and other details of nucleus production can be found in work by Steinberger and Bishop,[3] Lax and Feshbach,[5] Marshak,[6] and Littauer and Walker.[7] The π^-/π^+ ratios measured here do not indicate any variation with energy or angle. However, the result of Littauer and Walker[7] at 135° and 65-Mev meson energy (1.06±0.02) indicates that such a variation exists. Other experimental π^-/π^+ ratios can be found in work by Peterson *et al.*,[2] Feld *et al.*,[4] and Carothers.[8]

The author wishes to thank Thomas Jenkins, Thomas Palfrey, and Professor Robert R. Wilson for their contributions to these results.

[5] M. Lax and H. Feshbach, Phys. Rev. **81**, 189 (1951).
[6] R. E. Marshak, *Meson Physics* (McGraw-Hill Book Company, Inc., New York, 1952), Chap. 3.
[7] R. M. Littauer and D. Walker, Phys. Rev. **86**, 838 (1952).
[8] J. Carothers, Phys. Rev. **92**, 538 (1953).

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Synchrotron Oscillations Induced by Radiation Fluctuations*

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Phase oscillations of electrons in a high-energy synchrotron are induced by the radiation of quanta. These induced oscillations set a limit to the damping of electron bunches. This limiting bunch size is sufficient to influence the radial aperture and the radio-frequency voltage required at low beam intensities, and to reduce energy loss by coherent radiation at high intensities.

I. INTRODUCTION

THE radiation of electromagnetic energy by high-energy electrons in a synchrotron[1–5] not only requires that the radio-frequency system supply energy much in excess of that required for acceleration, but also causes a damping of the phase oscillations in normal guide fields.[2,3,5] According to the results of Bohm and Foldy,[5] in a synchrotron with a guide field which increases linearly with time, phase oscillations present at an energy E_0 will have their amplitude reduced by the factor $\exp[-(E^4-E_0^4)/E_d^4]$ at an energy E. The energy E_d may be several hundred Mev for a 1.5-Bev synchrotron with typical design parameters. Such strong damping above E_d may lead to electron bunches so small that energy loss due to coherent radiation[4,6,7] would set a serious limit to the maximum attainable beam intensity.

We present here a calculation which shows that the radiation damping will not decrease the spread in phase of an electron group below a certain minimum. This minimum is determined by the statistical fluctuations in the radiated energy loss—because of the emission of quanta. The effect is, fortunately, sufficiently large to preclude, in general, serious difficulty from coherent radiation.

We may use the following qualitative arguments to estimate the magnitude of the energy fluctuations produced by photon emission from the circulating electrons. The square of the energy fluctuation ΔE is approximately equal to the product of the mean square quantum energy ϵ_γ^2 of the radiation and the average number u of quanta emitted in one damping-time interval of the energy (phase) oscillations. This average number is ratio of the total energy radiated in one damping period to the average quantum energy. Using the relations of reference 5, we find

$$u \approx f(E/\epsilon_\gamma),$$

where E is the electron energy and f is a number which depends on the properties of the guide field and will in general be ≈ 1. We have then that

$$(\Delta E/E)^2 \approx f(\epsilon_\gamma/E).$$

From reference 13 the typical quantum energy may be taken as

$$\epsilon_\gamma \approx (\hbar c/r)(E/mc^2)^3,$$

where r is the orbit radius. We have then that

$$(\Delta E/E)^2 \approx f(\hbar/mc)(1/r)(E/mc^2)^2.$$

The oscillations in radius and phase are simply related to the energy fluctuations.

We present below more detailed calculations, including also the effects of harmonic accelerating fields and noncircular guide fields. We consider specifically only synchrotrons with constant-gradient ("weak-focusing") guide fields.

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[1] D. Iwanenko and I. Pomeranchuk, Phys. Rev. **65**, 343 (1944).
[2] J. Schwinger, Phys. Rev. **70**, 798 (1946).
[3] N. Frank, Phys. Rev. **70**, 177 (1946).
[4] L. Schiff, Rev. Sci. Instr. **17**, 6 (1946).
[5] D. Bohm and L. Foldy, Phys. Rev. **70**, 249 (1946).
[6] J. Nodvick and D. Saxon, University of California at Los Angeles Technical Report No. 21, May 1954; Phys. Rev. **96**, 180 (1954).
[7] J. Schwinger (1945, unpublished).

II. EQUATIONS OF MOTION

We consider a magnetic guide field composed of circular sectors connected by field-free regions of total length L. In the circular sectors the synchronous electrons move with a radius r_s and an angular velocity ω_s determined by the (angular) frequency of the accelerating system $\omega_0=k\omega_s/\lambda$, where k is an integer (the harmonic number), and $\lambda=1+L/2\pi r_s$. We take further that the guide field at the synchronous orbit increases linearly with time and that the field varies near r_s as $(r/r_s)^{-n}$.

Following Blachman and Courant,[8] we may write the equations which determine the phase oscillations in terms of small deviations from synchronous values as

$$\Delta E/E_s=(1-n)(\Delta r/r_s);\tag{1}$$

$$\Delta r/r_s=-\Delta\omega/\omega_s=-(\lambda^2/k)(1/\omega_s)(d\phi/dt),\tag{2}$$

where ϕ is the phase angle of the electron with respect to the accelerating field, and E is the energy of the electron which is assumed to be much greater than mc^2.

Differentiating and combining Eqs. (1), (2), we have

$$\frac{d^2\phi}{dt^2}+\frac{k\omega_s}{(1-n)\lambda^2E_s}\frac{d}{dt}(\Delta E)+\frac{\dot{E}_s}{E_s}\frac{d\phi}{dt}=0,\tag{3}$$

where $\dot{E}_s$ is the constant rate of increase of the synchronous energy.

The rate of change of the energy of the electron is given by the power received from the radio-frequency system and from the variation of flux linking the orbit, less the power lost by radiation.

$$(d/dt)\Delta E=P_\alpha(\phi,d\phi/dt)-P_\gamma(d\phi/dt,t)-\dot{E}_s,\tag{4}$$

where the t-dependence in P_γ represents the variations —due to quantum emission—in the rate of energy loss. As the period of a phase oscillation is long compared to a revolution period, we follow the usual procedure and take for dE/dt its average over one revolution.

The power received by the electron is then (correct to first order terms)[9]

$$P_\alpha(\phi,d\phi/dt)=P_{\alpha s}+P_{\alpha s}\Delta\phi\cot\phi_s+(P_{\alpha s}-\dot{E}_s)(\lambda/k\omega_s)(d\phi/dt),\tag{5}$$

where ϕ_s is the phase of the synchronous electron and $P_{\alpha s}$ is the power received by it from the radio-frequency system. We have for convenience taken the flux linking the equilibrium orbit to be zero. Equation (9), below, would remain unchanged in any case.

We represent the power radiated by the electron as the sum of its average value over one revolution, which varies only slowly with time, and a remaining term $p(t)$ which contains the rapid variations due to quantum emission. We ignore the dependence of $p(t)$ on $d\phi/dt$ using instead the function pertinent to the synchronous electron:

$$P_\gamma(d\phi/dt,t)=\langle P_\gamma(d\phi/dt)\rangle_{\text{Av}}+p(t).\tag{6}$$

The justification—and necessity—for such a procedure in problems of this type is given by Chandrasekhar.[10]

The average power[1,2] is given by

$$\left\langle P_\gamma\left(\frac{d\phi}{dt}\right)\right\rangle_{\text{Av}}=\frac{4\pi}{3}\frac{e^2}{r}\left(\frac{E}{mc^2}\right)^4\frac{c}{2\pi r+L},\tag{7}$$

where c is the velocity of light. We may expand this for energies near E_s and, using (1) and (2), obtain

$$\left\langle P_\gamma\left(\frac{d\phi}{dt}\right)\right\rangle_{\text{Av}}=P_{\gamma s}\left[1-\left(3-4n-\frac{1}{\lambda}\right)\frac{\lambda^2}{k\omega_s}\frac{d\phi}{dt}\right],\tag{8}$$

where $P_{\gamma s}$ is the average power loss (at the energy E_s) by the synchronous particle.

Combining Eqs. (3), (4), (5), (6), and (8), we have the differential equation of the phase oscillations.

$$\frac{d^2\phi}{dt^2}+[(3-4n)P_{\gamma s}+(1-n)\dot{E}_s]\frac{1}{(1-n)E_s}\frac{d\phi}{dt}+\frac{k}{\lambda^2}\frac{\omega_s}{(1-n)E_s}[P_{\alpha s}\cot\phi_s\Delta\phi-p(t)]=0.\tag{9}$$

In Eq. (9) the coefficients of $d\phi/dt$ and $\Delta\phi$ depend on E_s which varies slowly with time. We may treat the coefficients as constant provided that the change in E_s is small in one damping time constant (i.e., a time equal to the reciprocal of the coefficient of $d\phi/dt$). Since $P_{\gamma s}$ increases as E_s^4 such a treatment will be valid for energies appreciably above a critical energy E_c defined by

$$P_{\gamma s}(E_c)=(1-n)\dot{E}_s/(3-4n).\tag{10}$$

For energies below E_c the equation may be treated by the methods of references 5 or 8, with which one obtains the usual adiabatic damping of the phase oscillations.

Restricting ourselves to energies above E_c we may neglect in Eq. (9) the terms in $\dot{E}_s$. We have then

$$d^2\psi/dt^2+\rho d\psi/dt+\Omega^2\psi=g(t),\tag{11}$$

where

$$\psi=\Delta\phi=\phi-\phi_s,$$

$$\rho=\frac{3-4n}{1-n}\frac{P_{\gamma s}}{E_s},$$

$$\Omega^2=aP_{\alpha s}\cot\phi_s,$$

$$g(t)=ap(t),$$

$$a=\frac{k}{\lambda^2}\frac{\omega_s}{(1-n)E_s}.$$

[8] N. Blachman and E. Courant, Rev. Sci. Instr. **20**, 596 (1949).

[9] L. J. Laslett kindly pointed out that in the original manuscript the term in $\dot{E}_s$ had been omitted from this equation.

[10] S. Chandrasekhar, Revs. Modern Phys. **15**, 1 (1943).

III. FLUCTUATION EFFECTS

We now ignore any oscillations in phase which may be present initially (and which would be rapidly damped) and determine the phase oscillations which arise due to the fluctuation in the energy loss as represented by $g(t)$. We compute only the equilibrium value of these oscillations. It may be expected that they will reach this equilibrium value in a time about $1/\rho$.

The driving function $g(t)$ may be expressed as an infinite sum, each term of which represents the emission of a quantum of energy ϵ_i assumed to be emitted instantaneously.

$$g(t)=a\sum\epsilon_i\delta(t-t_i)-\bar{g}.$$

The resulting phase oscillations will be given by

$$\psi(t)=a\sum\epsilon_iH(t-t_i)-\phi_s,$$

where $H(t)$ is the response to a unit impulse of the system described by Eq. (11).

Let $u(\epsilon)d\epsilon$ represent the number of quanta per unit of time of energy between ϵ and $\epsilon+d\epsilon$. Applying Campbell's theorem,[11] the contribution of these quanta to the mean square fluctuation in phase is

$$d\langle\psi^2\rangle_{\mathrm{Av}}=u(\epsilon)d\epsilon\int_{-\infty}^{+\infty}a^2\epsilon^2H^2(t)dt, \tag{12}$$

if it is assumed that the quanta are emitted statistically independently.[12] The total fluctuation in phase is

$$\langle\psi^2\rangle_{\mathrm{Av}}=a^2\int_0^{\infty}\epsilon^2u(\epsilon)d\epsilon\int_{-\infty}^{+\infty}H^2(t)dt. \tag{13}$$

The integral over t is conveniently expressed as

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{d\sigma}{|\Omega^2-\sigma^2+i\rho\sigma|^2}=\frac{1}{2\Omega^2\rho}. \tag{14}$$

The quantum distribution function $u(\epsilon)$ is related to the power spectrum $p(\epsilon)$ of the radiation by $\epsilon u(\epsilon)=p(\epsilon)$. Taking the power spectrum given by Schwinger,[13]

$$u(\epsilon)=\frac{3^{5/2}}{8\pi}\frac{P_{\gamma s}}{\epsilon_c^2}\int_{\epsilon/\epsilon_c}^{\infty}K_{5/3}(\eta)d\eta dx. \tag{15}$$

with

$$\epsilon_c=(3/2)\hbar\omega_s(E_s/mc^2)^3. \tag{16}$$

The integral over ϵ in Eq. (13) is then

$$(3^{5/2}/8\pi)P_{\gamma s}\epsilon_c\int_0^{\infty}x^2\int_x^{\infty}K_{5/3}(\eta)d\eta dx. \tag{17}$$

Evaluating the definite integral,[14] we obtain

$$\int_0^{\infty}\epsilon^2u(\epsilon)d\epsilon=(55/2^33^{\frac{3}{2}})P_{\gamma s}\epsilon_c. \tag{18}$$

Combining the results of (13), (14), and (18), and recalling the definitions of (11), we obtain finally:

$$\langle\psi^2\rangle_{\mathrm{Av}}=\frac{55\sqrt{3}}{2^6}\frac{\hbar c}{e^2}\frac{k}{\lambda}\frac{\tan\phi_s}{(3-4n)}\frac{mc^2}{E_s}. \tag{19}$$

where, in keeping with our approximations, we have set $P_{\alpha s}=P_{\gamma s}$. We shall use this result later to compute the energy loss due to coherent radiation by the electron bunch.

It will be recalled that (19) is applicable for energies greater than E_c defined by (10). We may obtain an explicit expression for E_c, using [from (7)]

$$P_{\gamma s}=\frac{2}{3}\frac{c}{\lambda}\frac{e^2}{r_s^2}\left(\frac{E_s}{mc^2}\right)^4, \tag{20}$$

whence

$$\frac{E_c}{mc^2}=\left[\frac{3}{2}\frac{\lambda}{c}\frac{r_s^2}{e^2}\frac{(1-n)}{(3-4n)}\dot{E}_s\right]^{\frac{1}{4}}. \tag{21}$$

It is perhaps more convenient to express (21) in terms of the net energy gain per turn,

$$\Delta E_0=\lambda(2\pi r_s/c)\dot{E}_s.$$

Then

$$\frac{E_c}{mc^2}=\left[\frac{3}{4\pi}\frac{(1-n)}{(3-4n)}\frac{r_s}{r_0}\frac{\Delta E_0}{mc^2}\right]^{\frac{1}{4}}. \tag{22}$$

where r_0 is the classical electron radius. It may be expected that for energies above E_c the relative error in (19) will be of the order of $\Delta E_0/\Delta E_\gamma$, where ΔE_γ is the energy radiated in one turn.

The oscillations in radius associated with the fluctuations in phase may also be of some interest. From (2) we have

$$\left\langle\frac{(\Delta r)^2}{r_s^2}\right\rangle_{\mathrm{Av}}=\frac{\lambda^4}{k^2}\frac{1}{\omega_s^2}\left\langle\left(\frac{d\psi}{dt}\right)^2\right\rangle_{\mathrm{Av}}.$$

The damping of the phase oscillations in one cycle is very small.[15] We may, therefore, write

$$\langle(d\psi/dt)^2\rangle_{\mathrm{Av}}=\Omega^2\langle\psi^2\rangle_{\mathrm{Av}}.$$

We have then, for the mean square fluctuation in radius,

$$\left\langle\frac{(\Delta r)^2}{r_s^2}\right\rangle_{\mathrm{Av}}=\frac{55}{2^5\sqrt{3}}\frac{\hbar c}{e^2}\frac{1}{(1-n)(3-4n)}\frac{r_0}{r_s}\left(\frac{E_s}{mc^2}\right)^2. \tag{23}$$

[11] See, e.g., S. O. Rice, Bell System Tech. J. **23**, 282 (1944).
[12] See appendix for further discussion of this point.
[13] J. Schwinger, Phys. Rev. **75**, 1912 (1949), Eqs. (11), (16).
[14] G. N. Watson, *Bessel Functions* (McMillan Company, New York, 1945), p. 388.
[15] The logarithmic decrement is $\approx(\Delta E_\gamma/E_s)^{\frac{1}{2}}$. See, e.g., reference 5.

IV. NUMERICAL EXAMPLES

Design parameters for a 1.5-Bev synchrotron might be

$$r_s=375\text{ cm},\quad k=4,\quad \lambda=1.25,$$
$$\Delta E_0=750\text{ volts},\quad n=0.6.$$

For these values the fluctuations will become significant above $E_c=380$ Mev [using (22)] and will become dominant by about $E_s=700$ Mev.

The mean square fluctuation in phase will then be given by

$$\langle\psi^2\rangle_{\text{Av}}=1080(mc^2/E_s)\tan\phi_s=(0.56\text{ Bev}/E_s)\tan\phi_s.$$

For 1.5 Bev the root-mean-square angle is

$$\langle\psi^2\rangle_{\text{Av}}^{\frac{1}{2}}=0.59(\tan\phi_s)^{\frac{1}{2}}. \qquad (24)$$

Thus, an equilibrium phase angle of about $\pi/6$ is required if electrons are not to be lost due to these fluctuations. The maximum voltage of the radio-frequency system must, therefore, exceed 200 kilovolts (since $\Delta E_\gamma=120$ kev for $E_s=1.5$ Bev).

The required radial aperture at high energies may be determined from (23). For the parameters chosen here,

$$\left\langle\frac{(\Delta r)^2}{r_s^2}\right\rangle_{\text{Av}}=4.2\times10^{-13}\left(\frac{E_s}{mc^2}\right)^2=10^{-4}\left(\frac{E_s}{8\text{ Bev}}\right)^2.$$

For $E_s=1.5$ Bev the root-mean-square spread in radii is

$$\langle(\Delta r)^2\rangle_{\text{Av}}^{\frac{1}{2}}=1.9\times10^{-3}r_s=0.7\text{ cm}. \qquad (25)$$

The effective magnetic aperture must be many (≈10) times this value if loss of electrons is to be avoided.

V. COHERENT RADIATION

Several authors[4,6,7] have considered the increase in the rate of energy loss, above that given by P_γ, due to the coherent effects of all the electrons in a bunch. Schiff[4,6] gives expressions for the case in which the electrons are spread in a Gaussian distribution about the equilibrium phase. Since the important contribution to the coherent radiation comes from long wavelengths, the results depend on the proximity of metallic surfaces. If we assume that the mean square fluctuation in the phase of a single electron represents the longitudinal spread of a group of electrons, then the additional loss in one revolution *for each electron* (due to coherent radiation) is, in the absence of metallic shielding,[4]

$$\Delta E_{\text{coh}}=3^{1/6}\Gamma^2(2/3)(r_0/r_s)k^{\frac{1}{3}}N\langle\psi^2\rangle_{\text{Av}}^{-\frac{2}{3}}mc^2,$$

where N is the total number of electrons accelerated. Using the results above [Eq. (24)] and taking $\phi_s=\pi/4$, we see that the coherent loss per electron is, at 1.5 Bev,

$$\Delta E_{\text{coh}}=2.5\times10^{-9}N\text{ electron volts.}$$

Typical intensities involve the acceleration of from 10^8 to 10^{11} electrons. Even for $N=10^{12}$, the coherent energy loss is only 2500 ev, which is negligible in comparison with the incoherent losses.[16]

If the electron orbits are within a vacuum chamber which has conducting walls whose separation is less than $(r_s/k)\langle\psi^2\rangle_{\text{Av}}^{\frac{1}{2}}$ (as would normally be) the coherent loss is less than the above value. In a typical case the shielded coherent loss is computed (following reference 6) to be less than 1/50 the nonshielded loss.

This work profitted from discussions with M. Bloch, Leverett Davis, Jr., R. P. Feynman, R. V. Langmuir, Using the results above [(24)] and taking $\phi_s=\pi/4$, Robert L. Walker, and Kenneth M. Watson. L. Jackson Laslett kindly pointed out the existence of some numerical errors in the original manuscript.

APPENDIX[17]

The justification for the addition of quantum recoil effects to a "classical" system is contained essentially in the work of Bloch and Nordsieck and others.[18] These authors show that emissions of individual quanta from a classical radiating system are statistically independent.

Since we are here interested in the fluctuations in energy loss in a time interval much longer than time of emission of a typical quantum, the uncertainty in the definition of the electron energy is insignificant compared with the calculated energy fluctuations.

Watson[19] points out that a complete quantum-mechanical calculation can be performed by introducing into the equation of the phase oscillation the interaction of the electron with the radiation field. This results in replacing $g(t)$ in Eq. (11) by $a(e\mathbf{v}\cdot\boldsymbol{\mathcal{E}}-e\mathbf{v}\cdot\langle\boldsymbol{\mathcal{E}}\rangle_{\text{Av}})$, where $\mathbf{v}$ is the electron velocity, $\boldsymbol{\mathcal{E}}$ is the radiation field strength, and $\langle\boldsymbol{\mathcal{E}}\rangle_{\text{Av}}$ is its expectation value. An evaluation of the quantum-mechanical expectation value of ψ^2 then yields the result obtained here for the time average.

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[16] For intensities as large as 10^{12} electrons, spreading of the bunch due to space-charge forces may begin to become significant.

[17] I am indebted to K. M. Watson for the illuminating discussions which led to the following remarks.

[18] F. Bloch and A. Nordsieck, Phys. Rev. **52**, 54 (1937); W. Pauli and M. Fierz, Nuovo cimento **15**, 167 (1938); R. Glauber, Phys. Rev. **84**, 395 (1951).

[19] Kenneth M. Watson (private communication).