VIII. *On the Radiation of Light from the Boundaries of Diffracting Apertures.*

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[Plates III. & IV.]

I. *Introduction.*

IN his famous memoir on the mathematical theory of diffraction, Sommerfeld † has given a rigorous treatment of the effect of a semi-infinite perfectly reflecting screen on the propagation of plane waves of light through the medium. One of the most important results indicated by his investigation is that the diffraction effect due to the screen may be regarded as due to cylindrical waves emitted by its edge, the intensity of which is different in different directions, these waves alone being operative in the region of shadow, but in the other regions appearing superposed upon and interfering with the reflected and transmitted waves. Among the more recent writers who have observed and studied the phenomena of the luminosity of a diffracting edge experimentally may be mentioned Gouy ‡, Wien §, E. Maey ‖, and Kalaschnikow ¶.

The present paper deals experimentally and theoretically with the problem of the emission of light by the boundary of a diffracting *aperture* of limited area and of specified form. I consider the actual form of the luminous fringes seen at the boundary of a diffracting aperture when it is viewed solely by the diffracted light. To give the problem definiteness we have to assume that the aperture is viewed through a telescope focussed on its plane, the object-glass of the telescope being itself covered by a screen containing one or more apertures through which the diffracted light enters the field of view. In these circumstances the luminosity appears practically confined to certain more or less well-defined regions lying near the boundary of the aperture. When the wave entering the first aperture is convergent and the object-glass of the observing telescope is placed in the focal plane, the mathematical treatment becomes analogous to that given in a recent paper on the theory of Foucault's test by Lord Rayleigh *. The present investigation was, in fact, suggested by a perusal of Lord Rayleigh's paper, and the photographs reproduced in Plate IV. were taken with an arrangement analogous to that used in Foucault's test, though I have also made visual observations in the cases in which the incident waves do not converge to a focus in the plane of the second screen.

* Communicated by Prof. C. V. Raman.

† *Math. Ann.* Bd. xlvii. (1896).

‡ Gouy, *Ann. d. Phys. et de Chim.* (6), (8), p. 145 (1886).

§ Wien, Inaug. Diss., Berlin, 1886.

‖ E. Maey, Wiedemann's *Annalen*, xlix. (1893).

¶ Kalaschnikow, *Journ. Russ. Phys. Ges.* xliv. (1912).

The most remarkable result found in the course of the experimental work (and which, so far as I know, has not been noticed by any previous writer) is that in all cases in which the apertures in the focal plane through which the diffracted rays pass (whatever be their actual form) are *symmetrically* disposed about the centre of the field, the latter itself being excluded, the image of the boundary of the diffracting surface appears as *a perfectly black line* surrounded on either side by luminous bands. This is irrespective of the actual form of the boundary itself, that is, whether it is circular or of any other shape whatsoever. A similar result is also found when the screen with the apertures is placed symmetrically in any plane either in advance of or behind the focus. When the apertures on the screen are wide enough to admit a large portion of the diffraction pattern formed at the screen into the field of view of the observing telescope, this black line is characterized by extreme fineness and is surrounded on either side by broad luminous bands. Narrowing the apertures is, however, attended by an increase in the width of this black line and the appearance of a large number of well-defined fringes on either side. This remarkable feature is explained in a general way if we regard each element of the edge of a diffracting aperture as sending out two streams of light in directions more or less normal to itself (one on each side of the wave-normal), and that these streams are in opposite phases. This is distinctly suggested by Sommerfeld's well-known investigation on the diffraction of plane waves of light by a semi-infinite screen. The expression given by him for the radiation emitted by the edge is

$$\frac{1}{4\pi}\cos\left\{2\pi\left(\frac{r}{\lambda}-\frac{t}{\mathrm{T}}\right)+\frac{\pi}{4}\right\}\sqrt{\frac{\lambda}{r}}\left[-\frac{\pm 1}{\cos\frac{\phi+\phi'}{2}}-\frac{1}{\cos\frac{\phi-\phi'}{2}}\right],$$

where (r, ϕ) are the coordinates of a point in the medium

* Lord Rayleigh, "On Methods for detecting small Optical Retardations, and on the Theory of Foucault's Test," Phil. Mag. Feb. 1917.

and ϕ' is the angle which the incident beam makes with the screen. This is valid everywhere except in two extremely limited regions lying in the neighbourhood of the two directions $\phi=\pi-\phi'$ and $\phi=\pi+\phi'$, which indicate the planes of transition between the regions of shadow and of transmission and between the regions of transmission and reflexion respectively. It will be noticed that in the neighbourhood of these two planes of transition, one of the two terms within the square bracket is very small in comparison with the other, and may therefore be neglected. The term which is retained changes sign when we pass from one side of the plane of transition to the other. It is thus seen that the phase of the radiation emitted by the edge changes by π when we move from the region of shadow into the region of light. If similarly we assume that each element of the boundary of a diffracting *aperture* emits radiations, the phase of which differs by π on the two sides of the wave-normal passing through it, then the phenomena described by me would be qualitatively explained. The detailed mathematical treatment will, however, be given in the course of the paper for the case of the circular and rectangular boundaries.

2. *Case of the Circular Boundary.*

The optical surfaces examined by the well-known "knife-edge" test due to Foucault are most frequently limited by a circular aperture*, the illumination being that due to a point source. Fig. 9 (Pl. III.) reproduces a photograph of the luminosity observed at the boundary when the knife-edge is put in horizontally into the focal plane from below so as to cut off most of the light. It will be noticed that the luminosity is a maximum on the upper and the lower boundaries, and diminishes to zero at the ends of a horizontal diameter. In order to give definiteness to a discussion of this effect, it is necessary to postulate some specified forms for the boundaries of the apertures in the focal plane which admit the diffracted rays into the field of view of the observing telescope. For instance, we may assume that a horizontal slit is placed in the focal plane below the centre of the field. Fig. 12, Plate III., reproduces the beautiful lunette-shaped diffraction-fringes that appear on either side of the boundary of the circular aperture with this arrangement. The luminosity, as in the case of the simple knife-edge test, tends to zero at the ends of a horizontal diameter. The explanation of this fact and of the peculiar form of the fringes will appear later.

* See the memoirs by Draper and Ritchey "On the Construction of a Silvered Glass Telescope," Smithsonian Contributions to Knowledge, vol. xxxiv. (1904).

More striking still are the interference phenomena obtained when the boundary is observed through a pair of apertures of the same form placed in the focal plane. With two horizontal slits placed *on the same side* of the centre of the field, lunette-shaped *interference* fringes are observed, the central fringe which coincides with the boundary being white (Pl. III. fig. 15). But when two horizontal slits are placed on opposite sides of the centre of the field, *the central fringe is black*—in other words, the boundary of the aperture is itself non-luminous but appears surrounded on either side by luminous bands (Pl. III. fig. 6). As has been remarked in the introduction, this remarkable fact is one of great generality. Fig. 16 in Pl. III. represents the appearance of the circular boundary when a horizontal wire is placed across the centre of the focal plane. A fine black line may be seen running through the luminous arcs and dividing them into two. A case that admits of detailed mathematical treatment is that in which the arrangement is completely symmetrical about the axis. Figs. 7 & 10 in Pl. III. show the results obtained when the central part of the field at the focal plane is blocked out by a circular disk and only the diffracted rays passing through an annulus of greater or less width surrounding it enter the observing telescope. It will be seen that in both photographs the boundary appears as a perfectly black circle, with luminous rings on either side of it.

Let R be the radius of the circular aperture of the lens, and assume that in the focal plane there is a screen containing an annular aperture, R_1, R_2 being the radii of the circles defining the annulus. Let ϕ be the angle of diffraction of parallel rays which meet at any point Q in the focal plane. Since the path-difference between the rays leaving a point (r, θ) and the centre of the diffracting aperture is evidently $r\cos\theta\sin\phi$, the diffracted disturbance at a point in the focal plane due to an area $r\,d\theta\,dr$ can be written in the form

$$r\sin 2\pi\left(\frac{t}{T}-\frac{r\cos\theta\sin\phi}{\lambda}\right)d\theta\,dr.$$

The total disturbance at Q in the focal plane is therefore

$$\int_0^{2\pi}\int_0^{R} r\sin 2\pi\left(\frac{t}{T}-\frac{r\cos\theta\sin\phi}{\lambda}\right)d\theta\,dr. \quad . \quad . \quad . \quad (1)$$

The rays from the various elements $r'd\theta'dr'$ of the second aperture may be regarded as meeting in the field of the observing telescope proceeding at an angle ϕ' with the axis and producing the observed effect. The total disturbance in this direction due to the second aperture is therefore,

$$\int_0^{2\pi}\int_{R_1}^{R_2}\int_0^{2\pi}\int_0^{R} rr'\sin 2\pi\left(\frac{t}{T}-\frac{r\cos\theta\sin\phi}{\lambda}-\frac{r'\cos\theta'\sin\phi'}{\lambda}\right)dr\,d\theta\,dr'\,d\theta'. \quad (2)$$

Since ϕ and ϕ' are small quantities, the above expression can be written as

$$\int_0^{2\pi}\int_{R_1}^{R_2}\int_0^{2\pi}\int_0^{R} rr'\sin 2\pi\left(\frac{t}{T}-\frac{\phi\,.\,r\cos\theta}{\lambda}-\frac{\phi'\,.\,r'\cos\theta'}{\lambda}\right)dr\,d\theta\,dr'\,d\theta'. \quad (3)$$

This expression can be reduced to the form

$$\int_0^{2\pi}\int_{R_1}^{R_2}\left[r'\sin 2\pi\left(\frac{t}{T}-\frac{\phi'\,.\,r'\cos\theta'}{\lambda}\right)dr'\,d\theta' \times\int_0^{2\pi}\int_0^{R} r\cos\left(2\pi\frac{\phi\,.\,r\cos\theta}{\lambda}\right)dr\,d\theta\right], \quad (4)$$

the other integral being zero on account of symmetry of the diffracting aperture.

But the integral

$$\int_0^{2\pi}\int_0^{R} r\cos\left(2\pi\,.\,\frac{\phi\,.\,r\cos\theta}{\lambda}\right)dr\,d\theta=\frac{2}{\pi}\cdot\frac{R\lambda}{2\pi\phi}J_1\left(\frac{2\pi}{\lambda}R\phi\right).$$

Therefore the expression (4) becomes

$$\int_0^{2\pi}\int_{R_1}^{R_2} r'\,d\theta'\,dr'\sin 2\pi\left(\frac{t}{T}-\frac{\phi'\,.\,r'\cos\theta'}{\lambda}\right).\frac{1}{\phi}J_1\left(\frac{2\pi}{\lambda}R\phi\right), \quad (5)$$

neglecting a constant factor.

If f is the focal length of the lens, then $\phi=\frac{r'}{f}$. The expression (5) can be written in the form

$$\sin 2\pi\frac{t}{T}\int_0^{2\pi}\int_{R_1}^{R_2} r'\,d\theta'\,dr'\cos\left(2\pi\frac{\phi'\,.\,r'\cos\theta'}{\lambda}\right)\frac{f}{r'}J_1\left(\frac{2\pi R}{\lambda f}\,.\,r'\right)$$

$$+\cos 2\pi\frac{t}{T}\int_0^{2\pi}\int_{R_1}^{R_2} r'\,d\theta'\,dr'\sin\left(2\pi\frac{\phi'\,.\,r'\cos\theta'}{\lambda}\right)\frac{f}{r'}J_1\left(\frac{2\pi R}{\lambda f}\,.\,r'\right).$$

Since the second aperture is also symmetrical about the axis, the second integral is zero, for the elements of it arising from two points situated at equal distances on opposite sides of a diameter are equal and of opposite signs. Therefore the intensity as viewed in the direction ϕ' is

$$I=\left[\int_0^{2\pi}\int_{R_1}^{R_2} d\theta'\,dr'\cos\left(2\pi\frac{\phi'\,.\,r'\cos\theta'}{\lambda}\right)J_1\left(\frac{2\pi R}{\lambda f}\,.\,r'\right)\right]^2.$$

Integrating with respect to ϕ', we get

$$I=\left[\int_{R_1}^{R_2} J_0\left(\frac{2\pi\phi'}{\lambda}\,.\,r'\right)J_1\left(\frac{2\pi R}{\lambda f}\,.\,r'\right)dr'\right]^2$$

(neglecting a constant factor).

If the angular semi-diameter of the lens be denoted by ψ, then $R=f\psi$. The expansion for the intensity can therefore be written in the form

$$I=\left[\int_{R_1}^{R_2} J_0\left(\frac{2\pi\phi'}{\lambda}\,.\,r'\right)J_1\left(\frac{2\pi\psi}{\lambda}\,.\,r'\right)dr'\right]^2.$$

Since $\frac{1}{\lambda}$ is a very large quantity, it is convenient to use semi-convergent expansions for J_0 and J_1. We have

$$J_0(x)=\sqrt{\frac{2}{\pi x}}\left[\cos\left(x-\frac{\pi}{4}\right)\left\{1-\frac{1^2.\,3^2}{2!\,(8x)^2}+\frac{1^2.\,3^2.\,5^2.\,7^2}{4!\,(8x)^4}-\dots\right\}+\sin\left(x-\frac{\pi}{4}\right)\left\{\frac{1}{8x}-\frac{1^2.\,3^2.\,5^2}{3!\,(8x)^3}+\frac{1^2.\,3^2.\,5^2.\,7^2.\,9^2}{5!\,(8x)^5}-\dots\right\}\right],$$

$$J_1(x)=\sqrt{\frac{2}{\pi x}}\left[\sin\left(x-\frac{\pi}{4}\right)\left\{1+\frac{3\,.\,5\,.\,1}{8\,.\,16}\left(\frac{1}{x}\right)^2-\frac{3\,.\,5\,.\,7\,.\,9\,.\,1\,.\,3\,.\,5}{8\,.\,16\,.\,24\,.\,32}\left(\frac{1}{x}\right)^4+\dots\right\}+\cos\left(x-\frac{\pi}{4}\right)\left\{\frac{3}{8}\cdot\frac{1}{x}-\frac{3\,.\,5\,.\,7\,.\,1\,.\,3}{8\,.\,16\,.\,24}\left(\frac{1}{x}\right)^3+\frac{3\,.\,5\,.\,7\,.\,9\,.\,11\,.\,1\,.\,3\,.\,5\,.\,7}{8\,.\,16\,.\,24\,.\,32\,.\,40}\left(\frac{1}{x}\right)^5-\dots\right\}\right].$$

Thus, when x is large,

$$J_0(x)=\sqrt{\frac{2}{\pi x}}\left[\cos\left(x-\frac{\pi}{4}\right)+\frac{1}{8}\,\frac{\sin\left(x-\frac{\pi}{4}\right)}{x}\right],$$

$$J_1(x)=\sqrt{\frac{2}{\pi x}}\left[\sin\left(x-\frac{\pi}{4}\right)+\frac{3}{8}\,\frac{\cos\left(x-\frac{\pi}{4}\right)}{x}\right].$$

Therefore

$$I=\left[\int_{R_1}^{R_2}J_0\left(\frac{2\pi\phi'}{\lambda}x\right)J_1\left(\frac{2\pi\psi}{\lambda}x\right)dx\right]^2$$

$$=\frac{1}{\psi\phi'}\left[\frac{\lambda}{2\pi^2}\int_{R_1}^{R_2}\frac{\sin\frac{2\pi}{\lambda}(\psi-\phi')x}{x}dx-\frac{\lambda}{2\pi^2}\int_{R_1}^{R_2}\frac{\cos\frac{2\pi}{\lambda}(\psi+\phi')}{x}dx\right.$$

$$+\frac{\lambda^2}{32\pi^3}\left(\frac{1}{\phi'}+\frac{3}{\psi}\right)\int_{R_1}^{R_2}\frac{\cos\frac{2\pi}{\lambda}(\psi-\phi')x}{x}dx$$

$$-\frac{\lambda^2}{32\pi^3}\left(\frac{1}{\phi'}-\frac{3}{\psi}\right)\int_{R_1}^{R_2}\frac{\sin\frac{2\pi}{\lambda}(\psi+\phi')x}{x}dx$$

$$\left.+\text{ terms involving higher powers of }\lambda\right]^2.$$

Taking $R_1=\frac{3\lambda}{2\pi\psi}$, $R_2=\frac{50\lambda}{2\pi\psi}$, we obtain (neglecting a constant factor),

$$I=\frac{1}{\phi'\psi}\left[\left\{\mathrm{Si}\,50\left(1-\frac{\phi'}{\psi}\right)-\mathrm{Si}\,3\left(1-\frac{\phi'}{\psi}\right)\right\}\right.$$

$$-\left\{\mathrm{Ci}\,50\left(1+\frac{\phi'}{\psi}\right)-\mathrm{Ci}\,3\left(1+\frac{\phi'}{\psi}\right)\right\}$$

$$\left.+\lambda(\ldots\quad\ldots)+\ldots\right]^2.$$

Calculating the values of this expression for different values of $\frac{\phi'}{\psi}$, we construct the following table (Table I.).

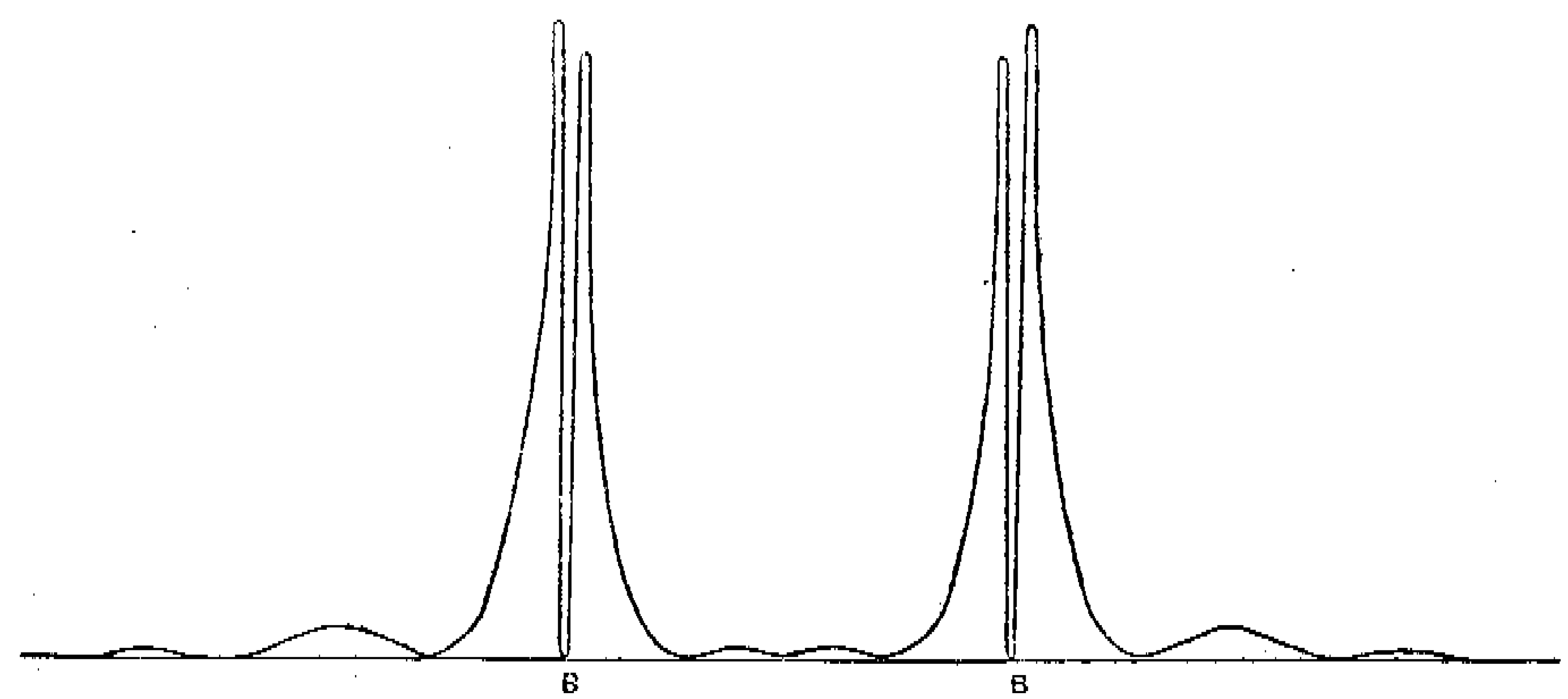
Plotting the values, we obtain a curve (fig. 1) representing the distribution of intensity along any given diameter. The fringes that appear on either side of the boundary are clearly shown, the most remarkable feature being the extreme rapidity with which the intensity falls practically to zero on

TABLE I.

ϕ'/ψ.	$\sqrt{I}$.	I (×const. factor).	ϕ'/ψ.	$\sqrt{I}$.	I (×const. factor).
0·00	− ·162	262	1·02	− ·889	7905
0·20	− ·219	478	1·04	−1·527	23307
0·40	− ·119	142	1·06	−1·701	28911
0·60	+ ·255	650	1·08	−1·549	23909
0·80	+ ·819	6711	1·10	−1·269	16096
0·90	+1·139	12902	1·20	−1·047	10964
0·92	+1·417	20076	1·40	− ·359	1294
0·94	+1·662	27613	1·60	+ ·058	34
0·96	+1·405	19754	1·80	+ ·254	645
0·98	+ ·815	6649	2·00	+ ·358	1281
0·99	+ ·396	1571	2·40	+ ·101	102
1·00	+ ·073	53	2·60	− ·054	29
1·01	− ·524	2746			

the boundary itself (BB in the figure) from a large value on either side of it. This feature depends on the inner radius R_1 of the annulus being small and the outer radius R_2 being very large, and is entirely confirmed by observations under these conditions.

Fig. 1.



If the radii R_1 and R_2 of the annulus in the focal plane do not differ much, or if they are both large, the brightness falls off to zero on the boundary, but not very suddenly. A large number of well-defined fringes also appear on either side of the boundary in the former case (see, for instance, Pl. III. fig. 13). This will be shown from the following calculations based on the data obtained from an actual experiment. The data were :

$$R = \cdot 94 \text{ mm.}, \quad f = 272 \text{ mm.}, \quad R_1 = 1\cdot 72 \text{ mm.}, \quad R_2 = 2\cdot 61 \text{ mm.}, \quad \lambda = \cdot 00045 \text{ mm.}$$

We thus obtain for this case

$$I = \frac{1}{\phi'\psi}\Big[\Big\{\text{Si}\,122\cdot 67\left(1-\frac{\phi'}{\psi}\right)-\text{Si}\,80\cdot 84\left(1-\frac{\phi'}{\psi}\right)\Big\} - \Big\{\text{Ci}\,122\cdot 67\left(1+\frac{\phi'}{\psi}\right)-\text{Ci}\,80\cdot 84\left(1+\frac{\phi'}{\psi}\right)\Big\} + \lambda\left(\ \ldots\ \ \ldots\ \right) + \&\text{c.}\Big]^2.$$

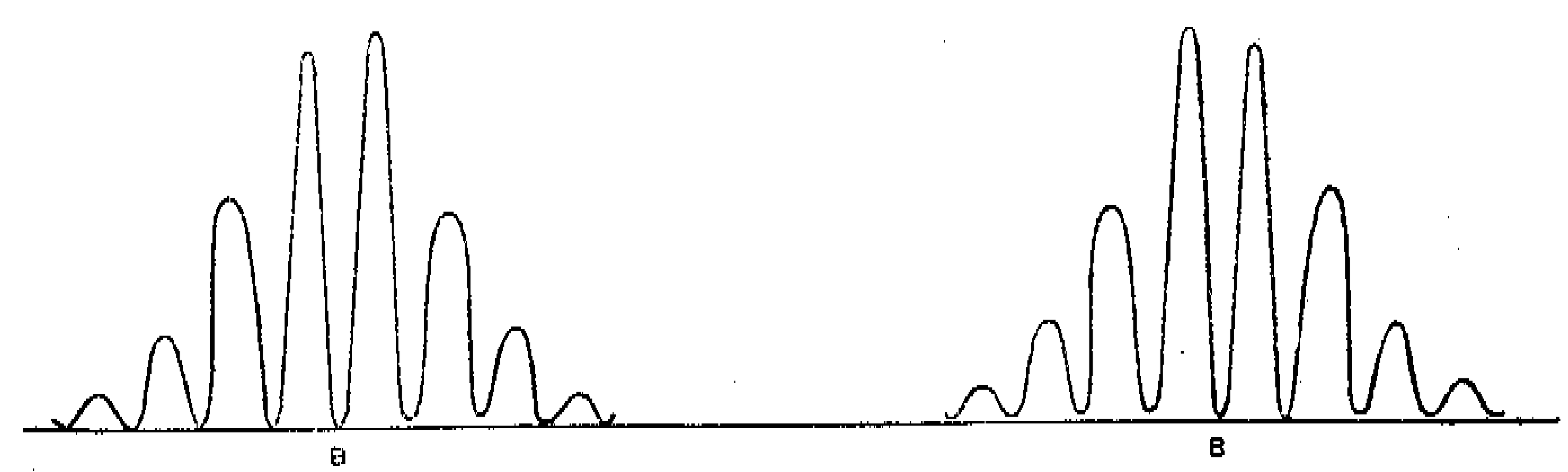
The values of this expression for different values of ϕ'/ψ are shown in Table II., in which the calculated and the observed values of the ratio ϕ'/ψ at which the illumination is a maximum or a minimum are given for comparison. It will be seen from the table that $\sqrt{I}$ changes sign at the minima, and these are therefore absolute zeros. The values for I shown in Table II. have been plotted in fig. 2. It will be seen from Tables I. and II. that $\sqrt{I}$ changes sign as it passes through its values at the boundary $\phi'/\psi=1$, showing that the radiations from the edge on the two sides of the boundary differ in phase by π.

The investigation given above may be modified to suit the case in which we have one or more slits (instead of an annulus) in the focal plane by suitably altering the limits of integration in the expression (5) given above. The writer hopes to give the detailed numerical calculations in a later paper. It is possible, however, to understand in a general way the reason for the peculiar configuration of the fringes shown in Pl. III. figs. 6, 12, & 15. If instead of a slit we have in the focal plane a small aperture at (r', θ') through which the boundary is viewed, the luminosity of the latter appears confined to certain regions lying in the neighbourhood of the two points $r=R$, $\theta=\theta'$, and $\theta=\pi+\theta'$, and which are more or less well-defined according to the size and the position of the aperture. The further the aperture is from the centre of the focal plane, the feebler is the luminosity observed through it. Accordingly, if we regard the horizontal slit placed in the focal plane as consisting of a number of elements along its length, the vanishing of the luminosity of the boundary at the ends of a horizontal diameter is seen to follow as a consequence. At

TABLE II.

ϕ'/ψ.	$\sqrt{I}$.	I (×const. factor).	Calculated values of ϕ'/ψ for maximum or minimum.	Observed values of ϕ'/ψ for maximum or minimum.
0·880	−·0536	2,873	0·885 min.	
0·890	−·1257	15,801	0·892 max.	0·894
0·900	−·0804	6,464	0·904 min.	0·901
0·910	+·0908	8,245		
0·920	+·2179	47,481	0·922 max.	0·930
0·930	+·1543	23,808		
0·940	−·0779	6,068	0·935 min.	0·940
0·950	−·3149	99,168	0·953 max.	0·957
0·960	−·2761	76,231		
0·970	+·0599	3,588	0·971 min.	0·971
0·975	+·2423	58,709		
0·980	+·3779	142,809		
0·985	+·4268	182,158	0·983 max.	0·985
0·990	+·3548	125,883		
0·995	+·2244	50,355		
1·000	+·0053	28	1·000 min.	1·000
1·005	−·2134	45,539		
1·010	−·3448	118,889		
1·015	−·4158	172,889	1·017 max.	1·015
1·020	−·3673	134,908		
1·025	−·2209	48,797		
1·030	−·0491	2,411	1·028 min.	1·030
1·040	+·2866	82,139		
1·050	+·3251	105,690	1·047 max.	1·046
1·060	+·0875	7,656	1·063 min.	1·061
1·070	−·1478	21,844		
1·080	−·2141	45,839	1·077 max.	1·078
1·090	−·0859	7,379		
1·100	+·0854	7,293	1·095 min.	1·096
1·110	+·1354	18,333	1·107 max.	1·109
1·120	+·0596	3,552	1·125 min.	1·132

Fig. 2.



and near these points also the radial width of the luminosity should obviously be the least, as the latter is, roughly speaking, in inverse proportion to the corresponding radial width of the aperture in the focal plane. This gives us a qualitative explanation of the lunette-shaped form of the fringes in these cases. Figs. 5, 8, 11, & 14 in Pl. III. illustrate the remarks made above regarding the localization of the luminosity of the boundary observed in certain cases. Fig. 5 represents the effect observed when there were two small circular apertures in the focal plane not lying on the same radius vector. Accordingly we have on the boundary four separate regions of luminosity. Fig. 8 represents a photograph obtained when a ring of six circular holes was placed symmetrically in the focal plane. Each of the six spots seen along the boundary is crossed by very fine fringes, due to the interference of the effects produced by the pair of apertures at the end of each diameter. Fig. 14 was obtained when the ring of holes was slightly displaced in the focal plane. Twelve spots appear on the boundary. Fig. 11 represents the effect observed when the screen in the focal plane was so placed that two out of the three pairs of apertures fell on lines passing through the centre of the field. Accordingly only eight spots are seen, the four larger ones being crossed by fine interference fringes.

3. *Case of the Rectangular Boundary.*

It has been shown by Lord Rayleigh* that when an optical surface bounded by parallel straight edges and illuminated by a linear source of light is examined by the "knife-edge" test, the intensity of the field as viewed in the direction ϕ is given by

$$I = \left[\mathrm{Si}\left\{\frac{2\pi}{\lambda}(\theta+\phi)\xi_2\right\} - \mathrm{Si}\left\{\frac{2\pi}{\lambda}(\theta+\phi)\xi_1\right\} + \mathrm{Si}\left\{\frac{2\pi}{\lambda}(\theta-\phi)\xi_2\right\} - \mathrm{Si}\left\{\frac{2\pi}{\lambda}(\theta-\phi)\xi_1\right\}\right]^2$$
$$+ \left[\mathrm{Ci}\left\{\frac{2\pi}{\lambda}(\theta-\phi)\xi_2\right\} - \mathrm{Ci}\left\{\frac{2\pi}{\lambda}(\theta-\phi)\xi_1\right\} - \mathrm{Ci}\left\{\frac{2\pi}{\lambda}(\theta+\phi)\xi_2\right\} + \mathrm{Ci}\left\{\frac{2\pi}{\lambda}(\theta+\phi)\xi_1\right\}\right]^2,$$

where θ is the angular semi-aperture of the lens, ξ_1 denotes

* *Loc. cit.*

the extent to which the knife-edge has been advanced in the focal plane beyond the centre of the field, and ξ_2 defines an upper limit for the aperture in the focal plane. If $\xi_1=0$ and ξ_2 is very large, this expression has the value

$$I = \tfrac{1}{4}\pi + \left(\log\frac{2\pi\theta}{\lambda}\xi_2\right)^2$$

at the boundaries $\phi/\theta=\pm 1$, and becomes logarithmically infinite with ξ_2. If ξ_1 and ξ_2 are both finite, the intensity at the boundaries is given by

$$I = \left[\log\frac{\xi_2}{\xi_1} - \mathrm{Ci}\frac{4\pi\theta}{\lambda}\xi_2 + \mathrm{Ci}\frac{4\pi\theta}{\lambda}\xi_1\right]^2 + \left[\mathrm{Si}\frac{4\pi\theta}{\lambda}\xi_2 - \mathrm{Si}\frac{4\pi\theta}{\lambda}\xi_1\right]^2,$$

which is also very large compared with the intensity of the other parts of the field.

Fig. 23 (Pl. IV.) represents the luminosity observed at the edges of a rectangular diffracting aperture in Foucault's test. In taking this photograph, ξ_1 was small and ξ_2 large. The luminosity accordingly appears highly condensed at the edges. Fig. 20 reproduces a photograph obtained when ξ_1, ξ_2 did not differ very considerably. Diffraction fringes are clearly seen on either side of the boundary in this case. Fig. 17 reproduces a photograph of the aperture obtained with two parallel slits in the focal plane on the same side. It will be noticed that the central fringe which coincides with each boundary is *white*. Figs. 21 and 24 represent photographs obtained when the central band and a few fringes on either side of the diffraction-pattern at the focal plane were cut off by a wire parallel to the edges of the aperture. It will be observed that the positions of the boundaries in these two photographs appear as *fine black lines* with luminous bands on either side. The same feature, but with the dark lines at the boundaries much broader, is shown in figs. 18 & 25, which were secured by placing two parallel slits symmetrically in the focal plane—that is, one on either side of the centre of the field.

We proceed to consider the explanation of the black lines marking the positions of the boundaries in the four photographs mentioned in the preceding paragraph. In the focal plane we have two apertures extending from ξ_1 to ξ_2 and from $-\xi_1$ to $-\xi_2$ respectively. On account of the symmetry, the Ci-functions disappear from the expression for the intensity of the field as viewed in the direction ϕ, which may be written in the form

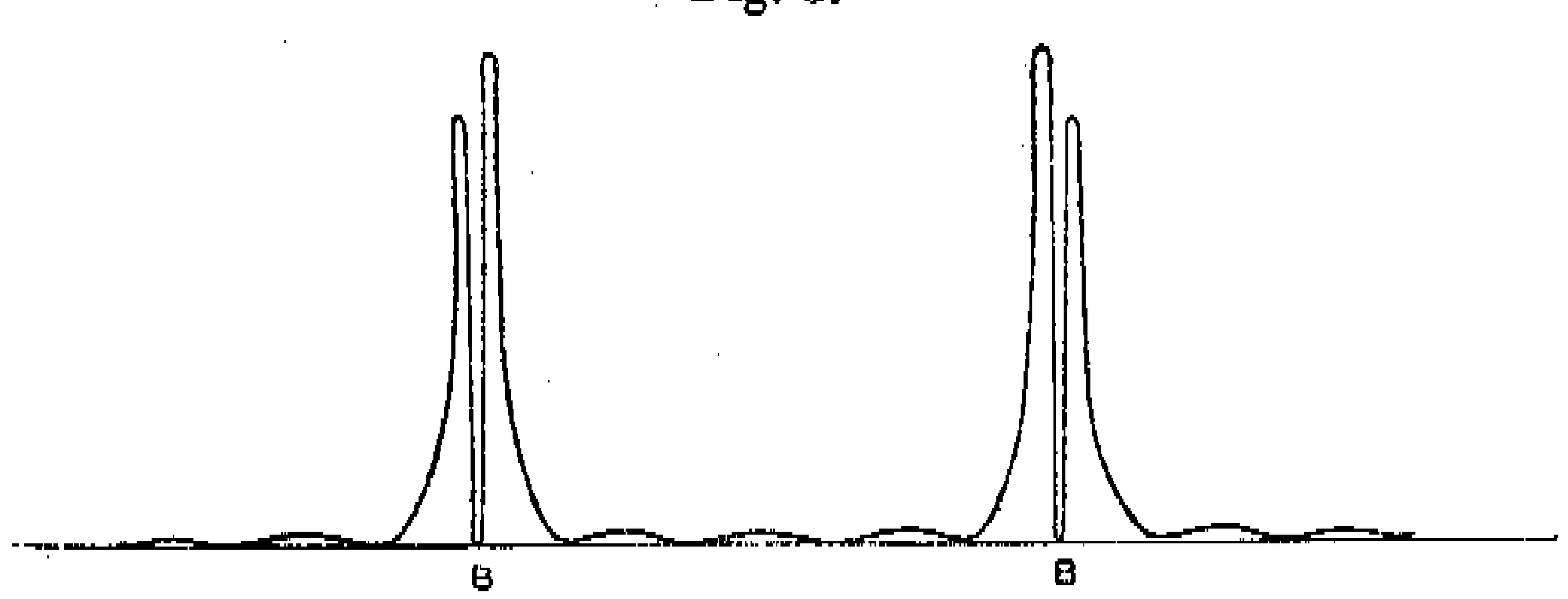
$$I = 4\left[\mathrm{Si}\left\{\frac{2\pi\theta}{\lambda}\left(1+\frac{\phi}{\theta}\right)\xi_2\right\} - \mathrm{Si}\left\{\frac{2\pi\theta}{\lambda}\left(1+\frac{\phi}{\theta}\right)\xi_1\right\}\right.$$
$$\left. + \mathrm{Si}\left\{\frac{2\pi\theta}{\lambda}\left(1-\frac{\phi}{\theta}\right)\xi_2\right\} - \mathrm{Si}\left\{\frac{2\pi\theta}{\lambda}\left(1-\frac{\phi}{\theta}\right)\xi_1\right\}\right]^2,$$

where θ is the angular semi-aperture of the lens. When $\phi/\theta = \pm 1$—that is, at the boundaries—this expression becomes very small, but the suddenness with which the illumination falls to zero at these points depends very much on the magnitudes of ξ_1 and ξ_2. To illustrate this statement, I have calculated the distribution of intensity for a hypothetical case in which $\xi_1 = \frac{\lambda}{\theta}$ and $\xi_2 = \frac{50\lambda}{\theta}$. The values are shown in Table III. (in which the factor 4 in the

Table III.

ϕ/θ.	I.	ϕ/θ.	I.
1·000	0·0045	1·000	0·0045
1·002	1·7956	0·998	2·2201
1·005	1·5129	0·995	1·8496
1·010	2·9584	0·990	3·4225
1·020	1·5376	0·980	2·1609
1·030	2·0164	0·970	2·4025
1·100	0·8281	0·900	1·0050
1·300	0·0020	0·700	0·0000
1·500	0·0605	0·500	0·0357
1·800	0·0000	0·200	0·0057
2·000	0·0182	0·000	0·0853

Fig. 3.



expression for the intensity has been neglected), and these have been plotted in fig. 3.

Another case, in which the disparity between ξ_1 and ξ_2 was much smaller, was chosen for experimental verification of the position of the diffraction maxima and minima given by theory. It is found that the illumination falls off to

Table IV.

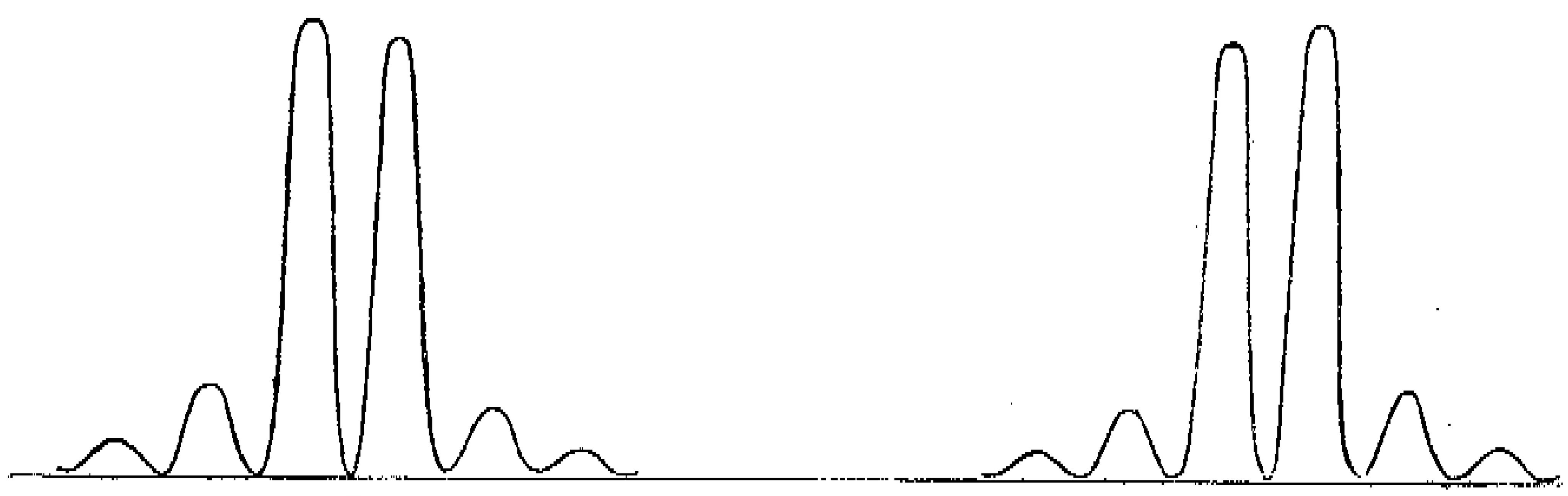
$\xi_1 = ·654$ mm., $\xi_2 = ·963$ mm., $f = 13·4$ cm., $\lambda = ·00045$ mm.,

width of the aperture $= 0·955$ mm.

ϕ/θ.	$\sqrt{\mathrm{I}}$.	I (×const. factor).	Calculated values of ϕ/θ for maximum or minimum.	Observed values of ϕ/θ for maximum or minimum.
·760	−·0002	0	·760 min.	·756
·780	+·1468	204		
·800	+·2123	449	·800 max.	·802
·820	+·1463	201		
·840	−·0018	0	·840 min.	·843
·860	−·1601	256		
·880	−·3084	957	·888 max.	·882
·900	−·2369	559		
·920	−·0065	0	·923 min.	·927
·930	+·1337	179		
·940	+·2386	567		
·950	+·3416	1167		
·960	+·6707	4498	·960 max.	·961
·970	+·3431	1177		
·980	+·2398	575		
·990	+·1350	182		
·995	+·0620	38		
1·000	−·0145	2	1·000 min.	1·000
1·005	−·0918	84		
1·010	−·1528	233		
1·020	−·2587	669		
1·030	−·3701	1370		
1·040	−·6898	4758	1·04 max.	1·038
1·050	−·3648	1331		
1·060	−·2517	634		
1·070	−·1513	229		
1·080	−·0059	0	1·080 min.	1·076
1·100	+·2431	589		
1·120	+·3196	1021	1·114 max.	1·115
1·140	+·1713	293		
1·160	−·0024	0	1·16 min.	1·157
1·180	−·1642	270		
1·200	−·2128	453	1·198 max.	1·205
1·220	−·1478	218		
1·240	−·0002	0	1·24 min.	1·238

practically zero-value on the boundaries, but much less suddenly than in the case previously discussed. The experimental data, the calculated intensities of the illumination, and the theoretical and the experimentally observed positions of maxima and minima are shown in Table IV. The agreement is fairly satisfactory. For comparison with the preceding case, the illumination curve has been plotted in fig. 4. As in the case of the circular boundary, the

Fig. 4.



minima of illumination are absolute zeros. It will be noticed also that $\sqrt{I}$ changes sign as it passes through its value at the boundary ($\phi/\theta = \pm 1$), showing that the radiations emitted by the edge on the two sides of the boundary differ in phase by π.

4. *Other Forms of Boundary.*

The cases in which the surface is bounded by forms of apertures other than those considered previously are of interest from the point of view of the general theory of diffraction. Figs. 19 and 22 (Pl. IV.) represent photographs of the effect observed when the surface is bounded by quadrilateral and triangular apertures respectively. These photographs were obtained using a point source of light and an annular aperture placed symmetrically in the focal plane. It will be noticed that the boundaries appear as black lines with luminous fringes on either side, thus showing a complete analogy with the case of the circular and rectangular boundaries previously considered.

In order more fully to study the luminosity at the boundaries of the triangular, quadrilateral, and other forms of aperture, an arrangement was devised in which a screen containing a small circular hole could be placed excentrically in the focal plane and rotated in this plane. This hole

comes successively over different parts of the diffraction-pattern formed at the focus, and the luminosity at the boundaries observed through it undergoes a series of changes. For instance, with a triangular aperture, it is known that the diffraction-pattern at the focus consists of a six-rayed "star," the "rays" being perpendicular to the three sides of the triangle respectively. When the hole comes over any one of the rays the corresponding boundary appears luminous, but in other cases it becomes practically invisible. Similarly, with a quadrilateral aperture, the diffraction-pattern is a "star" with eight rays perpendicular to its four sides, and each of these appears luminous when the excentrically-placed hole in the focal plane comes over the corresponding ray of the pattern.

Whether any particular part of the boundary appears luminous or not seems in general to depend on the normal to the boundary at that point being parallel to the radius vector from the centre of the focal plane to the aperture in the screen through which it is viewed. This is stated here as an experimental fact, the detailed mathematical explanation of which is deferred till a future occasion. An interesting illustration of its generality is furnished by the observation that minute irregularities on the boundary often appear luminous when the adjoining parts which are straight are invisible from any given point in the focal plane. A discussion of the cases in which the boundary is a complicated figure such as a grating or a series of parallel apertures is also reserved for a future occasion.

5. *On the Flow of Energy in a Diffraction Field.*

The phenomena described in the preceding sections suggest two important problems for study. In the ordinary Fresnel-Kirchhoff treatment of diffraction problems, the disturbance at any point of the field is expressed as an integral taken over a surface bounded by the diffracting aperture. It is a subject for investigation whether, in any circumstances, the surface integral can be resolved either wholly or partially to a line integral taken over the boundary, and whether the disturbance in the region of shadow could be expressed practically in terms of the line-integral alone. Another interesting problem which also suggests itself for investigation is the determination of the forms of the lines of flow of energy in the optical field due to rectangular or circular boundaries in convergent light. The shapes of these lines of flow of energy would be specially interesting in the neighbourhood of the focus. As a first step towards a detailed study of these problems, a few experimental observations have been made by placing a narrow screen (such as a needle or a plate of very small dimensions) or else a wide screen containing a narrow aperture in some selected part of the field and tracing the phenomena observed in its rear. Some very striking results have been obtained, especially with a circular aperture illuminated by a point source of light. If a small screen is placed in the focal plane so as to cut off the entire geometrical cone of rays, a bright image of the source may be traced along the axis behind the screen and for a considerable distance beyond. This effect is entirely due to the light diffracted by the boundary of the circular aperture*. With a narrow screen placed in the diffraction field due to a rectangular aperture illuminated by a linear source of light, two independent shadows each bordered by diffraction fringes and differing in intensity have been observed behind the screen, these being formed respectively by the two luminous edges of the aperture. If the narrow screen be placed in the region of shadow close to the geometrical pencil of rays, the two shadows differ considerably in their intensity, and the difference becomes less and less as the screen is moved more and more into the region of shadow. The writer hopes to take up the fuller study of these problems at an early opportunity.

The investigation described in this paper was carried out in the Laboratory of the Indian Association for the Cultivation of Science. The writer has much pleasure in gratefully acknowledging the helpful interest taken by Prof. C. V. Raman during the progress of the work.

Calcutta,
February 8th, 1918.

* Porter and Hufford have about the same time observed that the rays diffracted by a circular disk can form an optical image of the source along the axis of symmetry (see Phil. Mag. and Phys. Rev. April 1913). The phenomenon observed by me is somewhat analogous, but differs from that observed by these writers, as the image in this case is formed by the rays diffracted by the boundary of a circular *aperture*.

IX. *Note on the Effects of Grid Currents in Three-Electrode Ionic Tubes.* *By* E. V. Appleton, *M.A.*, *B.Sc.*, *St. John's College, Cambridge**.

In the interpretation of the functioning of three-electrode ionic tubes, which are now used so extensively in the reception of wireless signals, we have to consider simultaneous effects in two electrical circuits, which may be defined as follows:—

(*a*) The grid or input circuit, which is completed inside the tube by the space between the incandescent filament and the perforated grid, and

(*b*) the plate or output circuit, which is completed inside the tube by the space between the filament and the metal plate.

The variables to be considered are thus as follows:— Plate current, plate voltage, grid current, grid voltage, and filament temperature, the values of which may be represented by the letters I, V, i, v, and θ respectively. The complete working of the tube can only be interpreted when the static characteristic surfaces representing the relations between any three of the above quantities are known. In the case of a tube acting as a relay or amplifier, the main surface to be considered is that represented by the equation $f(\mathrm{I}, \mathrm{V}, v)=0$. Vallauri† has shown how this may be expressed approximately as $\mathrm{I}=av+b\mathrm{V}+c$, where a, b, and c are constants, and has applied it to the elucidation of many wireless circuits. Hazeltine‡ has recently made substantial additions to that part of Vallauri's work dealing with oscillating audion circuits. In both cases the effects of the currents flowing in the grid circuit were neglected.

The magnitude of the grid currents involved is known if we know the contour of the (i, v, V) surface. In general, this surface cannot be represented by a simple expression, but for small changes of the quantities concerned we may write

$$\Delta i=\kappa_1\Delta v+\kappa_2\Delta \mathrm{V}, \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where κ_1 is the slope of the $(v, i)_{\mathrm{V}=\mathrm{V}_0}$ curve and κ_2 is the slope of the $(\mathrm{V}, i)_{v=v_0}$ curve.

The quantity κ_1 may be either positive or zero in a tube of

* Communicated by the Author.
† G. Vallauri, *L'Elettrotecnica*, iv. 3, p. 43 (1917).
‡ L. A. Hazeltine, Proc. Inst. Rad. Eng., April 1918.