Discussion of results.

From the results so far obtained it is difficult to avoid the conclusion that the long-range atoms arising from collision of α particles with nitrogen are not nitrogen atoms but probably atoms of hydrogen, or atoms of mass 2. If this be the case, we must conclude that the nitrogen atom is disintegrated under the intense forces developed in a close collision with a swift α particle, and that the hydrogen atom which is liberated formed a constituent part of the nitrogen nucleus. We have drawn attention in paper III. to the rather surprising observation that the range of the nitrogen atoms in air is about the same as the oxygen atoms, although we should expect a difference of about 19 per cent. If in collisions which give rise to swift nitrogen atoms, the hydrogen is at the same time disrupted, such a difference might be accounted for, for the energy is then shared between two systems.

It is of interest to note, that while the majority of the light atoms, as is well known, have atomic weights represented by $4n$ or $4n+3$ where n is a whole number, nitrogen is the only atom which is expressed by $4n+2$. We should anticipate from radioactive data that the nitrogen nucleus consists of three helium nuclei each of atomic mass 4 and either two hydrogen nuclei or one of mass 2. If the H nuclei were outriders of the main system of mass 12, the number of close collisions with the bound H nuclei would be less than if the latter were free, for the α particle in a collision comes under the combined field of the H nucleus and of the central mass. Under such conditions, it is to be expected that the α particle would only occasionally approach close enough to the H nucleus to give it the maximum velocity, although in many cases it may give it sufficient energy to break its bond with the central mass. Such a point of view would explain why the number of swift H atoms from nitrogen is less than the corresponding number in free hydrogen and less also than the number of swift nitrogen atoms. The general results indicate that the H nuclei, which are released, are distant about twice the diameter of the electron (7×10^{-13} cm.) from the centre of the main atom. Without a knowledge of the laws of force at such small distances, it is difficult to estimate the energy required to free the H nucleus or to calculate the maximum velocity that can be given to the escaping H atom. It is not to be expected, *a priori*, that the velocity or range of the H atom released from the nitrogen atom should be identical with that due to a collision in free hydrogen.

Taking into account the great energy of motion of the α particle expelled from radium C, the close collision of such an α particle with a light atom seems to be the most likely agency to promote the disruption of the latter; for the forces on the nuclei arising from such collisions appear to be greater than can be produced by any other agency at present available. Considering the enormous intensity of the forces brought into play, it is not so much a matter of surprise that the nitrogen atom should suffer disintegration as that the α particle itself escapes disruption into its constituents. The results as a whole suggest that, if α particles—or similar projectiles—of still greater energy were available for experiment, we might expect to break down the nucleus structure of many of the lighter atoms.

I desire to express my thanks to Mr. William Kay for his invaluable assistance in counting scintillations.

University of Manchester,
April 1919.

LV. *The Rotational Oscillation of a Cylinder in a Viscous Liquid.* By D. Coster *.

THIS problem has been dealt with by Stokes † for the purpose of numerical calculations to determine the viscosity of the air. Still, I think it interesting to publish another solution of the problem which gives more opportunity of discussing the different cases, though it is perhaps less adapted to precise calculations.

The method to be followed will be in the main the same as that used by Prof. Verschaffelt in the analogous case of the sphere ‡. We consider the rotational swings about its axis of an infinitely long cylinder which executes a forced vibration. Our object will be to ascertain the motion in the liquid which will establish itself after an infinite time (in practice after a relatively short time §) in order to compute the frictional moment of forces exerted on the cylinder by the liquid. The calculations will be referred to a height of 1 cm.

The motion of the cylinder may be represented by $\alpha = a\cos pt$ where α is the angle of rotation. An obvious assumption to be made is that the liquid will be set in motion in coaxial cylindrical shells each of which will execute its oscillations as a whole. On this assumption it is not difficult

* Communicated by Prof. G. N. Watson, M.A., D.Sc. First published in the Amsterdam Proc. May 1918, vol. xxi. p. 193.

† Math. Papers, vol. v. p. 207.

‡ *Cf.* Amst. Proc. vol. xviii. p. 840; Comm. Leiden, 148 C.

§ *Cf.* Comm. Leiden, p. 22, footnote.

to establish the differential equation of the motion of the liquid.

Let ρ be the density,

μ the viscosity of the liquid.

ω the angular velocity of a cylindrical shell.

r the radius of the shell.

The frictional force per unit area of one of the shells will then be $F = r\mu\frac{\partial\omega}{\partial r}$ and the frictional couple on a cylindrical surface of radius r will be $2\pi r^3\mu\frac{\partial\omega}{\partial r}$.

Taking a shell of thickness dr its equation of motion will be

$$2\pi r^3 dr\rho\frac{\partial\omega}{\partial t} = \frac{\partial}{\partial r}\left\{2\pi r^3\mu\frac{\partial\omega}{\partial r}\right\}dr,$$

which reduces to

$$\frac{\rho}{\mu}\frac{\partial\omega}{\partial t} = \frac{\partial^2\omega}{\partial r^2} + \frac{3}{r}\frac{\partial\omega}{\partial r}. \quad \quad (1)$$

For an infinitely long time of vibration, *i. e.* for uniform rotation, (1) simplifies to

$$0 = \frac{d^2\omega}{dr^2} + \frac{3}{r}\frac{d\omega}{dr}. \quad \quad (2)$$

The solution of (2) is $\omega = \frac{c_1}{r_2} + c_2$, c_1 and c_2 being constants of integration. If the solid cylinder (radius R) rotates with uniform speed Ω in an infinite liquid, the result will be $\omega = \frac{R^2\Omega}{r^2}$, giving for the frictional couple, as is well known, the expression

$$-4\pi\mu R^2\Omega. \quad \quad (2')$$

In order to arrive at a possible solution of (1) we have to make our assumption regarding the motion of the liquid a little more definite by assuming that the angular displacement of each shell is represented by

$$a_r = f(r)\cos\{pt - \phi(r)\}. \quad \quad (3)$$

We may also consider (3) as the real part of the complex function ue^{ipt}, where u is a function of r the modulus of which gives the amplitude of the oscillation and the argument the phase-shift $\phi(r)$. Remembering that $\omega = \frac{\partial\alpha}{\partial t}$ equation (1) may be reduced to

$$\frac{d^2u}{dr^2} + \frac{3}{r}\frac{du}{dr} - \frac{i\rho pu}{\mu} = 0. \quad \quad (4)$$

Equation (4) is closely related to the differential equation of the cylindrical functions. Indeed by the substitution $y = zv$, Bessel's equation of the first order

$$\frac{d^2y}{dz^2} + \frac{1}{z}\frac{dy}{dz} + \left(1 - \frac{1}{z^2}\right)y = 0$$

changes to

$$\frac{d^2v}{dz^2} + \frac{3}{z}\frac{dv}{dz} + v = 0.$$

It follows that the general solution of equation (4) is

$$u = \frac{1}{r}\{AJ_1(cr) + BN_1(cr)\}, \quad \quad (5)$$

where $c = \sqrt{\frac{-i\rho p}{\mu}}$, A and B being complex constants of integration. J_1 is the cylindrical function of the first kind and first order, N_1 that of the second kind and first order *.

As regards c an agreement must be come to. We shall choose the root with the negative imaginary part, i. e., $c = ke^{-\frac{i\pi}{4}}$, where $k = |c| = \left|\sqrt{\frac{\rho p}{\mu}}\right|$.

As a first boundary-condition we have $\lim\limits_{r=\infty} ra_r = 0$. As this relation must hold for all values of t, it follows that $\lim\limits_{r=\infty} ru = 0$.

The cylindrical functions with complex argument all become infinite at infinity with the exception of the so-called functions of the third kind, or Hankel's functions $H_p^{(1)}$ and $H_p^{(2)}$. Of these $H_p^{(1)}$ disappears at infinity in the positive imaginary half-plane and on the contrary becomes infinite in the negative half, whereas the opposite is true for $H_p^{(2)}$. By our choice of c in the negative imaginary half we are led to the function $H_1^{(2)}$. For the constants of integration in equation (5) this gives the relation † $B = -iA$, so that (5) becomes

$$u = \frac{A}{r}H_1^{(2)}(cr). \quad \quad (6)$$

For the determination of A we have to use the second

* *Cf.* Gray and Mathews, 'Bessel Functions.' Nielsen, *Cylinderfunktionen*; Jahnke und Emde, *Funktiontafeln.* Instead of N, Gray and Mathews use the symbol Y.

† Between J, N, and H a linear relation holds. *Cf.* Jahnke u. Emde, p. 95.

boundary-condition $a_R = a\cos pt$, R being the radius of the cylinder. We therefore assume that there is no slipping along the wall.

Hence

$$A = \frac{aR}{H_1^{(2)}(cR)},$$

so that

$$a_r = \mathbf{R}\,\frac{aR}{H_1^{(2)}(cR)}\,\frac{H_1^{(2)}(cr)}{r}\,e^{ipt}. \quad . \quad . \quad . \quad . \quad . \qquad (7)$$

The symbol $\mathbf{R}$ means that the real part has to be taken of the function which stands after it.

If we had chosen for c the root with the positive imaginary part, we should have had to utilize the function $H_1^{(1)}$. It is quite easy to verify that this would not have made any essential change in the solution (7).

For large values of x (real and positive) $H_1^{(2)}(x\sqrt{-i})$ approaches asymptotically to

$$-\frac{e^{-\frac{x}{\sqrt{2}}}}{\sqrt{\frac{1}{2}\pi x}}\,e^{-i\left(\frac{x}{\sqrt{2}}-\frac{\pi}{8}\right)};$$

therefore for (kR) sufficiently large:

$$a_r \approx -\frac{aR}{|H_1^{(2)}(cR)|}\,\frac{e^{-\frac{kr}{\sqrt{2}}}}{\sqrt{\frac{1}{2}\pi k r^{1\frac{1}{2}}}}\cos\left(pt-\frac{kx}{\sqrt{2}}+\frac{\pi}{8}-\phi\right), \quad . \qquad (8)$$

where $\phi = \arg H_1^{(2)}(cR)$.

From (8) it appears that damped waves are propagated from the cylinder to infinity, the velocity of propagation being

$$v = \frac{p}{k/\sqrt{2}} = \frac{p\sqrt{2}}{k} = \sqrt{\frac{2p\mu}{\rho}},$$

and the wave-length

$$\lambda = vT = \frac{2\pi v}{p} = \frac{2\pi\sqrt{2}}{k} = 2\pi\sqrt{\frac{2\mu}{\rho p}}. \quad . \quad . \quad . \quad . \qquad (8')$$

The frictional moment on the wall of the vibrating cylinder is $2\pi\mu R^3\left[\frac{\partial\omega}{\partial r}\right]_R$ where $\omega = \frac{\partial\alpha}{\partial t}$. First we determine $\left[\frac{\partial\alpha_r}{\partial r}\right]_R$ from (7):

$$\left[\frac{\partial\alpha_r}{\partial r}\right]_R = \mathbf{R}\left[-\frac{a}{R}\,e^{ipt} + ac\,\frac{H_1^{(2)\prime}(cR)}{H_1^{(2)}(cR)}\,e^{ipt}\right]. \quad . \qquad (9)$$

For the reduction of the second part on the right-hand side of (9) we make use of the well-known recurrence-formula of the cylindrical functions:

$$\frac{dH_1^{(2)}(z)}{dz} = H_0^{(2)}z - \frac{1}{z}\,H_1^{(2)}(z).$$

By its application (9) assumes the form:

$$\left[\frac{\partial\alpha_r}{\partial r}\right]_R = \mathbf{R}\left[-\frac{2a}{R}\,e^{ipt} + ac\,\frac{H_0^{(2)}(cR)}{H_1^{(2)}(cR)}\,e^{ipt}\right] \qquad (10)$$

giving for the frictional couple:

$$K = 2\pi\mu R^3\left[\frac{\partial\omega}{\partial r}\right]_R = -4\pi\mu R^2\omega + \mathbf{R}\,\frac{d}{dt}\left[2\pi\mu R^3 ac\,\frac{H_0^{(2)}(cR)}{H_1^{(2)}(cR)}\,e^{ipt}\right]. \quad (11$$

For an infinite time of swing, i.e., $p=0$, but with a rotational velocity differing from 0, $|c| = \sqrt{\frac{\rho p}{\mu}}$ becomes 0. In that case the second term on the right of (11) disappears on two grounds: first, because $c=0$, secondly, $\lim\limits_{cR=0}\frac{H_0^{(2)}(cR)}{H_1^{(2)}(cR)} = 0$; only the first term then remains, which agrees with (2′).

Moreover*:

$$\lim_{cR=\infty}\frac{H_0^{(2)}(cR)}{H_1^{(2)}(cR)} = -i.$$

It appears from the accompanying graphs † of the modulus and argument of $\frac{H_0^{(2)}(cR)}{H_1^{(2)}(cR)}$ that this limiting value is practically reached at

$$|cR| = k.R = 10, \quad . \quad . \quad . \quad . \quad . \quad . \qquad (12)$$

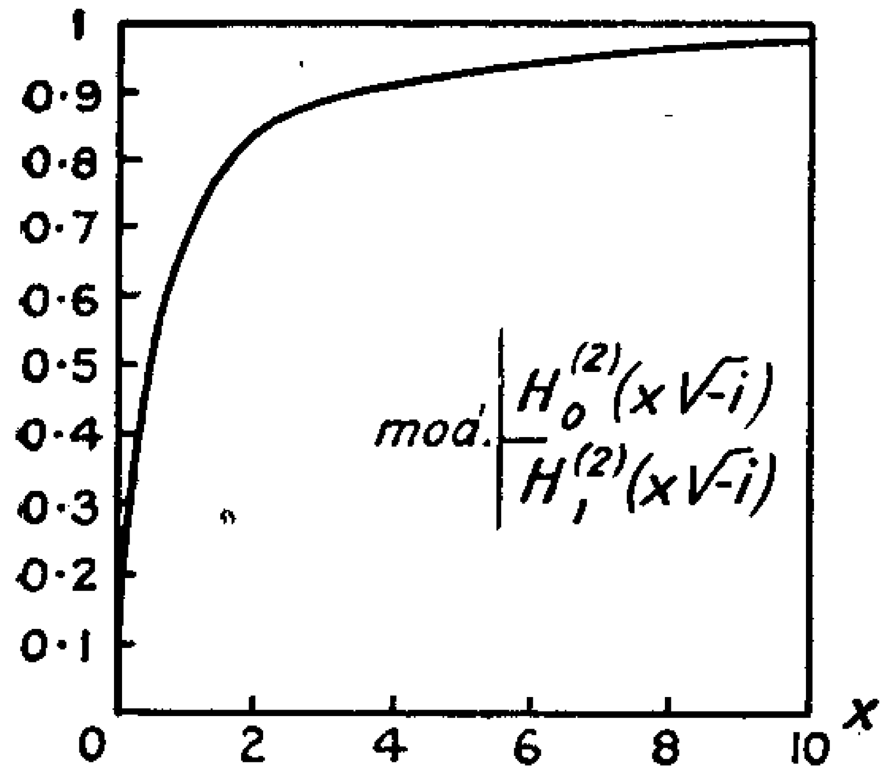
where

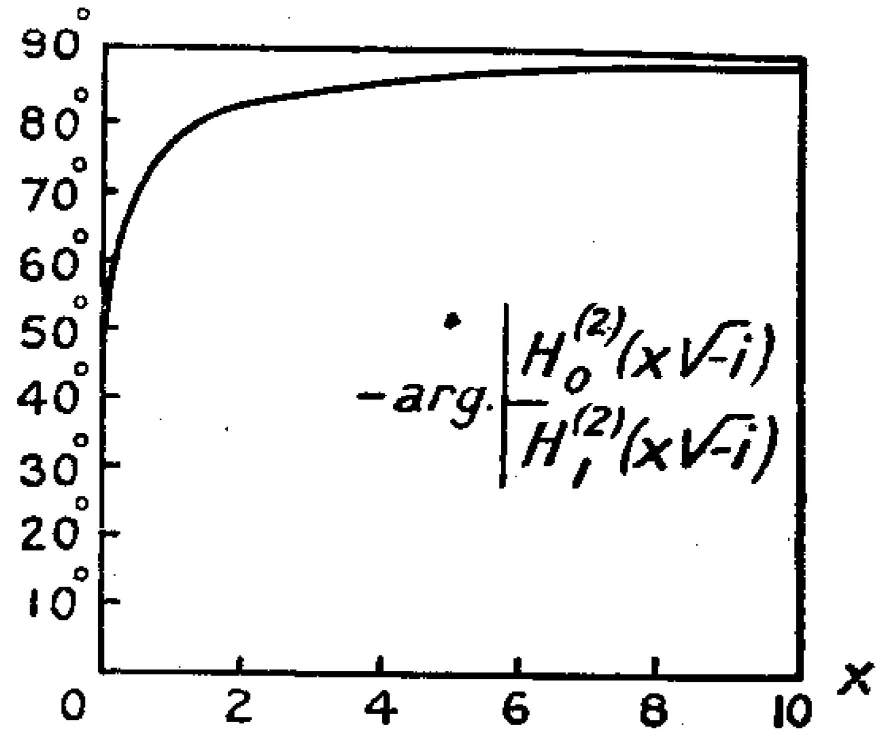
$$|c| = k = \frac{2\pi\sqrt{2}}{\lambda}\ (cf.\ 8').$$

* *Cf.* Jahnke u. Emde, l. c.

† Tables for $H_0^{(1)}$ and $H_0^{(2)}$ will be found Jahnke u. Emde, pp. 139, 140.

The condition $|cR|\geqq 10$ means that the radius of the cylinder must be about equal to or larger than the wave-length. If R is small compared with λ the second part of





the frictional couple is negligible. For $|cR|\geqq 10$ the second term on the right-hand side of (10) becomes:

$$-acie^{ipt} = -ake^{i\left(pt+\frac{\pi}{4}\right)} \left(\text{since } c = ke^{-\frac{i\pi}{4}}\right).$$

Hence equation (11) now becomes:

$$K = -4\pi\mu R^2\omega - 2\pi\mu k R^3 \frac{d}{dt}\left(a\cos\left(pt+\frac{\pi}{4}\right)\right) \quad . \quad (13)$$

where $$\omega = \frac{d}{dt}(a\cos pt).$$

The frictional couple thus divides into two parts, one of which does not contain the density of the liquid, and another in which it occurs and which therefore refers to the emission of waves. In the transition to the limit of uniform rotation the first part only remains.

In the discussion of the second part of the frictional moment the quantity $k=\sqrt{\frac{p\rho}{\mu}}$ is an important factor. If we take a time of oscillation of 2π seconds, so that $p=1$, we have $k=\sqrt{\frac{\rho}{\mu}}$.

This gives the following values for k:

	ρ.	μ.	$k=\sqrt{\frac{\rho}{\mu}}$ $(p=1)$.
Water 16°	1	0·011	9·5
Atm. air 0°	0·0013	0·000171	2·8
Air 0·01 atm.*			0·28
Air 0·001 atm.*			0·09
Hydrogen 1 atm. 0° ...	0·0000898	0·000085	1

* At these pressures μ has not become much smaller. *Cf.* Kundt u. Warburg, Pogg. *Ann.* 1875, Band clv.

From this table it appears that, except for dilute gases, R has to be relatively small in order that the second part may be neglected with respect to the first. For instance, for atmospheric air with R=0·5 cm. kR=1·4 and $\left|\frac{H_0^{(2)}(cR)}{H_1^{(2)}(cR)}\right| = 0\cdot 80$, so that the amplitude of the second term of the frictional couple is still 56 per cent. of that of the first (see equation (11)), in which everything is calculated for a time of oscillation of 2π seconds.

There is a further special limiting case of equation (13), which is of some interest. Let R become infinite, and let a at the same time disappear, in such a manner that Ra converges to a finite limit b. We thus approach the one-dimensional problem of the oscillation of an unlimited flat plate in its own plane in an infinitely extended liquid. The frictional force per unit of area is found from (13) to be

$$F = -\mu k \frac{d}{dt}\left(b\cos\left(pt+\frac{\pi}{4}\right)\right), \quad . \quad . \quad . \quad (14)$$

a formula which is well known from hydrodynamics*. A term analogous to $-4\pi\mu R^2\omega$ does not occur in the one-dimensional problem, the reason evidently being that with a uniform translation of the plate a condition of equilibrium does not arise, until the whole liquid away to infinity proceeds with the velocity of the plate.

Finally it is of importance to ascertain for what frequency the amplitude of the forced vibration becomes a maximum, in other words, to what frequency the system cylinder-liquid resounds, if the cylinder is urged back to the position of equilibrium by a quasi-elastic force.

The differential equation for the forced oscillation in complex notation is as follows:

$$\theta\frac{d^2a}{dt^2} + L\frac{da}{dt} + Ma = Ee^{ipt}. \quad . \quad . \quad . \quad (15)$$

Here in our case L is a complex quantity $L = L' + iL''$, where

$$L' = (4\pi\mu R^2 + \sqrt{2}\pi\mu k R^3)$$
$$L'' = \sqrt{2}\pi\mu k R^3.$$

If we only concern ourselves with the particular solution

* *Cf.* Lamb, 'Hydrodynamics,' 3rd edition, 1905, p. 559.

of (15) which gives the forced oscillation, we can also write (15) in the form :

$$\left(\theta+\frac{L''}{p}\right)\frac{d^2a}{dt^2}+L'\frac{da}{dt}+Ma=Ee^{ipt}. \quad . \quad . \quad (16)$$

We see, therefore, that in consequence of the motion of the liquid an apparent increase of the moment of inertia arises.

Putting
$$\theta+\frac{L''}{p}=\theta'$$

the particular solution of (16) becomes :

$$a=\frac{E}{\sqrt{(M-\theta' p^2)^2+L'^2p^2}}e^{i(pt-\phi)}$$

in which the phase-angle ϕ is determined by the constants of the differential equation.

Resonance occurs for $M-\theta' p^2=0$

or
$$\theta p^2+L''p-M=0 \quad . \quad . \quad . \quad . \quad . \quad (17)$$

Now L'' is proportional to k and $k=\sqrt{\frac{p\rho}{\mu}}$, so that we may conveniently write $L''=Np^{\frac{1}{2}}$, N being a constant.

(17) is now replaced by

$$\theta p^2+Np^{\frac{3}{2}}-M=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

This equation, which is bi-quadratic in $\sqrt{p}$, determines the frequencies to which the system resounds. On closer examination there appears to be but one resonance-frequency. Naturally we are only concerned with the real roots p of equation (18). There are found to be two such roots, one for which $\sqrt{p}$ is positive, and another for which $\sqrt{p}$ is negative. Now it follows from our calculation that we have assumed $\sqrt{p}$, which occurs in k, to be essentially positive. For if we substitute a negative value for $\sqrt{p}$ in our equations, we obtain a system of waves which moves from infinity towards the cylinder. But the amplitude of this system is infinite at infinity, so that our first boundary-condition would not be satisfied.

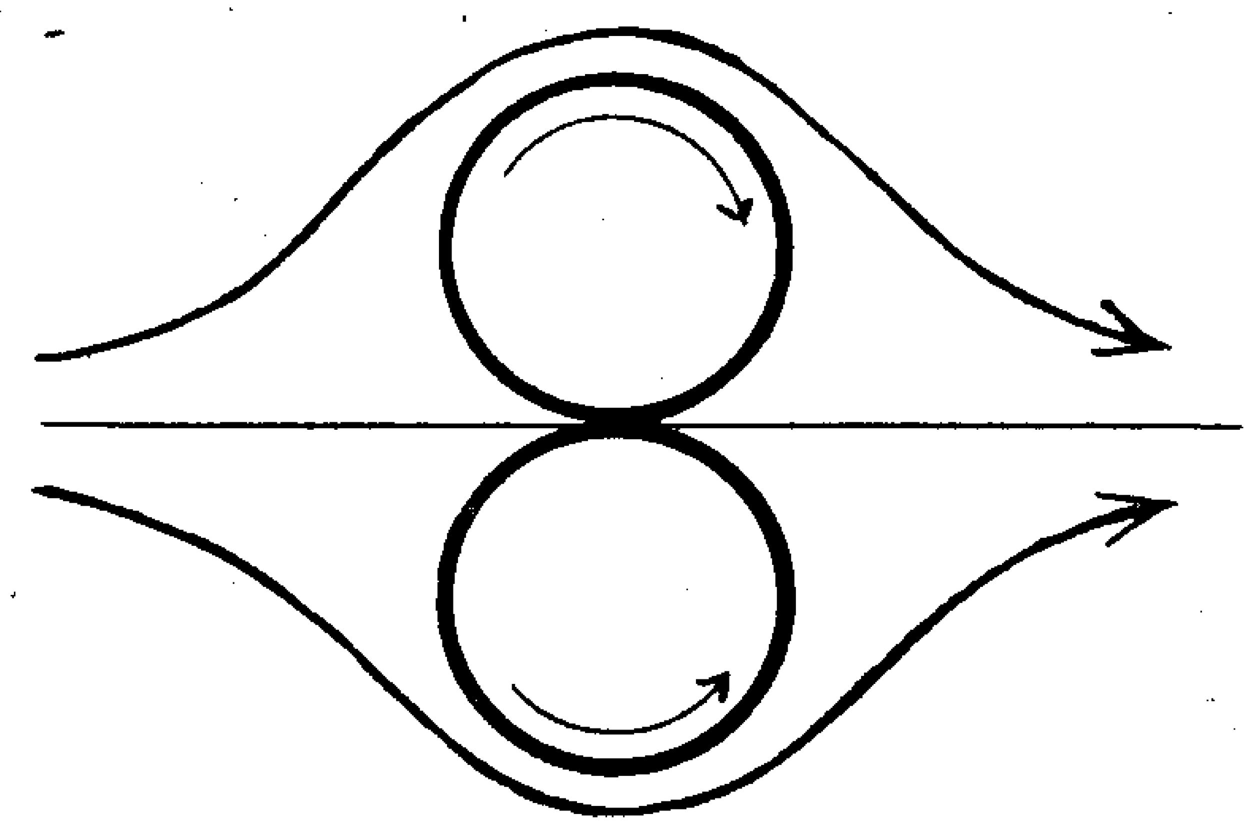
Delft (Holland),
March, 1919.

LVI. *A proposed Hydraulic Experiment.*

To the Editors of the Philosophical Magazine.

GENTLEMEN,—

IN the issue of this Journal for October 1918 (p. 315) Lord Rayleigh has proposed an experiment on the flow of a liquid between two cylinders, standing side by side, and on the influence of a rotation of these cylinders on the form of the stream-lines. May I draw your attention to a remark by Prof. F. Prandtl of Göttingen, put forward in a discussion at a meeting of November 1911 and published in the *Zeitschrift für Flugtechnik und Motorluftschiffahrt*, iii. p. 32 (1912), on an experiment which is only slightly different from that proposed by Lord Rayleigh? It is stated there that no vortices (eddies) arise if care has been taken that everywhere the parts of the walls go faster than the adjacent fluid. If two cylinders, standing side by side, very near to each other,



rotate in opposite directions, it is possible to make the stream-lines close perfectly behind the cylinders. The arrangement differs from that as proposed by Lord Rayleigh only as far as the flow is directed along the exterior sides of the cylinders, and not between them. Photographs seem to have been taken of the form of the stream-lines; however, they have not been published.

Delft (Holland),
4 Dec. 1918.

Yours truly,
J. M. BURGERS.