Hydrogen-Antihydrogen Collisions

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Matter-antimatter interactions are investigated using hydrogen-antihydrogen collisions as an example. Cross sections for elastic scattering and for the antihydrogen loss (either through the rearrangement reaction, resulting in formation of protonium and positronium according to $\mathrm{H} + \bar{\mathrm{H}} \rightarrow p\bar{p} + e^+e^-$, or via annihilation in flight) are calculated for the first time in a fully quantum mechanical approach. Implications for experiments intending to trap and cool antihydrogen are discussed.

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How stable is antimatter in contact with matter? In this Letter we consider the atom-antiatom interactions using the collisional reaction between hydrogen and antihydrogen as an example.

The Universe within our present observational horizon is charge asymmetric, and therefore the atomic antimatter is difficult to study experimentally. However, recent advances in producing, trapping, and cooling antiprotons and positrons open the possibility of antihydrogen formation in the laboratory [1,2]. This may allow studies of antimatter and tests of fundamental physical principles such as charge-parity-time invariance and the weak equivalence principle for antiparticles. Such experiments are planned at CERN AD (Antiproton Decelerator) within the ASACUSA, ATRAP, and ATHENA collaborations [1,2].

To study the matter-antimatter interactions in general, to explore the existence of antistars, and in particular to understand the mechanisms that allow trapping and cooling of antihydrogen, knowledge of the rates for elastic and inelastic atom-antiatom collisions is of paramount importance. However, the previous treatments of the problem have been very scarce [3–5]. The elastic cross section is responsible for cooling and the inelastic one, particularly the cross section for rearrangement resulting in formation of protonium, is responsible for the loss of antihydrogen (via annihilation during the cascade in protonium).

In this paper we focus on the p-$\bar{p}$ annihilation during H-$\bar{\mathrm{H}}$ collisions at low (down to ultracold) temperatures. In particular, we have calculated the rates for the collisional rearrangement reaction

$$\mathrm{H} + \bar{\mathrm{H}} \rightarrow p\bar{p} + e^+e^-, \tag{1}$$

which inevitably leads to the annihilation of antiparticles from the bound states of protonium ($\mathrm{Pn} \equiv p\bar{p}$) and/or positronium ($\mathrm{Ps} \equiv e^+e^-$) formed in the final channel.

The cross section for the rearrangement process in Eq. (1) is

$$\sigma^{\mathrm{rearr}} = \frac{(2\pi)^4}{k_i^2} \sum_f \delta(E_f - E_i)\,|T_{fi}|^2, \tag{2}$$

where f represents the complete set of quantum numbers specifying the final states of protonium and positronium. The transition matrix element T_{fi} is given by

$$T_{fi} = \langle \Upsilon_{\mathbf{k}_f} | \hat{\mathrm{V}}_f | \Psi^{(+)}_{\mathbf{k}_i} \rangle . \tag{3}$$

The initial state wave function $\Psi^{(+)}_{\mathbf{k}_i}$ represents the hydrogen and antihydrogen atoms oncoming with the initial momentum $\mathbf{k}_i$, and is a scattering solution of the eigenvalue problem with respect to the complete Hamiltonian $\hat{\mathrm{H}}$ describing an *interacting* hydrogen-antihydrogen system with the total initial energy E_i. Separation of leptonic and hadronic motions (in the spirit of the adiabatic approximation) leads to a factorized form of $\Psi^{(+)}_{\mathbf{k}_i}$,

$$\Psi^{(+)}_{\mathbf{k}_i}(\mathbf{R}; \mathbf{r}_e, \mathbf{r}_{\bar{e}}) = \psi_i(R; \mathbf{r}_e, \mathbf{r}_{\bar{e}}) \chi_{\mathbf{k}_i}(\mathbf{R}), \tag{4}$$

where $\mathbf{r}_e \equiv \mathbf{r}_{pe}$, $\mathbf{r}_{\bar{e}} \equiv \mathbf{r}_{\bar{p}\bar{e}}$, $R \equiv r_{p\bar{p}}$, and ψ_i is a solution to the leptonic eigenvalue equation

$$\hat{\mathrm{H}}^{\mathrm{lep}} \psi_i = V_i^{\mathrm{lep}}(R) \psi_i \tag{5}$$

with the leptonic Hamiltonian given by

$$\hat{\mathrm{H}}^{\mathrm{lep}} = -\frac{1}{2}\nabla_e^2 - \frac{1}{2}\nabla_{\bar{e}}^2 - \frac{1}{r_{pe}} + \frac{1}{r_{\bar{p}e}} + \frac{1}{r_{p\bar{e}}} - \frac{1}{r_{\bar{p}\bar{e}}} - \frac{1}{r_{e\bar{e}}} . \tag{6}$$

The hadronic wave function $\chi_{\mathbf{k}_i}$ is a scattering solution to the wave equation for the proton-antiproton motion in the leptonic potential

$$\left(-\frac{1}{2\mu_i}\nabla_R^2 + V_i(R) \right) \chi_{\mathbf{k}_i}(\mathbf{R}) = \epsilon_i \chi_{\mathbf{k}_i}(\mathbf{R}), \tag{7}$$

where $V_i(R) = V_i^{\mathrm{lep}}(R) - 1/R$, μ_i is the reduced mass of the two hadrons, and $\epsilon_i = k_i^2/2\mu_i$.

The final-state wave function $\Upsilon_{\mathbf{k}_f}(\mathbf{R}, \mathbf{r}_e, \mathbf{r}_{\bar{e}})$ is a free-wave solution to the eigenvalue problem with respect to the final-channel Hamiltonian $\hat{\mathrm{H}}_f$ describing the *noninteracting* positronium-protonium pair having reduced mass μ_f and moving with the relative (recoil) energy $\epsilon_f = k_f^2/2\mu_f$. This corresponds to a partitioning of the complete Hamiltonian $\hat{\mathrm{H}}$ according to $\hat{\mathrm{H}} = \hat{\mathrm{H}}_f + \hat{\mathrm{V}}_f$, where

$$\hat{\mathrm{V}}_f = -\frac{1}{r_{pe}} + \frac{1}{r_{p\bar{e}}} + \frac{1}{r_{\bar{p}e}} - \frac{1}{r_{\bar{p}\bar{e}}} \tag{8}$$

describes the interaction between protonium and positronium. The final-state wave function can be expressed in a product form,

$$\Upsilon_{k_f}(\mathbf{R}, \mathbf{r}_e, \mathbf{r}_{\bar{e}}) = \Phi_E(\mathbf{r}_e, \mathbf{r}_{\bar{e}})\tilde{\phi}_{\mathrm{NLM}}(\mathbf{R}), \tag{9}$$

where the hadronic part $\tilde{\phi}_{\mathrm{NLM}}(\mathbf{R}) = R^{-1}\tilde{u}_{\mathrm{NL}}(R)Y_{\mathrm{LM}}(\Omega_R)$ is a hydrogenic wave function describing the bound protonium, and the leptonic part $\Phi_E(\mathbf{r}_e, \mathbf{r}_{\bar{e}})$ is the wave function describing a (freely) moving positronium in the state nlm with internal energy $\mathcal{E}_n^{\mathrm{Ps}}$ and momentum $\mathbf{k}_f$, i.e.,

$$\Phi_E(\mathbf{r}_e, \mathbf{r}_{\bar{e}}) = \frac{1}{(2\pi)^{3/2}}\sqrt{\frac{\mu_f k_f}{\hbar^2}}\, e^{i\mathbf{k}_f \mathbf{r}_{e\bar{e}}^{\mathrm{cm}}}\phi_{nlm}(\mathbf{r}_{e\bar{e}}), \tag{10}$$

where the vector $\mathbf{r}_{e\bar{e}}^{\mathrm{cm}}$ points to the center of mass of the positronium. The index E of $\Phi_E(\mathbf{r}_e, \mathbf{r}_{\bar{e}})$ labels the *total* energy of the positronium, i.e., $E = \epsilon_f + \mathcal{E}_n^{\mathrm{Ps}}$.

The transition matrix element T_{fi} defined in Eq. (3) can now be rewritten with the aid of Eqs. (4) and (9) as

$$T_{fi} = \langle \Phi_E(\mathbf{r}_e, \mathbf{r}_{\bar{e}})\tilde{\phi}_{NLM}(\mathbf{R})\,|\hat{\mathrm{V}}_f|\psi_i(R; \mathbf{r}_e, \mathbf{r}_{\bar{e}})\chi_{\mathbf{k}_i}(\mathbf{R})\rangle . \tag{11}$$

To obtain a factorized form of T_{fi} we apply the partial-wave expansion

$$\chi_{\mathbf{k}_i}(\mathbf{R}) = \frac{1}{R}\sum_{L_i} f_{L_i}(k_i, R)Y_{L_i 0}(\Omega_R), \tag{12}$$

where $f_{L_i}(\epsilon_i, R)/R$ is the radial solution of Eq. (7) and Y_{LM} denotes a spherical harmonic. Using Eq. (12) and assuming low collisional energy leading to s-wave scattering, one obtains

$$T_{fi} = \delta_{L,0}\delta_{M,0}\int_0^\infty \tilde{u}_{N0}(R)t_{E,i}(R)f_0(k_i, R)\, dR \tag{13}$$

which clearly demonstrates the role of the leptonic transition matrix element defined as

$$t_{E,i}(R) = \langle \Phi_E(\mathbf{r}_e, \mathbf{r}_{\bar{e}})\,|\hat{\mathrm{V}}_f|\psi_i(R; \mathbf{r}_e, \mathbf{r}_{\bar{e}})\rangle . \tag{14}$$

The leptonic eigenvalue problem (5) is solved in prolate spheroidal coordinates by means of the variational method, using an explicitly correlated basis set of Hylleraas functions. The potential $V_i(R)$ governing the motion of the hadrons in the initial channel is presented in Fig. 1. As seen in the figure, the leptonic ground state potential is purely attractive; i.e., there are no potential wells/barriers which could temporarily trap the atom-antiatom system, preventing the hadrons from coming close to each other, and, ultimately, annihilate. Beyond $R = 13$ a.u. we calculated the potential as the asymptotic effective interaction of the form $\lim_{R\to\infty} V_i(R) = C_6/R^6 + C_8/R^8$ following from perturbation theory.

The radial part of the hadronic wave function $\chi_{\mathbf{k}_i}(\mathbf{R})$, i.e., $f_0(k_i, R)$, at the initial collision energy ϵ_i has been obtained by the numerical integration of Eq. (7). The radial wave function $f_0(k_i, R)$ is presented in Fig. 1 together with the hadronic wave function in the final channel (i.e., the excited bound state of protonium).

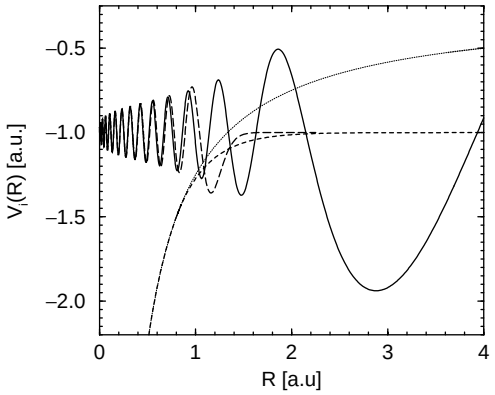

FIG. 1. The hadronic wave functions superimposed on the leptonic ground state potential. Solid line: the radial part of the hadronic scattering wave function $f_0(k_i, R)$ for $\epsilon_i = 10^{-10}$ a.u.; long-dashed line: the radial part $\tilde{u}_{24,0}(R)$ of the final Pn state; dashed line: ground-state interaction potential $V_i(R)$; dotted line: (for comparison) the proton-antiproton attraction.

The leptonic wave function $\Phi_E(\mathbf{r}_e, \mathbf{r}_{\bar{e}})$ [see Eqs. (9) and (10)] is obtained as the eigensolution to the leptonic part of the final state Hamiltonian

$$\hat{\mathrm{H}}_f^{\mathrm{lep}}\Phi_E = E\Phi_E, \tag{15}$$

where $\hat{\mathrm{H}}_f^{\mathrm{lep}} = \hat{\mathrm{H}}_f - \hat{\mathrm{H}}_{\mathrm{Pn}}$. To facilitate the calculation of the transition matrix elements between the H-$\bar{\mathrm{H}}$ and Ps-Pn channels, the eigenvalue problem (15) is also expressed in prolate spheroidal coordinates and solved in the matrix representation using an explicitly correlated basis. This leads to the discretization of the continuous spectrum of the *moving* positronium, resulting in $\mathcal{L}^2$ eigensolutions corresponding to a discretized set of eigenvalues E_j. The resulting eigenfunctions Φ_{E_j} represent the moving positronium and are square-integrable approximations to the analytic solutions given in Eq. (10). They are, however, expanded in terms of the basis functions and therefore transformed to the same prolate-spheroidal coordinate system which has been used for the calculation of the leptonic wave function in the initial channel, which greatly facilitates the calculation of $t_{E,i}(R)$.

The leptonic transition matrix element $t_{E,i}(R)$ needs to be calculated with respect to the leptonic final-state wave function Φ_E at the energy E, satisfying the energy conservation during the collision, $E = E_i - \mathcal{E}_N^{\mathrm{Pn}}$. It is extracted as the imaginary part of the expectation value of the operator $\hat{\mathrm{V}}_f\hat{\mathrm{G}}_f\hat{\mathrm{V}}_f$ with respect to the leptonic ground state in the initial channel

$$|t_{E,i}(R)|^2 = \frac{1}{\pi}\,\mathrm{Im}\langle\psi_i|\hat{\mathrm{V}}_f\hat{\mathrm{G}}_f(E)\hat{\mathrm{V}}_f|\psi_i\rangle, \tag{16}$$

where $\hat{\mathrm{G}}_f(E)$ is the resolvent operator of the leptonic final-channel Hamiltonian

$$\hat{G}_f(E) = \lim_{\epsilon \to 0} (\hat{H}_f^{\rm lep} - E - i\epsilon)^{-1}. \quad (17)$$

To handle correctly the singular nature of Eq. (16) and the presence of degenerate continua describing the moving positronium-protonium pair in various energetically possible combinations of excited states, we apply the method based on the use of complex coordinates [6]. The transition matrix element $t_{E,i}(R)$ can then be obtained from the dilated expression introduced in Eq. (16),

$$|t_{E,i}(R)|^2 = |\langle \Phi_E | \hat{V}_f | \psi_i \rangle|^2 = \frac{1}{\pi} \mathrm{Im} \sum_j \frac{\langle \psi_i^{\theta^*} | \hat{V}_f^\theta | \Phi_{\mathcal{E}_j}^\theta \rangle \langle \Phi_{\mathcal{E}_j}^{\theta^*} | \hat{V}_f^\theta | \psi_i^\theta \rangle}{\mathcal{E}_j - E}, \quad (18)$$

where $\mathcal{E}_j$ and $\Phi_{\mathcal{E}_j}^\theta$ are the complex eigenvalues and associated eigenvectors obtained by solving the pair of adjoint matrix eigenvalue problems obtained by dilation of Eq. (15). The summation in Eq. (18) includes all branches of the dilated continuous spectrum, and therefore the transition probability contains contributions from all energetically open leptonic final channels for a given final state of protonium. We could also perform a summation over all final states of protonium, but our calculation shows that the transition probabilities to states with $N \neq 24$ are very small. For $N = 24$ only one leptonic channel (with $n = 1$) is open.

In the evaluation of T_{fi} [Eq. (13)] we notice that for $R < 1$ a.u. the leptonic matrix element $t_{E,i}(R)$ approaches 0, while for larger R the protonium wave function vanishes (see Fig. 1). At about $R = 1$ a.u. the leptonic matrix element changes only very little. Therefore, generally T_{fi} is determined by the overlap of the initial and final hadronic wave functions, the protonium state with $N = 24$, and the low-energy proton-antiproton scattering wave function f_0, both rapidly oscillating in this region (see Fig. 1).

Our results for the rearrangement cross section are presented in Fig. 2. For low energies $\sigma^{\rm rearr} = 0.35/\sqrt{\epsilon_i}$ a.u., confirming the energy dependence expected from Wigner's threshold law.

In addition to the loss of antihydrogen caused by the rearrangement collisions, some losses will also occur via proton-antiproton annihilation in flight, i.e., *without* the formation of an intermediate protonium state according to

$$\mathrm{H} + \bar{\mathrm{H}} \rightarrow e^+ + e^- + \text{decay products}. \quad (19)$$

The cross section for this process is taken to be

$$\sigma_a^{p\bar{p}} = \frac{(2\pi)^3}{k_i^2} A^{p\bar{p}} |\chi_{\mathbf{k}_i}(0)|^2, \quad (20)$$

where the constant $A^{p\bar{p}}$ has been calculated using the experimental width of the ground state of protonium from Ref. [7], $\Gamma_{1s} = 1130$ eV. The resulting cross section is presented in Fig. 2. It is fairly small as compared to $\sigma^{\rm rearr}$, and for low energies shows the behavior characteristic for inelastic scattering dominated by the effective long range forces ($\sigma_a^{p\bar{p}} = 0.14/\sqrt{\epsilon_i}$ a.u.).

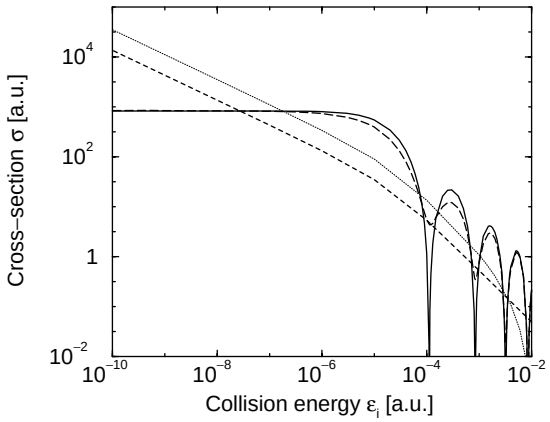

FIG. 2. Cross sections for H-$\bar{\mathrm{H}}$ collisions, elastic cross section obtained from the real part of the phase shift only (solid line), elastic cross section including correction for inelasticity (long-dashed line), rearrangement cross section (dotted line), and proton-antiproton annihilation in flight (dashed line).

The elastic cross section has been calculated using the relation

$$\sigma^{\rm el}(k_i) = \frac{\pi}{k_i^2} |1 - S_{ii}(k_i)|^2, \quad (21)$$

where S_{ii} is the scattering matrix element related to the phase shift δ_0 via

$$S_{ii}(k_i) = \exp[2i\,\delta_0(k_i)]. \quad (22)$$

In the presence of inelastic scattering the phase shift δ_0 is a complex quantity, whose real part has been determined by fitting the numerical radial solutions $f_0(k_i, R)$ at large R to the form $N \sin(k_i R + \mathrm{Re}\{\delta_0(k_i)\})$. The imaginary part of the phase shift has been extracted from the rearrangement cross section,

$$\sigma^{\rm rearr} = \frac{\pi}{k_i^2}(1 - |S_{ii}|^2) = \frac{\pi}{k_i^2}(1 - e^{-4\,\mathrm{Im}\,\delta_0}). \quad (23)$$

We have then corrected the elastic cross section using the results for the inelastic scattering cross section. The s-wave elastic cross section with and without the correction for inelasticity is presented in Fig. 2. The effect of the inelasticity is small, except for the dips in the cross section at energies where the real part of the phase shift goes through zero, which are smoothed out by the presence of inelastic scattering. We note that, as expected from general scattering theory, at low energies the elastic cross section becomes constant. Its limiting value has been obtained in terms of the (complex) scattering length $a = \alpha - i\beta$,

$$\lim_{k_i \to 0} \sigma^{\rm el}(k_i) = 4\pi |a|^2 = 4\pi(\alpha^2 + \beta^2). \quad (24)$$

The real part α of the scattering length a, together with the effective range $r_{\rm eff}$, was obtained by fitting the real

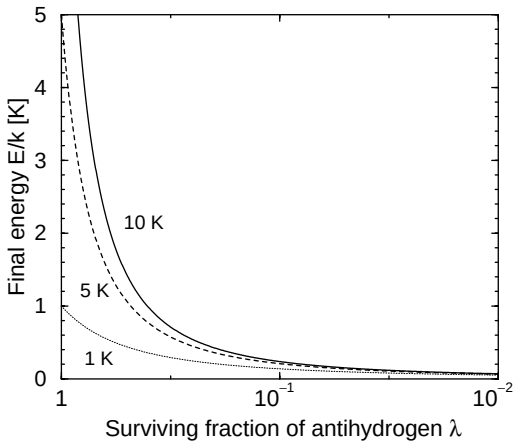

FIG. 3. Cooling of antihydrogen atoms due to elastic collisions with ultracold hydrogen as a function of the loss of antihydrogen atoms. Initial collision energy of the antihydrogen $E_0 = 10$ K (solid line), $E_0 = 5$ K (dashed line), and $E_0 = 1$ K (dotted line).

part of the phase shift in the energy region $10^{-7} \leq \varepsilon_i \leq 10^{-10}$ a.u. to the low-energy expansion [8]

$$k_i \cot\delta_0(k_i) = -\frac{1}{\alpha} + \frac{1}{2} r_{\text{eff}} k_i^2 + \ldots . \tag{25}$$

This procedure gave $\alpha = 8.1$ a.u. and $r_{\text{eff}} = 7.1$ a.u. The imaginary part of the scattering length has been obtained from its relation to the inelastic cross section in the low-energy limit

$$\beta = \frac{k_i}{4\pi} \sigma^{\text{rearr}} \tag{26}$$

which gives $\beta = 1.2$ a.u. With these values of α and β the elastic cross section is found to tend to a constant value $\sigma^{\text{el}} = 840$ a.u. By comparing the elastic and the rearrangement cross sections (cf. Fig. 2), we see that the latter exceeds the former at energies smaller than 2×10^{-7} a.u. This means, for instance, that sympathetic cooling of antihydrogen via collisions with cold hydrogen will become ineffective below this threshold energy (which, converted to the rest frame of hydrogen and divided by the Boltzmann's constant k, corresponds to 0.1 K).

The cooling efficiency can be found by solving the rate equations describing the losses of the kinetic energy and the density of antihydrogen atoms through repeated elastic/rearrangement collisions with hydrogen atoms,

$$\frac{dE}{dt} = -n^{\text{H}} v \sigma^{\text{el}} \frac{E}{2}; \qquad \frac{dn^{\bar{\text{H}}}}{dt} = -\sigma^{\text{rearr}} v n^{\text{H}} n^{\bar{\text{H}}}, \tag{27}$$

where n^{H} is the density of hydrogen, $n^{\bar{\text{H}}}$ the density of antihydrogen, and $\sigma^{\text{rearr}} v = 4\pi\hbar\beta/\mu_i$ is the rearrangement rate constant. With $E = 2\epsilon_i$ and $v = (2E/m_{\bar{\text{p}}})^{1/2}$ referring to the laboratory frame, the solution can be written in terms of the fractional loss of antihydrogen $\lambda(t) = n^{\bar{\text{H}}}(t)/n^{\bar{\text{H}}}(0)$ as

$$E(\lambda) = E_0 \left\{ 1 - \frac{\sigma^{\text{el}} \sqrt{E_0 \mu_i}}{16\pi\hbar\beta} \ln\lambda \right\}^{-2}. \tag{28}$$

By using the values of σ^{el} and σ^{rearr} obtained above, we have (in Fig. 3) plotted the decrease of collisional energy as a function of the loss of antihydrogen for initial energies $E_0 = 1$, 5, and 10 K. We see that, largely independent of the initial energy, the antihydrogen is cooled down to 0.1–0.2 K while losing 90% of the atoms. Surprisingly, starting with $n^{\text{H}} = n^{\bar{\text{H}}} = 10^7$ cm^{-3} and energy <10 K, it takes a whole 17 min for the mixture of equal amounts of H and $\bar{\text{H}}$ to lose half of all atoms.

In conclusion, the coexistence of hydrogen and antihydrogen is quite viable for energies larger than $\sim$0.1 K, while below this threshold the annihilation takes over. Because of the logarithmic dependence on the density loss and the unfavorable ratio $\sigma^{\text{el}}/\sigma^{\text{rearr}}$, the sympathetic cooling below this threshold energy occurs only at the expense of a high loss of antihydrogen atoms. Encouragingly, the rearrangement cross sections in $\bar{\text{H}}$-H are *smaller* than in $\bar{p}$-H collisions [9]. The vacuum requirements for storing antihydrogen might therefore be less severe than for bare antiprotons.

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