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A new derivation of the quadratic equation for the masses of the proton and electron

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1. In this paper I introduce some improvements of method which will, I think, bring about a considerable simplification of most of the physical problems treated in Part II of my book, *Relativity theory of protons and electrons* (hereinafter referred to as *P. and E.*). They also facilitate the extension of the theory to other problems awaiting solution. Here I confine myself to one of the problems, namely, the derivation of the fundamental quadratic $10m^2 - 136mm_0 + m_0^2 = 0$ for the mass m of a proton or electron. The derivation in *P. and E.* is, I believe, valid, though it now appears clumsy. But a proof giving more physical insight was greatly to be desired, since a thorough understanding of the underlying meaning of this relation is the first condition for a general advance.

A general acquaintance with the relativistic theory in *P. and E.* must be presumed; but, as it happens, there is little occasion to refer to the more specialized parts of the theory.

2. In relativity theory *density* is a component T_{44} of a tensor $T_{\mu\nu}$; on the other hand, *mass* (or its equivalent *energy*) is not normally part of a tensor. If the mass of a body is defined as the integral of T_{44} over a three-dimensional volume, the result is not a component of a tensor unless the space is assumed to be flat; but to assume flat space in treating $T_{\mu\nu}$ is inconsistent with the fundamental equation $-8\pi\kappa T_{\mu\nu} = G_{\mu\nu} - \frac{1}{2}g_{\mu\nu}G$ which determines the curvature in terms of $T_{\mu\nu}$. A more significant quantity is obtained by defining the mass of a system to be that of an equivalent point-particle which would produce the same gravitational field at great distances. This is found to be equal to the integral of $T_{44} + t_{44}$, where t_{44} is the potential energy; but the expression $t_{\mu\nu}$ (the potential energy pseudo-tensor) of which t_{44} is a component is not a tensor.

This difficulty associated with integration is especially pronounced in applications to quantum theory, because "the mass of a particle" is first introduced as a density distribution over a large (nominally infinite) wave front.

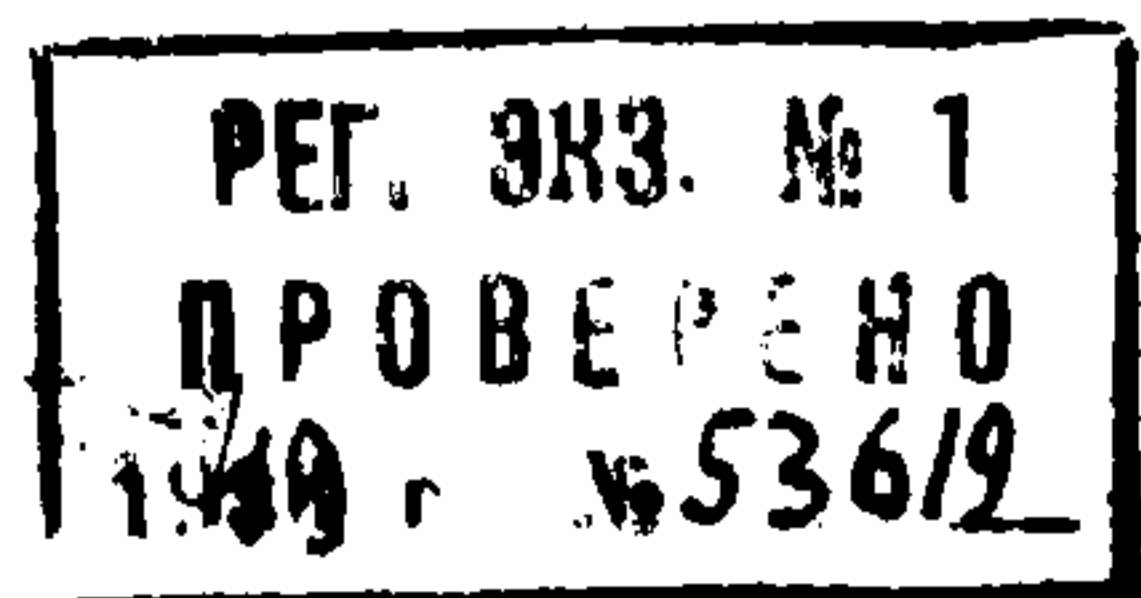
Accordingly, in extending relativity theory to microscopic physics, our starting-point must be the representation of density in wave mechanics

rather than the representation of mass or energy. By the general principles of wave analysis the whole density of the system is considered to be the sum of contributions from a set of elementary wave functions. By a survival of classical terminology we often describe the density associated with a particular wave function as due to a particle occupying the state which the wave function describes. This circumlocution is difficult to avoid, though it often involves using the term "particle" in a sense remote from ordinary conception. If we assign the density and other physical characteristics directly to the wave function, we have to reword the exclusion principle appropriately. The density contributed by an elementary wave function has a saturation value which cannot be exceeded; the actual contribution may be any fraction p ($0 \leq p \leq 1$) of the saturation value. A wave function will be said to be "fully occupied" if the density has the saturation value; and p will be called the "degree of occupation" or "occupation factor".

Our point of entry into wave mechanics is therefore the *density of a fully occupied wave function* in contrast to elementary quantum theory which begins with the *mass (or energy) of a particle*.

Many of the wave functions in current quantum theory are self-normalizing; that is to say, the functions themselves define a saturated density distribution which, if collected together, amounts to the mass of the particle. Quite early in the development of quantum theory it is customary to restrict wave functions by this condition. But the condition is not made retrospective, and the most elementary wave functions are not self-normalizing. Moreover, self-normalizing wave functions, when they are used, account for only part of the energy of the system; the remainder is left to be represented by functions of the elementary type. Thus we cannot evade the problem of normalizing functions which are not self-normalizing.

The functions which cause the difficulty are the "infinite plane waves". Quantum theory assigns a mass to the particle associated with the wave, but does not indicate any particular saturation density. This gap must be filled before we can connect the relativistic description of matter by an energy tensor with the quantum description of it by masses and momenta of so-called particles; and the form of treatment which follows is largely determined by this requirement. Clearly we must find for these unbounded functions a "natural normalization volume" such that the mass of the particle divided by the volume gives the saturation density. This would be a straightforward matter were it not for *degeneracy*—a complication due to symmetrical conditions and therefore the more liable to occur the more we simplify the system under consideration. Degeneracy is treated rigorously in current quantum theory when it arises in the course of its developments;



but current theory does not seem to have found occasion to study the degeneracy brought into it at the beginning by the simplest elements with which it deals.

The work thus falls into two stages. We first develop the theory of the natural normalization volume for non-degenerate unbounded wave functions. In § 10 we begin the discussion of degeneracy.

3. We shall use a "natural system" of units of length, time and mass chosen so that

$$c = 1, \quad \kappa h^2 = 1, \quad (1)$$

where κ is the constant of gravitation. The former is a common convention which eliminates the troublesome and unnecessary distinction between mass and energy; the latter is chosen primarily to simplify the relation between mass and density, which is the problem that will occupy us. Although the particular conventions are arbitrary, the fixing of definite relations between the units of length, time and mass by two such conventions is not an arbitrary procedure; and it is rather important to understand why the retention of triply-variable units in this branch of physics would be obstructive.

The purpose of the conditions (1) is to reduce the redundant fluidity of description of a physical system occasioned by referring it to three independent standards; it is *redundant* because, in so far as fluidity of description is advantageous, physics nowadays employs a much more comprehensive transformation theory (tensor calculus, etc.) better adapted to theoretical needs. The meaning of conditions such as (1) is that we employ in our formulae the numerical ratio of a physical magnitude to another comparable physical magnitude *covered by the theory* instead of its ratio to a physical magnitude outside the theory. It is clearly necessary to make a distinction in theory between a standard defined by a theoretical description and a standard (such as the kilogram) defined by pointing at it. Reference to the latter kind of standard means that we have come to the end of coherent speech and can only gesticulate. When we ascribe the dimensions ML^2T^{-2} to energy the symbols M , L , T stand for gesticulations. We cannot sacrifice the development of physics to the curious tradition that at the end of any statement it is necessary to point three times to make clear what we mean. Normally, in relativity theory and quantum theory what we want to say is of a form which makes it only necessary to point once. We retain *one* extrinsic standard because that corresponds to the conditions postulated in the problems usually treated; for, so long as the system is supposed to be

observed from outside, the "observables" contain reference to something not comprised in the theoretical description of the system.

The extrinsic standard may be taken to be length, time or mass, or indeed any physical quantity which implies reference to a standard. We shall here take it to be mass, since that is the standard most directly concerned in our problem. By the conditions (1) every physical quantity will have a dimension-index specifying the way in which it varies with the extrinsic unit of mass.

By the formula $G = 8\pi\kappa T = 8\pi\kappa\rho_0$, $\kappa\rho$ has the same dimensions as G , namely L^{-2} . (The condition $c = 1$ is understood to be already employed, so that L stands for length or time.) By the quantum expression for momentum $(-i\hbar/2\pi)\partial/\partial x$, M/\hbar has dimensions L^{-1} . Hence, eliminating L , $\kappa\hbar^2\rho$ has the dimensions M^2 . Then, since $\kappa\hbar^2 = 1$, ρ has the dimensions M^2 .

We shall accordingly write the density of a fully occupied wave function in the form

$$\rho = m_1 m_2 f(x, y, z, t), \quad (2)$$

where m_1, m_2 are masses, and $f(x, y, z, t)$ is a pure number.

It may be noted that in our units length has the dimensions $M^{-\frac{1}{2}}$, so that the reciprocal of a volume is a mass. The ordinary definition of density is obtained by interpreting the second mass in (2) as the reciprocal of a volume. But we shall proceed with the investigation of (2) unprejudiced by this interpretation.

4. Since all components of the energy tensor have the same dimension-index, (2) can be extended to

$$T_{\mu\nu} = m_1 m_2 f_{\mu\nu}(x, y, z, t), \quad (3)$$

where $f_{\mu\nu}$ is a purely numerical tensor function.

By the theory of the relation of space tensors and wave tensors a symmetrical tensor of the second rank $f_{\mu\nu}$ is derived from double wave vectors (*P. and E.* § 10.2); that is to say, the components of $f_{\mu\nu}$ are the appropriate matrix components of the product $\Psi_{\alpha\beta} X_{\gamma\delta}$ of two double wave vectors, or of a sum of such products. The wave tensor $\Psi_{\alpha\beta} X_{\gamma\delta}$ has other components besides those which form the energy tensor; their interpretation is treated in *P. and E.* § 11.4, where it is shown that the Riemann-Christoffel tensor is also included. Only one of the double vectors Ψ, X need be specified, the other being derivable from it by the reality conditions of the theory; this is well known in Dirac's theory which employs Hermitic reality conditions, and H. T. Corben (in an unpublished investigation) has shown that the

form of reality conditions obtained in my own relativistic theory yields the same result.

We can now see the appropriateness of the *two* masses in (3). In special cases $\Psi_{\alpha\beta}$ may be a product $\psi_\alpha\phi_\beta$, and the reality conditions then show that $X_{\gamma\delta}$ will also be a product $\chi_\gamma\omega_\delta$. The complete space tensor is then the outer product of two complete space vectors $J_1 = \psi_\alpha\chi_\gamma$ and $J_2 = \phi_\beta\omega_\delta$. Thus, when the numerical tensor $f_{\mu\nu}$ is the product of two *numerical* vectors J_1, J_2 , the energy tensor is the product of two *mass-like* vectors $M_1 = m_1J_1$ and $M_2 = m_2J_2$, which may be regarded as the momentum vectors of two particles of proper masses m_1, m_2 . In this way our approach to quantum theory via the density or energy tensor begins to be connected with the usual elementary treatment which starts with masses or momentum vectors. I shall call the entity analysed in this way into two particles a *bi-particle*.

No serious difference is made when Ψ is not factorizable so long as we confine attention to the case $f_{\mu\nu} = \text{constant}$. If Ψ is not a simple product it can be represented as the sum of products, and elementary quantum theory has provided for this by recognizing that a "particle" need not have an exact momentum vector, but may be in a number of superposed states with different momentum vectors; and similarly the bi-particle represented by a double vector need not be an exact pair of particles, but may be a superposition of exact pairs of particles. But a *function* $\Psi_{\alpha\beta}(x, y, z, t)$ cannot in general be represented as the sums of products of functions $\psi_\alpha(x, y, z, t)$, $\phi_\beta(x, y, z, t)$; and in this respect the attempt to establish a universal association of density with particles characterized by momentum vectors, exact or inexact, breaks down.

This fulfils our anticipations in § 2 where we saw that integration destroys the tensor character of $T_{\mu\nu}$, so that either mass is not rigorously the component of a vector (and accordingly the entity possessing it is not characterized by a momentum vector) or it is not exactly the integral of density. The introduction of field energy (potential energy), not localizable in individual particles, is part of the complication ensuing from this breakdown of elementary conceptions.

We shall find that the functions $f_{\mu\nu}$ required for our purposes are constant over the domain to which they apply, so that the complication will not arise. Nor does it arise if $f_{\mu\nu}$ is an algebraic wave function (*P. and E.* p. 120).

5. That two particles are concerned in an element of density is obvious from the principle of relativity. What we observe is, not the particles, but relations between the particles. An observable, such as density, can therefore only be associated with the relations between pairs of particles, which

are contained in the double wave functions specifying their combined probability distributions. But alongside the relativistic outlook we have to keep in mind the ordinary conception of density as made up of contributions of individual particles. The principle of this change of conception is that, of the two particles which furnish the contribution, m_2 is understood to be a standard comparison particle providing a reference frame for the "object-particle" m_1 to which the density is attributed. In order to connect with relativity theory the quantities (such as the masses of the proton and electron) which appear in the ordinary treatment, we must follow the conventions of the ordinary treatment where they are permissible, and we shall therefore distinguish the two masses m_1 , m_2 as the masses of an object-particle and a comparison particle. The ordinary treatment goes further and treats the physical reference frame furnished by the comparison particle as a purely geometrical Galilean frame, ignoring the combined uncertainty of position and velocity dependent on m_2 ; so that m_2 is forgotten about except in so far as it reappears in the empirical values of the constants of nature. Naturally we cannot follow the conventions of the ordinary treatment which are unpermissible—the mistakes—so we shall take due account of the mass m_2 of the comparison particle.

We shall now examine more closely the way in which one particle furnishes a reference frame for another.

Considering the co-ordinates x_1, y_1, z_1, t_1 and x_2, y_2, z_2, t_2 of two particles, let

$$\left. \begin{aligned} X &= \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}, & Y &= \frac{m_1 y_1 + m_2 y_2}{m_1 + m_2}, & Z &= \frac{m_1 z_1 + m_2 z_2}{m_1 + m_2}, & T &= \frac{m_1 t_1 + m_2 t_2}{m_1 + m_2}, \\ \xi &= x_1 - x_2, & \eta &= y_1 - y_2, & \zeta &= z_1 - z_2, & \tau &= t_1 - t_2. \end{aligned} \right\} \quad (4)$$

Then a function $\Psi_{\alpha\beta}$ of $x_1, y_1, z_1, t_1, x_2, y_2, z_2, t_2$ can also be expressed as a function $\Phi_{\alpha\beta}$ of $X, Y, Z, T, \xi, \eta, \zeta, \tau$. The formal theory of Φ is the same as the formal theory of Ψ in § 4; that is to say, we can (under the same conditions) represent the energy tensor as the product of the momentum vectors of two particles associated respectively with the co-ordinates X, Y, Z, T and ξ, η, ζ, τ which we call the *external particle* and the *internal particle*. The proper masses of the internal and the external particle are well known to be

$$\mu_1 = m_1 m_2 / (m_1 + m_2), \quad \mu_2 = m_1 + m_2.$$

Hence

$$\mu_1 \mu_2 = m_1 m_2. \quad (5)$$

The internal and external particles are commonly said to be fictitious; but

it is difficult to say in what sense one circumlocution can be more fictitious than another.

6. The internal wave function specifies the probability distribution of the *relative* co-ordinates ξ, η, ζ, τ . Since these co-ordinates express spatio-temporal relations between two ordinary physical particles, we usually regard the observable properties of a system as concentrated in its internal wave function. But in saying that the co-ordinates ξ, η, ζ of m_1 measured from a comparison particle m_2 are observable, whereas the co-ordinates x_1, y_1, z_1 measured from a geometrical origin are not, we pay attention only to the origin. An origin does not suffice to define a frame of co-ordinates. In substituting ξ, η, ζ, τ for x_1, y_1, z_1, t_1 we eliminate only the part of the unobservability of the reference frame which refers to the origin, and not the part which refers to the orientation of the axes. In particular the orientation of the time-axis which determines the velocity of the frame is not replaced by anything observable. Now in practice in studying internal wave functions, e.g. those of a hydrogen atom, we choose a frame such that the momentum of the system as a whole is zero; that is to say, the time-axis is chosen to coincide with the external momentum vector. When therefore we ascribe an observable energy to an internal wave function, our reference frame is (so far as four-dimensional orientation is concerned) not a Galilean frame but the physical frame defined by the momentum vector of the external particle. Relative to a Galilean frame the physical frame has the same uncertainty of velocity as the external particle.

Just as the position vector of the particle m_2 provides a physical origin from which positions of the object-particle m_1 can be measured, so the momentum vector of the external particle μ_2 provides a physical time-axis or standard of rest from which the momenta of the internal particle μ_1 can be measured. Properly the resulting momentum should be considered to belong equally to the internal and external particles; but the same kind of convention which leads us to assign the observable co-ordinates to m_1 , treating m_2 as a comparison particle, leads us to assign the observable momenta to μ_1 treating μ_2 as a comparison particle.

Since we are here concerned with momentum vectors, we must adopt the resolution into external and internal particles which provides for the reference of one momentum vector to a standard of rest provided by the other. That we cannot provide simultaneously for the reference of the position of one particle to an origin provided by the other is intelligible; because, when the momentum vectors are exact, the positions are entirely uncertain and there is nothing more to be said about them.

7. A certain conflict of conception must be noticed. We have just seen that the object-particles to which the observed mechanical characteristics are transferred are internal particles, the external particles constituting the physical reference frame. This gives us the ordinary picture of matter as a collection of particles with mass and momentum. But in the ordinary picture these are *genuine* particles—protons and electrons—whereas internal particles are generally labelled *fictitious*. This conflict merely emphasizes that the terms “genuine” and “fictitious” do not mean anything in particular as applied to particles in quantum theory. What we have shown is that in the present approach, via the energy tensor, the elementary mass-bearing particles appear as internal particles. Elementary quantum theory starts with these as its ordinary particles; so that the internal particles occurring in it (e.g. the particle associated with the internal wave function of a hydrogen atom) are, as it were, doubly internal.

In transferring the mechanical characteristics of matter from the relations between particles to the particles themselves we necessarily modify the concept of a particle, since we attribute to the particle characteristics which, according to the primary concept, it could not possibly possess. We shall call this modified concept the *secondary concept*, and the corresponding particles *secondary particles*.

We cannot simultaneously transfer the geometrical characteristics of the relations between particles to individual particles, since that would involve converting m_1, m_2 instead of μ_1, μ_2 into secondary particles. It is this incoherency which makes the secondary particles an unsuitable starting-point for theory, and drives us back to the primary particles for enlightenment.

To embody the conclusions of § 6 we should rewrite (3) as

$$T_{\mu\nu} = \mu_1\mu_2 f_{\mu\nu}(X, Y, Z, T). \quad (6)$$

By (5), $f_{\mu\nu}$ is not altered, but it is now made clear that its arguments are those of the comparison (external) particle μ_2 . The object-particle μ_1 , abstracted from its comparison particle, *has no position in space or time*. Its co-ordinates ξ, η, ζ, τ do not associate it with one region of space-time rather than another. What they signify as applied to a particle is difficult to describe, since we have transferred to a particle properties which cannot be pictured without invoking two particles, but we may perhaps call it a *polarization*. The distribution over ξ, η, ζ, τ may be called the *internality* of the object-particle. It is only connected with the energy tensor as an integrated whole.* If the internal wave function is self-normalizing, its

* Including integration over τ —an important point, but not required in the present discussion.

“density” is integrated into a mass; then, by associating the object-particle with its comparison particle, the mass is distributed along with the comparison particle over the co-ordinates x, y, z, t in space-time. For infinite wave functions a natural normalization volume must exist in both sets of co-ordinates; otherwise there is no systematic way of collecting density from one set and distributing it over the other set, and the link with the energy tensor is broken.

The whole energy tensor of a piece of matter is made up of a number of contributions such as we have been analysing. We shall speak of the comparison particles collectively as the *comparison fluid* (*P. and E.* § 11.1) and the object-particles as the *object-system*. Current theory concerns itself solely with the object-system, the comparison fluid being represented partly in the empirical constants and partly by the mistakes which become prominent when it attempts to deal with energies of the order of magnitude of the mass of a comparison particle ($136mc^2$). We are concerned especially with two of the empirical constants (the masses of the proton and electron) which are no longer empirical when the comparison fluid is studied as part of the system.

8. Comparison particles and object-particles play an unsymmetrical part in the description of phenomena. We shall now formulate this difference mathematically. Consider the momentum vectors $\mu_1 J_1, \mu_2 J_2$ of an internal and external particle. When the latter is adopted as comparison particle, so that its momentum vector defines the time-axis, J_2 automatically becomes the matrix E_{45} associated with the time-direction. Thus the form of the momentum vector of a comparison particle is limited to $\mu_2 E_{45}$; whereas in the momentum vector $\mu_1 J_1$ of an object-particle (an “ordinary” particle of current theory) J_1 may be any unitary matrix.

I present here a rather cut-and-dried picture of the comparison fluid, because I have a definite problem in hand, and the requirement in this (as in a large class of problems) is that the comparison fluid shall provide a definite standard of rest from which momentum can be measured. Various modifications of the scheme of separation into an object-system and comparison fluid can be made to meet the requirements in other problems. The point that we have to notice is that, whereas an object-particle can have a multiply infinite variety of relationship to a comparison particle, a comparison particle has only one possible relationship to itself—that of identity. They have therefore different numbers of degrees of freedom in the reference frame furnished by the comparison particle. We shall find later that this difference is expressed in the statement that the phase space of an object-

particle is ten-dimensional and the phase space of a comparison particle is one-dimensional.

We have seen that a different separation of comparison particles and object-particles is needed according as we require the former to furnish an origin for the measurement of position or a standard of rest for the measurement of momentum. This scarcely exhausts the possibilities; and it may well be that other types of problems will arise in which the requirement is a standard of non-rotation for the measurement of angular momentum. For this we should require "spin-comparison-particles". In a general way one would expect nuclear problems to be of this type.

9. As initially derived the comparison particles have exact momentum vectors $\mu_2 E_{45}$ and therefore complete uncertainty of position. Each particle fills all space. But the particles have no individuality (the term "particle" being an epithet associated with a fully occupied wave function), so that the analysis of the comparison fluid into individual particles follows the analysis into eigenfunctions and is changed if the system of eigenfunctions is changed. By this generalization the term "comparison particle" means "as much comparison fluid as would amount to one of the original particles if distributed in the same way". The simplest reanalysis is to divide the comparison fluid into non-overlapping volumes V , each containing one comparison particle. We shall call this *vertical section* in contrast to the original *horizontal section* into particles each extending over all space. It should be added that (except in horizontal section) the comparison particles are delineated in, not detached from, the comparison fluid; for, if detached, they would have uncertainty of momentum corresponding to their limitation of extension. This is expressed more formally by noticing that the momentum vector $\mu_2 E_{45}$ has ceased to be a true vector, since by nominating its particle as a comparison particle we have made it impossible for it to have a component in any direction other than E_{45} . Thus what we are really reanalysing is a scalar which gives no field of application for the uncertainty principle.

The functional factor in $f_{\mu\nu}$ belongs primarily to the comparison particle, its arguments x, y, z, t being the co-ordinates of that particle. But as the object-particle acquires position in space by its association with a comparison particle, so it acquires a distribution of position by association with a comparison particle which has that distribution of position. Thus the "delineation" of the comparison particle referred to in the preceding paragraph is also the distribution of the object-particle associated with it; and in practice it is more conveniently handled in the latter aspect. Naturally, current quantum theory before proceeding to neglect the comparison

particles transfers as much as possible of their attributes to the object-system; accordingly, the functional factor is attached to the object-vector instead of to the comparison vector when the two factors of the double vector are separated. The comparison particle must retain its momentum vector (since the object-particle has one already and cannot take over another); except for this, the comparison fluid has only those qualities which exist also in the geometrical frame of space-time for which it is commonly mistaken.

The consideration of functional factors, however, takes us beyond the point at which we wish to make contact with current quantum theory. We are concerned with plane waves for which the momentum vector is constant; so there is no functional factor to be transferred. The waves are nominally infinite, but in actual application normalized so that there is one object-particle in a prescribed volume. It is important to remember that the condition is one particle in a given volume, say q^{-1} , not (as is sometimes loosely stated) q particles per unit volume; for it is an elementary principle that q particles are represented by a q -tuple wave function. The wave front is therefore divided into cells, each cell containing one particle uniformly distributed. Since the positional distribution of an object-particle is that of the comparison particle to which it belongs, this corresponds to the "vertical section" of the comparison fluid in which the comparison particles occupy non-overlapping volumes V . Thus the volume V is the natural normalization volume, and the formally infinite wave function is applicable only within the volume V .

Since vertical section seems a very artificial device, I would point out that our reason for considering it is that the formulae of elementary quantum theory (which we have to connect with our own relativistic formulae) are expressed in terms of vertical section, since it is unable to deal with overlapping particles except by multiple wave functions.

By extending the same wave function over a number of cells, we should represent a number of particles having the same characteristics. This is forbidden by the exclusion principle. The continuation can only be approximate, and the complete momentum vector for each particle must be slightly different. This difference is evidently the quantum representation of the curvature introduced into space when it is occupied by mass and momentum. The exclusion principle is in fact a quantum adaptation of the equation $G_{\mu\nu} - \frac{1}{2}g_{\mu\nu}G = -8\pi\kappa T_{\mu\nu}$. The cellular division gives a polyhedral space closely approximating to the continuously curved space contemplated in relativity theory. The so-called cosmological developments of relativistic quantum theory transform the unnatural vertical section, imposed by the

limitations of elementary quantum theory, into a more nearly horizontal section associated with the eigenfunctions of spherical space. The corresponding development within quantum theory itself is the introduction of *second quantization* which enables us to treat particles overlapping in space without multiple wave functions.

When we decide the curvature of the space in which this cellular arrangement is to be constructed, we make provision for a density corresponding to the curvature. "Full occupation" normally means that the density is that for which provision has been made. We here encounter a feature of quantum theory which has no counterpart in relativity theory. Quantum theory erects a skeleton frame of eigenfunctions, leaving itself free to settle and vary afterwards the extent to which they are occupied. The framework is supposed to be there before anything is put into it. That is a survival of the classical outlook. Relativity theory admits no such flexibility; we cannot put anything into space without remodelling its curvature. This makes its problems so difficult that the exact solution, even of the problem of two bodies, has not yet been found. The more flexible method of quantum theory is, of course, an approximation—sometimes a bad approximation—but it is the beginning of a systematic method of successive approximation (the method of the "self-consistent field") so that ultimately it fulfills its purpose. When applied to a system of many particles, it avoids a vast number of individual adjustments most of which would in practice cancel out. The difference is that, if the content of the system differs from that for which provision has been made, relativity theory demands immediate readjustment of the reference frame, whereas quantum theory postpones it to a second approximation. Thus an exact comparison of the formulae of the two theories can only be based on the content for which provision has been made, i.e. for full occupation.

If our physical reference frame (the comparison fluid) is uniform, the volumes V of the comparison particles will all be equal. If the physical frame is non-uniform, the geometrical frame will be correspondingly non-uniform and represent a space of non-uniform curvature; but it will remain true that equal natural volumes correspond to equal numbers of comparison particles, for otherwise volumes which are observationally congruent according to relativity theory would not be observationally congruent according to quantum theory. The natural normalization volume V is thus a universal constant; its evaluation in terms of better known constants belongs to the cosmological branch of the theory.

10. I return now to the concept of a fully occupied wave function. This presents no difficulty when the eigenstates are discrete, but a complication

arises when they form a continuum. We have to distinguish continuity which arises from symmetry or indistinguishability (degenerate continuity) from continuity which arises as the limit of close packing (non-degenerate continuity). Broadly speaking, we can trust commonsense to deal with continuity from close packing, but continuity from symmetry is likely to be a pitfall. *Degeneracy*, which is the continuity arising from symmetry, does not signify a close packing of the eigenstates, in any one complete set of eigenstates, but a multiplicity of the admissible systems of analysis into complete sets of eigenstates. Thus if in unsymmetrical conditions we have two eigenstates associated correspondingly with an x -axis and a y -axis, when there is circular symmetry these will become similar (except for direction) and will be connected by a continuum of eigenstates associated with every direction in the xy plane, since there are now no distinctive directions in that plane. We call the number of original eigenstates amalgamated by symmetry into a continuum the *degree of degeneracy*. Evidently an $(n - 1)$ -dimensional continuum corresponds to the degree of degeneracy n .

Degeneracy is especially prominent in ideal elementary systems, because they are abstracted from the environment which in actual physical systems would always be in some degree asymmetrical. Current quantum theory (through a mistaken analogy with macroscopic systems) regards them as having *no* environment other than a geometrical frame of space-time. This idealization goes too far, since it leaves the system without observable properties. The ideal environment (for which the elementary equations and definitions are valid) is a physical environment, but it has the same symmetry as a space-time frame.

The local symmetry of space-time (which includes also *uniformity*, i.e. symmetry about a centre of curvature) is described by the group of its relativistic transformations. These will also apply to the idealized physical environment which has the same symmetry. The wave functions of a system in this environment will therefore have a degree of degeneracy ascertainable from a study of the group of relativistic rotations of the vectors or tensors defining the system. If the tensor has $n - 1$ independent relativity rotations, the eigenstates will fill an $(n - 1)$ -dimensional continuum, and the degree of degeneracy will be n . A full analytical study of the relativity rotations of wave vectors and tensors (and of the corresponding complete space vectors and tensors) is contained in my development of wave tensor calculus, where n appears as the number of dimensions of the "phase space" (*P. and E.* §§ 7-2, 10-6). I am now able to add in § 17 some further explanation, which I think will render this part of the subject

less obscure from the physical point of view. Meanwhile I shall use the results of the analytical investigation, namely that n is 10 for a simple wave tensor and 136 for a double wave tensor. The wave tensor of a comparison particle is necessarily non-degenerate, being restricted as explained in § 8; hence for a comparison particle $n = 1$.

One of the most striking achievements of ordinary quantum theory is the device by which it is able to employ rigorously these conceptions of elementary particles and systems, which by their definition require symmetrical environment (or, as it mistakenly supposes, no environment at all), in problems which postulate unsymmetrical conditions. Relativity theory provides no means of describing the perturbation of a system by its environment except as a change of its structure; but quantum theory describes small perturbations as changes of the degree of occupation of the various eigenstates, the structure and tensor characteristics of the eigenstates themselves remaining unchanged. This fundamental difference in the treatment of unsymmetrical environment makes it impossible to correlate the formulae of relativity theory and quantum theory except in the most symmetrical conditions, i.e. in uniform space (spherical or flat). Here we should notice that, according to the method of quantum theory, a degenerate wave vector or tensor does not cease to exist in unsymmetrical conditions; it becomes non-uniformly occupied. In certain cases new wave functions are introduced to describe permanent or semi-permanent non-uniformity of occupation.

Besides varying in orientation by relativistic rotation, a momentum vector $M = mJ$ can vary in magnitude. Since J is understood to be normalized, this is provided for by varying m . The eigenvalues of m can only become continuous by close packing; there is no question of degeneracy, since change of m is not a relativistic transformation. We shall take the eigenvalues of m to be discrete; the continuous case can then be included (if necessary) by forming the limit in the usual way.

Although cosmological considerations are outside the scope of this paper, it will be useful to have in mind the results of the cosmological theory in this connexion. There must, of course, be a vast number of possible values of m for an elementary object-particle; if there were only two eigenvalues, there could be at most twenty particles in the universe. But the values are extremely close packed; so that unless we treat aggregations of particles on an astronomical scale the differences from the standard values are insignificant. In very large systems, in order to provide enough particles, we have to include eigenvalues of m sensibly lower than the standard value for small systems. This is the quantum representation of gravitational potential

energy, which makes the total energy or mass of a system sensibly lower than the sum of the standard values of the masses of its particles.

11. Although current quantum theory does not concern itself with the degeneracy of its ultimate particles, it deals with degeneracy in other connexions. In radiation problems it is found that the degeneracy factor represents the *weight* of an oscillator for the purpose of calculating the intensity of the emitted radiation. This is a problem which connects atomic quantities, described by momentum vectors, with field quantities (electromagnetic waves), described by an energy tensor; and it is not difficult to verify that in our problem, which also connects atomic quantities with the energy tensor, degeneracy has the same effect as a weight. Thus we can treat degenerate particles as simple particles provided that we multiply their contributions to the energy tensor by the weight n . This rule is all that we require for the calculations later in this paper. But the degeneracy factors have so wide an importance in all branches of relativistic quantum theory that it is desirable at the outset to remove all obscurity connected with them.

In the radiation problem a degenerate state is capable of occupation by n particle-units. But here we are dealing with the initial degeneracy of the vector wave functions by which elementary particles are defined. That which fully occupies a complete momentum vector is called an electron (or proton). To square with the practice in later applications of degeneracy, we should say rather that it is a union of ten sub-electrons (or sub-protons). This is a mere formalism; but it is useful to remember that, according to the same system, the bi-particle is a union of 136 sub-bi-particles, and the comparison particle is just one comparison sub-particle. It is the three kinds of sub-particle that have the most simple analytical connexion with one another, since they (if they existed) would be free from degeneracy and would be represented by discrete wave functions.

The degenerate wave function replaces n discrete wave functions amalgamated by symmetry. The amalgamation is different from ordinary combination, because wave functions combine by multiplication. Instead of a discrete n -tuple wave function occupying a $3n$ -dimensional volume, we have a simple degenerate wave function occupying a space volume V , *but also extended over a continuum of orientation*. The volume is V because references to the "object-particle" in our previous work must be understood to mean the *whole* particle occupying the momentum vector M_1 .

The important point to notice is that the amalgamation involves a partial replacement of the usual multiplicative occupation factors (or probabilities) by additive occupation factors. The occupation factor of the multiple wave

function is the product $p_1 p_2 \dots p_n$ of the occupation factors of the n discrete wave functions composing it. This is now replaced by a simple degenerate wave function whose total occupation is distributed *additively* over the continuum of orientation. The original set of n discrete wave functions is obtained by concentrating the distribution at n orthogonal points in the continuum; but the occupation factors are now such that the total occupation or probability is $p_1 + p_2 + \dots + p_n$. Thus if discrete wave functions are regarded as having an occupation factor 1 when fully occupied, the degenerate wave function must be regarded as having an occupation factor n when fully occupied. This would be a departure from the original conception of degree of occupation; so we prefer to call n a "weight factor", recognizing, however, that it has the same effect as an occupation factor.

A case of special importance is when the probability distribution of the degenerate wave function is limited to a very small solid angle of orientation, so that the corresponding vector or tensor is "almost exact"; we shall call this a *pseudo-discrete* wave function. There is no sensible difference between a pseudo-discrete and a discrete wave function except that the former has a weight n —as if it were a discrete function occupied n times over.

Thus if we calculate the contribution to the energy tensor as if the momentum vector were discrete and fully occupied, we must multiply the result by n to obtain the contribution of the fully occupied pseudo-discrete momentum vector. This is the rule already given at the beginning of this section.

It will be noticed that in the pseudo-discrete wave function the concentration of the probability in orientation is greater than that obtainable from a set of n orthogonal discrete functions; and it might perhaps be thought that such excessive concentration would be forbidden. But the degenerate function represents a physical particle or bi-particle, and there is certainly nothing to forbid the allocation of its whole probability to a single eigenstate. In saying that the degenerate wave function replaces n discrete wave functions, we do not mean that it is merely a mathematical substitution. The replacement is occasioned by the fact that symmetry of environment affects the nature of our observational knowledge of a system, so that a new form of description must be introduced. Owing to the absence of landmarks for the determination of orientation, the simple direct method of measurements of the characteristics of a system (postulated in the most elementary quantum formulae) is inapplicable. A change in the method of observation implies a different system of separation into object-system and comparison fluid (see § 15).

It is worth noticing explicitly that a bi-particle (which fully occupies a

double vector) yields one object-particle and a comparison particle; but, expressed in terms of sub-particles, 136 sub-bi-particles yield 10 object-sub-particles and a comparison particle. Thus a weight-transformation factor 13.6 commonly appears in passing from the primary to the secondary concept of particles.

To see how degeneracy affects the problem on which we are engaged, we must return to the factorization of the original double vector $\Psi_{\alpha\beta}X_{\gamma\delta}$ into two complete vectors M_1 and M_2 . We have identified them as the momentum vectors of certain particles, but that only means that our terminology has introduced particles ("genuine" or "fictitious") for them to belong to. Primarily they are wave functions which happen to be constant over the domain to which they apply. We have treated the double vector as discrete; for otherwise we could not have factorized it without taking account of the element of solid angle over which its orientation is distributed. In my earlier derivation of the masses of the proton and electron I have taken account of the elements of solid angle associated with the double vector and with M_1 (M_2 being discrete). Progress was difficult because the solid angles are in domains with different numbers of dimensions; but the difficulty was not insuperable. Now, however, we have a much simpler method of dealing with "almost exactness". The double vector and the object-vector are pseudo-discrete, but we may continue treating them as discrete provided that we multiply their contributions to the energy tensor by their respective degeneracy factors in accordance with the rule stated.

12. We began with the expression for the energy tensor in terms of primary particles. We have now to examine its constitution in terms of secondary particles. We consider an *elementary* object-particle, i.e. an entity fully specified by one complete momentum vector $M = mJ$. Its contribution to the energy tensor is required to be a space tensor of the second rank. In principle there is only one way of forming such a tensor; we must take the outer square $M \times M$. For if another vector M' were introduced, the contribution would be jointly associated with particles specified by M and M' . It would, in fact, be a reversion to the primary concept which associates each element of density with two particles.

For accuracy it must be stated that the reality conditions of the theory may require that the second factor is not M but a vector uniquely determined by M , e.g. its complex conjugate. This, however, is merely a convention of symbolism. To guard ourselves formally, it must be understood that the operator \times includes the transformation (if any) imposed by the reality conditions.

The most general expression for the contribution of the fully occupied wave function of an elementary particle to the energy tensor is accordingly $a(M \times M)$, where a is a pure number. Since there are no pure numbers included in the specification of the particle, a must be a constant of nature.

This step in the argument is of great importance, because we shall find that there are two kinds of elementary secondary particles (protons and electrons), and it shows that a must be the same for both. Elementary theory seems to suggest that they ought to have different coefficients a_p and a_e proportional to the reciprocals of their masses; but this is excluded by the fact that we cannot work a_p and a_e into the specification of these particles. If we try to introduce the same difference solely in terms of the momentum vectors specifying the particles by taking the density equal to

$$(M \times M) \div \text{quarterspur } M,$$

we obtain a quantity of the wrong dimensions. We do not say that there can be no physical entity whose specification requires a pure number in addition to a momentum vector; but such an entity does not fulfil the definition of an elementary object-particle.

A comparison particle is an example of an entity whose specification includes a pure number as well as a momentum vector; for we have to qualify the momentum vector by the epithet "discrete", which means that it has only $\frac{1}{10}$ the degeneracy factor of a true vector. Thus if the momentum vector of a comparison particle is M_0 , we must not assume that its contribution to the energy tensor is $a(M_0 \times M_0)$. But we have learned in §11 the precise way in which the pure number $\frac{1}{10}$ appears in the energy tensor. Writing $a = 10b$, so as to show the degeneracy coefficient explicitly, the energy tensors of the comparison particle and object-particle are respectively

$$b(M_0 \times M_0), \quad 10b(M \times M). \quad (7)$$

The original expression for the energy tensor (according to the primary concept of particles) was the product $M \times M_0$ of the two momentum vectors (§4). In order to postpone the discussion of the scale in which this is expressed, we shall write it more generally as

$$136c(M \times M_0), \quad (8)$$

since we already know that it includes the degeneracy factor 136 of a double vector.

The expression of the energy tensor of a secondary particle as the square of its momentum vector accords in principle with macroscopic relativity theory which sets the part of $T_{\mu\nu}$ due to a particle of proper mass m equal to

$p_\mu p_\nu/m$, where $p_\mu = m dx_\mu/ds$ is the momentum vector. We have already remarked that our result omits the divisor m . The macroscopic expression is, of course, no definite guide to the individual contributions of protons and electrons, and the difference is not to be looked upon as a discordance.

13. We can now express the condition that in the replacement of the bi-particles by secondary object-particles and comparison particles the energy tensor is not altered. The sum of the energy tensors of the object-particles and comparison particles must be equal to the sum of the energy tensors of the bi-particles. Accordingly, by (7) and (8),

$$\Sigma\{b(M_0 \times M_0) + 10b(M \times M)\} = \Sigma 136c(M \times M_0). \quad (9)$$

If a double vector is partially occupied, the occupation factor p is attached to the vector of the object-particle in the factorization, the comparison particle being understood to be a permanent constituent of the physical reference frame and therefore incapable of reduced occupation. Thus the factor p would appear in two of the terms of (9) but not in the term $b(M_0 \times M_0)$. Clearly the equation (9), corresponding to full occupation, will not remain satisfied when arbitrary occupation factors are inserted. This was to be expected, because we have seen that the difference in method of relativity and quantum theory in treating variation of the content of the reference frame makes it essential to connect their respective formulae on the basis of full occupation (§ 9).

When we come to consider how it is possible to satisfy equation (9) certain difficulties appear. But it is necessary to remember that our discussion has been confined to the most elementary material of quantum theory—independent elementary particles in uniform motion—and the equation applies to object-systems composed solely of such material. Interaction energy, field energy, reaction at the boundary of the volume considered, would introduce extra terms whose form we have not yet discussed. Nor have we considered how a variation of the degrees of occupation of a wave function would affect the energy tensor, except in so far as a quasi-static approximation is sufficient.

The most conspicuous difficulty is that, whereas the left-hand side of (9) will generally yield a pressure component T_{11} , there is no provision for pressure components on the right. But a little reflexion shows that this was to be expected. If pressure is treated in the usual elementary way as a boundary effect, it corresponds to exchange of particles between the region W considered and the region outside W , and the corresponding exchange-energy must be introduced to represent it. If the pressure is treated as an

internal distribution, it is represented by the collision forces between the particles which are also a variety of exchange effect. Finally, if there are no collisions and no boundary exchange, so that the system is dispersing freely, terms representing the progressive evacuation dp/dt must be inserted.

To avoid a long incursion into other branches of relativistic quantum theory in which these matters are treated, we must find a simple application of (9) which steers clear of them. With this limitation the only problem we can tackle is that of particles at rest, since motion immediately raises the evacuation problem.* We can omit the summation, since there is now no advantage in considering more than one double vector and its factors.

Accordingly, taking the object-particle at rest and considering the double-time component T_{44} of the tensor, (9) gives

$$bm_0^2 + 10bm^2 = 136cmm_0, \quad (10)$$

where m , m_0 are the proper masses of the object-particle and comparison particle.

It is already clear from (10) that there will be two possible values of the ratio m/m_0 , so that there are elementary particles of two different masses.

14. The factors 1, 10, 136 in the three terms of (10) are the degeneracy factors of the respective wave tensors. By omitting them we obtain

$$b(m_0^2 + m^2) = cmm_0, \quad (11)$$

which would have been the result if degeneracy had been neglected. In order to find the ratio b/c , we examine how this simplified equation would be interpreted, weight factors being disregarded. We are considering a normalization volume V in which from the ordinary point of view there is only one particle whose mass is m . The two sides of the equation give two expressions for the density. The right-hand side corresponds to the ordinary expression $\rho = m/V$, so that we must have

$$cm_0 = 1/V. \quad (12)$$

The reciprocal of a three-dimensional volume is a vector normal to the volume; and, since in our units it has the dimensions of a mass, it has appeared in our analysis as the vector cM_0 which has a time-component cm_0 , as expressed in (12).

Relativity theory, however, insists that an observable such as density

* We should have to express in terms of occupation factors the condition (obvious from relativity considerations) that a space component of the momentum vector causes a particle to "move", i.e. become associated progressively with different elements of the comparison fluid.

cannot be associated with one particle in empty space; so that the second vector, whether written as cM_0 or V^{-1} , must represent a physical system and not a mere geometrical abstraction. Accordingly, the left-hand side of (11) exhibits the same density divided between the particle which we recognize and a comparison particle. It shows that if the ordinary particle were removed, so that the space became technically empty, there would still be a density bm_0^2 and therefore a mass bm_0^2V in the volume. Since this must be the mass m_0 of the comparison particle, we have

$$m_0 = bm_0^2V = bm_0^2/cm_0$$

by (12). Hence $b = c$.

Using this result, (10) becomes

$$m_0^2 + 10m^2 = 136mm_0, \quad (13)$$

which is the equation for the masses of the proton and electron (Eddington 1931). It equates the density of the *mutual energy* of particles (on the right) to that of the equivalent *self-energies* of particles (on the left).

15. If m_1, m_2 are the masses of the proton and electron, i.e. the roots of equation (13), the masses μ, M of the internal and external particles of the hydrogen atom are

$$\left. \begin{aligned} \mu &= m_1m_2/(m_1+m_2) = m_0/136, \\ M &= m_1+m_2 = 136m_0/10. \end{aligned} \right\} \quad (14)$$

It is instructive to see how these masses can be derived directly instead of via the proton and electron. If the foregoing arguments were applicable to internal and external particles without modification, we should have had

$$m_0^2 + (10\mu^2) = 136\mu m_0, \quad (15)$$

$$(m_0^2) + 10M^2 = 136Mm_0. \quad (16)$$

The brackets show the terms which must be omitted to agree with (14), and therefore call attention to the points at which the argument becomes modified.

A peculiar feature of the internal particle is that it has no rest-mass. The proper mass of the system is M , not $M + \mu$. Alternatively, we can say that the particle has a rest-mass equal to its mass-constant μ , but the zero of energy-reckoning has been lowered by a constant μ . Thus the density $10\mu^2$, corresponding to a mass μ , is either absent or cancelled, and must be omitted on the left-hand side of (15).

For the external particle the term m_0^2 is omitted. The argument for inserting this term was that when the self-energy of the object-particle is

removed there remains the self-energy of the comparison particle. This no longer applies, because the self-energy of the comparison particle is provided out of the internal mutual energy $136\mu m_0$ as shown in equation (15). The difference is that when we remove the self-energy of a proton we have to remember to make provision for the energy of the comparison particle; but when we remove the self-energy of an external particle we leave an internal particle which undertakes the provision of the energy of the comparison particle.

It will be seen that when internal and external wave functions are employed there is only one comparison particle instead of two. We have reached this conclusion by deduction; but the physical reason for it should also be considered. It turns on the fact that ξ is not the difference of two co-ordinates measured from the physical origin, but a direct measurement of one particle from the other as origin. The transformation to internal and external wave functions is not simply an analytical transformation but a change of the method of observation, the new method being one which *causes less disturbance to the physical frame*; for every measurement from the physical origin communicates an uncertain momentum to the physical frame just as it does to the particle that is being located. This lessened disturbance is represented by providing one comparison particle instead of two to take up the reaction from the measurements.

16. We are now in a position to treat more fully the transformation (4). In § 5 it was applied to primary particles; but we here consider the same transformation applied to object-particles so as to replace the wave functions of a proton and electron by internal and external wave functions. Being an initial step towards other developments, the transformation is limited to field-free non-interacting particles represented by "infinite plane waves". After this step has been taken, interaction can be introduced in the internal wave function, which then fulfils its normal practical purpose of specifying the *correlation* of the proton and electron distributions. The transformation, like any mathematical substitution, is always valid, but its utility is limited to uncorrelated distributions.

A well-known property of (4) is that the Jacobian $\partial(\xi, \mathbf{x})/\partial(x_1, x_2)$ is unity; this has the effect of eliminating any difference between polarization space ξ, η, ζ and ordinary space as regards normalization volumes.

Another well-known result is

$$\frac{1}{m_1} \frac{\partial^2}{\partial x_1^2} + \frac{1}{m_2} \frac{\partial^2}{\partial x_2^2} = \frac{1}{M} \frac{\partial^2}{\partial \mathbf{X}^2} + \frac{1}{\mu} \frac{\partial^2}{\partial \xi^2}, \quad (17)$$

with similar equations for the other three co-ordinates. Hence

$$m_1^{-1} \square_1 + m_2^{-1} \square_2 = M^{-1} \square_x + \mu^{-1} \square_\xi. \quad (18)$$

The second order (relativistic) wave equation for a field-free particle is

$$\{(-i\hbar/2\pi)^2 \square - m'^2\} \psi = 0, \quad (19)$$

where m' is considered to be the proper mass of the particle. Hence, for discrete eigenfunctions, (18) reduces to

$$m_1'^2/m_1 + m_2'^2/m_2 = M'^2/M + \mu'^2/\mu. \quad (20)$$

This reduction is not quite so simple as it seems, because the four \square operators do not usually reduce to eigenvalues simultaneously. But it is justified by the familiar theory of invariance of the density of the eigenstates in different modes of analysis, provided that the eigenstates are non-degenerate. When, however, non-scalar wave functions are employed, we must take account of their innate degeneracy. If in (19) ψ is a wave vector, it describes an amalgamation of ten discrete states and only $\frac{1}{10}$ of the eigenvalue m'^2 can be allotted to one discrete state. Thus the general form of (20) is

$$m_1'^2/m_1 n_1 + m_2'^2/m_2 n_2 = M'^2/MN + \mu'^2/\mu\nu, \quad (21)$$

where n_1, n_2, N, ν are the degeneracy factors. When, as in the formation of a hydrogen atom, the wave functions are all simple wave vectors, all four factors are 10 and we regain equation (20); but the form (21) should be kept in reserve for future developments.

Equation (20) or (21) expresses the condition that the pairs of wave systems ψ_1, ψ_2 and ψ_M, ψ_μ , satisfying wave equations of the form (19), describe the same object-system. We can use it in two ways: (1) to show how the eigen-masses M', μ' must be chosen in order that the condition may be satisfied, or (2) if M', μ' are assigned by convention in a way which does not satisfy the condition, we can find the nominal change of the energy of the object-system in passing from one mode of description to the other.

We must adopt $m_1' = m_1, m_2' = m_2$, since that is essentially the definition of the transformation we are applying. The practice of quantum theory commits us to the identification $\mu' = \mu$. Then if the condition (20) is satisfied, we have

$$M'^2 = M(M - \mu). \quad (22)$$

It is not impracticable to accord the external wave function the exceptional treatment which (22) demands, but it is confusing in conception. But a more illuminating point of view is obtained by following up alternative (2). We then adopt $M' = M$, making no distinction between the external particle

and the other particles. Condition (20) is then not satisfied, but the discrepancy is regarded as a change μ in the zero-point of our reckoning of energy when we pass from a description of the object-system in terms of ordinary particles to a description in terms of internal and external particles.

A change of zero-point is justified physically by the fact that the transformation is not merely analytical but implies a different procedure of observation—a procedure which introduces one comparison particle instead of two (§ 15), so that a different separation of the object-system from the comparison fluid is involved. I may add that the change of zero-point is not the direct result of the elimination of a comparison particle (for our method of analysis has provided for that); it is due to the elimination of its *interchange energy* with the comparison particle which is retained (*P. and E.* § 15·8).

17. The derivation of the numbers 10 and 136, representing degree of degeneracy or number of dimensions of phase space, is given in *P. and E.* pp. 95, 165. The following is a more elementary physical derivation.* The mechanical characteristics of a particle consist of its energy, momentum and spin-momentum. The momentum and energy constitute a 4-vector $(m_{15}, m_{25}, m_{35}, m_{45})$ and the spin momentum is a 6-vector $(m_{23}, m_{31}, m_{12}, m_{14}, m_{24}, m_{34})$, so that ten quantities are required to specify the mechanical characteristics. If, however, we are considering the possible variations of the characteristics of a particle of *fixed* proper mass m , there are only nine independent variations. In the postulated symmetrical conditions these form a nine-dimensional continuum of degeneracy; hence the degree of degeneracy is 10.

It is commonly assumed that the spin-momentum has only three components m_{23}, m_{31}, m_{12} ; but this assumes that the particle is at rest, since a Lorentz transformation would introduce the other components of the 6-vector. Bearing in mind that the particle is in general a superposition of elementary states with different momenta, we see that the full 6-vector must be specified, and each component is capable of independent variation. This applies even when we are dealing with a single elementary state as ordinarily understood, because the momentum vector is not exact (discrete) but almost exact (pseudo-discrete).

To exhibit the connexion of this treatment with phase space, we have to consider the strain vectors corresponding to the above space vectors. These are $(S_{14}, S_{24}, S_{34}, S_{16})$ and $(S_1, S_2, S_3, S_{15}, S_{25}, S_{35})$, together forming the ten

* This is the result of discussions with H. T. Corben as to the physical identification of the components of a strain vector.

space-like components of the complete strain vector. Starting with $S_{16} = m$, the other components being 0, infinitesimal variations of the other components give nine dimensions of phase space. To these we have to add the algebraic phase conjugate to S_{16} , making ten dimensions in all (*P. and E.* § 7·3).

An interesting point is that only *mechanical* characteristics of the particle are taken into account. The electromagnetic characteristics (distinguished by their change of sign in passing from a right-handed to a left-handed frame) correspond to the six time-like components of the complete strain vector, and are not represented in phase-space. The meaning of this is that electromagnetic characteristics have no physical significance if the particle is alone in a Riemannian frame; on the other hand mechanical characteristics have a significance, since the $g_{\mu\nu}$ constitute an inertial frame.

Passing to the bi-particle, the ten mechanical characteristics of each component form 100 combinations. These are all capable of independent variation, because the bi-particle is not necessarily in one separable state; it may be a superposition of separable states. In addition, electromagnetic characteristics of one particle in combination with those of another particle have a physical significance. The six dormant electromagnetic characteristics of each particle give 36 combinations representing electric stresses, etc., in the bi-particle. Naturally, the specification is more complicated than that of a macroscopic electromagnetic field which is much simplified by averaging. The total number of significant characteristics is $100 + 36 = 136$. There are also 120 combinations of mechanical characteristics of one particle with electromagnetic characteristics of the other which are not physically significant when the bi-particle is isolated in space; these are distinguished analytically as time-like components of the double strain vector.

Strictly speaking, one more degree of degeneracy is introduced into the bi-particle when we nominate one of the two components as the comparison particle. Exchange of the nomination is clearly a relativistic transformation, so that it forms an additional dimension of the continuum of degeneracy. I have taken no account of this in the foregoing discussion, because the subject of interchange is better treated as a whole. If we neglect interchange in this derivation of primitive material, and then hand over the material to current quantum theory which continues to neglect interchange, we obtain a consistent theory into which the complicated effect of interchange can be introduced systematically. But if we take account of interchange in the preliminary stages and then join our results to current theory, we obtain a mixture which is not only incorrect but difficult to amend. I consider

therefore that the non-interchange masses in equation (13) should be regarded as the standard masses of the proton and electron. It appears from the theory of interchange that the mass of the internal particle of a hydrogen atom (and therefore approximately the mass of an electron) derived by applying the formulae of current quantum theory to observation is 137/136 times the standard mass (*P. and E.* § 15·8).

In many other parts of my theory the formulae are more easily derived and understood when we interpret the numerical coefficients as degrees of degeneracy. Besides the coefficients 10, 136, 137 the coefficients most frequently occurring are 4 expressing the degeneracy of an ordinary (not a *complete*) space vector, and 3 for the three-dimensional vectors in static problems. But other combinations can arise according to the circumstances of the problem under consideration.

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The masses of the neutron and mesotron

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THE NEUTRON

1. The development of relativistic wave mechanics in the preceding paper makes it possible to calculate the mass of the neutron. References, unless otherwise stated, are to the sections of that paper. We have only to express the fact that a proton and electron by emitting a neutrino yield a neutron.

We consider the proton and electron initially without interaction, since the interaction energy would in any case have to be recalculated after the emission of the neutrino. They are equivalent to an external and an internal particle of masses $M = m_1 + m_2$, $\mu = m_1 m_2 / (m_1 + m_2)$. These, like the electron and proton, are specified by complete momentum vectors having the full degree of degeneracy 10. I take it that the emission of a neutrino is a way of saying that one of the particles loses its spin. Accordingly its complete