aligned}\right\} \ldots\ldots (37),$$

in all $i-1$ equations to determine the ratios $d\psi : d\phi : d\theta :\ldots$ From these, and (21), we find

$$\frac{d\psi}{\dot\psi} = \frac{d\phi}{\dot\phi} = \frac{d\theta}{\dot\theta} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (38)$$

Complete integral of characteristic equation.

[since (37) are the same as the equations which we obtain by differentiating (21) and (23) with reference to α, β,... successively, only that they have $d\psi$, $d\phi$, $d\theta$,... in place of $\dot\psi$, $\dot\phi$, $\dot\theta$,...].

General solution derived from complete integral.

A perfectly general solution of the partial differential equation, that is to say, an expression for A including every function of ψ, ϕ, θ,... which can satisfy (21), may of course be found, by the regular process, from the complete integral (34), by eliminating A_0, α, β,... from it by means of an arbitrary equation

$$f(A_0, \alpha, \beta, \ldots) = 0,$$

and the $(i-1)$ equations

$$\frac{1}{\dfrac{df}{dA_0}} = \frac{\dfrac{dF}{d\alpha}}{\dfrac{df}{d\alpha}} = \frac{\dfrac{dF}{d\beta}}{\dfrac{df}{d\beta}} = \ldots$$

where f denotes an arbitrary function of the i elements A_0, α, β,... now made to be variables depending on ψ, ϕ,... But the full meaning of the general solution of (21) will be better understood in connexion with the physical problem if we first go back to the Hamiltonian solution, and then from it to the general. Thus, first, let the equations (35) of the course be assumed to be satisfied for each of two sets ψ, ϕ, θ,..., and ψ', ϕ', θ',..., of the co-ordinates. They will give $2(i-1)$ equations for determining the $2(i-1)$ constants α, β,..., $\mathfrak{A}$, $\mathfrak{B}$,..., in terms of ψ, ϕ,..., ψ', ϕ',..., to fulfil these conditions. Using the values of α, β,..., so found, and assigning A_0 so that A shall vanish when $\psi = \psi'$, $\phi = \phi'$, etc., we have the Hamiltonian expression for A in terms of ψ, ϕ, ..., ψ', ϕ', ..., and E, which is therefore equivalent to a "complete integral" of the partial differential equation (21). Now let ψ', ϕ', ..., be connected by any single arbitrary equation

$$\mathbf{f}(\psi', \phi', \ldots) = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (39),$$

and by means of this equation and the following $(i-1)$ equations, let their values be determined in terms of ψ, ϕ, ..., and E:—

$$\frac{\dfrac{dA}{d\psi'}}{\dfrac{d\mathbf{f}}{d\psi'}} = \frac{\dfrac{dA}{d\phi'}}{\dfrac{d\mathbf{f}}{d\phi'}} = \frac{\dfrac{dA}{d\theta'}}{\dfrac{d\mathbf{f}}{d\theta'}} = \text{etc.} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (40).$$

Substituting the values thus found for ψ', ϕ', θ', etc., in the Hamiltonian A, we have an expression for A, which is the general

General solution derived from complete integral.

solution of (21). For we see immediately that (40) expresses that the values of A are equal for all configurations satisfying (39), that is to say, we have

$$\frac{dA}{d\psi'}d\psi' + \frac{dA}{d\phi'}d\phi' + \ldots = 0$$

when ψ', ϕ', etc., satisfy (39) and (40). Hence when, by means of these equations, ψ', ϕ', ..., are eliminated from the Hamiltonian expression for A, the complete Hamiltonian differential

$$dA = \left(\frac{dA}{d\psi}\right)d\psi + \left(\frac{dA}{d\phi}\right)d\phi + \ldots + \frac{dA}{d\psi'}d\psi' + \frac{dA}{d\phi'}d\phi' + \ldots\ldots \quad (41)$$

becomes merely

$$dA = \left(\frac{dA}{d\psi}\right)d\psi + \left(\frac{dA}{d\phi}\right)d\phi + \ldots\ldots\ldots\ldots\ldots\ldots (42),$$

where $\left(\frac{dA}{d\psi}\right)$, etc., denote the differential coefficients in the Hamiltonian expression. Hence, A being now a function of ψ, ϕ, etc., both as these appear in the Hamiltonian expression and as they are introduced by the elimination of ψ', ϕ', etc., we have

$$\frac{dA}{d\psi} = \left(\frac{dA}{d\psi}\right), \quad \frac{dA}{d\phi} = \left(\frac{dA}{d\phi}\right), \text{ etc.}\ldots\ldots\ldots\ldots\ldots (43):$$

and therefore the new expression satisfies the partial differential equation (21). That it is a completely general solution we see, because it satisfies the condition that the action is equal for all configurations fulfilling an absolutely arbitrary equation (39).

For the case of a single free particle, the interpretation of (39) is that the point (x', y', z') is on an arbitrary surface, and of (40) that each line of motion cuts this surface at right angles. Hence

Practical interpretation of the complete solution of the characteristic equation.

332. The most general possible solution of the quadratic, partial, differential equation of the first order, which Hamilton showed to be satisfied by his Characteristic Function (either terminal configuration alone varying), when interpreted for the case of a single free particle, expresses the action up to any point (x, y, z), from some point of a certain arbitrarily given surface, from which the particle has been projected, in the direction of the normal, and with the proper velocity to make the sum of the potential and actual energies have a given value. In other

words, the physical problem solved by the most general solution of that partial differential equation, is this:—

Properties of surfaces of equal action.

Let free particles, not mutually influencing one another, be projected normally from all points of a certain arbitrarily given surface, each with the proper velocity to make the sum of its potential and kinetic energies have a given value. To find, for the particle which passes through a given point (x, y, z), the "action" in its course from the surface of projection to this point. The Hamiltonian principles stated above, show that the surfaces of equal action cut the paths of the particles at right angles; and give also the following remarkable properties of the motion:—

If, from all points of an arbitrary surface, particles not mutually influencing one another be projected with the proper velocities in the directions of the normals; points which they reach with equal actions lie on a surface cutting the paths at right angles. The infinitely small thickness of the space between any two such surfaces corresponding to amounts of action differing by any infinitely small quantity, is inversely proportional to the velocity of the particle traversing it; being equal to the infinitely small difference of action divided by the whole momentum of the particle.

Let λ, μ, ν be the direction cosines of the normal to the surface of equal action through (x, y, z). We have

$$\lambda = \frac{\dfrac{dA}{dx}}{\left(\dfrac{dA^2}{dx^2} + \dfrac{dA^2}{dy^2} + \dfrac{dA^2}{dz^2}\right)^{\frac{1}{2}}}, \text{ etc. }\ldots\ldots\ldots\ldots\ldots (1).$$

But $\frac{dA}{dx} = m\dot{x}$, etc., and, if q denote the resultant velocity,

$$mq = \left(\frac{dA^2}{dx^2} + \frac{dA^2}{dy^2} + \frac{dA^2}{dz^2}\right)^{\frac{1}{2}} \ldots\ldots\ldots\ldots\ldots\ldots (2).$$

Hence $$\lambda = \frac{\dot{x}}{q}, \quad \mu = \frac{\dot{y}}{q}, \quad \nu = \frac{\dot{z}}{q},$$

which proves the first proposition. Again, if δA denote the in-

Properties of surfaces of equal action.

finitely small difference of action from (x, y, z) to any other point $(x+\delta x, y+\delta y, z+\delta z)$, we have

$$\delta A = \frac{dA}{dx}\delta x + \frac{dA}{dy}\delta y + \frac{dA}{dz}\delta z.$$

Let the second point be at an infinitely small distance, e, from the first, in the direction of the normal to the surface of equal action; that is to say, let

$$\delta x = e\lambda,\ \ \delta y = e\mu,\ \ \delta z = e\nu.$$

Hence, by (1), $$\delta A = e\left(\frac{dA^2}{dx^2} + \frac{dA^2}{dy^2} + \frac{dA^2}{dz^2}\right)^{\frac{1}{2}} \text{.....(3)};$$

whence, by (2), $$e = \frac{\delta A}{mq} \text{ .....(4)},$$

which is the second proposition.

Examples of varying action.

333. Irrespectively of methods for finding the "characteristic function" in kinetic problems, the fact that any case of motion whatever can be represented by means of a single function in the manner explained in § 331, is most remarkable, and, when geometrically interpreted, leads to highly important and interesting properties of motion, which have valuable applications in various branches of Natural Philosophy. One of the many applications of the general principle made by Hamilton* led to a general theory of optical instruments, comprehending the whole in one expression.

Some of its most direct applications; to the motions of planets, comets, etc., considered as free points, and to the celebrated problem of perturbations, known as the Problem of Three Bodies, are worked out in considerable detail by Hamilton (*Phil. Trans.*, 1834-35), and in various memoirs by Jacobi, Liouville, Bour, Donkin, Cayley, Boole, etc. The now abandoned, but still interesting, corpuscular theory of light furnishes a good and exceedingly simple illustration. In this theory light is supposed to consist of material particles not mutually influencing one another, but subject to molecular forces from the particles of bodies—not sensible at sensible distances, and therefore not causing any deviation from uniform rectilinear motion in a homogeneous medium, except within an indefinitely small dis-

* *On the Theory of Systems of Rays.* Trans. R. I. A., 1824, 1830, 1832.

Examples of varying action.

tance from its boundary. The laws of reflection and of single refraction follow correctly from this hypothesis, which therefore suffices for what is called geometrical optics.

Application to common optics,

We hope to return to this subject, with sufficient detail, in treating of Optics. At present we limit ourselves to state a theorem comprehending the known rule for measuring the magnifying power of a telescope or microscope (by comparing the diameter of the object-glass with the diameter of pencil of parallel rays emerging from the eye-piece, when a point of light is placed at a great distance in front of the object-glass), as a particular case.

or kinetics of a single particle

334. Let any number of attracting or repelling masses, or perfectly smooth elastic objects, be fixed in space. Let two stations, O and O', be chosen. Let a shot be fired with a stated velocity, V, from O, in such a direction as to pass through O'. There may clearly be more than one natural path by which this may be done; but, generally speaking, when one such path is chosen, no other, not considerably diverging from it, can be found; and any infinitely small deviation in the line of fire from O, will cause the bullet to pass infinitely near to, but not through, O'. Now let a circle, with infinitely small radius r, be described round O as centre, in a plane perpendicular to the line of fire from this point, and let—all with infinitely nearly the same velocity, but fulfilling the condition that the sum of the potential and kinetic energies is the same as that of the shot from O—bullets be fired from all points of this circle, all directed infinitely nearly parallel to the line of fire from O, but each precisely so as to pass through O'. Let a target be held at an infinitely small distance, a', beyond O', in a plane perpendicular to the line of the shot reaching it from O. The bullets fired from the circumference of the circle round O, will, after passing through O', strike this target in the circumference of an exceedingly small ellipse, each with a velocity (corresponding of course to its position, under the law of energy) differing infinitely little from V', the common velocity with which they pass through O'. Let now a circle, equal to the former, be described round O', in the plane perpendicular to the central path through O', and let bullets be fired from points in its circumference, each

Application to common optics, or kinetics of a single particle.

with the proper velocity, and in such a direction infinitely nearly parallel to the central path as to make it pass through O. These bullets, if a target is held to receive them perpendicularly at a distance $a = a'\frac{V}{V'}$, beyond O, will strike it along the circumference of an ellipse equal to the former and placed in a "corresponding" position; and the points struck by the individual bullets will correspond; according to the following law of "correspondence":—Let P and P' be points of the first and second circles, and Q and Q' the points on the first and second targets which bullets from them strike; then if P' be in a plane containing the central path through O' and the position which Q would take if its ellipse were made circular by a pure strain (§ 183); Q and Q' are similarly situated on the two ellipses.

For, let XOY, $X'OY'$, be planes perpendicular to the central path through O and through O'. Let A be the "action" from O to O', and ϕ the action from a point $P(x, y, z)$, in the neighbourhood of O, specified with reference to the former axes of co-ordinates, to a point $P'(x', y', z')$, in the neighbourhood of O', specified with reference to the latter.

The function $\phi - A$ vanishes, of course, when $x=0$, $y=0$, $z=0$, $x'=0$, $y'=0$, $z'=0$. Also, for the same values of the co-ordinates, its differential coefficients $\frac{d\phi}{dx}$, $\frac{d\phi}{dy}$, and $\frac{d\phi}{dx'}$, $\frac{d\phi}{dy'}$, must vanish, and $\frac{d\phi}{dz}$, $-\frac{d\phi}{dz'}$ must be respectively equal to V and V', since, for any values whatever of the co-ordinates, $\frac{d\phi}{dx}$ and $\frac{d\phi}{dy}$ are the component velocities parallel to the two lines OX, OY, of the particle passing through P, when it comes from P', and $-\frac{d\phi}{dx'}$ and $-\frac{d\phi}{dy'}$ are the components parallel to OX', OY', of the velocity through P' directed so as to reach P. Hence by Taylor's (or Maclaurin's) theorem we have

$$\begin{aligned}\phi - A = &- V'z' + Vz \\ &+ \tfrac{1}{2}\{(X, X)x^2 + (Y, Y)y^2 + (X', X')x'^2 + (Y', Y')y'^2 + \dots \\ &\quad + 2(Y, Z)yz + \dots + 2(Y', Z')y'z' + \dots \\ &\quad + 2(X, X')xx' + 2(Y, Y')yy' + 2(Z, Z')zz' \\ &\quad + 2(X, Y')xy' + 2(X, Z')xz' + \dots + 2(Z, Y')zy'\} + R \dots (1),\end{aligned}$$

Application to common optics, or kinetics of a single particle.

where (X, X), (X, Y), etc., denote constants, viz., the values of the differential coefficients $\frac{d^2\phi}{dx^2}$, $\frac{d^2\phi}{dxdy}$, etc., when each of the six co-ordinates x, y, z, x', y', z' vanishes; and R denotes the remainder after the terms of the second degree. According to Cauchy's principles regarding the convergence of Taylor's theorem, we have a rigorous expression for $\phi - A$ in the same form, without R, if the coefficients (X, X), etc., denote the values of the differential coefficients with some variable values intermediate between 0 and the actual values of x, y, etc., substituted for these elements. Hence, provided the values of the differential coefficients are infinitely nearly the same for any infinitely small values of the co-ordinates as for the vanishing values, R becomes infinitely smaller than the terms preceding it, when x, y, etc., are each infinitely small. Hence when each of the variables x, y, z, x', y', z' is infinitely small, we may omit R in the expression (1) for $\phi - A$. Now, as in the proposition to be proved, let us suppose z and z' each to be rigorously zero: and we have

$$\frac{d\phi}{dx} = (X, X)x + (X, Y)y + (X, X')x' + (X, Y')y';$$

$$\frac{d\phi}{dy} = (Y, Y)y + (X, Y)x + (Y, X')x' + (Y, Y')y'.$$

These expressions, if in them we make $x=0$, and $y=0$, become the component velocities parallel to OX, OY, of a particle passing through O having been projected from P'. Hence, if ξ, η, ζ denote its co-ordinates, an infinitely small time, $\frac{a}{V}$, after it passes through O, we have $\zeta = a$, and

$$\xi = \{(X, X')x' + (X, Y')y'\}\frac{a}{V},\quad \eta = \{(Y, X')x' + (Y, Y')y'\}\frac{a}{V} \dots (2).$$

Here ξ and η are the rectangular co-ordinates of the point Q' in which, in the second case, the supposed target is struck. And by hypothesis

$$x'^2 + y'^2 = r^2 \dots\dots\dots\dots\dots\dots (3).$$

If we eliminate x', y' between these three equations, we have clearly an ellipse; and the former two express the relation of the "corresponding" points. Corresponding equations with x and y for x' and y'; with ξ', η' for ξ, η; and with $-(X, X')$, $-(Y, X')$, $-(X, Y')$, $-(Y, Y')$, in place of (X, X'), (X, Y'),

Application to common optics, or kinetics of a single particle.

(Y, X'), (Y, Y'), express the first case. Hence the proposition, as is most easily seen by choosing OX and $O'X'$ so that (X, Y') and (Y, X') may each be zero.

Application to common optics.

335. The most obvious optical application of this remarkable result is, that in the use of any optical apparatus whatever, if the eye and the object be interchanged without altering the position of the instrument, the magnifying power is unaltered. This is easily understood when, as in an ordinary telescope, microscope, or opera-glass (Galilean telescope), the instrument is symmetrical about an axis, and is curiously contradictory of the common idea that a telescope "diminishes" when looked through the wrong way, which no doubt is true if the telescope is simply reversed about the middle of its length, eye and object remaining fixed. But if the telescope be removed from the eye till its eye-piece is close to the object, the part of the object seen will be seen enlarged to the same extent as when viewed with the telescope held in the usual manner. This is easily verified by looking from a distance of a few yards, in through the object-glass of an opera-glass, at the eye of another person holding it to his eye in the usual way.

The more general application may be illustrated thus:—Let the points, O, O' (the centres of the two circles described in the preceding enunciation), be the optic centres of the eyes of two persons looking at one another through any set of lenses, prisms, or transparent media arranged in any way between them. If their pupils are of equal sizes in reality, they will be seen as similar ellipses of equal apparent dimensions by the two observers. Here the imagined particles of light, projected from the circumference of the pupil of either eye, are substituted for the projectiles from the circumference of either circle, and the retina of the other eye takes the place of the target receiving them, in the general kinetic statement.

Application to system of free mutually influencing particles.

336. If instead of one free particle we have a conservative system of any number of mutually influencing free particles, the same statement may be applied with reference to the initial position of one of the particles and the final position of another, or with reference to the initial positions or to the final positions

Application to system of free mutually influencing particles, and to generalized system.

of two of the particles. It serves to show how the influence of an infinitely small change in one of those positions, on the direction of the other particle passing through the other position, is related to the influence on the direction of the former particle passing through the former position produced by an infinitely small change in the latter position. A corresponding statement, in terms of generalized co-ordinates, may of course be adapted to a system of rigid bodies or particles connected in any way. All such statements are included in the following very general proposition:—

The rate of increase of any one component momentum, corresponding to any one of the co-ordinates, per unit of increase of any other co-ordinate, is equal to the rate of increase of the component momentum corresponding to the latter per unit increase or diminution of the former co-ordinate, according as the two co-ordinates chosen belong to one configuration of the system, or one of them belongs to the initial configuration and the other to the final.

Let ψ and χ be two out of the whole number of co-ordinates constituting the argument of the Hamiltonian characteristic function A; and ξ, η the corresponding momentums. We have [§ 330 (18)]

$$\frac{dA}{d\psi} = \pm\xi, \quad \frac{dA}{d\chi} = \pm\eta,$$

the upper or lower sign being used according as it is a final or an initial co-ordinate that is concerned. Hence

$$\frac{d^2A}{d\psi d\chi} = \pm\frac{d\xi}{d\chi} = \pm\frac{d\eta}{d\psi},$$

and therefore

$$\frac{d\xi}{d\chi} = \frac{d\eta}{d\psi},$$

if both co-ordinates belong to one configuration, or

$$\frac{d\xi}{d\chi} = -\frac{d\eta}{d\psi},$$

if one belongs to the initial configuration, and the other to the final, which is the second proposition. The geometrical interpretation of this statement for the case of a free particle, and two co-ordinates both belonging to one position, its final position, for

Application to system of free mutually influencing particles, and to generalized system.

instance, gives merely the proposition of § 332 above, for the case of particles projected from one point, with equal velocities in all directions; or, in other words, the case of the arbitrary surface of that enunciation, being reduced to a point. To complete the set of variational equations derived from § 330 we have $\frac{dt}{d\chi} = \pm \frac{d\eta}{dE}$ which expresses another remarkable property of conservative motion.

Slightly disturbed equilibrium.

337. By the help of Lagrange's form of the equations of motion, § 318, we may now, as a preliminary to the consideration of stability of motion, investigate the motion of a system infinitely little disturbed from a position of equilibrium, and left free to move, the velocities of its parts being initially infinitely small. The resulting equations give the values of the independent co-ordinates at any future time, provided the displacements *continue* infinitely small; and the mathematical expressions for their values must of course show the nature of the equilibrium, giving at the same time an interesting example of the *coexistence of small motions*, § 89. The method consists simply in finding what the equations of motion, and their integrals, become for co-ordinates which differ infinitely little from values corresponding to a configuration of equilibrium—and for an infinitely small initial kinetic energy. The solution of these differential equations is always easy, as they are linear and have constant coefficients. If the solution indicates that these differences *remain infinitely small*, the position is one of stable equilibrium; if it shows that one or more of them may *increase indefinitely*, the result of an infinitely small displacement from or infinitely small velocity through the position of equilibrium may be a finite departure from it—and thus the equilibrium is unstable.

Since there is a position of equilibrium, the kinematic relations must be invariable. As before,

$$T = \tfrac{1}{2}\{(\psi, \psi)\,\dot{\psi}^2 + (\phi, \phi)\,\dot{\phi}^2 + 2\,(\psi, \phi)\,\dot{\psi}\dot{\phi} + \text{etc.}\ldots\}\ldots(1),$$

which cannot be negative for any values of the co-ordinates. Now, though the values of the coefficients in this expression are not generally constant, they are to be taken as constant in the approximate investigation, since their variations, depending on

Slightly disturbed equilibrium.

the infinitely small variations of ψ, ϕ, etc., can only give rise to terms of the third or higher orders of small quantities. Hence Lagrange's equations become simply

$$\frac{d}{dt}\left(\frac{dT}{d\dot{\psi}}\right) = \Psi,\ \frac{d}{dt}\left(\frac{dT}{d\dot{\phi}}\right) = \Phi,\ \text{etc.}\ldots\ldots\ldots\ldots\ldots\ldots(2),$$

and the first member of each of these equations is a linear function of $\ddot{\psi}$, $\ddot{\phi}$, etc., with constant coefficients.

Now, since we may take what origin we please for the generalized co-ordinates, it will be convenient to assume that ψ, ϕ, θ, etc., are measured from the position of equilibrium considered; and that their values are therefore always infinitely small.

Hence, infinitely small quantities of higher orders being neglected, and the forces being supposed to be independent of the velocities, we shall have linear expressions for Ψ, Φ, etc., in terms of ψ, ϕ, etc., which we may write as follows:—

$$\left.\begin{array}{l} \Psi = a\psi + b\phi + c\theta + \ldots \\ \Phi = a'\psi + b'\phi + c'\theta + \ldots \\ \quad\text{etc.} \qquad\quad \text{etc.} \end{array}\right\}\ldots\ldots\ldots\ldots\ldots\ldots(3).$$

Equations (2) consequently become linear differential equations of the second order, with constant coefficients; as many in number as there are variables ψ, ϕ, etc., to be determined.

The regular processes explained in elementary treatises on differential equations, lead of course, independently of any particular relation between the coefficients, to a general form of solution (§ 343 below). But this form has very remarkable characteristics in the case of a conservative system; which we therefore examine particularly in the first place. In this case we have

$$\Psi = -\frac{dV}{d\psi},\quad \Phi = -\frac{dV}{d\phi},\ \text{etc.}$$

where V is, in our approximation, a homogeneous quadratic function of ψ, ϕ, ... if we take the origin, or configuration of equilibrium, as the configuration from which (§ 273) the potential energy is reckoned. Now, it is obvious*, from the theory

Simultaneous transformation of two quadratic functions to sums of squares.

* For in the first place any such assumption as

$$\begin{array}{c} \psi = A\psi_{,} + B\phi_{,} + \ldots \\ \phi = A'\psi_{,} + B'\phi_{,} + \ldots \\ \text{etc., etc.} \end{array}$$

gives equations for $\dot{\psi}$, $\dot{\phi}$, etc., in terms of $\dot{\psi}_{,}$, $\dot{\phi}_{,}$, etc., with the same coefficients, A, B, etc., if these are independent of t. Hence (the co-ordinates being i in

Slightly disturbed equilibrium.

of the transformation of quadratic functions, that we may, by a determinate linear transformation of the co-ordinates, reduce the

Simultaneous transformation of two quadratic functions to sums of squares.

number) we have i^2 quantities $A, A', A'', \ldots B, B', B'', \ldots$ etc., to be determined by i^2 equations expressing that in $2T$ the coefficients of $\dot{\psi}_{,}^2, \dot{\phi}_{,}^2$, etc. are each equal to unity, and of $\dot{\psi}_{,}\dot{\phi}_{,}$ etc. each vanish, and that in V the coefficients of $\psi_{,}\phi_{,}$, etc. each vanish. But, particularly in respect to our dynamical problem, the following process in two steps is instructive:—

(1) Let the quadratic expression for T in terms of $\dot{\psi}^2, \dot{\phi}^2, \dot{\psi}\dot{\phi}$, etc., be reduced to the form $\dot{\psi}_{,}^2+\dot{\phi}_{,}^2+\ldots$ by proper assignment of values to A, B, etc. This may be done arbitrarily, in an infinite number of ways, without the solution of any algebraic equation of degree higher than the first; as we may easily see by working out a synthetical process algebraically according to the analogy of finding first the conjugate diametral plane to any chosen diameter of an ellipsoid, and then the diameter of its elliptic section, conjugate to any chosen diameter of this ellipse. Thus, of the $\frac{i(i-1)}{2}$ equations expressing that the coefficients of the products $\dot{\psi}_{,}\dot{\phi}_{,}$, $\dot{\psi}_{,}\dot{\theta}_{,}$, $\dot{\phi}_{,}\dot{\theta}_{,}$, etc. vanish in T, take first the one expressing that the coefficient of $\dot{\psi}_{,}\dot{\phi}_{,}$ vanishes, and by it find the value of one of the B's, supposing all the A's and all the B's but one to be known. Then take the two equations expressing that the coefficients of $\dot{\psi}_{,}\dot{\theta}_{,}$ and $\dot{\phi}_{,}\dot{\theta}_{,}$ vanish, and by them find two of the C's supposing all the C's but two to be known, as are now all the A's and all the B's: and so on. Thus, in terms of all the A's, all the B's but one, all the C's but two, all the D's but three, and so on, supposed known, we find by the solution of linear equations the remaining B's, C's, D's, etc. Lastly, using the values thus found for the unassumed quantities, B, C, D, etc., and equating to unity the coefficients of $\psi_{,}^2, \phi_{,}^2, \theta_{,}^2$, etc. in the transformed expression for $2T$, we have i equations among the squares and products of the $\frac{(i+1)i}{2}$ assumed quantities, (i) A's, $(i-1)$ B's, $(i-2)$ C's, etc., by which any one of the A's, any one of the B's, any one of the C's, and so on, are given immediately in terms of the $\frac{i(i-1)}{2}$ ratios of the others to them. Thus the thing is done, and $\frac{i(i-1)}{2}$ disposable ratios are left undetermined.

(2) These quantities may be determined by the $\frac{i(i-1)}{2}$ equations expressing that also in the transformed quadratic V the coefficients of $\psi_{,}\phi_{,}$, $\psi_{,}\theta_{,}$, $\phi_{,}\theta_{,}$, etc. vanish.

Or, having made the first transformation as in (1) above, with assumed values for $\frac{i(i-1)}{2}$ disposable ratios, make a second transformation determinately thus:

Generalized orthogonal transformation of co-ordinates.

—Let

$$\psi_{,}=l\psi_{,,}+m\phi_{,,}+\ldots$$
$$\phi_{,}=l'\psi_{,,}+m'\phi_{,,}+\ldots$$
$$\text{etc., etc.}$$

where the i^2 quantities $l, m, \ldots, l', m', \ldots$ satisfy the $\frac{1}{2}i(i+1)$ equations

$$ll'+mm'+\ldots=0,\quad l'l''+m'm''+\ldots=0,\ \text{etc.},$$

Simplified expressions for the kinetic and potential energies.

expression for $2T$, which is essentially positive, to a sum of squares of generalized component velocities, and at the same time V to a sum of the squares of the corresponding co-ordinates, each multiplied by a constant, which may be either positive or negative, but is essentially real. [In the case of an equality or of any number of equalities among the values of these constants (α, β, etc. in the notation below), roots as they are of a determinantal equation, the linear transformation ceases to be wholly determinate; but the degree or degrees of indeterminacy which supervene is the reverse of embarrassing in respect to either the process of obtaining the solution, or the interpretation and use of it when obtained.] Hence $\psi, \phi, \ldots$ may be so chosen that

$$T=\tfrac{1}{2}(\dot{\psi}^2+\dot{\phi}^2+\text{etc.})\ \ldots\ldots\ldots\ldots\ldots\ldots(4),$$

and

$$V=\tfrac{1}{2}(\alpha\psi^2+\beta\phi^2+\text{etc.})\ldots\ldots\ldots\ldots\ldots\ldots(5),$$

α, β, etc., being real positive or negative constants. Hence Lagrange's equations become

$$\ddot{\psi}=-\alpha\psi,\quad \ddot{\phi}=-\beta\phi,\quad \text{etc.}\ldots\ldots\ldots\ldots\ldots\ldots(6).$$

The solutions of these equations are

Integrated equations of motion, expressing the fundamental modes of vibration.

$$\psi=A\cos(t\sqrt{\alpha}-e),\quad \phi=A'\cos(t\sqrt{\beta}-e'),\ \text{etc.}\ \ldots\ldots(7),$$

A, e, A', e', etc., being the arbitrary constants of integration. Hence we conclude the motion consists of a simple harmonic variation of each co-ordinate, provided that α, β, etc., are all positive. This condition is satisfied when V is a true minimum at the configuration of equilibrium; which, as we have seen (§ 292), is necessarily the case when the equilibrium is stable. If any one or more of $\alpha, \beta, \ldots$ vanishes, the equilibrium might

Simultaneous transformation of two quadratic functions to sums of squares.

and

$$l^2+m^2+\ldots=1,\quad l'^2+m'^2+\ldots=1,\ \text{etc.},$$

leaving $\frac{1}{2}i(i-1)$ disposables.

We shall still have, obviously, the same form for $2T$, that is:—

$$2T=\dot{\psi}_{,,}^2+\dot{\phi}_{,,}^2+\ldots$$

And, according to the known theory of the transformation of quadratic functions, we may determine the $\frac{1}{2}i(i-1)$ disposables of $l, m, \ldots, l', m', \ldots$ so as to make the $\frac{1}{2}i(i-1)$ products of the co-ordinates $\psi_{,,}, \phi_{,,}$, etc. disappear from the expression for V, and give

$$2V=\alpha\psi_{,,}^2+\beta\phi_{,,}^2+\ldots,$$

where α, β, γ, etc., are the roots, necessarily real, of an equation of the ith degree of which the coefficients depend on the coefficients of the squares and products in the expression for V in terms of $\psi_{,}, \phi_{,}$, etc. Later [(7′), (8) and (9) of § 343 f], a *single process* for carrying out this investigation will be worked out.

Integrated equations of motion, expressing the fundamental modes of vibration;

or of falling away from configuration of unstable equilibrium.

be either stable or unstable, or neutral; but terms of higher orders in the expansion of V in ascending powers and products of the co-ordinates would have to be examined to test it; and if it were stable, the period of an infinitely small oscillation in the value of the corresponding co-ordinate or co-ordinates would be infinitely great. If any or all of $\alpha, \beta, \gamma, \ldots$ are negative, V is not a minimum, and the equilibrium is (§ 292) essentially unstable. The form (7) for the solution, for each co-ordinate for which this is the case, becomes imaginary, and is to be changed into the exponential form, thus; for instance, let $-\alpha = p$, a positive quantity. Thus

Infinitely small disturbance from unstable equilibrium.

$$\psi = C\epsilon^{+t\sqrt{p}} + K\epsilon^{-t\sqrt{p}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (8),$$

which (unless the disturbance is so adjusted as to make the arbitrary constant C vanish) indicates an unlimited increase in the deviation. This form of solution expresses the approximate law of falling away from a configuration of unstable equilibrium. In general, of course, the approximation becomes less and less accurate as the deviation increases.

Potential and Kinetic energies expressed as functions of time.

We have, by (5), (4), (7) and (8),

$$\left.\begin{array}{ll} & V = \frac{1}{4}\alpha A^2 [1 + \cos 2 (t\sqrt{\alpha} - e)] + \text{etc.} \\ \text{or} & V = -\frac{1}{2}p\,[2CK + C^2\epsilon^{2t\sqrt{p}} + K^2\epsilon^{-2t\sqrt{p}}] - \text{etc.} \end{array}\right\} \ldots\ldots (9),$$

$$\left.\begin{array}{ll} \text{and} & T = \frac{1}{4}\alpha A^2 [1 - \cos 2 (t\sqrt{\alpha} - e)] + \text{etc.} \\ \text{or} & T = \frac{1}{2}p\,[-2CK + C^2\epsilon^{2t\sqrt{p}} + K^2\epsilon^{-2t\sqrt{p}}] + \text{etc.} \end{array}\right\} \ldots\ldots (10);$$

and, verifying the constancy of the sum of potential and kinetic energies,

$$\left.\begin{array}{ll} & T + V = \frac{1}{2}(\alpha A^2 + \beta A'^2 + \text{etc.}) \\ \text{or} & T + V = -2\,(pCK + qC'K' + \text{etc.}) \end{array}\right\} \ldots\ldots\ldots\ldots\ldots (11).$$

Example of fundamental modes.

One example for the present will suffice. Let a solid, immersed in an infinite *liquid* (§ 320), be prevented from any motion of rotation, and left only freedom to move parallel to a certain fixed plane, and let it be influenced by forces subject to the conservative law, which vanish in a particular position of equilibrium. Taking any point of reference in the body, choosing its position when the body is in equilibrium, as origin of rectangular co-ordinates OX, OY, and reckoning the potential energy from it, we shall have, as in general,

$$2T = A\dot{x}^2 + B\dot{y}^2 + 2C\dot{x}\dot{y}; \quad 2V = ax^2 + by^2 + 2cxy,$$

Example of fundamental modes.

the principles stated in § 320 above, allowing us to regard the co-ordinates x and y as fully specifying the system, provided always, that if the body is given at rest, or is brought to rest, the whole liquid is at rest (§ 320) at the same time. By solving the obviously determinate problem of finding that pair of conjugate diameters which are in the same directions for the ellipse

$$Ax^2 + By^2 + 2Cxy = \text{const.},$$

and the ellipse or hyperbola,

$$ax^2 + by^2 + 2cxy = \text{const.},$$

and choosing these as oblique axes of co-ordinates (x_1, y_1), we shall have

$$2T = A_1\dot{x}_1^2 + B_1\dot{y}_1^2, \text{ and } 2V = a_1x_1^2 + b_1y_1^2.$$

And, as A_1, B_1 are essentially positive, we may, to shorten our expressions, take $x_1\sqrt{A_1} = \psi$, $y_1\sqrt{B_1} = \phi$; so that we shall have

$$2T = \dot{\psi}^2 + \dot{\phi}^2, \quad 2V = \alpha\psi^2 + \beta\phi^2,$$

the normal expressions, according to the general forms shown above in (4) and (5).

The interpretation of the general solution is as follows:—

General theorem of fundamental modes of infinitely small motion about a configuration of equilibrium.

338. If a conservative system is infinitely little displaced from a configuration of stable equilibrium, it will ever after vibrate about this configuration, remaining infinitely near it; each particle of the system performing a motion which is composed of simple harmonic vibrations. If there are i degrees of freedom to move, and we consider any system (§ 202) of generalized co-ordinates specifying its position at any time, the deviation of any one of these co-ordinates from its value for the configuration of equilibrium will vary according to a complex harmonic function (§ 68), composed of i simple harmonics generally of incommensurable periods, and therefore (§ 67) the whole motion of the system will not in general recur periodically through the same series of configurations. There are, however, i distinct displacements, generally quite determinate, which we shall call *the normal displacements*, fulfilling the condition, that if any one of them be produced alone, and the system then left to itself for an instant at rest, this displacement will diminish and increase periodically according to a simple harmonic func-

Normal displacements from equilibrium.

Fundamental modes of vibration.

tion of the time, and consequently every particle of the system will execute a simple harmonic movement in the same period. This result, we shall see later (Vol. II.), includes cases in which there are an infinite number of degrees of freedom; as for instance a stretched cord; a mass of air in a closed vessel; waves in water, or oscillations of water in a vessel of limited extent, or of an elastic solid; and in these applications it gives the theory of the so-called "fundamental vibration," and successive "harmonics" of a cord or organ-pipe, and of all the different possible simple modes of vibration in the other cases. In all these cases it is convenient to give the name "fundamental mode" to any one of the possible simple harmonic vibrations, and not to restrict it to the gravest simple harmonic mode, as has been hitherto usual in respect to vibrating cords and organ-pipes.

Theorem of kinetic energy; of potential energy.

The whole kinetic energy of any complex motion of the system is [§ 337 (4)] equal to the sum of the kinetic energies of the fundamental constituents; and [§ 337 (5)] the potential energy of any displacement is equal to the sum of the potential energies of its normal components.

Infinitesimal motions in neighbourhood of configuration of unstable equilibrium.

Corresponding theorems of normal constituents and fundamental modes of motion, and the summation of their kinetic and potential energies in complex motions and displacements, hold for motion in the neighbourhood of a configuration of *unstable* equilibrium. In this case, some or all of the constituent motions are fallings away from the position of equilibrium (according as the potential energies of the constituent normal vibrations are negative).

Case of equality among periods.

339. If, as may be in particular cases, the periods of the vibrations for two or more of the normal displacements are equal, any displacement compounded of them will also fulfil the condition of being a normal displacement. And if the system be displaced according to any one such normal displacement, and projected with velocity corresponding to another, it will execute a movement, the resultant of two simple harmonic movements in equal periods. The graphic representation of the variation of the corresponding co-ordinates of the system, laid down as two rectangular co-ordinates in a plane diagram, will consequently (§ 65) be a circle or an ellipse; which will therefore,

Graphic representation.

Graphic representation.

of course, be the form of the orbit of any particle of the system which has a distinct direction of motion, for two of the displacements in question. But it must be remembered that some of the principal parts [as for instance the body supported on the fixed axis, in the illustration of § 319, *Example* (C)] may have only one degree of freedom; or even that each part of the system may have only one degree of freedom, as for instance if the system is composed of a set of particles each constrained to remain on a given line, or of rigid bodies on fixed axes, mutually influencing one another by elastic cords or otherwise. In such a case as the last, no particle of the system can move otherwise than in one line; and the ellipse, circle, or other graphical representation of the composition of the harmonic motions of the system, is merely an aid to comprehension, and is not the orbit of a motion actually taking place in any part of the system.

Dissipative systems.

340. In nature, as has been said above (§ 278), every system uninfluenced by matter external to it is conservative, when the ultimate molecular motions constituting heat, light, and magnetism, and the potential energy of chemical affinities, are taken into account along with the palpable motions and measurable forces. But (§ 275) practically we are obliged to admit forces of friction, and resistances of the other classes there enumerated, as causing losses of energy, to be reckoned, in abstract dynamics, without regard to the equivalents of heat or other molecular actions which they generate. Hence when such resistances are to be taken into account, forces opposed to the motions of various parts of a system must be introduced into the equations. According to the approximate knowledge which we have from experiment, these forces are independent of the velocities when due to the friction of solids: but are simply proportional to the velocities when due to fluid viscosity directly, or to electric or magnetic influences; with corrections depending on varying temperature, and on the varying configuration of the system. In consequence of the last-mentioned cause, the resistance of a *real liquid* (which is always more or less viscous) against a body moving rapidly enough through it, to leave a great deal of irregular motion, in the shape of

Views of Stokes on resistance to a solid moving through a liquid.

"eddies," in its wake, seems, when the motion of the solid has been kept long enough uniform, to be nearly in proportion to the square of the velocity; although, as Stokes has shown, at the lowest speeds the resistance is probably in simple proportion to the velocity, and for all speeds, after long enough time of one speed, may, it is probable, be approximately expressed as the sum of two terms, one simply as the velocity, and the other as the square of the velocity. If a solid is started from rest in an incompressible fluid, the initial law of resistance is no doubt simple proportionality to velocity, (however great, if suddenly enough given;) until by the gradual growth of eddies the resistance is increased gradually till it comes to fulfil Stokes' law.

Stokes' probable law.

Friction of solids.

341. The effect of friction of solids rubbing against one another is simply to render impossible the *infinitely* small vibrations with which we are now particularly concerned; and to allow any system in which it is present, to rest balanced when displaced, within certain finite limits, from a configuration of frictionless equilibrium. In mechanics it is easy to estimate its effects with sufficient accuracy when any practical case of finite oscillations is in question. But the other classes of dissipative agencies give rise to resistances simply as the velocities, without the corrections referred to, when the motions are infinitely small; and can never balance the system in a configuration deviating to any extent, however small, from a configuration of equilibrium. In the theory of infinitely small vibrations, they are to be taken into account by adding to the expressions for the generalized components of force, proper (§ 343 *a*, below) linear functions of the generalized velocities, which gives us equations still remarkably amenable to rigorous mathematical treatment.

Resistances varying as velocities.

The result of the integration for the case of a single degree of freedom is very simple; and it is of extreme importance, both for the explanation of many natural phenomena, and for use in a large variety of experimental investigations in Natural Philosophy. Partial conclusions from it are as follows:—

If the resistance per unit velocity is less than a certain critical value, in any particular case, the motion is a simple

harmonic oscillation, with amplitude decreasing in the same ratio in equal successive intervals of time. But if the resistance equals or exceeds the critical value, the system when displaced from its position of equilibrium, and left to itself, returns gradually towards its position of equilibrium, never oscillating through it to the other side, and only reaching it after an infinite time.

Resistances varying as velocities.

In the unresisted motion, let n^2 be the rate of acceleration, when the displacement is unity; so that (§ 57) we have $T = \frac{2\pi}{n}$: and let the rate of retardation due to the resistance corresponding to unit velocity be k. Then the motion is of the oscillatory or non-oscillatory class according as $k^2 < (2n)^2$ or $k^2 > (2n)^2$. In the first case, the period of the oscillation is increased by the resistance from T to $T \frac{n}{(n^2 - \frac{1}{4}k^2)^{\frac{1}{2}}}$, and the rate at which the Napierian logarithm of the amplitude diminishes per unit of time is $\frac{1}{2}k$. If a negative value be given to k, the case represented will be one in which the motion is assisted, instead of resisted, by force proportional to the velocity: but this case is purely ideal.

Effect of resistance varying as velocity in a simple motion.

The differential equation of motion for the case of one degree of motion is

$$\ddot{\psi} + k\dot{\psi} + n^2\psi = 0;$$

of which the complete integral is

$$\psi = \{A \sin n't + B \cos n't\}\epsilon^{-\frac{1}{2}kt}, \text{ where } n' = \sqrt{(n^2 - \tfrac{1}{4}k^2)},$$

or, which is the same,

$$\psi = (C\epsilon^{-n_{,}t} + C'\epsilon^{n_{,}t})\epsilon^{-\frac{1}{2}kt}, \text{ where } n_{,} = \sqrt{(\tfrac{1}{4}k^2 - n^2)},$$

A and B in one case, or C and C' in the other, being the arbitrary constants of integration. Hence the propositions above. In the case of $k^2 = (2n)^2$ the general solution is $\psi = (C + C't)\,\epsilon^{-\frac{1}{2}kt}$.

Case of equal roots.

342. The general solution [§ 343 *a* (2) and § 345'] of the problem, to find the motion of a system having any number, i, of degrees of freedom, when infinitely little disturbed from a position of stable equilibrium, and left to move subject to resistances proportional to velocities, shows that the whole motion may be resolved, in general determinately, into $2i$ different motions each

Infinitely small motion of a dissipative system.

Infinitely small motion of a dissipative system.

either simple harmonic with amplitude diminishing according to the law stated above, or non-oscillatory and consisting of equi-proportionate diminutions of the components of displacement in equal successive intervals of time.

Cycloidal system defined.

Easy and instructive lecture illustration.

343. It is now convenient to cease limiting our ideas to infinitely small motions of an absolutely general system through configurations infinitely little different from a configuration of equilibrium, and to consider any motions large or small of a system so constituted that the positional* forces are proportional to displacements and the motional* to velocities, and that the kinetic energy is a quadratic function of the velocities with constant coefficients. Such a system we shall call a cycloidal† system; and we shall call its motions cycloidal motions. A good and instructive illustration is presented in the motion of one two or more weights in a vertical line, hung one from another, and the highest from a fixed point, by spiral springs.

343 *a*. If now instead of ψ, ϕ,... we denote by ψ_1, ψ_2,... the generalized co-ordinates, and if we take 11, 12, 21, 22..., 11, 12, 21, 22,... to signify constant coefficients (not numbers as in the ordinary notation of arithmetic), the most general equations of motions of a cycloidal system may be written thus:

Positional and Motional Forces.

* Much trouble and verbiage is to be avoided by the introduction of these adjectives, which will henceforth be in frequent use. They tell their own meanings as clearly as any definition could.

† A single adjective is needed to avoid a sea of troubles here. The adjective 'cycloidal' is already classical in respect to any motion with one degree of freedom, curvilineal or rectilineal, lineal or angular (Coulomb-torsional, for example), following the same law as the cycloidal pendulum, that is to say:—*the displacement a simple harmonic function of the time.* The motion of a particle on a cycloid with vertex up may as properly be called cycloidal; and in it the displacement is an imaginary simple harmonic, or a real exponential, or the sum of two real exponentials of the time

$$\left(C\epsilon^{t\sqrt{\frac{g}{l}}}+C'\epsilon^{-t\sqrt{\frac{g}{l}}}\right).$$

In cycloidal motion as defined in the text, each component of displacement is proved to be a sum of exponentials $(C\epsilon^{\lambda t}+C'\epsilon^{\lambda' t}+\text{etc.})$ real or imaginary, reducible to a sum of products of real exponentials and real simple harmonics $[C\epsilon^{mt}\cos(nt-e)+C'\epsilon^{m't}\cos(n't-e')+\text{etc.}]$.

Differential equations of complex cycloidal motion.

$$\left.\begin{array}{l}\dfrac{d}{dt}\left(\dfrac{dT}{d\dot\psi_1}\right)+11\dot\psi_1+12\dot\psi_2+\ldots+11\psi_1+12\psi_2+\ldots=0\\ \dfrac{d}{dt}\left(\dfrac{dT}{d\dot\psi_2}\right)+21\dot\psi_1+22\dot\psi_2+\ldots+21\psi_1+22\psi_2+\ldots=0\\ \qquad\text{etc.}\qquad\qquad\text{etc.}\end{array}\right\}\ldots\ldots(1).$$

Positional forces of the non-conservative class are included by *not* assuming $12=21$, $13=31$, $23=32$, etc.

The theory of simultaneous linear differential equations with constant coefficients shows that the general solution for each co-ordinate is the sum of particular solutions, and that every particular solution is of the form

$$\psi_1=a_1\epsilon^{\lambda t},\ \psi_2=a_2\epsilon^{\lambda t}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

Their solution.

Assuming, then, this to be a solution, and substituting in the differential equations, we have

$$\left.\begin{array}{l}\lambda^2\dfrac{d\mathfrak{T}}{da_1}+\lambda(11a_1+12a_2+\ldots)+11a_1+12a_2+\ldots=0\\ \lambda^2\dfrac{d\mathfrak{T}}{da_2}+\lambda(21a_1+22a_2+\ldots)+21a_1+22a_2+\ldots=0\\ \qquad\text{etc.}\qquad\qquad\text{etc.}\end{array}\right\}\ldots\ldots(3),$$

where $\mathfrak{T}$ denotes the same homogeneous quadratic function of a_1, a_2..., that T is of $\dot\psi_1$, $\dot\psi_2$,.... These equations, i in number, determine λ by the determinantal equation

$$\begin{vmatrix}(11)\lambda^2+11\lambda+11, & (12)\lambda^2+12\lambda+12,\ldots\\ (21)\lambda^2+21\lambda+21, & (22)\lambda^2+22\lambda+22,\ldots\\ \ldots\ldots\ldots\ldots & \ldots\ldots\ldots\ldots\\ \ldots\ldots\ldots\ldots & \ldots\ldots\ldots\ldots\end{vmatrix}=0\ldots\ldots(4),$$

where (11), (22), (12), (21), etc. denote the coefficients of squares and doubled products in the quadratic, $2T$; with identities

$$(12)=(21),\ (13)=(31),\ \text{etc.}\ldots\ldots\ldots\ldots\ldots\ldots(5).$$

The equation (4) is of the degree $2i$, in λ; and if any one of its roots be used for λ in the i linear equations (3), these become harmonized and give the $i-1$ ratios a_2/a_1, a_3/a_1, etc.; and we have then, in (2), a particular solution with one arbitrary constant, a_1. Thus, from the $2i$ roots, when unequal, we have $2i$ distinct particular solutions, each with an arbitrary constant; and the addition of these solutions, as explained above, gives the general solution.

Solution of differential equations of complex cycloidal motion.

343 b. To show explicitly the determination of the ratios a_2/a_1, a_3/a_1, etc. put for brevity

$$(11)\,\lambda^2+11\lambda+11=1{\cdot}1,\quad (12)\,\lambda^2+12\lambda+12=1{\cdot}2,\ \text{etc.},$$
$$(32)\,\lambda^2+32\lambda+32=3{\cdot}2,\ \text{etc.}\ \ldots\ldots(5)';$$

and generally let $j{\cdot}k$ denote the coefficient of a_k in the jth equation of (3), or the kth term of the jth line of the determinant (to be called D for brevity) constituting the first member of (4).

Algebra of linear equations.

Let $M(j{\cdot}k)$ denote the factor of $j{\cdot}k$ in D so that $j{\cdot}k\,.\,M(j{\cdot}k)$ is the sum of all the terms of D which contain $j{\cdot}k$, and we have

$$D=\frac{1}{i}\sum_{j=1}^{j=i}\sum_{k=1}^{k=i} j{\cdot}k\,.\,M(j{\cdot}k)\ldots\ldots\ldots\ldots\ldots\ldots(5)'',$$

because in the sum $\Sigma\Sigma$ each term of D clearly occurs i times: and taking different groupings of terms, but each one only once, we have

$$\left.\begin{array}{l}
D=1{\cdot}1\,M(1{\cdot}1)+1{\cdot}2\,M(1{\cdot}2)+1{\cdot}3\,M(1{\cdot}3)+\text{etc.}\\
\quad=2{\cdot}1\,M(2{\cdot}1)+2{\cdot}2\,M(2{\cdot}2)+2{\cdot}3\,M(2{\cdot}3)+\text{etc.}\\
\quad=3{\cdot}1\,M(3{\cdot}1)+3{\cdot}2\,M(3{\cdot}2)+3{\cdot}3\,M(3{\cdot}3)+\text{etc.}\\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
\quad=1{\cdot}1\,M(1{\cdot}1)+2{\cdot}1\,M(2{\cdot}1)+3{\cdot}1\,M(3{\cdot}1)+\text{etc.}\\
\quad=1{\cdot}2\,M(1{\cdot}2)+2{\cdot}2\,M(2{\cdot}2)+3{\cdot}2\,M(3{\cdot}2)+\text{etc.}\\
\quad=1{\cdot}3\,M(1{\cdot}3)+2{\cdot}3\,M(2{\cdot}3)+3{\cdot}3\,M(3{\cdot}3)+\text{etc.}\\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{array}\right\}\ldots(5)'''$$

in all $2i$ different expressions for D.

Minor determinants.

Farther, by the elementary law of formation of determinants we see that

$$M(j-1{\cdot}k-1)=(-1)^{(i-1)(j+k)}\left|\begin{array}{ll}
j{\cdot}k, & j{\cdot}(k+1),\ j{\cdot}(k+2),\ldots j{\cdot}i,\ j{\cdot}1,\ j{\cdot}2,\ldots,\ j{\cdot}(k-2)\\
(j+1){\cdot}k, & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
(j+2){\cdot}k, & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
\ldots\ldots\ldots\ldots & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
\ldots\ldots\ldots\ldots & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
i{\cdot}k, & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
1{\cdot}k, & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
2{\cdot}k, & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
\ldots\ldots\ldots\ldots & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
\ldots\ldots\ldots\ldots & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
\ldots\ldots\ldots\ldots & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
(j-2){\cdot}k, & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,\quad (j-2){\cdot}(k-2)
\end{array}\right|$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5)^{\text{iv}}.$$

Minors of a determinant.

The quantities $M(1{\cdot}1)$, $M(1{\cdot}2)$, ... $M(j{\cdot}k)$, thus defined are what are commonly called the first minors of the determinant D, with just this variation from ordinary usage that the proper signs are given to them by the factor

$$(-1)^{(i-1)(j+k)}$$

in 5^{iv} so that in the formation of D the ordinary complication of alternate positive and negative signs when i is even and all signs positive when i is odd is avoided. In terms of the notation (5)′ the linear equations (3) become

$$\left.\begin{array}{l}
1{\cdot}1a_1+1{\cdot}2a_2+\ldots\ldots+1{\cdot}ia_i=0\\
2{\cdot}1a_1+2{\cdot}2a_2+\ldots\ldots+2{\cdot}ia_i=0\\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
i{\cdot}1a_1+i{\cdot}2a_2+\ldots\ldots+i{\cdot}ia_i=0
\end{array}\right\}\ldots\ldots\ldots\ldots(5)^{\text{v}},$$

and when $D=0$, which is required to harmonize them, they may be put under any of the following i different but equivalent forms,

$$\left.\begin{array}{ll}
 & \dfrac{a_1}{M(1{\cdot}1)}=\dfrac{a_2}{M(1{\cdot}2)}=\dfrac{a_3}{M(1{\cdot}3)}=\text{etc.}\\
\text{or} & \dfrac{a_1}{M(2{\cdot}1)}=\dfrac{a_2}{M(2{\cdot}2)}=\dfrac{a_3}{M(2{\cdot}3)}=\text{etc.}\\
\text{or} & \dfrac{a_1}{M(3{\cdot}1)}=\dfrac{a_2}{M(3{\cdot}2)}=\dfrac{a_3}{M(3{\cdot}3)}=\text{etc.}
\end{array}\right\}\ldots\ldots\ldots(5)^{\text{vi}},$$

from which we find

$$\left.\begin{array}{l}
\dfrac{a_2}{a_1}=\dfrac{M(1{\cdot}2)}{M(1{\cdot}1)}=\dfrac{M(2{\cdot}2)}{M(2{\cdot}1)}=\dfrac{M(3{\cdot}2)}{M(3{\cdot}1)}=\text{etc.}\\
\dfrac{a_3}{a_1}=\dfrac{M(1{\cdot}3)}{M(1{\cdot}1)}=\dfrac{M(2{\cdot}3)}{M(2{\cdot}1)}=\dfrac{M(3{\cdot}3)}{M(3{\cdot}1)}=\text{etc.}\\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{array}\right\}\ldots\ldots\ldots(5)^{\text{vii}}.$$

Relations among the minors of an evanescent determinant.

The remarkable relations here shown among the minors, due to the evanescence of the major determinant D, are well known in algebra. They are all included in the following formula,

$$M(j{\cdot}k)\,.\,M(l{\cdot}n)-M(j{\cdot}n)\,.\,M(l{\cdot}k)=0\ldots\ldots\ldots(5)^{\text{viii}},$$

which is given in Salmon's *Higher Algebra* (§ 33 Ex. 1), as a consequence of the formula

$$M(j{\cdot}k)\,.\,M(l{\cdot}n)-M(j{\cdot}n)\,.\,M(l{\cdot}k)=D\,.\,M(j,\,l{\cdot}k,\,n)\ldots(5)^{\text{ix}},$$

where $M(j,\,l{\cdot}k,\,n)$ denotes the second minor formed by suppressing the jth and l^{th} columns and the kth and n^{th} lines.

Relations among the minors of an evanescent determinant.

343 *c*. When there are equalities among the roots the problem has generally solutions of the form

$$\psi_1 = (c_1 t + b_1)\,\epsilon^{\lambda t}, \quad \psi_2 = (c_2 t + b_2)\,\epsilon^{\lambda t}, \quad \text{etc.} \ldots\ldots\ldots\ldots (6).$$

To prove this let λ, λ' be two unequal roots which become equal with some slight change of the values of some or all of the given constants (11), 11, 11, (12), 12, 12, etc.; and let

$$\psi_1 = A_1'\epsilon^{\lambda' t} - A_1\epsilon^{\lambda t}, \quad \psi_2 = A_2'\epsilon^{\lambda' t} - A_2\epsilon^{\lambda t}, \quad \text{etc.} \ldots\ldots\ldots\ldots (6)'$$

be a particular solution of (1) corresponding to these roots.

Now let

$$\left.\begin{array}{ll} c_1 = A_1'(\lambda'-\lambda), & c_2 = A_2'(\lambda'-\lambda), \quad \text{etc.} \\ \text{and} \quad b_1 = A_1' - A_1, & b_2 = A_2' - A_2, \quad \text{etc.} \end{array}\right\} \ldots\ldots\ldots (6)''.$$

Using these in (6)′ we find

$$\psi_1 = c_1\frac{\epsilon^{\lambda' t} - \epsilon^{\lambda t}}{\lambda' - \lambda} + b_1\epsilon^{\lambda t}, \quad \psi_2 = c_2\frac{\epsilon^{\lambda' t} - \epsilon^{\lambda t}}{\lambda' - \lambda} + b_2\epsilon^{\lambda t}, \quad \text{etc} \ldots (6)'''.$$

To find proper equations for the relations among $b_1, b_2, \ldots c_1, c_2, \ldots$ in order that (6)‴ may be a solution of (1), proceed thus:—first write down equations (3) for the λ' solution, with constants A_1', A_2', etc.: then subtract from these the corresponding equations for the λ solution: thus, and introducing the notation (6)″, we find

$$\left.\begin{array}{l} \{(11)\lambda'^2 + 11\lambda' + 11\}\,c_1 + \{(12)\lambda'^2 + 12\lambda' + 12\}\,c_2 + \text{etc.} = 0 \\ \{(21)\lambda'^2 + 21\lambda' + 21\}\,c_1 + \{(22)\lambda'^2 + 22\lambda' + 22\}\,c_2 + \text{etc.} = 0 \\ \qquad\text{etc.} \qquad\qquad\qquad\qquad \text{etc.} \end{array}\right\} \ldots (6)^{\text{iv}},$$

and

$$\left.\begin{array}{l} \{(11)\lambda'^2 + 11\lambda' + 11\}\,b_1 + \{(12)\lambda'^2 + 12\lambda' + 12\}\,b_2 + \text{etc.} \\ \quad = -[c_1 - b_1(\lambda'-\lambda)]\{(11)(\lambda+\lambda') + 11\} \\ \qquad\qquad - [c_2 - b_2(\lambda'-\lambda)]\{(12)(\lambda+\lambda') + 12\} - \text{etc.} \\ \{(21)\lambda'^2 + 21\lambda' + 21\}\,b_1 + \{(22)\lambda'^2 + 22\lambda' + 22\}\,b_2 + \text{etc.} \\ \quad = -[c_1 - b_1(\lambda'-\lambda)]\{(21)(\lambda+\lambda') + 21\} \\ \qquad\qquad - [c_2 - b_2(\lambda'-\lambda)]\{(22)(\lambda+\lambda') + 22\} + \text{etc.} \\ \qquad\text{etc.} \qquad\qquad\qquad\qquad \text{etc.} \end{array}\right\} \ldots (6)^{\text{v}}.$$

Case of equal roots.

Equations (6)^{iv} require that λ' be a root of the determinant, and $i-1$ of them determine $i-1$ of the quantities c_1, c_2, etc. in terms of one of them assumed arbitrarily. Supposing now c_1, c_2, etc. to be thus all known, the i equations (6)^{v} fail to determine the i quantities b_1, b_2, etc. in terms of the right-hand members because λ' is a root of the determinant. The two sets of equations (6)^{iv} and (6)^{v} require that λ be also a root of the determinant: and $i-1$ of the equations (6)^{v} determine $i-1$ of the

quantities b_1, b_2, etc. in terms of c_1, c_2, etc. (supposed already known as above) and a properly assumed value of one of the b's. Case of equal roots.

343 *d*. When λ' is infinitely nearly equal to λ, (6)‴ becomes infinitely nearly the same as (6), and (6)^{iv} and (6)^{v} become in terms of the notation (5)′

$$\left.\begin{array}{l} 1{\cdot}1\,c + 1{\cdot}2\,c_2 + \text{etc.} = 0 \\ 2{\cdot}1\,c_1 + 2{\cdot}2\,c_2 + \text{etc.} = 0 \\ \qquad\text{etc.} \qquad\qquad \text{etc.} \end{array}\right\} \ldots\ldots\ldots\ldots (6)^{\text{vi}},$$

$$\left.\begin{array}{l} 1{\cdot}1\,b_1 + 1{\cdot}2\,b_2 + \text{etc.} = -c_1\dfrac{d1{\cdot}1}{d\lambda} - c_2\dfrac{d1{\cdot}2}{d\lambda} - \text{etc.} \\ 2{\cdot}1\,b_1 + 2{\cdot}2\,b_2 + \text{etc.} = -c_1\dfrac{d2{\cdot}1}{d\lambda} - c_2\dfrac{d2{\cdot}2}{d\lambda} - \text{etc.} \\ \qquad\text{etc.} \qquad\qquad\qquad \text{etc.} \end{array}\right\} \ldots (6)^{\text{vii}}.$$

These, (6)^{vi}, (6)^{vii}, are clearly the equations which we find simply by trying if (6) is a solution of (1). (6)^{vi} requires that λ be a root of the determinant D; and they give by (5)^{vi} with c substituted for a the values of $i-1$ of the quantities c_1, c_2, etc. in terms of one of them assumed arbitrarily. And by the way we have found them we know that (6)^{vii} superadded to (6)^{vi} shows that λ must be a dual root of the determinant. To verify this multiply the first of them by $M(1{\cdot}1)$, the second by $M(2{\cdot}1)$, etc., and add. The coefficients of b_2, b_3, etc. in the sum are each identically zero in virtue of the elementary constitution of determinants, and the coefficient of b_1 is the major determinant D. Thus irrespectively of the value of λ we find in the first place,

$$Db_1 = -c_1\left\{M(1{\cdot}1)\frac{d1{\cdot}1}{d\lambda} + M(2{\cdot}1)\frac{d2{\cdot}1}{d\lambda} + \text{etc.}\right\}$$
$$\qquad - c_2\left\{M(1{\cdot}1)\frac{d1{\cdot}2}{d\lambda} + M(2{\cdot}1)\frac{d2{\cdot}2}{d\lambda} + \text{etc.}\right\} - \text{etc.} \ldots (6)^{\text{viii}}.$$

Now in virtue of (6)^{vi} and (5)^{v} we have

$$\frac{c_2}{c_1} = \frac{a_2}{a_1}, \quad \frac{c_3}{c_1} = \frac{a_3}{a_1}, \quad \text{etc.}$$

Using successively the several expressions given by (5)^{vii} for these ratios, in (6)^{viii}, and putting $D = 0$, we find

$$0 = \Sigma_1^i\,\Sigma_1^i\,M(j{\cdot}k)\frac{dj{\cdot}k}{d\lambda}, \quad \text{or } 0 = \frac{dD}{d\lambda},$$

which with $D = 0$ shows that λ is a double root.

Suppose now that one of the c's has been assumed, and the others found by (6)^{vi}: let one of the b's be assumed: the other

Case of equal roots.

$i-1$ b's are to be calculated by $i-1$ of the equations $(6)^{\text{vii}}$. Thus for example take $b_1=0$. In the first place use all except the first of equations $(6)^{\text{vii}}$ to determine b_2, b_3, etc.: we thus find

$$\left.\begin{aligned} M(1\cdot1)b_2 &= -\left\{M(1,2\cdot1,2)\frac{d2\cdot1}{d\lambda}+M(1,2\cdot1,3)\frac{d3\cdot1}{d\lambda}+\text{etc.}\right\}c_1 \\ &\quad -\left\{M(1,2\cdot1,2)\frac{d2\cdot2}{d\lambda}+M(1,2\cdot1,3)\frac{d3\cdot2}{d\lambda}+\text{etc.}\right\}c_2-\text{etc.} \\ M(1\cdot1)b_3 &=\text{etc.}\quad M(1\cdot1)b_4=\text{etc.}\qquad \text{etc.}\qquad \text{etc.} \end{aligned}\right\}\ldots(6)^{\text{ix}}.$$

Secondly, use all except the second of $(6)^{\text{vii}}$ to find b_2, b_3, etc.: we thus find

$$M(2\cdot1)b_2=\text{etc.},\quad M(2\cdot1)b_3=\text{etc.},\quad M(2\cdot1)b_4=\text{etc.}\ldots\ldots(6)^{\text{x}}.$$

Thirdly, by using all of $(6)^{\text{vii}}$ except the third, fourthly, all except the fourth, and so on, we find

$$M(3\cdot1)b_2=\text{etc.},\quad M(3\cdot1)b_3=\text{etc.},\quad M(3\cdot1)b_4=\text{etc.}\ldots\ldots(6)^{\text{xi}}.$$

343 *e*. In certain cases of equality among the roots (343 *m*) it is found that values of the coefficients (11), 11, 11, etc. differing infinitely little from particular values which give the equality give values of a_1 and a_1', a_2 and a_2', etc., which are not infinitely nearly equal. In such cases we see by $(6)''$ that b_1, b_2, etc. are finite, and c_1, c_2, etc. vanish: and so the solution does not contain terms of the form $t\epsilon^{\lambda t}$: but the requisite number of arbitrary constants is made up by a proper degree of indeterminateness in the residuary equations for the ratios b_2/b_1, b_3/b_1, etc.

Now when $c_1=0$, $c_2=0$, etc. the second members of equations $(6)^{\text{ix}}$, $(6)^{\text{x}}$, $(6)^{\text{xi}}$, etc. all vanish, and as b_2, b_3, b_4, etc. do not all vanish, it follows that we have

$$M(1\cdot1)=0,\quad M(2\cdot1)=0,\quad M(3\cdot1)=0,\quad \text{etc.}\ldots\ldots(6)^{\text{xii}}.$$

Case of equal roots and evanescent minors.

Hence by $(5)^{\text{vii}}$ or $(5)^{\text{viii}}$ we infer that all the first minors are zero for any value of λ which is doubly a root, and which yet does not give terms of the form $t\epsilon^{\lambda t}$ in the solution. This important proposition is due to Routh*, who, escaping the errors of previous writers (§ 343 *m* below), first gave the complete theory of equal roots of the determinant in cycloidal motion.

* *Stability of Motion* (Adams Prize Essay for 1877), chap. I. § 5.

Routh's theorem.

He also remarked that the factor t does not necessarily imply instability, as terms of the form $t\epsilon^{-pt}$, or $t\epsilon^{-pt}\cos(nt-e)$, when p is positive, do not give instability, but on the contrary correspond to non-oscillatory or oscillatory subsidence to equilibrium.

Case of no motional forces.

343 *f*. We fall back on the case of no motional forces by taking $11=0$, $12=0$, etc., which reduces the equations (3) for determining the ratios a_2/a_1, a_3/a_1, etc. to

$$\lambda^2\frac{d\mathfrak{T}}{da_1}+11a_1+12a_2+\text{etc.}=0,\quad \lambda^2\frac{d\mathfrak{T}}{da_2}+21a_1+22a_2+\text{etc.}=0,\ \text{etc.}\ (7),$$

or, expanded,

$$\left.\begin{aligned} [(11)\lambda^2+11]a_1+[(12)\lambda^2+12]a_2+\text{etc.}=0 \\ [(21)\lambda^2+21]a_1+[(22)\lambda^2+22]a_2+\text{etc.}=0 \end{aligned}\right\}\ldots(7').$$

The determinantal equation (4) to harmonize these simplified equations (7) or (7′) becomes

$$\begin{vmatrix} (11)\lambda^2+11, & (12)\lambda^2+12, & \ldots \\ (21)\lambda^2+21, & (22)\lambda^2+22, & \ldots \\ \ldots & \ldots & \ldots \end{vmatrix}=0\ldots\ldots\ldots\ldots(8).$$

This is of degree i, in λ^2: therefore λ has i pairs of oppositely signed equal values, which we may now denote by

$$\pm\lambda,\ \pm\lambda',\ \pm\lambda'',\ \ldots;$$

and for each of these pairs the series of ratio-equations (7′) are the same. Hence the complete solution of the differential equations of motion may be written as follows, to show its arbitraries explicitly:—

$$\left.\begin{aligned} \psi_1 &= (A\epsilon^{\lambda t}+B\epsilon^{-\lambda t})+(A'\epsilon^{\lambda' t}+B'\epsilon^{-\lambda' t})+(A''\epsilon^{\lambda'' t}+B''\epsilon^{-\lambda'' t})+\text{etc.} \\ \psi_2 &= \frac{a_2}{a_1}(A\epsilon^{\lambda t}+B\epsilon^{-\lambda t})+\frac{a_2'}{a_1'}(A'\epsilon^{\lambda' t}+B'\epsilon^{-\lambda' t})+\frac{a_2''}{a_1''}(A''\epsilon^{\lambda'' t}+B''\epsilon^{-\lambda'' t})+\text{etc.} \\ \psi_3 &= \frac{a_3}{a_1}(A\epsilon^{\lambda t}+B\epsilon^{-\lambda t})+\frac{a_3'}{a_1'}(A'\epsilon^{\lambda' t}+B'\epsilon^{-\lambda' t})+\frac{a_3''}{a_1''}(A''\epsilon^{\lambda'' t}+B''\epsilon^{-\lambda'' t})+\text{etc.} \\ &\quad\text{etc.}\qquad\qquad\text{etc.}\qquad\qquad\text{etc.} \end{aligned}\right\}\ldots\ldots(9),$$

where A, B; A', B'; A'', B''; etc. denote $2i$ arbitrary constants, and

$$\frac{a_2}{a_1},\ \frac{a_3}{a_1},\ \ldots,\ \frac{a_2'}{a_1'},\ \frac{a_3'}{a_1'},\ \ldots,\ \frac{a_2''}{a_1''},\ \frac{a_3''}{a_1''},\ \ldots,\ \text{etc.},$$

are i sets of $i-1$ ratios each, the values of which, when all the i roots of the determinantal equation in λ^2 have different values, are fully determined by giving successively these i values to λ^2 in (7′).

Case of no motional forces.

343 *g*. When there are equal roots, the solution is to be completed according to § 343 *d* or *e*, as the case may be. The case of a conservative system (343 *h*) necessarily falls under § 343 *e*, as is proved in § 343 *m*. The same form, (9), still represents the complete solutions when there are equalities among the roots, but with changed conditions as to arbitrariness of the elements appearing in it. Suppose $\lambda^2 = \lambda'^2$ for example. In this case any value may be chosen arbitrarily for a_2/a_1, and the remainder of the set a_3/a_1, a_4/a_1 ... are then fully determined by (7′); again another value may be chosen for a_2'/a_1', and with it a_3'/a_1', a_4'/a_1', ... are determined by a fresh application of (7′) with the same value for λ^2: and the arbitraries now are $A + A'$, $B + B'$,

$$\frac{a_2}{a_1}A + \frac{a_2'}{a_1'}A',\ \frac{a_2}{a_1}B + \frac{a_2'}{a_1'}B',\ A'',\ B'',\ A''',\ B''',\ \ldots A^{(i-1)},\ \text{and}\ B^{(i-1)},$$

numbering still $2i$ in all. Similarly we see how, beginning with the form (9), convenient for the general case of i different roots, we have in it also the complete solution when λ^2 is triply, or quadruply, or any number of times a root, and when any other root or roots also are double or multiple.

Cycloidal motion. Conservative positional, and no motional, forces.

343 *h*. For the case of a conservative system, that is to say, the case in which

$$12 = 21,\quad 13 = 31,\quad 23 = 32,\quad \text{etc., etc.} \ldots\ldots\ldots\ldots (10),$$

the differential equations of motion, (1), become

$$\frac{d}{dt}\left(\frac{dT}{d\dot\psi}\right) + \frac{dV}{d\psi} = 0,\quad \frac{d}{dt}\left(\frac{dT}{d\dot\phi}\right) + \frac{dV}{d\phi} = 0,\ \text{etc.} \ldots\ldots\ldots (10'),$$

and the solving linear algebraic equations, (3), become

$$\frac{d\mathfrak{T}}{da_1} + \frac{d\mathfrak{V}}{da_1} = 0,\quad \frac{d\mathfrak{T}}{da_2} + \frac{d\mathfrak{V}}{da_2} = 0 \ldots\ldots\ldots\ldots\ldots\ldots (10''),$$

where

$$V = \tfrac{1}{2}(11\psi_1^2 + 2.12\psi_1\psi_2 + \text{etc.}),\ \text{and}\ \mathfrak{V} = \tfrac{1}{2}(11a_1^2 + 2.12a_1a_2 + \text{etc.}) \ldots (10''').$$

In this case the i roots, λ^2, of the determinantal equation are the negatives of the values of α, β, ... of our first investigation; and thus in (10″), (8), and (9) we have the promised solution by one completely expressed process. From § 337 and its footnote we infer that in the present case the roots λ^2 are all real, whether negative or positive.

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In § 337 it was expressly assumed that T (as it must be in the dynamical problem) is essentially positive; but the investigation was equally valid for any case in which either of the quadratics T or V is incapable of changing sign for real values of the variables ($\dot\psi_1$, $\dot\psi_2$, etc. for T, or ψ_1, ψ_2, etc. for V). Thus we see that the roots λ^2 are all real when the relations (5) and (9) are satisfied, and when the magnitudes of the residual independent coefficients (11), (22), (12), ... and 11, 22, 12, ... are such that of the resulting quadratics, $\mathfrak{T}$, $\mathfrak{V}$, one or other is essentially positive or essentially negative. This property of the determinantal equation (7′) is very remarkable. A more direct algebraic proof is to be desired. Here is one:—

343 *k*. Writing out (7′) for λ^2, and for λ'^2, multiplying the first for λ^2 by $\frac{1}{2}a_1'$, the second by $\frac{1}{2}a_2'$, and so on, and adding; and again multiplying the first for λ'^2 by $\frac{1}{2}a_1$, the second by $\frac{1}{2}a_2$, and so on, and adding, we find

$$\left.\begin{aligned} \lambda^2\mathfrak{T}(a, a') + \mathfrak{V}(a, a') &= 0 \\ \lambda'^2\mathfrak{T}(a, a') + \mathfrak{V}(a, a') &= 0 \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots (11),$$

and

where

$$\left.\begin{aligned} \mathfrak{T}(a, a') &= \tfrac{1}{2}\{(11)\,a_1a_1' + (12)(a_1a_2' + a_2a_1') + \text{etc.}\} \\ \text{and}\quad \mathfrak{V}(a, a') &= \tfrac{1}{2}\{\,11\ a_1a_1' + 12\ (a_1a_2' + a_2a_1') + \text{etc.}\} \end{aligned}\right\} \ldots (12).$$

Remark that according to this (12) notation $\mathfrak{T}(a, a)$ means the same thing as $\mathfrak{T}$ simply, according to the notation of (3) etc. above, and $\mathfrak{T}(\dot\psi, \dot\psi)$ the same thing as T. Remark farther that $\mathfrak{T}(a, a')$ is a linear function of a_1, a_2, ... with coefficients each involving a_1', a_2', ... linearly; and that it is symmetrical with reference to a_1, a_1', and a_2, a_2', etc.; and that we therefore have

$$\left.\begin{aligned} &\mathfrak{T}(mp, p') = m\mathfrak{T}(p, p') = \mathfrak{T}(p, mp')\ \text{and} \\ &\mathfrak{T}(mp + nq, m'p + n'q) = mm'\,\mathfrak{T}(p, p) + (mn' + m'n)\,\mathfrak{T}(p, q) + nn'\,\mathfrak{T}(q, q) \end{aligned}\right\} \ldots (13).$$

Precisely similar statements and formulas hold for $\mathfrak{V}(a, a')$.

From (11) we infer that if λ^2 and λ'^2 be unequal we must have

$$\mathfrak{T}(a, a') = 0,\ \text{and}\ \mathfrak{V}(a, a') = 0 \ldots\ldots\ldots\ldots\ldots (14).$$

Now if there can be imaginary roots, λ^2, let $\lambda^2 = \rho + \sigma\sqrt{-1}$ and $\lambda'^2 = \rho - \sigma\sqrt{-1}$ be a pair of them, ρ and σ being real. And, p_1, q_1, p_2, q_2, etc. being all real, let $p_1 + q_1\sqrt{-1}$, $p_1 - q_1\sqrt{-1}$, be arbitrarily chosen values of a_1, a_1', and let

$$p_2 + q_2\sqrt{-1},\ p_3 + q_3\sqrt{-1},\ \ldots,\ p_2 - q_2\sqrt{-1},\ p_3 - q_3\sqrt{-1},\ \ldots$$

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be the *determinately deduced* values* of $a_2, a_3, \ldots, a_2', a_3', \ldots$ according to (7′); we have, by (13), with

$$m = m' = 1,\ n = \sqrt{-1},\ n' = -\sqrt{-1},$$

$$\left.\begin{aligned} \mathfrak{T}(a, a') &= \mathfrak{T}(p, p) + \mathfrak{T}(q, q) \\ \text{and}\quad \mathfrak{V}(a, a') &= \mathfrak{V}(p, p) + \mathfrak{V}(q, q) \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots (14').$$

Now by hypothesis either $\mathfrak{T}(x, x)$, or $\mathfrak{V}(x, x)$ is essentially of one sign for all real values of x_1, x_2, etc. Hence the second member of one or other of equations (14′) cannot be zero, because $p_1, p_2, \ldots$, and $q_1, q_2, \ldots$ are all real. But by (14) the first member of each of the equations (14′) is zero if λ^2 and λ'^2 are unequal: hence they are equal: hence either $p_1 = 0$, $p_2 = 0$, etc., or $q_1 = 0$, $q_2 = 0$, etc., that is to say the roots λ^2 are all necessarily real, whether negative or positive.

343 *l*. Farther we now see by going back to (11):—

(*a*) if for all real values of $x_1, x_2, \ldots$ the values of $\mathfrak{T}(x, x)$ and $\mathfrak{V}(x, x)$ have the same unchanging sign, the roots λ^2 are all negative;

(*b*) if for different real values of x_1, x_2, etc., one of the two $\mathfrak{T}(x, x)$, $\mathfrak{V}(x, x)$ has different signs (the other by hypothesis having always one sign), some of the roots λ^2 are negative and some positive;

(*c*) if the values of $\mathfrak{T}$ and $\mathfrak{V}$ have essentially opposite signs (and each therefore according to hypothesis unchangeable in sign), the roots λ^2 are all positive.

The (*a*) and (*c*) of this tripartite conclusion we see by taking $\lambda'^2 = \lambda^2$ in (11), which reduces them to

$$\lambda^2\mathfrak{T}(a, a) + \mathfrak{V}(a, a) = 0 \ldots\ldots\ldots\ldots\ldots\ldots (15),$$

and remarking that a_2, a_3, etc. are now all real if we please to give a real value to a_1. The (*b*) is proved in § 343 *o* below.

Case of equal roots.

343 *m*. From (14) we see that when two roots λ^2, λ'^2, are infinitely nearly equal there is no approach to equality between a_1 and a_1', a_2 and a_2', and therefore, when there are no motional forces, and when the positional forces are conservative, equality of roots essentially falls under the case of § 343 *e* above. This may be proved explicitly as follows:—let

$$\psi_1 = (a_1 t + b_1)\,\epsilon^{\lambda t},\quad \psi_2 = (a_2 t + b_2)\,\epsilon^{\lambda t}, \text{ etc.} \ldots\ldots\ldots (15)'$$

* Cases of equalities among the roots are disregarded for the moment merely to avoid circumlocutions, but they obviously form no exception to the reasoning and conclusion.

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be the complete solution corresponding to the root λ supposed to be a dual root. Using this in equations (1) and equating to zero in each equation so found the coefficients of $t\epsilon^{\lambda t}$ and of $\epsilon^{\lambda t}$, with the notation of (12) we find

$$\lambda^2\frac{d\mathfrak{T}(a, a)}{da_1} + \frac{d\mathfrak{V}(a, a)}{da_1} = 0,\quad \lambda^2\frac{d\mathfrak{T}(a, a)}{da_2} + \frac{d\mathfrak{V}(a, a)}{da_2} = 0, \text{ etc.} \ldots (15)'',$$

$$\left.\begin{aligned} &\lambda^2\frac{d\mathfrak{T}(b, b)}{db_1} + 2\lambda\frac{d\mathfrak{T}(a, a)}{da_1} + \frac{d\mathfrak{V}(b, b)}{db_1} = 0, \\ &\lambda^2\frac{d\mathfrak{T}(b, b)}{db_2} + 2\lambda\frac{d\mathfrak{T}(a, a)}{da_2} + \frac{d\mathfrak{V}(b, b)}{db_2} = 0, \quad \text{etc.} \end{aligned}\right\} \ldots (15)'''.$$

Multiplying the first, second, third, etc. of (15)″ by b_1, b_2, b_3, etc. and adding we find

$$\lambda^2\mathfrak{T}(a, b) + \mathfrak{V}(a, b) = 0 \ldots\ldots\ldots\ldots\ldots\ldots (15)^{\text{iv}};$$

and similarly from (15)‴ with multipliers a_1, a_2, etc.

$$\lambda^2\mathfrak{T}(a, b) + \mathfrak{V}(a, b) + 2\lambda\mathfrak{T}(a, a) = 0 \ldots\ldots\ldots\ldots (15)^{\text{v}}.$$

Subtracting $(15)^{\text{iv}}$ from $(15)^{\text{v}}$ we see that $\mathfrak{T}(a, a) = 0$. Hence we must have $a_1 = 0$, $a_2 = 0$, etc., that is to say there are no terms of the form $t\epsilon^{\lambda t}$ in the solution. It is to be remarked that the inference of $a_1 = 0$, $a_2 = 0$, etc. from $\mathfrak{T}(a, a) = 0$, is not limited to real roots λ because λ^2 in the present case is essentially real, and whether it be positive or negative the ratios a_2/a_1, a_3/a_1, etc., are essentially real.

It is remarkable that both Lagrange and Laplace fell into the error of supposing that equality among roots necessarily implies terms in the solution of the form $t\epsilon^{\lambda t}$ (or $t\cos\rho t$), and therefore that for stability the roots must be all unequal. This we find in the *Mécanique Analytique*, Seconde Partie, section VI. Art. 7 of the second edition of 1811 published three years before Lagrange's death, and repeated without change in the posthumous edition of 1853. It occurs in the course of a general solution of the problem of the infinitely small oscillations of a system of bodies about their positions of equilibrium, with conservative forces of position and no motional forces, which from the "Avertissement" (p. vi.) prefixed to the 1811 edition seems to have been first published in the 1811 edition, and not to have appeared in the original edition of 1788*. It would be

* Since this statement was put in type, the first edition of the *Mécanique Analytique* (which had been inquired for in vain in the University libraries of Cambridge and Glasgow) has been found in the University library of Edinburgh,

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curious if such an error had remained for twenty-three years in Lagrange's mind. It could scarcely have existed even during the writing and printing of the Article for his last edition if he had been in the habit of considering particular applications of his splendid analytical work: if he had he would have seen that a proposition which asserted that the equilibrium of a particle in the bottom of a frictionless bowl is unstable if the bowl be a figure of revolution with its axis vertical, cannot be true. No such obvious illustration presents itself to suggest or prove the error as Laplace has it in the *Mécanique Céleste* (Première Partie, Livre II. Art. 57) in the course of an investigation of the secular inequalities of the planetary system. But as [by a peculiarly simple case of the process of § 345[xiv] (54)] he has reduced his analysis of this problem virtually to the same as that of conservative oscillations about a configuration of equilibrium, the physical illustrations which abound for this case suffice to prove the error in Laplace's statement, different and comparatively recondite as its dynamical subject is. An error the converse of that of Laplace and Lagrange occurred in page 278 of our First Edition where it was said that "Cases in which "there are equal roots leave a corresponding number of degrees "of indeterminateness in the ratios $l : m$, $l : n$, etc., and so allow "the requisite number of arbitrary constants to be made up," without limiting this statement to the case of conservative positional and no motional forces, for which its truth is obvious from the nature of the problem, and for which alone it is obvious at first sight; although for the cases of adynamic oscillations, and of stable precessions, § 345[xiv], it is also essentially true. The correct theory of equal roots in the generalized problem of cycloidal motion has been so far as we know first given by Routh in his investigation referred to above (§ 343 *e*)*.

343 *n*. Returning to § 343 *l*, to make more of (*b*), and to understand the efficiency of the oppositely signed roots, λ^2, asserted in it, let $\sigma^2 = -\lambda^2$ in any case in which λ^2 is negative, and let

$$\psi_1 = r_1 \cos(\sigma t - e),\ \ \psi_2 = r_2 \cos(\sigma t - e),\ \text{etc.}\ldots\ldots(16),$$

be the corresponding particular solution in fully *realized* terms,

and it does contain the problem of infinitely small oscillations, with the remarkable error referred to in the text.

* [The error of Lagrange and Laplace was pointed out and corrected by Weierstrass in 1858. H. L.]

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as in § 337 (6) above but with somewhat different notation. By substituting in (1) and multiplying the first of the resulting equations by r_1, the second by r_2, and so on and adding, virtually as we found (15), we now find

$$-\sigma^2 \mathfrak{T}(r, r) + \mathfrak{V}(r, r) = 0 \ldots\ldots\ldots\ldots\ldots\ldots(17).$$

Adopting now the notation of (9) for the real positive ones of the roots λ^2, but taking, for brevity, $a_1 = 1$, $a_1' = 1$, $a_1'' = 1$, etc., we have for the complete solution when there are both negative and positive roots of the determinantal equation (7′);

$$\left.\begin{array}{l} \psi_1 = \ (A\epsilon^{\lambda t} + B\epsilon^{-\lambda t}) + \ (A'\epsilon^{\lambda' t} + B'\epsilon^{-\lambda' t}) + \text{etc.} + r_1 \cos(\sigma t - e) + r_1' \cos(\sigma' t - e') + \text{etc.} \\ \psi_2 = a_2(A\epsilon^{\lambda t} + B\epsilon^{-\lambda t}) + a_2'(A'\epsilon^{\lambda' t} + B'\epsilon^{-\lambda' t}) + \text{etc.} + r_2 \cos(\sigma t - e) + r_2' \cos(\sigma' t - e') + \text{etc.} \\ \psi_3 = \text{etc.},\qquad \psi_4 = \text{etc.}\qquad \text{etc.}\qquad \text{etc.} \end{array}\right\} \ldots(18).$$

343 *o*. Using this in the general expressions for T and V, with the notation (12), and remarking that the products $\epsilon^{\lambda t} \times \epsilon^{\lambda' t}$, etc. and $\epsilon^{\lambda t} \times \sin(\sigma t - e)$, etc., and $\sin(\sigma t - e) \times \sin(\sigma' t - e')$, etc., disappear from the terms in virtue of (11), we find

$$\left.\begin{array}{l} T = \lambda^2 \mathfrak{T}(a, a)(A\epsilon^{\lambda t} - B\epsilon^{-\lambda t})^2 + \lambda'^2 \mathfrak{T}(a', a')(A'\epsilon^{\lambda' t} - B'\epsilon^{-\lambda' t})^2 + \text{etc.} \\ \quad + \sigma^2 \mathfrak{T}(r, r) \sin^2(\sigma t - e) + \sigma'^2 \mathfrak{T}(r', r') \sin^2(\sigma' t - e') + \text{etc.} \end{array}\right\} (19),$$

and

$$\left.\begin{array}{l} V = \mathfrak{V}(a, a)(A\epsilon^{\lambda t} + B\epsilon^{-\lambda t})^2 + \mathfrak{V}(a', a')(A'\epsilon^{\lambda' t} + B'\epsilon^{-\lambda' t})^2 + \text{etc.} \\ \quad + \mathfrak{V}(r, r) \cos^2(\sigma t - e) + \mathfrak{V}(r', r') \cos^2(\sigma' t - e') + \text{etc.} \end{array}\right\} (20).$$

The factors which appear with

$$\mathfrak{T}(a, a),\ \mathfrak{T}(a', a'), \ldots \mathfrak{T}(r, r),\ \mathfrak{T}(r', r')$$

in this expression (19) for T are all essentially positive; and the same is true of $\mathfrak{V}$ in (20) for V. Now for every set of real co-ordinates and velocity-components the potential and kinetic energies are expressible by the formulas (20) and (19) because (18) is the complete solution with $2i$ arbitraries. Hence if the value of V can change sign with real values of the co-ordinates, the quantities $\mathfrak{V}(a, a)$, $\mathfrak{V}(a', a')$, etc., and $\mathfrak{V}(r, r)$, $\mathfrak{V}(r', r')$, etc., for the several roots must be some of them positive and some of them negative; and if the value of T could change sign with real values of the velocity-components, some of the quantities $\mathfrak{T}(a, a)$, $\mathfrak{T}(a', a')$, etc., and $\mathfrak{T}(r, r)$, $\mathfrak{T}(r', r')$, etc. would need to be positive and some negative. So much being learned from (20) and (19) we must now recal to mind that according to hypothesis one only of the two quadratics T and V can change sign, to conclude from (15) and (17) that there are both positive and negative roots λ^2 when either T or V can change sign. Thus (*b*) of the tripartite conclusion above is rigorously proved.

Cycloidal motion. Conservative positional, and no motional, forces.

343 *p*. A short algebraic proof of (*b*) could no doubt be easily given; but our somewhat elaborate discussion of the subject is important as showing in (15)...(20) the whole relation between the previous short algebraic investigation, conducted in terms involving quantities which are essentially imaginary for the case of oscillations about a configuration of stable equilibrium, and the fully realized solution, with formulas for the potential and kinetic energies realized both for oscillations and for fallings away from unstable equilibrium.

We now see definitively by (15) and (17) that, in *real* dynamics (that is to say T essentially positive) the factors $\mathfrak{V}(a, a)$, $\mathfrak{V}(a', a')$, etc., are all negative, and $\mathfrak{V}(r, r)$, $\mathfrak{V}(r', r')$, etc., all positive in the expression (20) for the potential energy. Adding (20) to (19) and using (15) and (17) in the sum, we find

Equation of energy in realized general solution.

$$\left.\begin{aligned} T + V = &-4AB\lambda^2\mathfrak{T}(a, a) - 4A'B'\lambda'^2\mathfrak{T}(a', a'), \text{ etc.} \\ &+ \sigma^2\mathfrak{T}(r, r) + \sigma'^2\mathfrak{T}(r', r') + \text{etc.} \end{aligned}\right\} \ldots(21).$$

It is interesting to see in this formula how the constancy of the sum of the potential and kinetic energies is attained in any solution of the form $A\epsilon^{\lambda t} + B\epsilon^{-\lambda t}$ [which, with $\lambda = \sigma\sqrt{-1}$, includes the form $r\cos(\sigma t - e)$], and to remark that for any single solution $a\epsilon^{\lambda t}$, or solution compounded of single solutions depending on unequal values of λ^2 (whether real or imaginary), the sum of the potential and kinetic energies is essentially zero.

Artificial or ideal accumulative system.

344. When the positional forces of a system violate the law of conservatism, we have seen (§ 272) that energy without limit may be drawn from it by guiding it perpetually through a returning cycle of configurations, and we have inferred that in every real system, not supplied with energy from without, the positional forces fulfil the conservative law. But it is easy to arrange a system artificially, in connexion with a source of energy, so that its positional forces shall be non-conservative; and the consideration of the kinetic effects of such an arrangement, especially of its oscillations about or motions round a configuration of equilibrium, is most instructive, by the contrasts which it presents to the phenomena of a natural system. The preceding formulas, (7)...(9) of § 343 *f* and § 343 *g*, express the general solution of the problem—to find the infinitely small motion of a cycloidal system, when, without motional forces, there is deviation from conservatism by the character of the positional forces.

Artificial or ideal accumulative system. Criterion of stability.

In this case [(10) not fulfilled,] just as in the case of motional forces fulfilling the conservative law (10), the character of the equilibrium as to stability or instability is discriminated according to the character of the roots of an algebraic equation of degree equal to the number of degrees of freedom of the system.

If the roots (λ^2) of the determinantal equation § 343 (8) are all real and negative, the equilibrium is stable: in every other case it is unstable.

345. But although, when the equilibrium is stable, no possible infinitely small displacement and velocity given to the system can cause it, when left to itself, to go on moving farther and farther away till either a finite displacement is reached, or a finite velocity acquired; it is very remarkable that stability should be possible, considering that even in the case of stability an endless increase of velocity may, as is easily seen from § 272, be obtained merely by *constraining* the system to a particular closed course, or circuit of configurations, nowhere deviating by more than an infinitely small amount from the configuration of equilibrium, and leaving it at rest anywhere in a certain part of this circuit. This result, and the distinct peculiarities of the cases of stability and instability, will be sufficiently illustrated by the simplest possible example, that of a material particle moving in a plane.

Let the mass be unity, and the components of force parallel to two rectangular axes be $ax + by$, and $a'x + b'y$, when the position of the particle is (x, y). The equations of motion will be

$$\ddot{x} = ax + by, \quad \ddot{y} = a'x + b'y \ldots\ldots\ldots\ldots\ldots\ldots(1).$$

Let $\quad \frac{1}{2}(a' + b) = c$, and $\frac{1}{2}(a' - b) = e$:

the components of the force become

$$ax + cy - ey, \text{ and } cx + b'y + ex,$$

or

$$-\frac{dV}{dx} - ey, \text{ and } -\frac{dV}{dy} + ex,$$

where

$$V = -\tfrac{1}{2}(ax^2 + b'y^2 + 2cxy).$$

The terms $-ey$ and $+ex$ are clearly the components of a force $e(x^2 + y^2)^{\frac{1}{2}}$, perpendicular to the radius-vector of the particle. Hence if we turn the axes of co-ordinates through any angle, the

Artificial or ideal accumulative system.

corresponding terms in the transformed components are still $-ey$ and $+ex$. If, therefore, we choose the axes so that

$$V = \tfrac{1}{2}(\alpha x^2 + \beta y^2) \text{.....} (2),$$

the equations of motion become, without loss of generality,

$$\ddot{x} = -\alpha x - ey, \quad \ddot{y} = -\beta y + ex.$$

To integrate these, assume, as in general [§ 343 (2)],

$$x = l\epsilon^{\lambda t}, \quad y = m\epsilon^{\lambda t}.$$

Then, as before [§ 343 (7)],

$$(\lambda^2 + \alpha)\, l + em = 0, \text{ and } -el + (\lambda^2 + \beta)\, m = 0.$$

Whence $$(\lambda^2 + \alpha)(\lambda^2 + \beta) = -e^2 \text{.....} (3),$$

which gives

$$\lambda^2 = -\tfrac{1}{2}(\alpha + \beta) \pm \{\tfrac{1}{4}(\alpha - \beta)^2 - e^2\}^{\frac{1}{2}}.$$

This shows that the equilibrium is stable if both $\alpha\beta + e^2$ and $\alpha + \beta$ are positive and $e^2 < \frac{1}{4}(\alpha - \beta)^2$ but unstable in every other case.

But let the particle be constrained to remain on a circle, of radius r. Denoting by θ its angle-vector from OX, and transforming (§ 27) the equations of motion, we have

$$\ddot{\theta} = -(\beta - \alpha)\sin\theta\cos\theta + e = -\tfrac{1}{2}(\beta - \alpha)\sin 2\theta + e \text{.....} (4).$$

If we had $e = 0$ (a conservative system of force) the positions of equilibrium would be at $\theta = 0$, $\theta = \frac{1}{2}\pi$, $\theta = \pi$, and $\theta = \frac{3}{2}\pi$; and the motion would be that of the quadrantal pendulum. But when e has any finite value less than $\frac{1}{2}(\beta - \alpha)$ which, for convenience, we may suppose positive, there are positions of equilibrium at

$$\theta = \vartheta, \quad \theta = \frac{\pi}{2} - \vartheta, \quad \theta = \pi + \vartheta, \text{ and } \theta = \frac{3\pi}{2} - \vartheta,$$

where ϑ is half the acute angle whose sine is $\dfrac{2e}{\beta - \alpha}$: the first and third being positions of stable, and the second and fourth of unstable, equilibrium. Thus it appears that the effect of the constant tangential force is to displace the positions of stable and unstable equilibrium forwards and backwards on the circle through angles each equal to ϑ. And, by multiplying (4) by $2\dot{\theta}dt$ and integrating, we have as the integral equation of energy

$$\dot{\theta}^2 = C + \tfrac{1}{2}(\beta - \alpha)\cos 2\theta + 2e\theta \text{.....} (5).$$

Artificial or ideal accumulative system.

From this we see that the value of C, to make the particle just reach the position of unstable equilibrium, is

$$C = -\tfrac{1}{2}(\beta - \alpha)\cos(\pi - 2\vartheta) - e(\pi - 2\vartheta),$$

$$= \sqrt{\frac{(\beta - \alpha)^2}{4} - e^2} - e\left(\pi - \sin^{-1}\frac{2e}{\beta - \alpha}\right),$$

and by equating to zero the expression (5) for $\dot{\theta}^2$, with this value of C substituted, we have a transcendental equation in θ, of which the least negative root, $\theta_{,}$, gives the limit of vibrations on the side reckoned backwards from a position of stable equilibrium. If the particle be placed at rest on the circle at any distance less than $\dfrac{\pi}{2} - 2\vartheta$ *before* a position of stable equilibrium, or less than $\vartheta - \theta_{,}$ *behind* it, it will vibrate. But if placed anywhere beyond those limits and left either at rest or moving with any velocity in either direction, it will end by flying round and round forwards with a periodically increasing and diminishing velocity, but increasing every half turn by equal additions to its squares.

If on the other hand $e > \frac{1}{2}(\beta - \alpha)$, the positions both of stable and unstable equilibrium are imaginary; the tangential force predominating in every position. If the particle be left at rest in any part of the circle it will fly round with continually increasing velocity, but periodically increasing and diminishing acceleration.

345[i]. Leaving now the ideal case of positional forces violating the law of conservatism, interestingly curious as it is, and instructive in respect to the contrast it presents with the positional forces of nature which are essentially conservative, let us henceforth suppose the positional forces of our system to be conservative and let us admit infringement of conservatism only as in nature through motional forces. We shall soon see (§ 345[vii] and [viii]) that we may have motional forces which do not violate the law of conservatism. At present we make no restriction upon the motional forces and no other restriction on the positional forces than that they are conservative.

Cycloidal system with conservative positional forces and unrestricted motional forces.

The differential equations of motion, taken from (1) of 343*a* above, with the relations (10), and with V to denote the potential energy, are,

Cycloidal system with conservative positional forces and unrestricted motional forces.

$$\left.\begin{array}{l} \dfrac{d}{dt}\left(\dfrac{dT}{d\dot{\psi}_1}\right) + 11\dot{\psi}_1 + 12\dot{\psi}_2 + \ldots + \dfrac{dV}{d\psi_1} = 0 \\ \dfrac{d}{dt}\left(\dfrac{dT}{d\dot{\psi}_2}\right) + 21\dot{\psi}_1 + 22\dot{\psi}_2 + \ldots + \dfrac{dV}{d\psi_2} = 0 \\ \quad\text{etc.} \qquad\qquad \text{etc.} \end{array}\right\} \ldots\ldots\ldots\ldots(1).$$

Multiplying the first of these by $\dot{\psi}_1$, the second by $\dot{\psi}_2$, adding and transposing, we find

$$\frac{d(T+V)}{dt} = -Q \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2),$$

where

$$Q = 11\dot{\psi}_1^2 + (12+21)\dot{\psi}_1\dot{\psi}_2 + 22\dot{\psi}_2^2 + (13+31)\dot{\psi}_1\dot{\psi}_3 + \text{etc.}\ldots\ldots(3).$$

Dissipativity defined.

345ⁱⁱ. The quadratic function of the velocities here denoted by Q has been called by Lord Rayleigh* the Dissipation Function. We prefer to call it *Dissipativity*. It expresses the rate at which the *palpable energy* of our supposed cycloidal system is lost, not, as we now know, annihilated but (§§ 278, 340, 341, 342) dissipated away into other forms of energy. *It is essentially positive when the assumed motional forces are such as can exist in nature.* That it is equal to a quadratic function of the velocities is an interesting and important theorem.

Lord Rayleigh's theorem of Dissipativity.

Integral equation of energy.

Multiplying (2) by dt, and integrating, we find

$$T + V = E_0 - \int_0^t Q dt \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4),$$

where E_0 is a constant denoting the sum of the kinetic and potential energies at the instant $t=0$. Now T and Q are each of them essentially positive except when the system is at rest, and then each of them is zero. Therefore $\int_0^t Q dt$ must increase to infinity unless the system comes more and more nearly to rest as time advances. Hence either this must be the case, or V must diminish to $-\infty$. It follows that when V is positive for all real values of the co-ordinates the system must as time advances come more and more nearly to rest in its zero-configuration, whatever may have been the initial values of the co-ordinates and velocities. Even if V is negative for some or for all values of the co-ordinates, the system may be projected from *some given*

* *Proceedings of the London Mathematical Society*, May, 1873; *Theory of Sound*, Vol. I. § 81.

configurations with such velocities that when $t=\infty$ it shall be at rest in its zero configuration: this we see by taking, as a particular solution, the terms of (9) § 345ⁱᵛ below, for which m is negative. But this equilibrium is essentially unstable, unless V is positive for all real values of the co-ordinates. To prove this imagine the system placed in any configuration in which V is negative, and left there either at rest or with any motion of kinetic energy less than or at the most equal to $-V$: thus E_0 will be negative or zero; $T+V$ will therefore have increasing negative value as time advances; therefore V must always remain negative; and therefore the system can never reach its zero configuration. It is clear that $-V$ and T must each on the whole increase though there may be fluctuations, of T diminishing for a time, during which $-V$ must also diminish so as to make the excess $(-V)-T$ increase at the rate equal to Q per unit of time according to formula (2).

Cycloidal system with conservative positional forces and unrestricted motional forces.

345ⁱⁱⁱ. To illustrate the circumstances of the several cases let $\lambda = m + n\sqrt{-1}$ be a root of the determinantal equation, m and n being both real. The corresponding realized solution of the dynamical problem is

$$\psi_1 = r_1\epsilon^{mt}\cos(nt - e_1),\ \psi_2 = r_2\epsilon^{mt}\cos(nt - e_2),\ \text{etc.}\ldots\ldots\ldots(5),$$

where the differences of epochs $e_2 - e_1$, $e_3 - e_1$, etc. and the ratios r_2/r_1, etc., in all $2i-2$ numerics*, are determined by the $2i$ simultaneous linear equations (3) of § 343 harmonized by taking for $\lambda = m + n\sqrt{-1}$, and again $\lambda = m - n\sqrt{-1}$. Using these expressions for ψ_1, ψ_2, etc. in the expressions for V, Q, T, we find,—

$$\left.\begin{array}{l} V = \epsilon^{2mt}(C + A\cos 2nt + B\sin 2nt) \\ Q = \epsilon^{2mt}(C' + A'\cos 2nt + B'\sin 2nt) \\ T = \epsilon^{2mt}(C'' + A''\cos 2nt + B''\sin 2nt) \end{array}\right\} \ldots\ldots\ldots\ldots(6),$$

* The term numeric has been recently introduced by Professor James Thomson to denote a number, or a proper fraction, or an improper fraction, or an incommensurable ratio (such as π or ϵ). It must also to be useful in mathematical analysis include imaginary expressions such as $m + n\sqrt{-1}$, where m and n are real numerics. "Numeric" may be regarded as an abbreviation for "numerical expression." It lets us avoid the intolerable verbiage of integer or proper or improper fraction which mathematical writers hitherto are so often compelled to use; and is more appropriate for mere number or ratio than the designation "quantity," which rather implies quantity of something than the mere numerical expression by which quantities of any measurable things are reckoned in terms of the unit of quantity.

Cycloidal system with conservative positional forces and unrestricted motional forces.

where $C, A, B, C', A', B', C'', A'', B''$, are determinate constants: and in order that Q and T may be positive we have

$$C' > +\sqrt{(A'^2 + B'^2)}, \text{ and } C'' > +\sqrt{(A''^2 + B''^2)} \quad \ldots\ldots\ldots\ldots (7).$$

Substituting these in (2), and equating coefficients of corresponding terms, we find

$$\left.\begin{aligned} 2m\,(C + C'') &= -C' \\ 2\,\{m\,(A + A'') + n\,(B + B'')\} &= -A' \\ 2\,\{m\,(B + B'') - n\,(A + A'')\} &= -B' \end{aligned}\right\} \ldots\ldots\ldots\ldots (8).$$

Real part of every root proved negative when V positive for all real co-ordinates;

The first of these shows that $C + C''$ and m must be of contrary signs. Hence if V be essentially positive [which requires that C be greater than $+\sqrt{(A^2+B^2)}$], every value of m must be negative.

345iv. If V have negative values for some or all real values of the co-ordinates, m must clearly be positive for some roots, but there must still, and always, be roots for which m is negative.

positive for some roots when V has negative values;

To prove this last clause let us instead of (5) take sums of particular solutions corresponding to different roots

$$\lambda = m \pm n\sqrt{-1}, \quad \lambda' = m' \pm n'\sqrt{-1}, \text{ etc.},$$

m and n denoting real numerics. Thus we have

$$\left.\begin{aligned} \psi_1 &= r_1\epsilon^{mt}\cos(nt - e_1) + r'_1\epsilon^{m't}\cos(n't - e'_1) + \text{etc.} \\ \psi_2 &= r_2\epsilon^{mt}\cos(nt - e_2) + r'_2\epsilon^{m't}\cos(n't - e'_2) + \text{etc.} \\ &\text{etc.} \end{aligned}\right\} \quad (9).$$

Suppose now m, m', etc. to be all positive: then for $t = -\infty$, we should have $\psi_1 = 0$, $\psi_2 = 0$, $\dot\psi_1 = 0$, $\dot\psi_2 = 0$, etc., and therefore $V = 0$, $T = 0$. Hence, for finite values of t, T would in virtue of (4) be less than $-V$ (which in this case is essentially positive): but we may place the system in any configuration and project it with any velocity we please, and therefore the amount of kinetic energy we may give it is unlimited. Hence, if (9) be the complete solution, it must include some negative value or values of m, and therefore of all the roots λ, λ', etc. there must be some of which the real part is negative. This conclusion is also obvious on purely algebraic grounds, because the coefficient of λ^{2i-1} in the determinant is obviously $11 + 22 + 33 + \ldots$, which is essentially positive when Q is positive for all real values of the co-ordinates.

but always negative for some roots.

345v. It is an important subject for investigation, interesting both in mere Algebra and in Dynamics, to find how many roots there are with m positive, or how many with m negative in any particular case or class of cases; also to find under what conditions n disappears [or the motion non-oscillatory (compare § 341)]. We hope to return to it in our second volume, and should be very glad to find it taken up and worked out fully by mathematicians in the mean time. At present it is obvious that if V be negative for all real values of ψ_1, ψ_2, etc., the motion must be non-oscillatory for every mode (or every value of λ must be real) if Q be but large enough: but as we shall see immediately with Q *not too large*, n may appear in some or in all the roots, even though V be negative for all real co-ordinates, when there are forces of the gyroscopic class [§ 319, Examp. (G) above and § 345x below). When the motional forces are wholly of the viscous class it is easily seen that n can only appear if V is positive for some or all real values of the co-ordinates: n must disappear if V is negative for all real values of the co-ordinates (again compare § 341).

Non-oscillatory subsidence to stable equilibrium, or falling away from unstable.

Oscillatory subsidence to stable equilibrium, or falling away from unstable.

Falling away from wholly unstable equilibrium is essentially non-oscillatory if motional forces wholly viscous.

345vi. A chief part of the substance of §§ 345ii ... 345v above may be expressed shortly without symbols thus:—When there is any dissipativity the equilibrium in the zero position is stable or unstable according as the same system with no motional forces, but with the same positional forces, is stable or unstable. The gyroscopic forces which we now proceed to consider may convert instability into stability, as in the gyrostat § 345x below, *when there is no dissipativity*:—but when there is any dissipativity gyroscopic forces may convert rapid falling away from an unstable configuration into falling by (as it were) exceedingly gradual spirals, but they cannot convert instability into stability if there be any dissipativity.

Stability of Dissipative system.

The theorem of Dissipativity [§ 345i, (2) and (3)] suggests the following notation,—

$$\left.\begin{aligned} &\tfrac{1}{2}(12 + 21) = [12] \text{ or } [21], \quad \tfrac{1}{2}(13 + 31) = [13] \text{ or } [31], \text{ etc.} \\ \text{and } &\tfrac{1}{2}(12 - 21) = 12] \text{ or } -21], \quad \tfrac{1}{2}(13 - 31) = 13] \text{ or } -31], \text{ etc.} \end{aligned}\right\} (10),$$

so that the symbols [12], [21], [13], etc., and 12], 21], 13], etc. denote quantities which respectively fulfil the following mutual relations,

$$\left.\begin{aligned} [12] = [21], \quad [13] = [31], \quad [23] = [32], \quad &\text{etc.} \\ 12] = -21], \quad 13] = -31], \quad 23] = -32], \quad &\text{etc.} \end{aligned}\right\} \ldots\ldots\ldots (11).$$

Thus (3) of § 345i becomes

$$Q = 11\dot\psi_1^2 + 2\,[12]\,\dot\psi_1\dot\psi_2 + 22\dot\psi_2^2 + 2\,[13]\,\dot\psi_1\dot\psi_3 + \text{etc.} \ldots\ldots (12),$$

and going back to (1), with (10) and (12) we have

$$\left.\begin{array}{l} \dfrac{d}{dt}\dfrac{dT}{d\dot\psi_1}+\dfrac{dQ}{d\dot\psi_1}+12]\dot\psi_2+13]\dot\psi_3+\text{etc.}+\dfrac{dV}{d\psi_1}=0 \\ \dfrac{d}{dt}\dfrac{dT}{d\dot\psi_2}+\dfrac{dQ}{d\dot\psi_2}+21]\dot\psi_1+23]\dot\psi_3+\text{etc.}+\dfrac{dV}{d\psi_2}=0 \\ \dfrac{d}{dt}\dfrac{dT}{d\dot\psi_3}+\dfrac{dQ}{d\dot\psi_3}+31]\dot\psi_1+32]\dot\psi_2+\text{etc.}+\dfrac{dV}{d\psi_3}=0 \\ \dots\dots\dots\dots\dots\dots\dots\dots \end{array}\right\}\dots\dots(13).$$

Various origins of gyroscopic terms.

In these equations the terms $12]\dot\psi_2$, $21]\dot\psi_1$, $13]\dot\psi_3$, $31]\dot\psi_1$, etc. represent what we may call gyroscopic forces, because, as we have seen in § 319, Ex. G, they occur when fly-wheels each given in a state of rapid rotation form part of the system by being mounted on frictionless bearings connected through framework with other parts of the system; and because, as we have seen in § 319, Ex. F, they occur when the motion considered is motion of the given system relatively to a rigid body revolving with a constrainedly constant angular velocity round a fixed axis. This last reason is especially interesting on account of Laplace's dynamical theory of the tides at the foundation of which it lies, and in which it is answerable for some of the most curious and instructive results, such as the beautiful vortex problem presented by what Laplace calls "Oscillations of the First Species*."

Equation of energy.

345vii. The gyrostatic terms disappear from the equation of energy as we see by § 345i, (2) and (3), and as we saw previously by § 319, Example G (19), and in § 319, Ex. F′ (*f*). Comparing § 319 (*f*) and (*g*), we see that in the case of motion relatively to a body revolving uniformly round a fixed axis it is not the equation of total absolute energy but the equation of energy of the *relative motion* that the gyroscopic terms disappear from, as (*f*) of § 319; and (2) and (3) of § 345i when the subject of their application is to such relative motion.

* The integrated equation for this species of tidal motions, in an ideal ocean equally deep over the whole solid rotating spheroid, is given in a form ready for numerical computation in "Note on the 'Oscillations of the First Species' in Laplace's Theory of the Tides" (W. Thomson), *Phil. Mag.* Oct. 1875.

Gyrostatic conservative system:

345viii. To discover something of the character of the gyroscopic influence on the motion of a system, suppose there to be no resistances (or viscous influences), that is to say let the dissipativity, Q, be zero. The determinantal equation (4) becomes

$$\begin{vmatrix} (11)\lambda^2 & +11, & (12)\lambda^2+12]\lambda+12,\dots \\ (21)\lambda^2+21]\lambda+21, & (22)\lambda^2 & +22,\dots \\ \dots\dots & \dots\dots & \dots\dots \\ \dots\dots & \dots\dots & \dots\dots \end{vmatrix}=0\dots\dots(14).$$

Now by the relations (12) = (21), etc., 12 = 21, etc., and $12]=-21]$, we see that if λ be changed into $-\lambda$ the determinant becomes altered merely by interchange of terms between columns and rows, and hence the value of the determinant remains unchanged. Hence the first member of (14) cannot contain odd powers of λ, and therefore its roots must be in pairs of oppositely signed equals. The condition for stability of equilibrium in the zero configuration is therefore that the roots λ^2 of the determinantal equation be each real and negative.

simplification of its equations.

345ix. The equations are simplified by transforming the co-ordinates (§ 337) so as to reduce T to a sum of squares with positive coefficients and V to a sum of squares with positive or negative coefficients as the case may be, or which is the same thing to adopt for co-ordinates those displacements which would correspond to "fundamental modes" (§ 338), if the positional forces were as they are and there were no motional forces*. Suppose farther the unit values of the co-ordinates to be so chosen that the coefficients of the squares of the velocities in $2T$ shall be each unity; and let us put ϖ_1, ϖ_2, ϖ_3, etc. instead of the coefficients 11, 22, 33, etc., remaining in $2V$. Thus we have

$$T=\tfrac{1}{2}(\dot\psi_1^2+\dot\psi_2^2+\text{etc.}),\text{ and }V=\tfrac{1}{2}(\varpi_1\psi_1^2+\varpi_2\psi_2^2+\text{etc.})\dots(15).$$

If now we omit the half brackets] as no longer needed to avoid ambiguity, and understand that $12=-21$, $13=-31$, $23=-32$, etc., the equations of motion are

$$\left.\begin{array}{l} \ddot\psi_1+12\dot\psi_2+13\dot\psi_3+\dots\dots+\varpi_1\psi_1=0 \\ \ddot\psi_2+21\dot\psi_1+23\dot\psi_3+\dots\dots+\varpi_2\psi_2=0 \\ \ddot\psi_3+31\dot\psi_1+32\dot\psi_2+\dots\dots+\varpi_3\psi_3=0 \\ \dots\dots\dots\dots\dots\dots\dots\dots \end{array}\right\}\dots\dots\dots(16),$$

* [The function V includes what is called the "kinetic potential" of the gyrostatic system, and the determination of the *quasi* normal coordinates in the text will therefore depend on, and vary with, the magnitudes of the various gyrostatic momenta. This has a bearing on subsequent results. H. L.]

and the determinantal equation becomes

$$\begin{vmatrix} \lambda^2+\varpi_1, & 12\lambda, & 13\lambda,\ldots \\ 21\lambda, & \lambda^2+\varpi_2, & 23\lambda,\ldots \\ 31\lambda, & 32\lambda, & \lambda^2+\varpi_3,\ldots \\ \ldots & \ldots & \ldots \end{vmatrix} = 0 \ldots\ldots\ldots (17).$$

Determinant of gyrostatic conservative system.

The determinant (which for brevity we shall denote by D) in this case is what has been called by Cayley a skew determinant. What it would become if zero were substituted for $\lambda^2+\varpi_1$, $\lambda^2+\varpi_2$, etc. in its principal diagonal is what is called a skew symmetric determinant. The known algebra of skew and skew symmetric determinants gives

$$\left.\begin{aligned} D = {} & (\lambda^2+\varpi_1)(\lambda^2+\varpi_2)\ldots(\lambda^2+\varpi_i) \\ & + \lambda^2\Sigma(\lambda^2+\varpi_3)(\lambda^2+\varpi_4)\ldots(\lambda^2+\varpi_i)\,12^2 \\ & + \lambda^4\Sigma(\lambda^2+\varpi_5)(\lambda^2+\varpi_6)\ldots(\lambda^2+\varpi_i)(12.34+31.24+23.14)^2 \\ & + \lambda^6\Sigma(\lambda^2+\varpi_7)(\lambda^2+\varpi_8)\ldots(\lambda^2+\varpi_i)(\Sigma 12.34.56)^2 + \text{etc.} \\ & + \lambda^i(\Sigma 12.34.56\ldots\ldots i-1,\ i)^2 \end{aligned}\right\} (18),$$

when i is even. For example see (30) below. When λ is odd the last term is

$$\lambda^{i-1}\Sigma(\lambda^2+\varpi_i)(\Sigma 12.34.56\ldots i-2,\ i-1)^2 \ldots\ldots\ldots\ldots (18'),$$

and no other change in the formula is necessary. In each case the small Σ denotes the sum of the products obtained by making every possible permutation of the numbers in the line of factors following it, with orders chosen acccording to a proper rule to render the sign of each product positive (Salmon's *Higher Algebra*, Lesson v. Art. 40). This sum is in each case the square root of a certain corresponding skew symmetric determinant.

Square roots of skew symmetrics.

An easy rule to find other products from any one given to begin with is this:—Invert the order in any one factor, and make a simple interchange of any two numbers in different factors. Thus, in the last Σ of (18) alter $i-1$, i to i, $i-1$, and interchange $i-1$ with 3: so we find $12.i-1$, $4.56\ldots\ldots i$, 3 for a term: similarly $12.64.53\ldots i$, $i-1$, and $62.14.53\ldots i-1$, i, for two others. The same number must not occur more than once in any one product. Two products differing only in the orders of the two numbers in factors are not admitted. If n be the number of factors in each term, the whole number of factors is clearly $1.3.5\ldots(2n-1)$, and they may be found in regular

Square roots of skew symmetrics.

progression thus: Begin with a single factor and single term 12. Then apply to it the factor 34, and permute to suit 24 instead of 34, and permute the result to suit 14 instead of 24. Thirdly, apply to the sum thus found the factor 56, and permute successively from 56 to 46, from 46 to 36, from 36 to 26, and from 26 to 16. Fourthly, introduce the factor 78; and so on. Thus we find

$$\left.\begin{aligned} & \sqrt{\begin{vmatrix} 0, & 12 \\ 21, & 0 \end{vmatrix}} = 12 \\ & \sqrt{\begin{vmatrix} 0, & 12, & 13, & 14 \\ 21, & 0, & 23, & 24 \\ 31, & 32, & 0, & 34 \\ 41, & 42, & 43, & 0 \end{vmatrix}} = 12.34+31.24+23.14 \\ & \sqrt{\begin{vmatrix} 0, & 12, & 13, & 14, & 15, & 16 \\ 21, & 0, & 23, & 24, & 25, & 26 \\ 31, & 32, & 0, & 34, & 35, & 36 \\ 41, & 42, & 43, & 0, & 45, & 46 \\ 51, & 52, & 53, & 54, & 0, & 56 \\ 61, & 62, & 63, & 64, & 65, & 0 \end{vmatrix}} = \begin{aligned} & (12.34+31.24+23.14)56 \\ & +(12.53+13.52+23.51)46 \\ & +(12.45+41.52+42.51)36 \\ & +(31.45+41.35+34.51)26 \\ & +(23.45+24.35+34.25)16 \end{aligned} \end{aligned}\right\} (19).$$

The second member of the last of these equations is what is denoted by $\Sigma 12.34.56$ in (18).

345ᶜ. Each term of the determinant D except

$$(\lambda^2+\varpi_1)(\lambda^2+\varpi_2)\ldots(\lambda^2+\varpi_i)$$

Gyrostatic system with two freedoms.

contains λ^2 as a factor. Hence, when all are expanded in powers of λ^2, the term independent of λ is $\varpi_1\varpi_2\ldots\varpi_i$. If this be negative there must be at least one real positive and one real negative root λ^2. Hence for stability either must all of ϖ_1, ϖ_2, ... ϖ_i be positive or an even number of them negative. Ex.:—Two modes of motion, x and y the co-ordinates. Let the equations of motion be

$$\left.\begin{aligned} I\ddot{x}+g\dot{y}+Ex=0 \\ J\ddot{y}-g\dot{x}+Fy=0 \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots (20),$$

and the determinantal equation is

$$(I\lambda^2+E)(J\lambda^2+F)+g^2\lambda^2=0.$$

If we put

$$x=\xi/\sqrt{I},\quad y=\eta/\sqrt{J} \ldots\ldots\ldots\ldots\ldots\ldots (21),$$

and

$$E=\varpi I,\quad F=\zeta J,\quad \text{and}\quad g=\gamma\sqrt{(IJ)} \ldots\ldots\ldots (22),$$

Gyrostatic system with two freedoms.

equations (20) and the determinantal equation become

$$\left.\begin{aligned}\ddot{\xi}+\gamma\dot{\eta}+\varpi\xi&=0\\ \ddot{\eta}-\gamma\dot{\xi}+\zeta\eta&=0\end{aligned}\right\}\text{.....(23)},$$

and $$(\lambda^2+\varpi)(\lambda^2+\zeta)+\gamma^2\lambda^2=0\text{ ..... (24)}.$$

The solution of this quadratic in λ^2 may be put under the following forms,—

$$\left.\begin{aligned}-\lambda^2&=\tfrac{1}{2}(\gamma^2+\varpi+\zeta)\pm\tfrac{1}{2}\{[\gamma^2+(\sqrt{\varpi}+\sqrt{\zeta})^2]\,[\gamma^2+(\sqrt{\varpi}-\sqrt{\zeta})^2]\}^{\frac{1}{2}}\\ -\lambda^2&=\tfrac{1}{2}(\gamma^2+\varpi+\zeta)\pm\tfrac{1}{2}\{[\gamma^2-(\sqrt{-\varpi}+\sqrt{-\zeta})^2][\gamma^2-(\sqrt{-\varpi}-\sqrt{-\zeta})^2]\}^{\frac{1}{2}}\end{aligned}\right\}\text{...(25)}.$$

To make both values of $-\lambda^2$ real and positive ϖ and ζ must be of the same sign. If they are both positive no farther condition is necessary. If they are both negative we must have

$$\gamma>\sqrt{-\varpi}+\sqrt{-\zeta}\text{ .....(26)}.$$

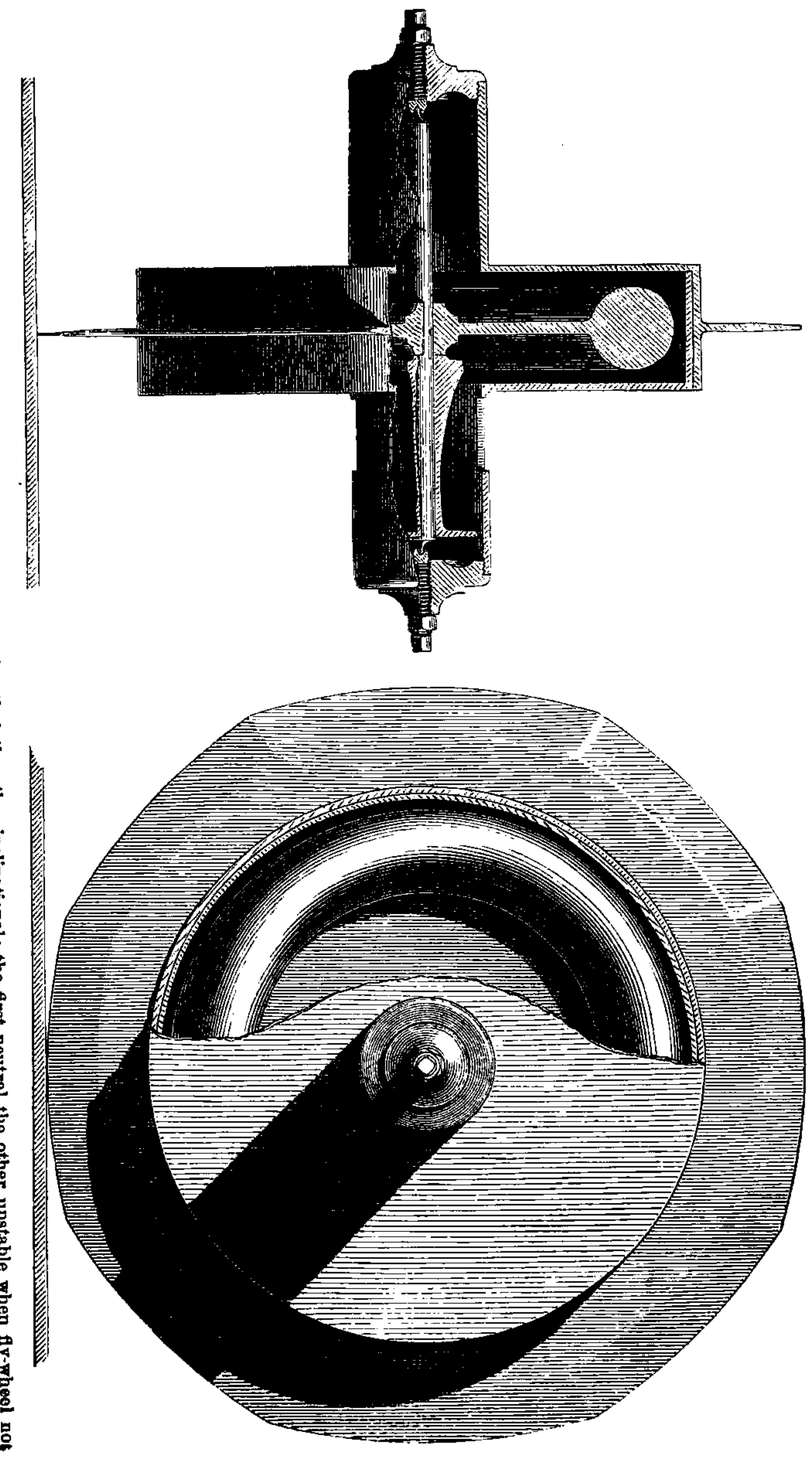
These are the conditions that the zero configuration may be stable. Remark that when (as practically in all the gyrostatic illustrations) γ^2 is very great in comparison with $\sqrt{(\varpi\zeta)}$, the greater value of $-\lambda^2$ is approximately equal to γ^2, and therefore (as the product of the two roots is exactly $\varpi\zeta$), the less is approximately equal to $\varpi\zeta/\gamma^2$. Remark also that $2\pi/\sqrt{\varpi}$ and $2\pi/\sqrt{\zeta}$ are the periods of the two fundamental vibrations of a system otherwise the same as the given system, but with $\gamma=0$. Hence, using the word irrotational to refer to the system with $g=0$, and gyroscopic, or gyrostatic, or gyrostat, to refer to the actual system;

Gyrostatic influence dominant.

From the preceding analysis we have the curious and interesting result that, in a system with two freedoms, two irrotational instabilities are converted into complete gyrostatic stability (each freedom stable) by sufficiently rapid rotation; but that with one irrotational stability the gyrostat is essentially unstable, with one of its freedoms unstable and the other stable, if there be one irrotational instability. Various good illustrations of gyrostatic systems with two, three, and four freedoms (§§ 345ˣ, ˣˡ and ˣˡˡ) are afforded by the several different modes of mounting shown in the accompanying sketches, applied to the ordinary gyrostat* (a rapidly rotating fly-wheel pivoted as finely as possible within a rigid case, having a convex curvilinear polygonal border, in the plane perpendicular to the axis through the centre of gravity of the whole).

Gyrostatic stability.

* *Nature*, No. 379, Vol. 15 (February 1, 1877), page 297.

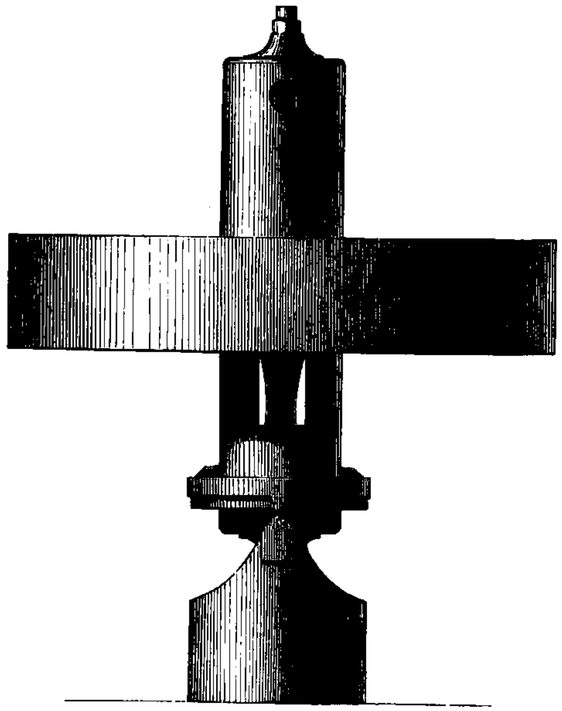


Ordinary gyrostats.

(Translational motions not considered) there are two freedoms, one azimuthal the other inclinational; the first neutral the other unstable when fly-wheel not rotating: the first still neutral the second stable when fly-wheel in rapid rotation. Equations (23) with $\zeta=0$ express the problem, and (24) and (25) its solution.

Gyrostats,

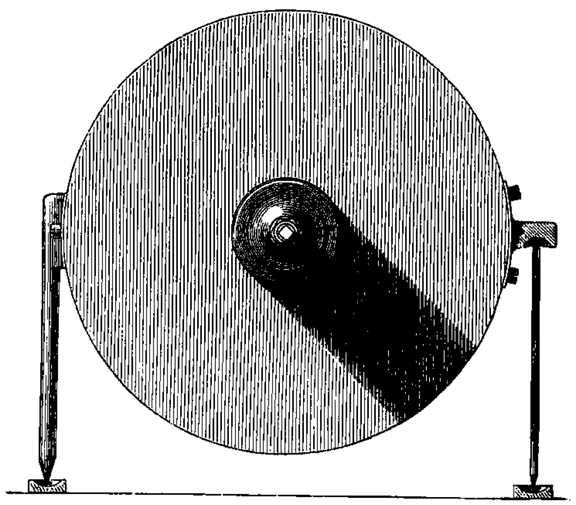
on gimbals;



on universal-flexure-joint (§ 109) in place of gimbals; constituting an inverted gyroscopic pendulum (§ 319, Ex. D).

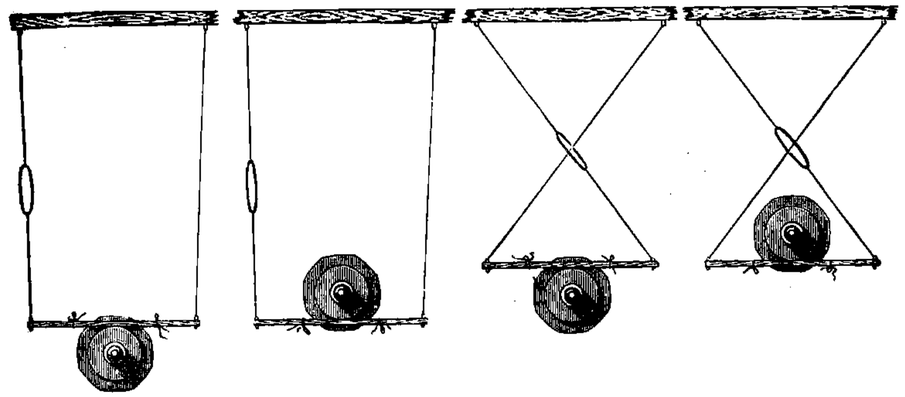
Gyrostat on knife-edge gimbals with its axis vertical. Two freedoms; each unstable without rotation of the fly-wheel; each stable when it is rotating rapidly. Neglecting inertia of the knife-edges and gimbal-ring we have $I=J$ in (20), and supposing the levels of the knife-edges to be the same, we have $E=F$. Thus its determinantal equation is $(I\lambda^2+E)^2+g^2\lambda^2=0$. A similar result, expressed by the same equations of motion, is obtained by supporting the gyrostat on a little elastic universal flexure-joint of, for example, thin steel pianoforte-wire one or two centimetres long between end clamps or solderings. A drawing is unnecessary.

on stilts;



Two freedoms, one azimuthal the other inclinational, both unstable without, both stable with, rapid rotation of the fly-wheel.

bifilarly slung in four ways.



No. 1. No. 2. No. 3. No. 4.

Four freedoms, reducible to three if desired by a third thread in each case, diagonal in the first and second, lateral in the third and fourth, the freedom thus annulled being in each case stable and independent of the rotation of the fly-wheel. Three modes essentially involved in the gyrostatic system in each case, two inclinational and one azimuthal.

No. 1.—Azimuthally stable without rotation; with rotation all three modes stable.

No. 2.—Azimuthally stable, one inclinational mode unstable the other stable without rotation; with rotation two unstable, one stable.

No. 3.—The azimuthal mode unstable, two inclinational modes stable the other unstable, without rotation; with rotation one azimuthal mode and one inclinational mode unstable, and one inclinational mode stable.

No. 4.—Azimuthally and one inclinational mode unstable, one inclinational mode stable, without rotation; with rotation all three stable.

Gyrostatic system with three freedoms.

345xi. Take for another example a system having three freedoms (that is to say, three independent co-ordinates ψ_1, ψ_2, ψ_3), (16) become

$$\left.\begin{aligned} \ddot{\psi}_1 + g_3\dot{\psi}_2 - g_2\dot{\psi}_3 + \varpi_1\psi_1 &= 0 \\ \ddot{\psi}_2 + g_1\dot{\psi}_3 - g_3\dot{\psi}_1 + \varpi_2\psi_2 &= 0 \\ \ddot{\psi}_3 + g_2\dot{\psi}_1 - g_1\dot{\psi}_2 + \varpi_3\psi_3 &= 0 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots (27),$$

where g_1, g_2, g_3 denote the values of the three pairs of equals 23 or -32, 31 or -13, 12 or -21. Imagine ψ_1, ψ_2, ψ_3 to be rectangular co-ordinates of a material point, and let the co-ordinates be transformed to other axes OX, OY, OZ, so chosen that OZ coincides with the line whose direction cosines relatively to the ψ_1-, ψ_2-, ψ_3- axes are proportional to g_1, g_2, g_3. The equations become

$$\left.\begin{aligned} \ddot{x} - 2\omega\dot{y} &= X \\ \ddot{y} + 2\omega\dot{x} &= Y \\ \ddot{z} \qquad &= Z \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (28),$$

Gyrostatic system with three freedoms:

where $\omega = \sqrt{(g_1^2 + g_2^2 + g_3^2)}$, and the force-components parallel to the fresh axes are denoted by X, Y, Z (instead of $-\frac{dV}{dx}$, $-\frac{dV}{dy}$, $-\frac{dV}{dz}$, because the present transformation is clearly independent of the assumption we have been making latterly that the positional forces are conservative). These (28) are simply the equations [§ 319, Ex. (E)] of the motion of a particle relatively to co-ordinates revolving with angular velocity ω round the axis OZ, if we suppose X, Y, Z to include the components of the centrifugal force due to this rotation.

reduced to a mere rotating system.

Hence the influence of the gyroscopic terms however originating in any system with three freedoms (and therefore also in any system with only two freedoms) may be represented by the motion of a material particle supported by massless springs attached to a rigid body revolving uniformly round a fixed axis. It is an interesting and instructive exercise to imagine or to actually construct mechanical arrangements for the motion of a material particle to illustrate the experiments described in § 345x.

345xii. Consider next the case of a system with four freedoms. The equations are

$$\left.\begin{aligned} \ddot{\psi}_1 + 12\dot{\psi}_2 + 13\dot{\psi}_3 + 14\dot{\psi}_4 + \varpi_1\psi_1 &= 0 \\ \ddot{\psi}_2 + 21\dot{\psi}_1 + 23\dot{\psi}_3 + 24\dot{\psi}_4 + \varpi_2\psi_2 &= 0 \\ \ddot{\psi}_3 + 31\dot{\psi}_1 + 32\dot{\psi}_2 + 34\dot{\psi}_4 + \varpi_3\psi_3 &= 0 \\ \psi_4 + 41\dot{\psi}_1 + 42\dot{\psi}_2 + 43\dot{\psi}_3 + \varpi_4\psi_4 &= 0 \end{aligned}\right\} \quad \ldots\ldots\ldots (29).$$

Denoting by D the determinant we have, by (18),

$$\left.\begin{aligned} D = {} & (\lambda^2+\varpi_1)(\lambda^2+\varpi_2)(\lambda^2+\varpi_3)(\lambda^2+\varpi_4) \\ & + \lambda^2\{34^2(\lambda^2+\varpi_1)(\lambda^2+\varpi_2) + 12^2(\lambda^2+\varpi_3)(\lambda^2+\varpi_4) + 42^2(\lambda^2+\varpi_1)(\lambda^2+\varpi_3) \\ & \quad + 13^2(\lambda^2+\varpi_4)(\lambda^2+\varpi_2) + 23^2(\lambda^2+\varpi_1)(\lambda^2+\varpi_4) + 14^2(\lambda^2+\varpi_2)(\lambda^2+\varpi_3)\} \\ & + \lambda^4(12\ 34 + 13\ 42 + 14\ 23)^2 \end{aligned}\right\} \ldots (30).$$

If ϖ_1, ϖ_2, ϖ_3, ϖ_4 be each zero, D becomes

$$\lambda^8 + (12^2+13^2+14^2+23^2+42^2+34^2)\lambda^6 + (12\ 34 + 13\ 42 + 14\ 23)^2\lambda^4.$$

This equated to zero and viewed as an equation for λ^2 has two

roots each equal to 0, and two others given by the residual quadratic

Quadruply free gyrostatic system without force.

$$\lambda^4 + (12^2+13^2+14^2+23^2+24^2+34^2)\lambda^2 + (12\ 34 + 13\ 42 + 14\ 23)^2 = 0 \ldots (31).$$

Now remarking that the solution of $z^2 + pz + q^2 = 0$ may be written

$$-z = \tfrac{1}{2}\{p \pm \sqrt{(p+2q)(p-2q)}\} = \tfrac{1}{4}\{\sqrt{(p+2q)} \pm \sqrt{(p-2q)}\}^2,$$

we have from (31)

$$\left.\begin{aligned} -\lambda^2 &= \tfrac{1}{2}(12^2+13^2+14^2+23^2+24^2+34^2 \pm \sqrt{s}) \\ &= \tfrac{1}{4}(r \pm s)^2 \end{aligned}\right\} \ldots\ldots (32),$$

$$\left.\begin{aligned} \text{where} \quad r &= \sqrt{\{(12+34)^2 + (13+42)^2 + (14+23)^2\}} \\ \text{and} \quad s &= \sqrt{\{(12-34)^2 + (13-42)^2 + (14-23)^2\}} \end{aligned}\right\} \ldots\ldots (33).$$

As 12, 34, 13, etc. are essentially real, r and s are real, and (unless $12\ 43 + 13\ 42 + 14\ 23 = 0$, when one of the values of λ^2 is zero, a case which must be considered specially, but is excluded for the present,) they are unequal. Hence the two values of $-\lambda^2$ given by (32) are real and positive. Hence two of the four freedoms are stable. The other two (corresponding to $-\lambda^2 = 0$) are neutral.

Excepted case of failing gyrostatic predominance.

345xiii. Now suppose ϖ_1, ϖ_2, ϖ_3, ϖ_4 to be not zero, but each very small. The determinantal equation will be a biquadratic in λ^2, of which two roots (the two which vanish when ϖ_1, etc. vanish) are approximately equal to the roots of the quadratic

Quadruply free cycloidal system, gyrostatically dominated.

$$\begin{aligned} (12\ 34 + 13\ 42 + 14\ 23)^2\lambda^4 + {} & (12^2\varpi_3\varpi_4 + 13^2\varpi_2\varpi_4 + 14^2\varpi_2\varpi_3 \\ & + 23^2\varpi_1\varpi_4 + 24^2\varpi_1\varpi_3 + 34^2\varpi_1\varpi_2)\lambda^2 + \varpi_1\varpi_2\varpi_3\varpi_4 = 0 \ldots (34), \end{aligned}$$

and the other two roots are approximately equal to those of the previous residual quadratic (31).

To solve equation (34), first write it thus :—

$$\left(\frac{1}{\lambda^2}\right)^2 + (12'^2+13'^2+14'^2+23'^2+24'^2+34'^2)\frac{1}{\lambda^2} + (12'34'+13'42'+14'23')^2 = 0 \quad \ldots\ldots\ldots (35),$$

where

$$12' = \frac{12}{\sqrt{(\varpi_1\varpi_2)}}, \quad 13' = \frac{13}{\sqrt{(\varpi_1\varpi_3)}}, \quad 23' = \frac{23}{\sqrt{(\varpi_2\varpi_3)}}, \quad \text{etc.} \ldots\ldots (36).$$

Quadruply free cycloidal system, gyrostatically dominated.

Thus, taken as a quadratic for λ^{-2}, it has the same form as (31) for λ^2, and so, as before in (32) and (33), we find

$$\frac{-1}{\lambda^2} = \tfrac{1}{4}(r' \pm s')^2 \quad\text{.....}\quad (37)$$

$$\left.\begin{aligned} \text{where}\quad r' &= \sqrt{\{(12'+34')^2+(13'+42')^2+(14'+23')^2\}} \\ \text{and}\quad s' &= \sqrt{\{(12'-34')^2+(13'-42')^2+(14'-23')^2\}} \end{aligned}\right\} \quad\text{.....}(38).$$

Four irrotational stabilities confirmed, four irrotational instabilities rendered stable, by gyrostatic links.

Now if $\varpi_1, \varpi_2, \varpi_3, \varpi_4$ be all four positive or all four negative, $12'$, $34'$, $13'$, etc. are all real, and therefore both the values of $-\dfrac{1}{\lambda^2}$ given by (37) are real and positive (the excluded case referred to at the end of § 345xii, which makes

$$12'34' + 13'42' + 14'23' = 0,$$

and therefore the smaller value of $-\dfrac{1}{\lambda^2} = 0$, being still excluded). Hence the corresponding freedoms are stable. But it is not *necessary* for stability that $\varpi_1, \varpi_2, \varpi_3, \varpi_4$ be all four of one sign: it *is* necessary that their product be positive: since if it were negative the values of λ^2 given by (34) would both be real, but one only negative and the other positive. Suppose two of them, ϖ_3, ϖ_4 for example, be negative, and the other two, ϖ_1, ϖ_2, positive: this makes $\varpi_1\varpi_3$, $\varpi_1\varpi_4$, $\varpi_2\varpi_3$, and $\varpi_2\varpi_4$ negative, and therefore $13'$, $14'$, $23'$, and $24'$ imaginary. Instead of four of the six equations (36), put therefore

$$13'' = \frac{13}{\sqrt{(-\varpi_1\varpi_3)}},\ 14'' = \frac{14}{\sqrt{(-\varpi_1\varpi_4)}},\ 23'' = \frac{23}{\sqrt{(-\varpi_2\varpi_3)}},\ 24'' = \frac{24}{\sqrt{(-\varpi_2\varpi_4)}} \quad (39).$$

Thus $13''$ etc. are real, and $13' = 13''\sqrt{-1}$ etc., and (38) become

$$\left.\begin{aligned} r' &= \sqrt{\{(12'+34')^2-(13''+42'')^2-(14''+23'')^2\}} \\ s' &= \sqrt{\{(12'-34')^2-(13''-42'')^2-(14''-23'')^2\}} \end{aligned}\right\} \quad\text{.....}(40).$$

Hence for stability it is necessary and sufficient that

$$\left.\begin{aligned} &(12'+34')^2 > (13''+42'')^2+(14''+23'')^2 \\ \text{and}\quad &(12'-34')^2 > (13''-42'')^2+(14''-23'')^2 \end{aligned}\right\} \quad\text{.....}(41).$$

Combined dynamic and gyrostatic stability gyrostatically counteracted.

If these inequalities are reversed, the stabilities due to ϖ_1, ϖ_2 and $34'$ are undone by the gyrostatic connexions $13''$, $42''$, $14''$ and $23''$.

345xiv. Going back to (29) we see that for the particular solution $\psi_1 = a_1\epsilon^{\lambda t}$, $\psi_2 = a_2\epsilon^{\lambda t}$, etc., given by the first pair of roots of (32), they become approximately

$$\left.\begin{aligned} \lambda a_1 + 12\,a_2 + 13\,a_3 + 14\,a_4 &= 0 \\ \lambda a_2 + 21\,a_1 + 23\,a_3 + 24\,a_4 &= 0 \\ \lambda a_3 + 31\,a_1 + 32\,a_2 + 34\,a_4 &= 0 \\ \lambda a_4 + 41\,a_1 + 42\,a_2 + 43\,a_3 &= 0 \end{aligned}\right\} \quad\text{.....}(42);$$

Completed solution.

being in fact the linear algebraic equations for the solution in the form $\epsilon^{\lambda t}$ of the simple simultaneous differential equations (53) below. And if we take

$$\psi_1 = \frac{b_1}{\sqrt{\varpi_1}}\epsilon^{\lambda t},\quad \psi_2 = \frac{b_2}{\sqrt{\varpi_2}}\epsilon^{\lambda t},\ \text{etc.} \quad\text{.....}(43),$$

for either particular approximate solution of (29) corresponding to (37), we find from (29) approximately

$$\left.\begin{aligned} \lambda^{-1}b_1 + 12'b_2 + 13'b_3 + 14'b_4 &= 0 \\ \lambda^{-1}b_2 + 21'b_1 + 23'b_3 + 24'b_4 &= 0 \\ \lambda^{-1}b_3 + 31'b_1 + 32'b_2 + 34'b_4 &= 0 \\ \lambda^{-1}b_4 + 41'b_1 + 42'b_2 + 43'b_3 &= 0 \end{aligned}\right\} \quad\text{.....}(44).$$

Remark that in (42) the coefficients of the first terms are imaginary and those of all the others real. Hence the ratios a_1/a_2, a_1/a_3, etc., are imaginary. To realize the equations put

$$\lambda = n\sqrt{-1},\ a_1 = p_1 + q_1\sqrt{-1},\ a_2 = p_2 + q_2\sqrt{-1},\ \text{etc.}\ldots(45),$$

Realization of completed solution.

and let p_1, q_1, p_2, etc. be real; we find, as equivalent to (42),

$$\left.\begin{aligned} &\left\{\begin{aligned} -nq_1 + 12\,p_2 + 13\,p_3 + 14\,p_4 &= 0 \\ np_1 + 12\,q_2 + 13\,q_3 + 14\,q_4 &= 0 \end{aligned}\right. \\ &\left\{\begin{aligned} -nq_2 + 21\,p_1 + 23\,p_3 + 24\,p_4 &= 0 \\ np_2 + 21\,q_1 + 23\,q_3 + 24\,q_4 &= 0 \end{aligned}\right. \\ &\qquad\text{etc.}\qquad\qquad\text{etc.} \end{aligned}\right\} \quad\text{.....}(46).$$

Eliminating q_1, q_2, etc. from the seconds by the firsts of these pairs, we find

$$\left.\begin{aligned} (n^2+11)p_1 + 12\,p_2 + 13\,p_3 + 14\,p_4 &= 0 \\ 21\,p_1 + (n^2+22)p_2 + 23\,p_3 + 24\,p_4 &= 0 \\ 31\,p_1 + 32\,p_2 + (n^2+33)\,p_3 + 34\,p_4 &= 0 \\ 41\,p_1 + 42\,p_2 + 43\,p_3 + (n^2+44)p_4 &= 0 \end{aligned}\right\} \quad (47);$$

and by eliminating p_1, p_2, etc. similarly we find similar equations

Realization of completed solution.

for the q's; with the same coefficients 11, 12, etc., given by the following formulas:—

$$\left.\begin{array}{l} 11 = 12\ 21 + 13\ 31 + 14\ 41 \\ 12 = 13\ 32 + 14\ 42 \\ 13 = 12\ 23 + 14\ 43 \\ 21 = 23\ 31 + 24\ 41 \\ \text{etc.} \qquad \text{etc.} \end{array}\right\} \ldots\ldots\ldots\ldots\ldots (48).$$

Remember now that

$$12 = -21,\ 13 = -31,\ 32 = -23,\ \text{etc.} \ldots\ldots\ldots\ldots (49),$$

and we see in (48) that

$$12 = 21,\ 13 = 31,\ 23 = 31,\ \text{etc.} \ldots\ldots\ldots\ldots\ldots\ldots (50);$$

and farther, that 11, 12, etc. are the negatives of the coefficients of $\frac{1}{2}a_1^2$, a_1a_2, etc. in the quadratic

$$\tfrac{1}{2}\{(12\,a_2 + 13\,a_3 + 14\,a_4)^2 + (21\,a_1 + 23\,a_3 + 24\,a_4)^2 + \text{etc.}\} \ldots (51)$$

Resultant motion reduced to motion of a conservative system with four fundamental periods equal two and two.

expanded. Hence if $G(aa)$ denote this quadratic, and $G(pp)$, $G(qq)$ the same of the p's and the q's, we may write (47) and the corresponding equations for the q's as follows:

$$\left.\begin{array}{ll} -n^2p_1 + \dfrac{dG(pp)}{dp_1} = 0, & -n^2p_2 + \dfrac{dG(pp)}{dp_2} = 0,\ \text{etc.} \\ -n^2q_1 + \dfrac{dG(qq)}{dq_1} = 0, & -n^2q_2 + \dfrac{dG(qq)}{dq_2} = 0,\ \text{etc.} \end{array}\right\} \quad (52).$$

These equations are harmonized by, and as is easily seen, only by, assigning to n^2 one or other of the two values of $-\lambda^2$ given in (32), above. Hence their determinantal equation, a biquadratic in n^2, has two pairs of equal real positive roots. We readily verify this by verifying that the square of the determinant of (42), with λ^2 replaced by $-n^2$, is equal to the determinant of (47) with 11, 12, etc. replaced by their values (48).

Algebraic theorem.

Hence (§ 343g) there is for each root an indeterminacy in the ratios p_1/p_2, p_1/p_3, p_1/p_4, according to which one of them may be assumed arbitrarily and the two others then determined by two of the equations (47); so that with two of the p's assumed arbitrarily the four are known: then the corresponding set of four q's is determined explicitly by the firsts of the pairs (46). Similarly the other root, n'^2, of the determinantal equation gives another solution with two fresh arbitraries. Thus we have the complete solution of the four equations

Details of realized solution

Details of realized solution

$$\left.\begin{array}{l} \dfrac{d\psi_1}{dt} + 12\,\psi_2 + 13\,\psi_3 + 14\,\psi_4 = 0 \\ \dfrac{d\psi_2}{dt} + 21\,\psi_1 + 23\,\psi_3 + 24\,\psi_4 = 0 \\ \text{etc.} \qquad \text{etc.} \end{array}\right\} \ldots\ldots\ldots\ldots\ldots (53),$$

with its four arbitraries. The formulas (46)...(52) are clearly the same as we should have found if we had commenced with assuming

$$\psi_1 = p_1 \sin nt + q_1 \cos nt,\quad \psi_2 = p_2 \sin nt + q_2 \cos nt,\ \text{etc.} \ldots (54),$$

as a particular solution of (53).

Orthogonalities proved between two components of one fundamental oscillation:

345xv. Important properties of the solution of (53) are found thus:—

(a) Multiply the firsts of (46) by p_1, p_2, p_3, p_4 and add: or the seconds by q_1, q_2, q_3, q_4 and add: either way we find

$$p_1q_1 + p_2q_2 + p_3q_3 + p_4q_4 = 0 \ldots\ldots\ldots\ldots\ldots\ldots (55).$$

(b) Multiply the firsts of (46) by q_1, q_2, q_3, q_4 and add: multiply the seconds by p_1, p_2, p_3, p_4 and add: and compare the results: we find

and equality of their energies.

$$n\Sigma p^2 = n\Sigma q^2 = \Sigma 12\,(p_2q_1 - p_1q_2) \ldots\ldots\ldots\ldots\ldots (56),$$

where Σ of the last member denotes a sum of such double terms as the sample without repetition of their equals, such as $21\,(p_1q_2 - p_2q_1)$.

Orthogonalities proved between different fundamental oscillations.

(c) Let n^2, n'^2 denote the two values of $-\lambda^2$ given in (32), and let (54) and

$$\psi_1 = p'_1 \sin n't + q'_1 \cos n't,\quad \psi_2 = p'_2 \sin n't + q'_2 \cos n't,\ \text{etc.} \ldots (57)$$

be the two corresponding solutions of (53). Imagine (46) to be written out for n'^2 and call them (46′): multiply the firsts of (46) by p'_1, p'_2, p'_3, p'_4 and add: multiply the firsts of (46′) by p_1, p_2, p_3, p_4 and add. Proceed correspondingly with the seconds. Proceed similarly with multipliers q for the firsts and p for the seconds. By comparisons of the sums we find that when n' is not equal to n we must have

$$\left.\begin{array}{l} \Sigma p'q = 0,\quad \Sigma 12\,(p'_1p_2 - p'_2p_1) = 0 \\ \Sigma q'p = 0,\quad \Sigma 12\,(q'_1q_2 - q'_2q_1) = 0 \\ \left.\begin{array}{l} \Sigma q'q = 0 \\ \Sigma p'p = 0 \end{array}\right\} \Sigma 12\,(q'_1p_2 - q'_2p_1) = 0,\quad \Sigma 12\,(p'_1q_2 - p'_2q_1) = 0 \end{array}\right\} (58).$$

Case of equal periods.

345xvi. The case of $n=n'$ is interesting. The equations $\Sigma q'q=0$, $\Sigma p'p=0$, $\Sigma p'q=0$, $\Sigma q'p=0$, when n differs however little from n', show (as we saw in a corresponding case in § 343m) that equality of n to n' does *not* bring into the solution terms of the form $Ct\cos nt$, and it must therefore come under § 343e. The condition to be fulfilled for the equality of the roots is seen from (32) and (33) to be

$$12=34,\ 13=42,\ \text{and}\ 14=23 \ldots\ldots\ldots\ldots(59):$$

and to give

$$n^2=12^2+13^2+14^2 \ldots\ldots\ldots\ldots(60)$$

for the common value of the roots. It is easy to verify that these relations reduce to zero each of the first minors of (42), as they must according to Routh's theorem (§ 343e), because each root, λ, of (42) is a double root. According to the same theorem all the first, second and third minors of (47) must vanish for each root, because each root, n^2, of (47) is a quadruple root: for this, as there are just four equations, it is necessary and sufficient that

$$11=22=33=44 \ \text{and}\ 12=0,\ 13=0,\ 14=0,\ 23=0,\ \text{etc.}\ldots(60'),$$

which we see at once by (48) is the case when (59) are fulfilled. In fact, these relations immediately reduce (51) to

$$G(aa)=\tfrac{1}{2}(12^2+13^2+14^2)(a_1^2+a_2^2+a_3^2+a_4^2)\ldots\ldots(61).$$

In this case one particular solution is readily seen from (52) and (46) to be

$$\left.\begin{array}{llll} p_1=1, & p_2=0, & p_3=0, & p_4=0 \\ q_1=0, & q_2=-\dfrac{12}{n}, & q_3=-\dfrac{13}{n}, & q_4=-\dfrac{14}{n} \\ \psi_1=\sin nt, & \psi_2=-\dfrac{12}{n}\cos nt, & \psi_3=-\dfrac{13}{n}\cos nt, & \psi_4=-\dfrac{14}{n}\cos nt \end{array}\right\}(62).$$

Completed solution for case of equal periods.

Hence the general solution, with four arbitraries p_1, p_2, p_3, p_4, is

$$\left.\begin{array}{l} \psi_1=p_1\sin nt+\dfrac{1}{n}(12p_2+13p_3+14p_4)\cos nt \\ \psi_2=p_2\sin nt+\dfrac{1}{n}(-12p_1+14p_3-13p_4)\cos nt \\ \psi_3=p_3\sin nt+\dfrac{1}{n}(-13p_1-14p_2+12p_4)\cos nt \\ \psi_4=p_4\sin nt+\dfrac{1}{n}(-14p_1+13p_2-12p_3)\cos nt \end{array}\right\}\ldots\ldots(63).$$

It is easy to verify that this satisfies the four differential equations (53).

Two higher, and two lower, of the four fundamental oscillations, similarly dealt with by solution of two similar quadratics.

345xvii. Quite as we have dealt with (42), (45), (53), (54) in § 345xiv, we may deal with (44) and the simple simultaneous equations for the solution of which they serve, which are

$$\left.\begin{array}{l} 12\dfrac{d\psi_2}{dt}+13\dfrac{d\psi_3}{dt}+14\dfrac{d\psi_4}{dt}+\varpi_1\psi_1=0, \\ 21\dfrac{d\psi_1}{dt}+23\dfrac{d\psi_3}{dt}+24\dfrac{d\psi_4}{dt}+\varpi_2\psi_2=0, \\ \quad\text{etc.}\qquad\qquad \text{etc.} \end{array}\right\}\ldots\ldots\ldots\ldots(64);$$

and all the formulas which we meet in so doing are real when $\varpi_1, \varpi_2, \varpi_3, \varpi_4$ are all of one sign, and therefore $12'$, $13'$, etc., all real. In the case of some of the ϖ's negative and some positive there is no difficulty in realizing the formulas, but the consideration of the simultaneous reduction of the two quadratics,

$$\left.\begin{array}{l} \tfrac{1}{2}\left\{\dfrac{(12\,a_2+13\,a_3+14\,a_4)^2}{\varpi_1}+\dfrac{(21\,a_2+23\,a_3+24\,a_4)^2}{\varpi_2}+\text{etc.}\right\} \\ \text{and}\qquad \tfrac{1}{2}(\varpi_1a_1^2+\varpi_2a_2^2+\varpi_3a_3^2+\varpi_4a_4^2) \end{array}\right\}(65),$$

to which we are led when we go back from the notation $12'$, etc. of (36), is not completely instructive in respect to stability, as was our previous explicit working out of the two roots of the determinantal equation in (37), (38), and (40).

provided that gyrostatic influence be fully dominant.

345xviii. The conditions to be fulfilled that the system may be dominated by gyrostatic influence are that the smaller value of $-\lambda^2$ found from (31) and the greater found from (34) be respectively very great in comparison with the greatest and very small in comparison with the smallest, of the four quantities $\varpi_1, \varpi_2, \varpi_3, \varpi_4$ irrespectively of their signs. Supposing ϖ_1 to be the greatest and ϖ_4 the smallest, these conditions are easily proved to be fulfilled when, and only when,

$$\frac{(12\,.\,34+13\,.\,42+14\,.\,23)^2}{12^2+13^2+14^2+34^2+42^2+23^2}>>\pm\varpi_1\ldots\ldots\ldots(66),$$

and

$$\frac{(12\,.\,34+13\,.\,42+14\,.\,23)^2}{12^2\varpi_3\varpi_4+13^2\varpi_4\varpi_2+14^2\varpi_2\varpi_3+34^2\varpi_1\varpi_2+42^2\varpi_1\varpi_3+23^2\varpi_1\varpi_4}>>\pm\varpi_4^{-1}\ (67),$$

where $>>$ denotes "*very great in comparison with.*" When these conditions are fulfilled, let 12, 13, 23, etc., be each increased in the ratio of N to 1. The two greater values of n (or $\lambda\sqrt{-1}$) will be increased in the same ratio, N to 1; and the two smaller

will be diminished each in the inverse ratio, 1 to N. Again, let $\sqrt{\pm\varpi_1}$, $\sqrt{\pm\varpi_2}$, $\sqrt{\pm\varpi_3}$, $\sqrt{\pm\varpi_4}$ be each diminished in the ratio M to 1; the two larger values of n will be sensibly unaltered; and the two smaller will be diminished in the ratio M^2 to 1.

Limits of smallest and second smallest of the four periods.

345^{xix}. Remark that

(*a*) When (66) is satisfied the two greater values of n are each

$$\left.\begin{aligned} &< \sqrt{\{(12^2+13^2+14^2+34^2+42^2+23^2)\}} \\ \text{and}\quad &> \frac{12\,.\,34+13\,.\,42+14\,.\,23}{\sqrt{(12^2+13^2+14^2+34^2+42^2+23^2)}} \end{aligned}\right\}\ldots\ldots(68);$$

and that when they are very unequal the greater is approximately equal to the former limit and the less to the latter.

(*b*) When (67) is satisfied, and when the equilibrium is stable, the two smaller values of n are each

$$\left.\begin{aligned} &< \frac{\sqrt{(12^2\varpi_3\varpi_4+13^2\varpi_4\varpi_2+14^2\varpi_2\varpi_3+34^2\varpi_1\varpi_2+42^2\varpi_1\varpi_3+23^2\varpi_1\varpi_4)}}{12\,.\,34+13\,.\,42+14\,.\,23} \\ &\text{and} \\ &> \frac{\sqrt{(\varpi_1\varpi_2\varpi_3\varpi_4)}}{\sqrt{\{(12^2\varpi_3\varpi_4+13^2\varpi_4\varpi_2+14^2\varpi_2\varpi_3+34^2\varpi_1\varpi_2+42^2\varpi_1\varpi_3+23^2\varpi_1\varpi_4)\}}} \end{aligned}\right\}(69),$$

Limits of the next greatest and greatest of the four periods.

and that when they are very unequal the greater of the two is approximately equal to the former limit, and the less to the latter.

345^{xx}. Both (66) and (67) must be satisfied in order that the four periods may be found approximately by the solution of the two quadratics (31), (34). If (66) is satisfied but not (67), the biquadratic determinant still splits into two quadratics, of which one is approximately (31) but the other is not approximately (34). Similarly, if (67) is satisfied but not (66), the biquadratic splits into two quadratics of which one is approximately (34) but the other not approximately (31).

Quadruply free cycloidal system with non-dominant gyrostatic influences.

345^{xxi}. When neither (66) nor (67) is fulfilled there is not generally any splitting of the biquadratic into two rational quadratics; and the conditions of stability, the determination of the fundamental periods, and the working out of the complete solution depend essentially on the roots of a biquadratic equation. When ϖ_1, ϖ_2, ϖ_3, ϖ_4 are all positive it is clear from the equation

of energy [345^{ii}, (4), with $Q=0$] that the motion is stable whatever be the values of the gyrostatic coefficients 12, 34, 13, etc. and therefore in this case each of the four roots λ^2 of the biquadratic is real and negative, a proposition included in the general theorem of § 345^{xxvi} below. To illustrate the interesting questions which occur when the ϖ's are not all positive put

Quadruply free cycloidal system with non-dominant gyrostatic influences.

$$12=_{12}g,\ 34=_{34}g,\ 13=_{13}g,\ \text{etc.}\ldots\ldots\ldots\ldots\ldots\ldots(70),$$

where $_{12}$, $_{34}$, $_{13}$, etc. denote any numerics whatever subject only to the condition that they do not make zero of

$$_{12}\,.\,_{34}+{}_{13}\,.\,_{42}+{}_{14}\,.\,_{23}.$$

When ϖ_1, ϖ_2, ϖ_3, ϖ_4, are all negative each root λ^2 of the biquadratic is as we have seen in § 345^{xiii} real and negative when the gyrostatic influences dominate. It becomes an interesting question to be answered by treatment of the biquadratic, how small may g be to keep all the roots λ^2 real and negative, and how large may g be to render them other than real and positive as they are when $g=0$? Similar questions occur in connexion with the case of two of the ϖ's negative and two positive, when the gyrostatic influences are so proportioned as to fulfil 345^{xiii} (41), so that when g is infinitely great there is complete gyrostatic stability, though when $g=0$ there are two instabilities and two stabilities.

Gyrostatic system with any number of freedoms.

345^{xxii}. Returning now to 345^{x} and 345^{vii}, 345^{viii}, and 345^{ix}, for a gyrostatic system with any number of freedoms, we see by 345^{vi} that the roots λ^2 of the determinantal equation (14) or (17) are necessarily real and negative when ϖ_1, ϖ_2, ϖ_3, ϖ_4, etc. are all positive. This conclusion is founded on the reasoning of § 345^{ii} regarding the equation of energy (4) applied to the case $Q=0$, for which it becomes $T+V=E_0$, or the same as for the case of no motional forces. It is easy of course to eliminate dynamical considerations from the reasoning and to give a purely algebraic proof that the roots λ^2 of the determinantal equation (14) of 345^{viii} are necessarily real and negative, provided both of the two quadratic functions $(11)\,a_1^2+2\,(12)\,a_1a_2+$ etc., and $11a_1^2+2\,12a_1a_2+$ etc. are positive for all real values of a_1, a_2, etc. But the equations (14) of § 343 (*k*), which we obtained and used in the course of the corresponding demonstration for the case of no motional forces, do not hold in our present case of gyrostatic motional forces. Still for this present case we have the con-

Case of equal roots with stability.

clusion of § 343 (m) that equality among the roots falls essentially under the case of § 343 (e) above. For we know from the consideration of energy, as in § 345^{ii}, that no particular solution can be of the form $t\epsilon^{\lambda t}$ or $t \sin \sigma t$, when the potential energy is positive for all displacements: yet [though there cannot be equal roots for the gyrostatic system of two freedoms (§ 345^{x}) as we see from the solution (25) of the determinantal equation for this case] there obviously may be equality of roots* in a quadruply free gyrostatic system, or in one with more than four freedoms. Hence, if both the quadratic functions have the same sign for all real values of a_1, a_2, etc., all the first minors

Application of Routh's theorem.

* Examples of this may be invented *ad libitum* by commencing with pairs of equations such as (23) and altering the variables by (generalized) orthogonal transformations. For one very simple example put $\zeta = \varpi$ and take (23) as one pair of equations of motion, and as a second pair take

$$\ddot{\xi}' + \gamma\dot{\eta}' + \varpi\xi' = 0,$$
$$\ddot{\eta}' - \gamma\dot{\xi}' + \varpi\eta' = 0.$$

The second of (23) and the first of these multiplied respectively by $\cos\alpha$ and $\sin\alpha$, and again by $\sin\alpha$ and $\cos\alpha$, and added and subtracted, give

$$\ddot{\psi}_2 - \gamma\cos\alpha\dot{\xi} + \gamma\sin\alpha\dot{\eta}' + \varpi\psi_2 = 0,$$

and

$$\ddot{\psi}_3 + \gamma\sin\alpha\dot{\xi} + \gamma\cos\alpha\dot{\eta}' + \varpi\psi_3 = 0,$$

where

$$\psi_2 = \xi'\sin\alpha + \eta\cos\alpha,$$

and

$$\psi_3 = \xi'\cos\alpha - \eta\sin\alpha.$$

Eliminating ξ' and η by these last equations, from the first and fourth of the equations of motion, and for symmetry putting ψ_1 instead of ξ, and ψ_4 instead of η', and for simplicity putting $\gamma\cos\alpha = g$, and $\gamma\sin\alpha = h$, and collecting the equations of motion in order, we have the following,—

$$\ddot{\psi}_1 + g\dot{\psi}_2 - h\dot{\psi}_3 + \varpi\psi_1 = 0,$$
$$\ddot{\psi}_2 - g\dot{\psi}_1 + h\dot{\psi}_4 + \varpi\psi_2 = 0,$$
$$\ddot{\psi}_3 + h\dot{\psi}_1 + g\dot{\psi}_4 + \varpi\psi_3 = 0,$$
$$\ddot{\psi}_4 - h\dot{\psi}_2 - g\dot{\psi}_3 + \varpi\psi_4 = 0,$$

for the equations of motion of a quadruply free gyrostatic system having two equalities among its four fundamental periods. The two different periods are the two values of the expression

$$2\pi/\{\sqrt{(\tfrac{1}{2}g^2 + \tfrac{1}{2}h^2)} \pm \sqrt{(\tfrac{1}{2}g^2 + \tfrac{1}{2}h^2 + \varpi)}\}.$$

When these two values are unequal the equalities among the roots *do not* give rise to terms of the form $t\epsilon^{\lambda t}$ or $t\cos\sigma t$ in the solution. But if $\varpi = -(\tfrac{1}{2}g^2 + \tfrac{1}{2}h^2)$, which makes these two values equal, and therefore all four roots equal, terms of the form $t\cos\sigma t$ *do appear* in the solution, and the equilibrium is unstable in the transitional case though it is stable if $-\varpi$ be less than $\tfrac{1}{2}g^2 + \tfrac{1}{2}h^2$ by ever so small a difference.

Application of Routh's theorem.

of the determinantal equation (14), § 345^{viii}, must vanish for each double, triple, or multiple root of the equation, if it has any such roots.

It will be interesting to find a purely algebraic proof of this theorem, and we leave it as an exercise to the student; remarking only that, when the quadratic functions have contrary signs for some real values of a_1, a_2, etc., there may be equality among the roots without the evanescence of all the first minors; or, in dynamical language, there may be terms of the form $t\epsilon^{\lambda t}$, or $t\sin\sigma t$, in the solution expressing the motion of a gyrostatic system, in transitional cases between stability and instability. It is easy to invent examples of such cases, taking for instance the quadruply free gyrostatic system, whether gyrostatically dominated as in § 345^{xiii}, but in this case with some of the four quantities negative, and some positive; or, as in § 345^{xxi}, not gyrostatically dominated, with either some or all of the quantities $\varpi_1, \varpi_2, \ldots, \varpi_4$ negative. All this we recommend to the student as interesting and instructive exercise.

Equal roots with instability in transitional cases between stability and instability.

Conditions of gyrostatic domination.

345^{xxiii}. When all the quantities $\varpi_1, \varpi_2, \ldots, \varpi_4$ are of the same sign it is easy to find the conditions that must be fulfilled in order that the system may be gyrostatically dominated. For if $\rho_1, \rho_2, \ldots, \rho_n$ are the roots of the equation

$$c_0 z^n + c_1 z^{n-1} + \ldots + c_{n-1} z + c_n = 0,$$

we have

$$-(\rho_1 + \rho_2 + \ldots + \rho_n) = \frac{c_1}{c_0}, \text{ and } -\left(\frac{1}{\rho_1} + \frac{1}{\rho_2} + \ldots + \frac{1}{\rho_n}\right) = \frac{c_{n-1}}{c_n}.$$

Hence if $-\rho_1, -\rho_2, \ldots -\rho_n$ be each positive, c_1/nc_0 is their arithmetic mean, and nc_n/c_{n-1} is their harmonic mean. Hence c_1/nc_0 is greater than nc_n/c_{n-1}, and the greatest of $-\rho_1, -\rho_2, \ldots, -\rho_n$ is greater than c_1/nc_0, and the least of them is less than nc_n/c_{n-1}. Take now the two following equations:

$$\lambda^i + \lambda^{i-2}\sum 12^2 + \lambda^{i-4}\sum(\Sigma 12 \cdot 34)^2 + \lambda^{i-6}\sum(\Sigma 12 \cdot 34 \cdot 56)^2 + \text{etc.} = 0 \quad \ldots\ldots\ldots (71),$$

$$\left(\frac{1}{\lambda}\right)^i + \left(\frac{1}{\lambda}\right)^{i-2}\sum 12'^2 + \left(\frac{1}{\lambda}\right)^{i-4}\sum(\Sigma 12' \cdot 34')^2 + \left(\frac{1}{\lambda}\right)^{i-6}\sum(\Sigma 12' \cdot 34' \cdot 56')^2 + \text{etc.} = 0 \quad (72),$$

where

$$12' = \frac{12}{\sqrt{(\varpi_1\varpi_2)}},\ 13' = \frac{13}{\sqrt{(\varpi_1\varpi_3)}},\ 34' = \frac{34}{\sqrt{(\varpi_3\varpi_4)}}, \ldots, i-1, i' = \frac{i-1, i}{\sqrt{(\varpi_{i-1}\varpi_i)}} \quad (73).$$

Cycloidal motion. Conditions of gyrostatic domination.

Suppose for simplicity i to be even. All the roots λ^2 of (71) are (§ 345xxvi below) essentially real and negative. So are those of (72) provided $\varpi_1, \varpi_2, \ldots, \varpi_i$ are all of one sign as we now suppose them to be. Hence the smallest root $-\lambda^2$ of (71) is less than

$$\frac{\frac{1}{2}i\,\Sigma\,(12\,.\,34\,.\,56,\ldots,\,i-1,\,i)^2}{\Sigma\,(\Sigma\,12\,.\,34\,.\,56,\ldots,\,i-3,\,i-2)^2}\ldots\ldots\ldots\ldots\ldots(74),$$

and the greatest root $-\lambda^2$ of (72) is greater than

$$\frac{\Sigma\,(\Sigma\,12'\,.\,34'\,.\,56',\ldots,\,i-3,\,i-2')^2}{\frac{1}{2}i\,\Sigma\,(12'\,.\,34'\,.\,56',\ldots,\,i-1,\,i')}\ldots\ldots\ldots\ldots(75).$$

Hence the conditions for gyrostatic domination are that (74) must be much greater than the greatest of the positive quantities $\pm\varpi_1$, $\pm\varpi_2, \ldots, \pm\varpi_i$, and that (75) must be very much less than the least of these positive quantities. When these conditions are fulfilled the i roots of (18) § 345ix equated to zero are separable into two groups of $\frac{1}{2}i$ roots which are infinitely nearly equal to the roots of equations (71) and (72) respectively, conditions of reality of which are investigated in § 345xxvi below. The interpretation leads to the following interesting conclusions:—

Gyrostatic links explained.

345xxiv. Consider a cycloidal system provided with non-rotating flywheels mounted on frames so connected with the moving parts as to give infinitesimal angular motions to the axes of the flywheels proportional to the motions of the system. Let the number of freedoms of the system exclusive of the ignored co-ordinates [§ 319, Ex. (G)] of the flywheels relatively to their frames be even. Let the forces of the system be such that when the flywheels are given at rest, when the system is at rest, the equilibrium is either stable for all the freedoms, or unstable for all the freedoms. Let the number and connexions of the gyrostatic links be such as to permit gyrostatic domination (§ 345xxi) when each of the flywheels is set into sufficiently rapid rotation. Now let the flywheels be set each into sufficiently rapid rotation to fulfil the conditions of gyrostatic domination (§ 345xxi): the equilibrium of the system becomes stable: with half the whole number i of its modes of vibration exceedingly rapid, with frequencies equal to the roots of a certain algebraic equation of the degree $\frac{1}{2}i$; and the other half of

Gyrostatically dominated system:

its modes of vibration very slow, with frequencies given by the roots of another algebraic equation of degree $\frac{1}{2}i$. The first class of fundamental modes may be called adynamic because they are the same as if no forces were applied to the system, or acted between its moving parts, except actions and reactions in the normals between mutually pressing parts (depending on the inertias of the moving parts). The second class of fundamental modes may be called precessional because the precession of the equinoxes, and the slow precession of a rapidly spinning top supported on a very fine point, are familiar instances of it. Remark however that the obliquity of the ecliptic should be infinitely small to bring the precession of the equinoxes precisely within the scope of the equations of our "cycloidal system."

Gyrostatically dominated system: its adynamic oscillations (very rapid); and precessional oscillations (very slow).

345xxv. If the angular velocities of all the flywheels be altered in the same proportion the frequencies of the adynamic oscillations will be altered in the same proportion directly, and those of the precessional modes in the same proportion inversely. Now suppose there to be either no inertia in the system except that of the flywheels round their pivoted axes and round their equatorial diameters, or suppose the effective inertia of the connecting parts to be comparable with that of the flywheels when given without rotation. The period of each of the adynamic modes is comparable with the periods of the flywheels. And the periods of the precessional modes are comparable with a third proportional to a mean of the periods of the flywheels and a mean of the irrotational periods of the system, if the system be stable when the flywheels are deprived of rotation. For the last mentioned term of the proportion we may, in the case of irrotational *instability*, substitute the time of increasing a displacement a thousandfold, supposing the system to be falling away from its configuration of equilibrium according to one of its fundamental modes of motion ($\epsilon^{\lambda t}$). The reciprocal of this time we shall call, for brevity, the rapidity of the system, for convenience of comparison with the frequency of a vibrator or of a rotator, which is the name commonly given to the reciprocal of its period.

Comparison between adynamic frequencies, rotational frequencies of the flywheels, precessional frequencies of the system, and frequencies or rapidities of the system, with flywheels deprived of rotation.

Proof of reality of adynamic and of precessional periods when system's irrotational periods are either all real or all imaginary.

345xxvi. It remains to prove that the roots λ^2 of (71), and of (72) also when $\varpi_1, \varpi_2, \ldots, \varpi_i$ are all of one sign, are essentially real and negative. (71) is the determinantal equation of § 345xiv (42) with any even number of equations instead of only four. The treatment of §§ 345xiv and 345xv is all directly applicable without change to this extension; and it proves that the roots λ^2 are real and negative by bringing the problem to that of the orthogonal reduction of the essentially positive quadratic function

$$G(aa)=\tfrac{1}{2}\{(12a_2+13a_3+\text{etc.})^2+(21a_1+23a_3+\text{etc.})^2+(31a_1+32a_2+\text{etc.})^2+\text{etc.}\} \quad (76):$$

it proves also the equalities of energies of (56), § 345xv, and the orthogonalities of (55), (58) § 345xv: also the curious algebraic theorem that the determinantal roots of the quadratic function consist of $\frac{1}{2}i$ pairs of equals.

Algebraic theorem.

Inasmuch as (72) is the same as (71) with λ^{-1} put for λ and $12'$, $13'$, $23'$, etc. for 12, 13, 23, etc., all the formulas and propositions which we have proved for (71) hold correspondingly for (72) when $12'$, $13'$, $23'$, etc. are all real, as they are when $\varpi_1, \varpi_2, \ldots \varpi_i$ are all of one sign.

345xxvii. Going back now to § 345viii, and taking advantage of what we have learned in § 345ix and the consequent treatment of the problem, particularly that in § 345xiv, we see now how to simplify equations (14) of § 345viii otherwise than was done in § 345ix, by a new method which has the advantage of being applicable also to materially simplify the general equations (13) of § 345vi. Apply orthogonal transformation of the co-ordinates to reduce to a sum of squares of simple co-ordinates, the quadratic function (76). Thus denoting by $G(\psi\psi)$ what $G(aa)$ becomes when ψ_1, ψ_2, etc. are substituted for a_1, a_2, etc.; and denoting by $n_1^2, n_2^2, \ldots, n_{\frac{1}{2}i}^2$ the values of the pairs of roots of the determinantal equation of degree i, which are simply the negative of the roots λ^2 of equation (71) of degree $\frac{1}{2}i$ in λ^2; and denoting by $\xi_1, \eta_1, \xi_2, \eta_2, \ldots \xi_{\frac{1}{2}i}\eta_{\frac{1}{2}i}$, the fresh co-ordinates, we have

$$G(\psi\psi)=\tfrac{1}{2}\{n_1^2(\xi_1^2+\eta_1^2)+n_2^2(\xi_2^2+\eta_2^2)+\ldots+n_{\frac{1}{2}i}^2(\xi_{\frac{1}{2}i}^2+\eta_{\frac{1}{2}i}^2)\}\ldots(77).$$

It is easy to see that the general equations of cycloidal motion (13) of § 345vi transformed to the ξ-co-ordinates come out in $\frac{1}{2}i$ pairs as follows:

Cycloidal motion.

$$\left.\begin{array}{l}
\left\{\begin{array}{l}\dfrac{d}{dt}\dfrac{dT}{d\dot{\xi}_1}+\dfrac{dQ}{d\dot{\xi}_1}+n_1\dot{\eta}_1+\dfrac{dV}{d\xi_1}=0\\[2ex] \dfrac{d}{dt}\dfrac{dT}{d\dot{\eta}_1}+\dfrac{dQ}{d\dot{\eta}_1}-n_1\dot{\xi}_1+\dfrac{dV}{d\eta_1}=0\end{array}\right.\\[2ex]
\left\{\begin{array}{l}\dfrac{d}{dt}\dfrac{dT}{d\dot{\xi}_2}+\dfrac{dQ}{d\dot{\xi}_2}+n_2\dot{\eta}_2+\dfrac{dV}{d\xi_2}=0\\[2ex] \dfrac{d}{dt}\dfrac{dT}{d\dot{\eta}_2}+\dfrac{dQ}{d\dot{\eta}_2}-n_2\dot{\xi}_2+\dfrac{dV}{d\eta_2}=0\end{array}\right.\\[2ex]
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\[2ex]
\left\{\begin{array}{l}\dfrac{d}{dt}\dfrac{dT}{d\dot{\xi}_{\frac{1}{2}i}}+\dfrac{dQ}{d\dot{\xi}_{\frac{1}{2}i}}+n_{\frac{1}{2}i}\dot{\eta}_{\frac{1}{2}i}+\dfrac{dV}{d\xi_{\frac{1}{2}i}}=0\\[2ex] \dfrac{d}{dt}\dfrac{dT}{d\dot{\eta}_{\frac{1}{2}i}}+\dfrac{dQ}{d\dot{\eta}_{\frac{1}{2}i}}-n_{\frac{1}{2}i}\dot{\xi}_{\frac{1}{2}i}+\dfrac{dV}{d\eta_{\frac{1}{2}i}}=0\end{array}\right.
\end{array}\right\}\ldots\ldots\ldots\ldots\ldots(78).$$

345xxviii. Considerations of space and time prevent us from detailed treatment at present of gyrostatic systems with odd numbers of degrees of freedom, but it is obvious from § 345xxvii and 345xi that the general equations (13) of § 345vi may, when i the number of freedoms is odd, by proper transformation from co-ordinates ψ_1, ψ_2, etc. to a set of co-ordinates ζ, ξ_1, $\eta_2, \ldots \xi_{\frac{1}{2}(i-1)}$, $\eta_{\frac{1}{2}(i-i)}$ be reduced to the following form:

$$\left.\begin{array}{l}
\left\{\begin{array}{l}\dfrac{d}{dt}\dfrac{dT}{d\dot{\xi}_1}+\dfrac{dQ}{d\dot{\xi}_1}+n_1\dot{\eta}_1+\dfrac{dV}{d\xi_1}=0\\[2ex] \dfrac{d}{dt}\dfrac{dT}{d\dot{\eta}_1}+\dfrac{dQ}{d\dot{\eta}_1}-n_1\dot{\xi}_1+\dfrac{dV}{d\eta_1}=0\end{array}\right.\\[2ex]
\left\{\begin{array}{l}\dfrac{d}{dt}\dfrac{dT}{d\dot{\xi}_2}+\dfrac{dQ}{d\dot{\xi}_2}+n_2\dot{\eta}_2+\dfrac{dV}{d\xi_2}=0\\[2ex] \dfrac{d}{dt}\dfrac{dT}{d\dot{\eta}_2}+\dfrac{dQ}{d\dot{\eta}_2}-n_2\dot{\xi}_2+\dfrac{dV}{d\eta_2}=0\end{array}\right.\\[2ex]
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\[2ex]
\left\{\begin{array}{l}\dfrac{d}{dt}\dfrac{dT}{d\dot{\xi}_{\frac{1}{2}(i-1)}}+\dfrac{dQ}{d\xi_{\frac{1}{2}(i-1)}}+n_{\frac{1}{2}(i-1)}\dot{\eta}_{\frac{1}{2}(i-1)}+\dfrac{dV}{d\xi_{\frac{1}{2}(i-1)}}=0\\[2ex] \dfrac{d}{dt}\dfrac{dT}{d\dot{\eta}_{\frac{1}{2}(i-1)}}+\dfrac{dQ}{d\dot{\eta}_{\frac{1}{2}(i-1)}}-n_{\frac{1}{2}(i-1)}\dot{\xi}_{\frac{1}{2}(i-1)}+\dfrac{dV}{d\eta_{\frac{1}{2}(i-1)}}=0\end{array}\right.\\[2ex]
\dfrac{d}{dt}\dfrac{dT}{d\dot{\zeta}}+\dfrac{dQ}{d\dot{\zeta}}+\dfrac{dV}{d\zeta}=0
\end{array}\right\}\ldots\ldots(79).$$

Kinetic stability.

346. There is scarcely any question in dynamics more important for Natural Philosophy than the stability or instability of motion. We therefore, before concluding this chapter, propose to give some general explanations and leading principles regarding it.

Conservative disturbance.

A "conservative disturbance of motion" is a disturbance in the motion or configuration of a conservative system, not altering the sum of the potential and kinetic energies. A conservative disturbance of the motion through any particular configuration is a change in velocities, or component velocities, not altering the whole kinetic energy. Thus, for example, a conservative disturbance of the motion of a particle through any point is a change in the direction of its motion, unaccompanied by change of speed.

Kinetic stability and instability discriminated.

347. The actual motion of a system, from any particular configuration, is said to be *stable* if every possible infinitely small conservative disturbance of its motion through that configuration may be compounded of conservative disturbances, any one of which would give rise to an alteration of motion which would bring the system again to some configuration belonging to the undisturbed path, in a finite time, and without more than an infinitely small digression. If this condition is not fulfilled, the motion is said to be *unstable**.

Examples.

348. For example, if a body, A, be supported on a fixed vertical axis; if a second, B, be supported on a parallel axis belonging to the first; a third, C, similarly supported on B, and so on; and if B, C, etc., be so placed as to have each its centre of inertia as far as possible from the fixed axis, and the whole set in motion with a common angular velocity about this axis, the motion will be stable, from every configuration, as is evident from the principles regarding the resultant centrifugal force on a rigid body, to be proved later. If, for instance, each of the bodies is a flat rectangular board hinged on one edge, it is obvious that the whole system will be kept stable by centrifugal force, when all are in one plane and as far out from the axis as possible. But if A consist partly of a shaft and crank, as a common spinning-wheel, or the fly-wheel and crank of a

* [This definition is somewhat narrow; according to it the vertical fall of a particle under gravity would be characterized as unstable. It does not appear to be easy to frame a perfectly general *and natural* definition of dynamical stability. In § 778″ *post* the term is used in a different sense. H. L.]

steam-engine, and if B be supported on the crank-pin as axis, and turned inwards (towards the fixed axis, or across the fixed axis), then, even although the centres of inertia of C, D, etc., are placed as far from the fixed axis as possible, consistent with this position of B, the motion of the system will be unstable. Examples.

349. The rectilinear motion of an elongated body lengthwise, or of a flat disc edgewise, through a fluid is unstable. But the motion of either body, with its length or its broadside perpendicular to the direction of motion, is stable. This is demonstrated for the ideal case of a perfect liquid (§ 320), in § 321, Example (2); and the results explained in § 322 show, for a solid of revolution, the precise character of the motion consequent upon an infinitely small disturbance in the direction of the motion from being exactly along or exactly perpendicular to the axis of figure; whether the infinitely small oscillation, in a definite period of time, when the rectilineal motion is stable, or the swing round to an infinitely nearly inverted position when the rectilineal motion is unstable. Observation proves the assertion we have just made, for real fluids, air and water, and for a great variety of circumstances affecting the motion. Several illustrations have been referred to in § 325; and it is probable we shall return to the subject later, as being not only of great practical importance, but profoundly interesting although very difficult in theory. Kinetic stability Hydrodynamic example.

350. The motion of a single particle affords simpler and not less instructive illustrations of stability and instability. Thus if a weight, hung from a fixed point by a light inextensible cord, be set in motion so as to describe a circle about a vertical line through its position of equilibrium, its motion is stable. For, as we shall see later, if disturbed infinitely little in direction without gain or loss of energy, it will describe a sinuous path, cutting the undisturbed circle at points successively distant from one another by definite fractions of the circumference, depending upon the angle of inclination of the string to the vertical. When this angle is very small, the motion is sensibly the same as that of a particle confined to one plane and moving under the influence of an attractive Circular simple pendulum.

force towards a fixed point, simply proportional to the distance; and the disturbed path cuts the undisturbed circle four times in a revolution. Or if a particle confined to one plane, move under the influence of a centre in this plane, attracting with a force inversely as the square of the distance, a path infinitely little disturbed from a circle will cut the circle twice in a revolution. Or if the law of central force be the nth power of the distance, and if $n+3$ be positive, the disturbed path will cut the undisturbed circular orbit at successive angular intervals, each equal to $\pi/\sqrt{n+3}$. But the motion will be unstable if n be negative, and $-n>3$.

Circular orbit.

Kinetic stability in circular orbit.

The criterion of stability is easily investigated for circular motion round a centre of force from the differential equation of the general orbit (§ 36),

$$\frac{d^2u}{d\theta^2}+u=\frac{P}{h^2u^2}.$$

Let the value of h be such that motion in a circle of radius a^{-1} satisfies this equation. That is to say, let $P/h^2u^2=u$, when $u=a$. Let now $u=a+\rho$, ρ being infinitely small. We shall have

$$u-\frac{P}{h^2u^2}=a\rho,$$

if a denotes the value of $\frac{d}{du}\left(u-\frac{P}{h^2u^2}\right)$ when $u=a$: and therefore the differential equation for motion infinitely nearly circular is

$$\frac{d^2\rho}{d\theta^2}+a\rho=0.$$

The integral of this is most conveniently written

$$\rho=A\sin(\theta\sqrt{a}+\beta)$$

when a is positive, and

$$\rho=C\epsilon^{\theta\sqrt{-a}}+C'\epsilon^{-\theta\sqrt{-a}}$$

when a is negative.

Hence we see that the circular motion is stable in the former case, and unstable in the latter.

Kinetic stability in circular orbit.

For instance, if $P=\mu r^n=\mu u^{-n}$, we have

$$\frac{d}{du}\left(u-\frac{P}{h^2u^2}\right)=1+(n+2)\frac{P}{h^2u^3};$$

and putting $\frac{P}{h^2u^2}=u=a$, in this we find $a=n+3$; whence the result stated above.

Or, taking Example (B) of § 319, and putting mP for P, and mh for h,

$$\frac{P}{h^2u^2}=\frac{m'}{m+m'}\left(\frac{g}{h^2}u^{-2}+u\right),$$

$$\frac{d}{du}\left(u-\frac{P}{h^2u^2}\right)=\frac{m+\frac{2m'g}{h^2u^3}}{m+m'}.$$

Hence, putting $u=a$, and making $h^2=gm'/ma^3$ so that motion in a circle of radius a^{-1} may be possible, we find

$$a=\frac{3m}{m+m'}.$$

Hence the circular motion is always stable; and the period of the variation produced by an infinitely small disturbance from it is

$$2\pi\sqrt{\frac{m+m'}{3m}}.$$

Kinetic stability of a particle moving on a smooth surface.

351. The case of a particle moving on a smooth fixed surface under the influence of no other force than that of the constraint, and therefore always moving along a geodetic line of the surface, affords extremely simple illustrations of stability and instability. For instance, a particle placed on the inner circle of the surface of an anchor-ring, and projected in the plane of the ring, would move perpetually in that circle, but unstably, as the smallest disturbance would clearly send it away from this path, never to return until after a digression round the outer edge. (We suppose of course that the particle is held to the surface, as if it were placed in the infinitely narrow space between a solid ring and a hollow one enclosing it.) But if a particle is placed on the outermost, or greatest,

Kinetic stability of a particle moving on a smooth surface.

circle of the ring, and projected in its plane, an infinitely small disturbance will cause it to describe a sinuous path cutting the circle at points round it successively distant by angles each equal to $\pi\sqrt{b/a}$, or intervals of time, $\pi\sqrt{b}/\omega\sqrt{a}$, where a denotes the radius of that circle, ω the angular velocity in it, and b the radius of the circular cross section of the ring. This is proved by remarking that an infinitely narrow band from the outermost part of the ring has, at each point, a and b for its principal radii of curvature, and therefore (§ 150) has for its geodetic lines the great circles of a sphere of radius $\sqrt{ab}$, upon which (§ 152) it may be bent.

352. In all these cases the undisturbed motion has been circular or rectilineal, and, when the motion has been stable, the effect of a disturbance has been *periodic*, or recurring with the same phases in equal successive intervals of time. An illustration of thoroughly stable motion in which the effect of a disturbance is not "periodic," is presented by a particle sliding down an inclined groove under the action of gravity. To take the simplest case, we may consider a particle sliding down along the lowest straight line of an inclined hollow cylinder. If slightly disturbed from this straight line, it will oscillate on each side of it perpetually in its descent, but not with a uniform periodic motion, though the durations of its excursions to each side of the straight line are all equal.

Kinetic stability. Incommensurable oscillations.

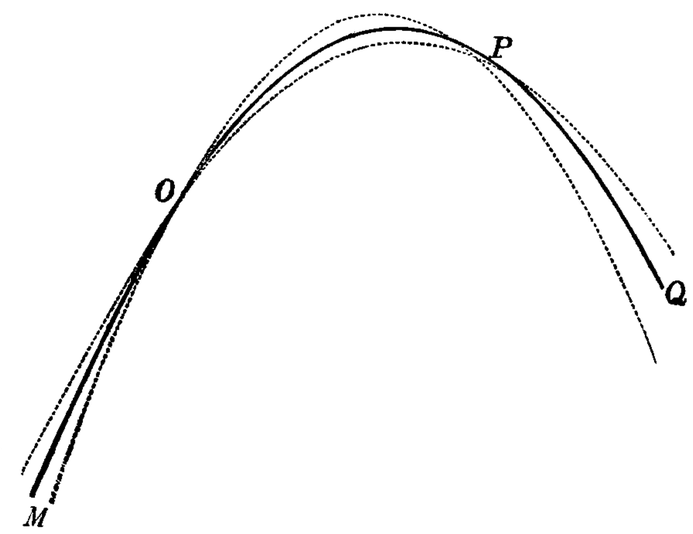
353. A very curious case of stable motion is presented by a particle constrained to remain on the surface of an anchor-ring fixed in a vertical plane, and projected along the great circle from any point of it, with any velocity. An infinitely small disturbance will give rise to a disturbed motion of which the path will cut the vertical circle over and over again for ever, at unequal intervals of time, and unequal angles of the circle; and obviously not recurring periodically in any cycle, except with definite particular values for the whole energy, some of which are less and an infinite number are greater than that which just suffices to bring the particle to the highest point of the ring. The full mathematical investigation of these

circumstances would afford an excellent exercise in the theory of differential equations, but it is not necessary for our present illustrations.

Oscillatory kinetic stability.

354. In this case, as in all of stable motion with only two degrees of freedom, which we have just considered, there has been stability throughout the motion; and an infinitely small disturbance from any point of the motion has given a disturbed path which intersects the undisturbed path over and over again at finite intervals of time. But, for the sake of simplicity at present confining our attention to two degrees of freedom, we have a *limited* stability in the motion of an unresisted projectile, which satisfies the criterion of stability only at points of its upward, not of its downward, path. Thus if $MOPQ$ be

Limited kinetic stability.



the path of a projectile, and if at O it be disturbed by an infinitely small force either way perpendicular to its instantaneous direction of motion, the disturbed path will cut the undisturbed infinitely near the point P where the direction of motion is perpendicular to that at O: as we easily see by considering that the line joining two particles projected from one point at the same instant with equal velocities in the directions of any two lines, will always remain perpendicular to the line bisecting the angle between these two lines.

Kinetic stability of a projectile.

General criterion.

355. The principle of varying action gives a mathematical criterion for stability or instability in every case of motion. Thus in the first place it is obvious, and it will be proved below (§§ 358, 361), that if the action is a true minimum in the motion of a system from any one configuration to the configuration reached at any other time, however much later, the motion is thoroughly unstable. For instance, in the motion of a particle constrained to remain on a smooth fixed surface, and uninfluenced by gravity, the action is simply the length of the path, multiplied by the constant velocity. Hence in the particular case of a particle uninfluenced by gravity, moving round the inner circle in the plane of an anchor-ring considered above, the action, or length of path, is clearly a minimum from any one point to the point reached at any subsequent time. (The action is not merely a minimum, but is the smaller of two minimums, when the course is from any point of the circular path to any other, through less than half a circumference of the circle.) On the other hand, although the path from any point in the greatest circle of the ring to any other at a distance from it along the circle, less than $\pi\sqrt{ab}$, is clearly least possible if along the circumference; the path of absolutely least length is not along the circumference between two points at a greater circular distance than $\pi\sqrt{ab}$ from one another, nor is the path along the circumference between them a minimum at all in this latter case. On any surface whatever which is everywhere anticlastic, or along a geodetic of any surface which passes altogether through an anticlastic region, the motion is thoroughly unstable. For if it were stable from any point O, we should have the given undisturbed path, and the disturbed path from O cutting it at some point Q;—two different geodetic lines joining two points; which is impossible on an anticlastic surface, inasmuch as the sum of the exterior angles of any closed figure of geodetic lines exceeds four right angles (§ 136) when the integral curvature of the enclosed area is negative, which (§§ 138, 128) is the case for every portion of surface thoroughly anticlastic. But, on the other hand, it is easily proved that if we have an endless rigid band of curved surface everywhere synclastic, with a geodetic line running through its

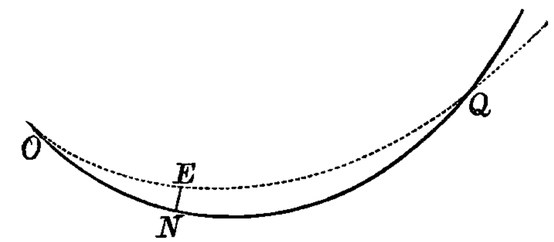
Examples.

Motion on an anticlastic surface proved unstable.

Motion of a particle on an anticlastic surface, unstable;

middle, the motion of a particle projected along this line will be stable throughout, and an infinitely slight disturbance will give a disturbed path cutting the given undisturbed path again and again for ever at successive distances differing according to the different specific curvatures of the intermediate portions of the surface. If from any point, N, of the undisturbed path, a perpendicular be drawn to cut the infinitely near disturbed path in E, the angles OEN and NOE must (§ 138) be together greater than a right angle by an amount equal to the integral curvature of the area EON. From this the differential equation of the disturbed path may be obtained immediately.

on a synclastic surface, stable.



Differential equation of disturbed path.

Let $\angle EON = \alpha$, $ON = s$, and $NE = u$; and let ϑ, a known function of s, be the specific curvature (§ 136) of the surface in the neighbourhood of N. Let also, for a moment, ϕ denote the complement of the angle OEN. We have

$$\alpha - \phi = \int_0^s \vartheta u ds.$$

Hence $$\frac{d\phi}{ds} = -\vartheta u.$$

But, obviously, $$\phi = \frac{du}{ds};$$

hence $$\frac{d^2u}{ds^2} + \vartheta u = 0.$$

When ϑ is constant (as in the case of the equator of a surface of revolution considered above, § 351), this gives

$$u = A \cos (s \sqrt{\vartheta} + E),$$

agreeing with the result (§ 351) which we obtained by development into a spherical surface.

The case of two or more bodies supported on parallel axes in the manner explained above in § 348, and rotating with the centre of inertia of the whole at the least possible distance from the fixed axis, affords a very good illustration also of this proposition which may be safely left as an exercise to the student.

General investigation of disturbed path.

356. To investigate the effect of an infinitely small conservative disturbance produced at any instant in the motion of any conservative system, may be reduced to a practicable problem (however complicated the required work may be) of mathematical analysis, provided the undisturbed motion is thoroughly known.

General equation of motion free in two degrees.

(*a*) First, for a system having but two degrees of freedom to move, let

$$2T = P\dot{\psi}^2 + Q\dot{\phi}^2 + 2R\dot{\psi}\dot{\phi} \quad \text{.....(1)},$$

where P, Q, R are functions of the co-ordinates not depending on the actual motion. Then

$$\left.\begin{aligned} \frac{dT}{d\dot{\psi}} &= P\dot{\psi} + R\dot{\phi}, \quad \frac{dT}{d\dot{\phi}} = Q\dot{\phi} + R\dot{\psi} \\ \frac{d}{dt}\frac{dT}{d\dot{\psi}} &= P\ddot{\psi} + R\ddot{\phi} + \frac{dP}{d\psi}\dot{\psi}^2 + \left(\frac{dP}{d\phi} + \frac{dR}{d\psi}\right)\dot{\psi}\dot{\phi} + \frac{dR}{d\phi}\dot{\phi}^2 \end{aligned}\right\} \ldots(2);$$

and the Lagrangian equations of motion [§ 318 (24)] are

$$\left.\begin{aligned} P\ddot{\psi} + R\ddot{\phi} + \tfrac{1}{2}\left\{\frac{dP}{d\psi}\dot{\psi}^2 + 2\frac{dP}{d\phi}\dot{\psi}\dot{\phi} + \left(2\frac{dR}{d\phi} - \frac{dQ}{d\psi}\right)\dot{\phi}^2\right\} &= \Psi \\ R\ddot{\psi} + Q\ddot{\phi} + \tfrac{1}{2}\left\{\left(2\frac{dR}{d\psi} - \frac{dP}{d\phi}\right)\dot{\psi}^2 + 2\frac{dQ}{d\psi}\dot{\psi}\dot{\phi} + \frac{dQ}{d\phi}\dot{\phi}^2\right\} &= \Phi \end{aligned}\right\} \ldots(3).$$

We shall suppose the system of co-ordinates so chosen that none of the functions P, Q, R, nor their differential coefficients $\frac{dP}{d\phi}$, etc., can ever become infinite.

(*b*) To investigate the effects of an infinitely small disturbance, we may consider a motion in which, at any time t, the co-ordinates are $\psi + p$ and $\phi + q$, p and q being infinitely small; and, by simply taking the variations of equations (3) in the usual manner, we arrive at two simultaneous differential equations of the second degree, linear with respect to

$$p,\ q,\ \dot{p},\ \dot{q},\ \ddot{p},\ \ddot{q},$$

but having variable coefficients which, when the undisturbed motion ψ, ϕ is fully known, may be supposed to be known functions of t. In these equations obviously none of the coefficients can at any time become infinite if the data correspond to a real dynamical problem, provided the system of co-ordinates is properly chosen (*a*); and the coefficients of $\ddot{p}$ and $\ddot{q}$ are the

values, at the time t, of P, R, and R, Q, respectively, in the order in which they appear in (3), P, Q, R being the coefficients of a homogeneous quadratic function (1) which is essentially positive. These properties being taken into account, it may be shown that in no case can an infinitely small interval of time be the solution of the problem presented (§ 347) by the question of kinetic stability or instability, which is as follows:— General investigation of disturbed path.

(*c*) The component velocities $\dot{\psi}$, $\dot{\phi}$ are at any instant changed to $\dot{\psi} + \alpha$, $\dot{\phi} + \beta$, subject to the condition of not changing the value of T. Then, α and β being infinitely small, it is required to find the interval of time until q/p first becomes equal to $\dot{\phi}/\dot{\psi}$.

(*d*) The differential equations in p and q reduce this problem, and in fact the full problem of finding the disturbance in the motion when the undisturbed motion is given, to a practicable form. But, merely to prove the proposition that the disturbed course cannot meet the undisturbed course until after some finite time, and to estimate a limit which this time must exceed in any particular case, it may be simpler to proceed thus:—

(*e*) To eliminate t from the general equations (3), let them first be transformed so as not to have t independent variable. We must put

$$\ddot{\psi} = \frac{dtd^2\psi - d\psi d^2t}{dt^3}, \quad \ddot{\phi} = \frac{dtd^2\phi - d\phi d^2t}{dt^3} \quad \text{.....(4)}.$$

And by the equation of energy we have

$$dt = \frac{(Pd\psi^2 + Qd\phi^2 + 2Rd\psi d\phi)^{\frac{1}{2}}}{\{2(E - V)\}^{\frac{1}{2}}} \quad \text{.....(5)},$$

it being assumed that the system is conservative. Eliminating dt and d^2t between this and the two equations (3), we find a differential equation of the second degree between ψ and ϕ, which is the differential equation of the course. For simplicity, let us suppose one of the co-ordinates, ϕ for instance, to be independent variable; that is, let $d^2\phi = 0$. We have, by (4),

$$d^2t = -\ddot{\phi}\frac{dt^3}{d\phi},$$

and therefore $$\dot{\psi}dt^2 = d^2\psi + \frac{d\psi}{d\phi}\ddot{\phi}dt^2,$$

General investigation of disturbed path.

and the result of the elimination becomes

$$(PQ-R^2)\frac{d^2\psi}{d\phi^2}+F\left(\frac{d\psi}{d\phi}\right)=\frac{\left(P\frac{d\psi^2}{d\phi^2}+2R\frac{d\psi}{d\phi}+Q\right)\left[\left(Q+R\frac{d\psi}{d\phi}\right)\Psi-\left(R+P\frac{d\psi}{d\phi}\right)\Phi\right]}{2(E-V)} \quad \ldots\ldots\ldots\ldots(6),$$

$F\left(\frac{d\psi}{d\phi}\right)$ denoting a function of $\frac{d\psi}{d\phi}$ of the third degree, with variable coefficients, none of which can become infinite as long as $E-V$, the kinetic energy, is finite.

(*f*) Taking the variation of this equation on the supposition that ψ becomes $\psi+p$, where p is infinitely small, we have

$$(PQ-R^2)\frac{d^2p}{d\phi^2}+L\frac{dp}{d\phi}+Mp=0 \quad \ldots\ldots\ldots\ldots(7),$$

where L and M denote known functions of ϕ, neither of which has any infinitely great value. This determines the deviation, p, of the course. Inasmuch as the quadratic (1) is essentially always positive, $PQ-R^2$ must be always positive. Hence, if for a particular value of ϕ, p vanishes, and $\frac{dp}{d\phi}$ has a given value which defines the disturbance we suppose made at any instant, ϕ must increase by a finite amount (and therefore a finite time must elapse) before the value of p can be again zero; that is to say, before the disturbed course can again cut the undisturbed course.

(*g*) The same proposition consequently holds for a system having any number of degrees of freedom. For the preceding proof shows it to hold for the system subjected to any frictionless constraint, leaving it only two degrees of freedom; including that particular frictionless constraint which would not alter either the undisturbed or the disturbed course. The full general investigation of the disturbed motion, with more than two degrees of freedom, takes a necessarily complicated form, but the principles on which it is to be carried out are sufficiently indicated by what we have done.

(*h*) If for $L/PQ-R^2$ we substitute a constant $2a$, less than its least value, irrespectively of sign, and for $M/PQ-R^2$, a

General investigation of disturbed path.

constant β greater algebraically than its greatest value, we have an equation

$$\frac{d^2p}{d\phi^2}+2a\frac{dp}{d\phi}+\beta=0 \quad \ldots\ldots\ldots\ldots(8).$$

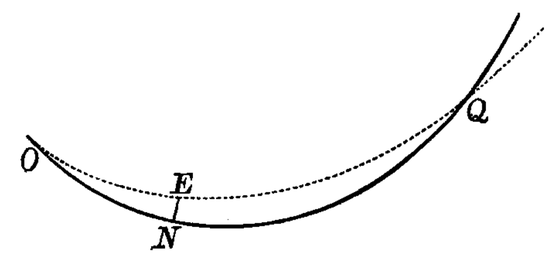
Here the value of p vanishes, for values of ϕ successively exceeding one another by $\pi/\sqrt{\beta-a^2}$, which is clearly less than the increase that ϕ must have in the actual problem before p vanishes a second time. Also, we see from this that if $a^2>\beta$ the actual motion is unstable. It might of course be unstable even if $a^2<\beta$; and the proper analytical methods for finding either the rigorous solution of (7), or a sufficiently near practical solution, would have to be used to close the criterion of stability or instability, and to thoroughly determine the disturbance of the course.

Differential equation of disturbed path of single particle in a plane.

(*i*) When the system is only a single particle, confined to a plane, the differential equation of the deviation may be put under a remarkably simple form, useful for many practical problems. Let N be the normal component of the force, per unit of the mass, at any instant, v the velocity, and ρ the radius of curvature of the path. We have (§ 259)

$$N=\frac{v^2}{\rho}.$$

Let, in the diagram, ON be the undisturbed, and OE the disturbed path. Let EN, cutting ON at right angles, be denoted by u, and ON by s. If further we denote by ρ' the radius of curvature in the disturbed path,



remembering that u is infinitely small, we easily find

$$\frac{1}{\rho'}=\frac{1}{\rho}+\frac{d^2u}{ds^2}+\frac{u}{\rho^2} \quad \ldots\ldots\ldots\ldots(9).$$

Hence, using δ to denote variations from N to E, we have

$$\delta N=\delta\frac{v^2}{\rho}=\frac{\delta(v^2)}{\rho}+v^2\left(\frac{d^2u}{ds^2}+\frac{u}{\rho^2}\right) \quad \ldots\ldots\ldots\ldots(10).$$

Differential equation of disturbed path of single particle in a plane.

But, by the equation of energy,

$$v^2 = 2(E - V),$$

and therefore

$$\delta(v^2) = -2\delta V = 2Nu = \frac{2v^2}{\rho}u.$$

Hence (10) becomes

$$\frac{d^2u}{ds^2} + \frac{3u}{\rho^2} - \frac{\delta N}{v^2} = 0 \quad\text{.....(11)},$$

or, if we denote by ζ the rate of variation of N, per unit of distance from the point N in the normal direction, so that $\delta N = \zeta u$,

$$\frac{d^2u}{ds^2} + \left(\frac{3}{\rho^2} - \frac{\zeta}{v^2}\right)u = 0 \quad\text{.....(12)}.$$

This includes, as a particular case, the equation of deviation from a circular orbit, investigated above (§ 350).

Kinetic foci.

357. If, from any one configuration, two courses differing infinitely little from one another have again a configuration in common, this second configuration will be called a kinetic focus relatively to the first: or (because of the reversibility of the motion) these two configurations will be called conjugate kinetic foci. Optic foci, if for a moment we adopt the corpuscular theory of light, are included as a particular case of kinetic foci in general. By § 356 (*g*) we see that there must be finite intervals of space and time between two conjugate foci in every motion of every kind of system, only provided the kinetic energy does not vanish.

Theorem of minimum action.

Action never a minimum in a course including kinetic foci.

358. Now it is obvious that, provided only a sufficiently short course is considered, the *action*, in any natural motion of a system, is less than for any other course between its terminal configurations. It will be proved presently (§ 361) that the first configuration up to which the action, reckoned from a given initial configuration, ceases to be a minimum, is the first kinetic focus; and conversely, that when the first kinetic focus is passed, the action, reckoned from the initial configuration, ceases to be a minimum; and therefore of course can never again be a minimum, because a course of shorter action, deviating infinitely little from it, can be found for a part, without altering the remainder of the whole, natural course.

Notation for configurations, courses, and action.

359. In such statements as this it will frequently be convenient to indicate particular configurations of the system by single letters, as O, P, Q, R; and any particular course, in which it moves through configurations thus indicated, will be called the course $O \ldots P \ldots Q \ldots R$. The *action* in any natural course will be denoted simply by the terminal letters, taken in the order of the motion. Thus OR will denote the action from O to R; and therefore $OR = -RO$. When there are more real natural courses from O to R than one, the analytical expression for OR will have more than one real value; and it may be necessary to specify for which of these courses the action is reckoned. Thus we may have

$$OR \text{ for } O \ldots E \ldots R,$$
$$OR \text{ for } O \ldots E' \ldots R,$$
$$OR \text{ for } O \ldots E'' \ldots R,$$

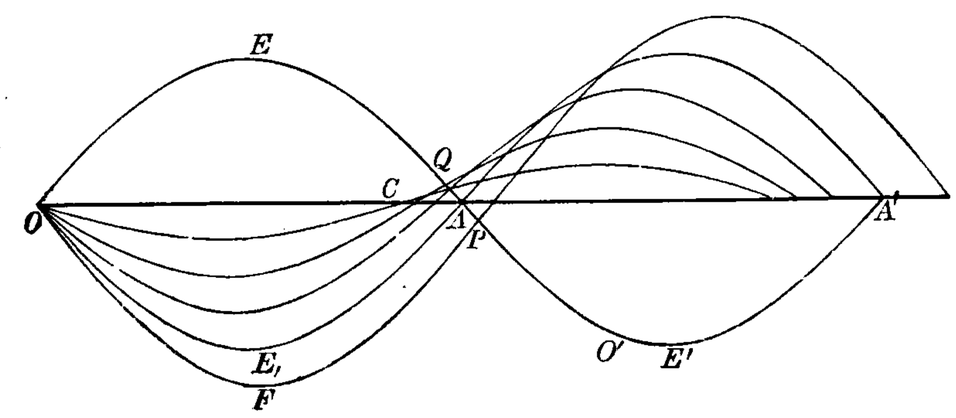
three different values of one algebraic irrational expression.

Theorem of minimum action.

360. In terms of this notation the preceding statement (§ 358) may be expressed thus:—If, for a conservative system, moving on a certain course $O \ldots P \ldots O' \ldots P'$, the first kinetic focus conjugate to O be O', the action OP', in this course, will be less than the action along any other course deviating infinitely little from it: but, on the other hand, OP' is greater than the actions in some courses from O to P' deviating infinitely little from the specified natural course $O \ldots P \ldots O' \ldots P'$.

Two or more courses of minimum action possible

361. It must not be supposed that the action along OP is necessarily *the least possible* from O to P. There are, in fact, cases in which the action ceases to be *least of all possible*, before



Two or more courses of minimum action possible.

a kinetic focus is reached. Thus if $OEAPO'E'A'$ be a sinuous geodetic line cutting the outer circle of an anchor-ring, or the equator of an oblate spheroid, in successive points O, A, A', it is easily seen that O', the first kinetic focus conjugate to O, must lie somewhat beyond A. But the length $OEAP$, although a *minimum* (a stable position for a stretched string), is not the shortest distance on the surface from O to P, as *this* must obviously be a line lying entirely on one side of the great circle. From O, to any point, Q, short of A, the distance along the geodetic $OEQA$ is clearly the least possible: but if Q be near enough to A (that is to say, between A and the point in which the envelope of the geodetics drawn from O, cuts OEA), there will also be two other geodetics from O to Q. The length of one of these will be a minimum, and that of the other not a minimum. If Q is moved forward to A, the former becomes OE_1A, equal and similar to OEA, but on the other side of the great circle: and the latter becomes the great circle from O to A. If now Q be moved on, to P, beyond A, the minimum geodetic $OEAP$ ceases to be the less of the two minimums, and the geodetic OFP lying altogether on the other side of the great circle becomes the least possible line from O to P. But until P is advanced beyond the point, O', in which it is cut by another geodetic from O lying infinitely nearly along it, the length $OEAP$ remains a minimum, according to the general proposition of § 358, which we now proceed to prove.

Case of two minimum, and one not minimum, geodetic lines between two points.

Difference between two sides and the third of a kinetic triangle.

(*a*) Referring to the notation of § 360, let P_1 be any configuration differing infinitely little from P, but not on the course $O\ldots P\ldots O'\ldots P'$; and let S be a configuration on this course, reached at some finite time after P is passed. Let ψ, ϕ,... be the co-ordinates of P, and ψ_1, ϕ_1,... those of P_1, and let

$$\psi_1 - \psi = \delta\psi,\ \phi_1 - \phi = \delta\phi, \ldots$$

Thus, by Taylor's theorem,

$$OP_1 + P_1S = OS + \left\{\frac{d(OP+PS)}{d\psi}\delta\psi + \frac{d(OP+PS)}{d\phi}\delta\phi + \ldots\right\}$$
$$+ \tfrac{1}{2}\left\{\frac{d^2(OP+PS)}{d\psi^2}(\delta\psi)^2 + 2\frac{d^2(OP+PS)}{d\psi d\phi}\delta\psi\delta\phi + \ldots\right\}$$
$$+ \text{etc.}$$

Difference between two sides and the third of a kinetic triangle.

But if ξ, η,... denote the components of momentum at P in the course $O\ldots P$, which are the same as those at P in the continuation, $P\ldots S$, of this course, we have [§ 330 (18)]

$$\xi = \frac{dOP}{d\psi} = -\frac{dPS}{d\psi},\quad \eta = \frac{dOP}{d\phi} = -\frac{dPS}{d\phi}, \ldots$$

Hence the coefficients of the terms of the first degree of $\delta\psi$, $\delta\phi$, in the preceding expression vanish, and we have

$$OP_1 + P_1S - OS = \tfrac{1}{2}\left\{\frac{d^2(OP+PS)}{d\psi^2}\delta\psi^2 + 2\frac{d^2(OP+PS)}{d\psi d\phi}\delta\psi\delta\phi + \ldots\right\} \quad (1).$$
$$+ \text{etc.}$$

(*b*) Now, assuming

$$\left.\begin{array}{l} x_1 = \alpha_1\delta\psi + \beta_1\delta\phi + \ldots \\ x_2 = \alpha_2\delta\psi + \beta_2\delta\phi + \ldots \\ \quad\text{etc.} \qquad \text{etc.} \end{array}\right\} \ldots\ldots\ldots\ldots\ldots (2),$$

according to the known method of linear transformations, let α_1, β_1,... α_2, β_2,... be so chosen that the preceding quadratic function be reduced to the form

$$A_1x_1^2 + A_2x_2^2 + \ldots + A_ix_i^2,$$

the whole number of degrees of freedom being i.

This may be done in an infinite variety of ways; and, towards fixing upon one particular way, we may take $\alpha_i = \dot\psi$, $\beta_i = \dot\phi$, etc.; and subject the others to the conditions

$$\dot\psi\alpha_1 + \dot\phi\beta_1 + \ldots = 0,\ \dot\psi\alpha_2 + \dot\phi\beta_2 + \ldots = 0, \text{ etc.}$$

This will make $A_i = 0$: for if for a moment we suppose P_1 to be on the course $O\ldots P\ldots O'$, we have

$$\frac{\delta\psi}{\dot\psi} = \frac{\delta\phi}{\dot\phi} = \ldots,$$

and therefore

$$x_i = \frac{\dot\psi}{\delta\psi}(\delta\psi^2 + \delta\phi^2 + \ldots),\ x_{i-1} = 0,\ \ldots x_2 = 0,\ x_1 = 0.$$

But in this case $OP_1 + P_1S = OS$; and therefore the value of the quadratic must be zero; that is to say, we must have $A_i = 0$. Hence we have

$$\left.\begin{array}{l} OP_1 + P_1S - OS = \tfrac{1}{2}(A_1x_1^2 + A_2x_2^2 + \ldots + A_{i-1}x_{i-1}^2) \\ \qquad\qquad + R \end{array}\right\} \ldots\ldots\ldots (3)$$

where R denotes a remainder consisting of terms of the third and higher degrees in $\delta\psi$, $\delta\phi$, etc., or in x_1, x_2, etc.

Difference between two sides and the third of a kinetic triangle.

(c) Another form, which will be used below, may be given to the same expression thus:—Let $(\xi_{,}, \eta_{,}, \zeta_{,}, \ldots)$ and $(\xi_{,}', \eta_{,}', \zeta_{,}', \ldots)$ be the components of momentum at $P_{,}$, in the courses $OP_{,}$ and $P_{,}S$ respectively. By § 330 (18) we have

$$\xi_{,} = \frac{dOP'}{d\psi_{,}},$$

and therefore by Taylor's theorem

$$\xi_{,} = \frac{dOP}{d\psi} + \frac{d^2OP}{d\psi^2}\delta\psi + \frac{d^2OP}{d\psi d\phi}\delta\phi + \ldots + \text{etc.}$$

Similarly,

$$-\xi_{,}' = \frac{dPS}{d\psi} + \frac{d^2PS}{d\psi^2}\delta\psi + \frac{d^2PS}{d\psi d\phi}\delta\phi + \ldots + \text{etc.};$$

and therefore, as $\dfrac{dOP}{d\psi} = -\dfrac{dPS}{d\psi}$,

$$\xi_{,}' - \xi_{,} = -\left\{\frac{d^2(OP+PS)}{d\psi^2}\delta\psi + \frac{d^2(OP+PS)}{d\psi d\phi}\delta\phi + \ldots\right\} + \text{etc.} \ldots (4),$$

and so for $\eta_{,}' - \eta_{,}$, etc. Hence (1) is the same as

$$\left.\begin{aligned} OP_{,} + P_{,}S - OS = -\tfrac{1}{2}\{(\xi_{,}' - \xi')\,\delta\psi + (\eta_{,}' - \eta_{,})\,\delta\phi + \ldots\} \\ + R \end{aligned}\right\} \ldots\ldots (5),$$

where R denotes a remainder consisting of terms of the third and higher degrees. Also the transformation from $\delta\psi$, $\delta\phi$, ... to x_1, x_2, ..., gives clearly

$$\left.\begin{aligned} \xi_{,}' - \xi_{,} &= -(A_1\alpha_1x_1 + A_2\alpha_2x_2 + \ldots + A_{i-1}\alpha_{i-1}x_{i-1}) \\ \eta_{,}' - \eta_{,} &= -(A_1\beta_1x_1 + A_2\beta_2x_2 + \ldots + A_{i-1}\beta_{i-1}x_{i-1}) \\ &\text{etc.} \qquad\qquad \text{etc.} \end{aligned}\right\} \ldots\ldots\ldots\ldots (6).$$

(d) Now for any infinitely small time the velocities remain sensibly constant; as also do the coefficients (ψ, ψ), (ψ, ϕ), etc., in the expression [§ 313 (2)] for T: and therefore for the action we have

$$\int 2T dt = \sqrt{2T}\int\sqrt{2T}dt$$

$$= \sqrt{2T}\{(\psi, \psi)(\psi - \psi_0)^2 + 2(\psi, \phi)(\psi - \psi_0)(\phi - \phi_0) + \text{etc.}\}^{\frac{1}{2}}$$

where $(\psi_0, \phi_0, \ldots)$ are the co-ordinates of the configuration from which the action is reckoned. Hence, if P, P', P'' be any three configurations infinitely near one another, and if Q, with the proper differences of co-ordinates written after it, be used to denote square roots of quadratic functions such as that in the preceding expression, we have

Difference between two sides and the third of a kinetic triangle.

$$\left.\begin{aligned} PP' &= \sqrt{2T}\,.\,Q\{(\psi - \psi'), (\phi - \phi'), \ldots\} \\ P'P'' &= \sqrt{2T}\,.\,Q\{(\psi' - \psi''), (\phi' - \phi''), \ldots\} \\ P''P &= \sqrt{2T}\,.\,Q\{(\psi'' - \psi'), (\phi'' - \phi'), \ldots\} \end{aligned}\right\} \ldots\ldots\ldots\ldots (7).$$

In the particular case of a single free particle, these expressions become simply proportional to the distances PP', $P'P''$, $P''P$; and by Euclid we have

$$P'P + PP'' < P'P''$$

unless P is in the straight line $P'P''$.

The verification of this proposition by the preceding expressions (7) is merely its proof by co-ordinate geometry with an oblique rectilineal system of co-ordinates, and is necessarily somewhat complicated. If $(\psi, \phi) = (\phi, \theta) = (\theta, \psi) = 0$, the co-ordinates become rectangular and the algebraic proof is easy. There is no difficulty, by following the analogies of these known processes, to prove that, for any number of co-ordinates, ψ, ϕ, etc., we have

$$P'P + PP'' > P'P'',$$

unless

$$\frac{\psi - \psi'}{\psi'' - \psi'} = \frac{\phi - \phi'}{\phi'' - \phi'} = \frac{\theta - \theta'}{\theta'' - \theta'} = \ldots$$

(expressing that P is on the course from P to P''), in which case

$$P'P + PP'' = P'P'',$$

$P'P$, etc., being given by (7). And further, by the aid of (1), it is easy to find the proper expression for $P'P + PP'' - P'P''$, when P is infinitely little off the course from P' to P'': but it is quite unnecessary for us here to enter on such purely algebraic investigations.

(e) It is obvious indeed, as has been already said (§ 358), that the action along any natural course is *the least possible between its terminal configurations* if only a sufficiently short course is included. Hence for all cases in which the time from O to S is less than some particular amount, the quadratic term in the expression (3) for $OP_{,} + P_{,}S - OS$ is necessarily positive, for all values of x_1, x_2, etc.; and therefore A_1, A_2, ... A_{i-1} must each be positive.

Actions on different courses infinitely near one another.

(f) Let now S be removed further and further from O, along the definite course $O \ldots P \ldots O'$, until it becomes O'. When it is O', let $P_{,}$ be taken on a natural course through O and O', de-

Between two conjugate kinetic foci, proved ultimately equal.

viating infinitely little from the course OPO'. Then, as $OP_{,}O'$ is a natural course,

$$\xi_{,}' - \xi_{,} = \eta_{,}' - \eta_{,}, \ldots = 0;$$

and therefore (5) becomes

$$OP_{,} + P_{,}O' - OO' = R,$$

which proves that the chief, or quadratic, term in the other expression (3) for the same, vanishes. Hence one at least of the coefficients A_1, A_2,... must vanish, and if one only, $A_{i-1} = 0$ for instance, we must have

$$x_1 = 0,\ x_2 = 0, \ldots x_{i-2} = 0.$$

These equations express the condition that $P_{,}$ lies on a natural course from O to O'.

If two sides, deviating infinitely little from the third, are together equal to it, they constitute an unbroken natural course.

(*g*) Conversely if one or more of the coefficients A_1, A_2, etc., vanishes, if for instance $A_{i-1} = 0$, S must be a kinetic focus. For if we take $P_{,}$ so that

$$x_1 = 0,\ x_2 = 0, \ldots x_{i-2} = 0,$$

we have, by (6),

$$\xi_{,}' - \xi_{,} = \eta_{,}' - \eta' = \ldots = 0.$$

(*h*) Thus we have proved that at a kinetic focus conjugate to O the action from O is not a minimum of the first order*, and that the last configuration, up to which the action from O *is* a minimum of the first order, is a kinetic focus conjugate to O.

(*i*) It remains to be proved that the action from O ceases to be a minimum when the first kinetic focus conjugate to O is passed. Let, as above (§ 360), $O \ldots P \ldots O' \ldots P'$ be a natural course extending beyond O', the first kinetic focus conjugate to O. Let P and P' be so near one another that there is no focus conjugate to either, between them; and let $O \ldots P_{,} \ldots O'$ be a natural course from O to O' deviating infinitely little from $O \ldots P \ldots O'$. By what we have just proved (*e*), the action OO' along $O \ldots P_{,} \ldots O'$ differs only by R, an infinitely small quantity of the third order, from the action OO' along $O \ldots P \ldots O'$, and therefore

Natural course proved not a course of minimum action, beyond a kinetic focus.

$$\begin{aligned} Ac.(O \ldots P \ldots O' \ldots P') &= Ac.(O \ldots P_{,} \ldots O') + O'P' + R \\ &= OP_{,} + P_{,}O' + O'P' + R. \end{aligned}$$

* A maximum or minimum "of the first order" of any function of one or more variables, is one in which the differential of the first degree vanishes, but not that of the second degree.

But, by a proper application of (*e*) we see that

Natural course proved not a course of minimum action, beyond a kinetic focus.

$$P_{,}O' + O'P' = P_{,}P' + Q$$

where Q denotes an infinitely small quantity of the second order, which is essentially positive. Hence

$$Ac(O \ldots P \ldots O' \ldots P') = OP_{,} + P_{,}P' + Q + R,$$

and therefore, as R is infinitely small in comparison with Q,

$$Ac(O \ldots P \ldots O' \ldots P') > OP_{,} + P_{,}P'.$$

Hence the broken course $O \ldots P_{,}$, $P_{,} \ldots P'$ has less action than the natural course $O \ldots P \ldots O' \ldots P'$, and therefore, as the two are infinitely near one another, the latter is not a minimum.

A course which includes no focus conjugate to either extremity includes no pair of conjugate foci.

362. As it has been proved that the action from any configuration ceases to be a minimum at the first conjugate kinetic focus, we see immediately that if O' be the first kinetic focus conjugate to O, reached after passing O, no two configurations on this course from O to O' can be kinetic foci to one another. For, the action from O just ceasing to be a minimum when O' is reached, the action between any two intermediate configurations of the same course is necessarily a minimum.

How many kinetic foci in any case.

363. When there are i degrees of freedom to move there are in general, on any natural course from any particular configuration, O, at least $i-1$ kinetic foci conjugate to O. Thus, for example, on the course of a ray of light emanating from a luminous point O, and passing through the centre of a convex lens held obliquely to its path, there are two kinetic foci conjugate to O, as defined above, being the points in which the line of the central ray is cut by the so-called "focal lines"* of a pencil of rays diverging from O and made convergent after passing through the lens. But some or all of these kinetic foci may be on the course previous to O; as for instance in the case of a common projectile when its course passes obliquely downwards through O. Or some or all may be lost; as when, in the optical illustration just referred to, the lens is only strong enough to produce convergence in one of the principal planes, or too weak to produce convergence in either. Thus

* In our second volume we hope to give all necessary elementary explanations on this subject.

How many kinetic foci in any case.

also in the case of the undisturbed rectilineal motion of a point, or in the motion of a point uninfluenced by force, on an anticlastic surface (§ 355), there are no real kinetic foci. In the motion of a projectile (not confined to one vertical plane) there can only be one kinetic focus on each path, conjugate to one given point; though there are three degrees of freedom. Again, there may be any number more than $i-1$, of foci in one course, all conjugate to one configuration, as for instance on the course of a particle uninfluenced by force, moving round the surface of an anchor-ring, along either the outer great circle, or along a sinuous geodetic such as we have considered in § 361, in which clearly there are an infinite number of foci each conjugate to any one point of the path, at equal successive distances from one another.

Referring to the notation of § 361 (f), let S be gradually moved on until first one of the coefficients, A_{i-1} for instance, vanishes; then another, A_{i-2}, etc.; and so on. We have seen that each of these positions of S is a kinetic focus: and thus by the successive vanishing of the $i-1$ coefficients we have $i-1$ foci. If none of the coefficients can ever vanish, there are no kinetic foci. If one or more of them, after vanishing, comes to a minimum, and again vanishes, as S is moved on, there may be any number more than $i-1$ of foci each conjugate to the same configuration, O.

Theorem of maximum action.

364. If $i-1$ distinct* courses from a configuration O, each differing infinitely little from a certain natural course

$$O \ldots E \ldots O_1 \ldots O_2 \ldots O_{i-1} \ldots Q,$$

cut it in configurations O_1, O_2, $O_3, \ldots O_{i-1}$, and if, besides these, there are not on it any other kinetic foci conjugate to O, between O and Q, and no focus at all, conjugate to E, between E and Q, the action in this natural course from O to Q is the maximum for all courses $O \ldots P_{,}$, $P_{,} \ldots Q$; $P_{,}$ being a configuration infinitely nearly agreeing with some configuration between E and O_1 of the standard course $O \ldots E \ldots O_1 \ldots O_2 \ldots O_{i-1} \ldots Q$, and $O \ldots P_{,}$, $P_{,} \ldots Q$

* Two courses are not called distinct if they differ from one another only in the absolute magnitude, not in the proportions of the components, of the deviations by which they differ from the standard course.

Theorem of maximum action.

denoting the natural courses between O and $P_{,}$, and $P_{,}$ and Q, which deviate infinitely little from this standard course.

In § 361 (i), let O' be any one, O_1, of the foci O_1, $O_2, \ldots O_{i-1}$, and let $P_{,}$ be called P_1 in this case. The demonstration there given shows that

$$OQ > OP_1 + P_1Q.$$

Hence there are $i-1$ different broken courses

$$O \ldots P_1,\ P_1 \ldots Q;\ O \ldots P_2,\ P_2 \ldots Q;\ \text{etc.},$$

in each of which the action is less than in the standard course from O to Q. But whatever be the deviation of $P_{,}$, it may clearly be compounded of deviations P to P_1, P to P_2, P to P_3, ..., P to P_{i-1}, corresponding to these $i-1$ cases respectively; and it is easily seen from the analysis that

$$OP_{,} + P_{,}Q - OQ = (OP_1 + P_1Q - OQ) + (OP_2 + P_2Q - OQ) + \ldots$$

Hence $OP_{,} + P_{,}Q < OQ$, which was to be proved.

Applications to two degrees of freedom.

365. Considering now, for simplicity, only cases in which there are but two degrees (§§ 195, 204) of freedom to move, we see that after any infinitely small conservative disturbance of a system in passing through a certain configuration, the system will first again pass through a configuration of the undisturbed course, at the first configuration of the latter at which the action in the undisturbed motion ceases to be a minimum. For instance, in the case of a particle, confined to a surface, and subject to any conservative system of force, an infinitely small conservative disturbance of its motion through any point, O, produces a disturbed path, which cuts the undisturbed path at the first point, O', at which the action in the undisturbed path from O ceases to be a minimum. Or, if projectiles, under the influence of gravity alone, be thrown from one point, O, in all directions with equal velocities, in one vertical plane, their paths, as is easily proved, intersect one another consecutively in a parabola, of which the focus is O, and the vertex the point reached by the particle projected directly upwards. The actual course of each particle from O is the course of least possible action to any point, P, reached before the enveloping parabola, but is not a course of minimum action to any point, Q, in its path after the envelope is passed.

Applications to two degrees of freedom.

366. Or again, if a particle slides round along the greatest circle of the smooth inner surface of a hollow anchor-ring, the "action," or simply the length of path, from point to point, will be least possible for lengths (§ 351) less than $\pi\sqrt{ab}$. Thus, if a string be tied round outside on the greatest circle of a perfectly smooth anchor-ring, it will slip off unless held in position by staples, or checks of some kind, at distances of not less than $\pi\sqrt{ab}$ from one another in succession round the circle. With reference to this example, see also § 361, above.

Or, of a particle sliding down an inclined cylindrical groove, the action from any point will be the least possible along the straight path to any other point reached in a time less than that of the vibration one way of a simple pendulum of length equal to the radius of the groove, and influenced by a force equal $g\cos i$, instead of g the whole force of gravity. But the action will not be a minimum from any point, along the straight path, to any other point reached in a longer time than this. The case in which the groove is horizontal ($i=0$) and the particle is projected along it, is particularly simple and instructive, and may be worked out in detail with great ease, without assuming any of the general theorems regarding action.

Hamilton's second form.

367. In the preceding account of the Hamiltonian principle, and of developments and applications which it has received, we have adhered to the system (§§ 328, 330) in which the initial and final co-ordinates and the constant sum of potential and kinetic energies are the elements of which the action is supposed to be a function. Another system was also given by Hamilton, according to which the action is expressed in terms of the initial and final co-ordinates and the *time prescribed for the motion;* and a set of expressions quite analogous to those with which we have worked, are established. For practical applications this method is generally less convenient than the other; and the analytical relations between the two are so obvious that we need not devote any space to them here.

Liouville's kinetic theorem.

368. We conclude by calling attention to a very novel analytical investigation of the motion of a conservative system, by Liouville (*Comptes Rendus*, June 16, 1856), which leads im-

Liouville's kinetic theorem.

mediately to the principle of least action, and the Hamiltonian principle with the developments by Jacobi and others; but which also establishes a very remarkable and absolutely new theorem regarding the amount of the action along any constrained course. For brevity we shall content ourselves with giving it for a single free particle, referring the reader to the original article for Liouville's complete investigation in terms of generalized co-ordinates, applicable to any conservative system whatever.

Let (x, y, z) be the co-ordinates of any point through which the particle may move: V its potential energy in this position: E the sum of the potential and kinetic energies of the motion in question: A the action, from any position (x_0, y_0, z_0) to (x, y, z) along any course arbitrarily chosen (supposing, for instance, the particle to be guided along it by a frictionless guiding tube). Then (§ 326), the mass of the particle being taken as unity,

$$A = \int v ds = \int \sqrt{2(E-V)}\sqrt{(dx^2+dy^2+dz^2)}.$$

Now let ϑ be a function of x, y, z, which satisfies the partial differential equation

$$\frac{d\vartheta^2}{dx^2}+\frac{d\vartheta^2}{dy^2}+\frac{d\vartheta^2}{dz^2} = 2(E-V).$$

Then

$$A = \int\sqrt{\left(\frac{d\vartheta^2}{dx^2}+\frac{d\vartheta^2}{dy^2}+\frac{d\vartheta^2}{dz^2}\right)(dx^2+dy^2+dz^2)}$$

$$= \int\sqrt{\left[\left(\frac{d\vartheta}{dx}dx+\frac{d\vartheta}{dy}dy+\frac{d\vartheta}{dz}dz\right)^2+\left(\frac{d\vartheta}{dz}dy-\frac{d\vartheta}{dy}dz\right)^2+\left(\frac{d\vartheta}{dx}dz-\frac{d\vartheta}{dz}dx\right)^2 + \left(\frac{d\vartheta}{dy}dx-\frac{d\vartheta}{dx}dy\right)^2\right]}.$$

But
$$\frac{d\vartheta}{dx}dx+\frac{d\vartheta}{dy}dy+\frac{d\vartheta}{dz}dz = d\vartheta,$$

and, if $\dot{x}, \dot{y}, \dot{z}$ denote the actual component velocities along the arbitrary path, and $\dot{\vartheta}$ the rate at which ϑ increases per unit of time in this motion,

$$dx = \dot{x}dt,\quad dy = \dot{y}dt,\quad dz = \dot{z}dt,\quad d\vartheta = \dot{\vartheta}dt.$$

Hence the preceding becomes

$$A = \int d\vartheta\sqrt{\left\{1+\frac{\left(\dot{y}\frac{d\vartheta}{dz}-\dot{z}\frac{d\vartheta}{dy}\right)^2+\left(\dot{z}\frac{d\vartheta}{dx}-\dot{x}\frac{d\vartheta}{dz}\right)^2+\left(\dot{x}\frac{d\vartheta}{dy}-\dot{y}\frac{d\vartheta}{dx}\right)^2}{\dot{\vartheta}^2}\right\}}.$$