Now, by Eq. (B–2)

$$\frac{d\mathbf{v}_L}{dt} \simeq \hat{e}_2' \frac{dV_2}{dt} + \hat{e}_3' \frac{dV_3}{dt} - \frac{V_1 V_2}{R_1} \hat{e}_1, \tag{B-13}$$

if we neglect the torsion, $\partial \hat{e}_3 / \partial x_1$. Choosing

$$\hat{e}_3' = \hat{e}_3 - \hat{e}_1 (V_1/\omega R_1), \quad \hat{e}_2' = \hat{e}_2, \tag{B-14}$$

Eq. (B–13) reduces to

$$\frac{d\mathbf{v}_L}{dt} = \hat{e}_2 \frac{dV_2}{dt} + \hat{e}_3 \frac{dV_3}{dt}. \tag{B-15}$$

($dV_3/dt \simeq -\omega V_2$, to first order in η.) Substituting Eq. (B–15) into Eq. (B–12), we obtain coupled differential equations for dV_2/dt and dV_3/dt. By means of the adiabatic theorem in mechanics, we can show that, keeping first-order terms in η, the solution is just Eqs. (20) with $V_\perp$ given by Eq. (22).

From Eqs. (B–11), we easily obtain

$$v_1 - V_1 = \left(\frac{V_2 V_3}{2\omega}\right)\left(\frac{1}{R_2} - \frac{1}{R_3}\right). \tag{B-16}$$

We have now defined all quantities except $\mathbf{v}_L'$ in Eq. (B–9). Substituting, this becomes

$$\mathbf{v}_L' = \mathbf{v}_L' \times \boldsymbol{\omega} + \frac{1}{R_1}\left[V_2 V_3 \hat{e}_3 + \frac{\hat{e}_2}{2}(V_2^2 - V_3^2)\right], \tag{B-17}$$

which has the solution (to order η)

$$\mathbf{v}_L' = \frac{1}{3\omega R_1}\left\{\hat{e}_2 V_2 V_3 - \frac{\hat{e}_3}{2}(V_2^2 - V_3^2)\right\}. \tag{B-18}$$

Thus we have all the quantities in Eq. (B–6), which is just Eq. (24).

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Use of the Boltzmann Equation for the Study of Ionized Gases of Low Density. II*

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The Boltzmann equation for ionized gases of low density in an external magnetic field is used to obtain approximate solutions in the nonstatic case. The Boltzmann and Maxwell equations are linearized by assuming small deviations from a static solution. It is shown that in the limit of a strong magnetic field ($\eta \ll 1$, as defined in the text), the motion transverse to the magnetic field is described by the conventional hydrodynamic equations. The variation along field lines is described by a one-dimensional (i.e., one space dimension and one velocity dimension) Boltzmann equation. Several applications are given, including an analysis of the Kruskal-Schwarzschild gravitational instability of a plasma.

I. INTRODUCTION

In Part I[1] we discussed on rather general grounds the behavior of an ionized gas of low density in a strong magnetic field. The properties of the static state were treated in detail. In the present paper we study further the dynamic and thermodynamic behavior of the gas.

We recall a few of the basic equations from Part I. The dynamical properties were described by the Boltzmann equation

$$\frac{\partial f}{\partial t} + \mathbf{c}\cdot\nabla f + \frac{e}{M}\left[\mathbf{E} + \frac{1}{C}\mathbf{c}\times\mathbf{B}\right]\cdot\nabla_c f = 0, \tag{1}$$

which applies to either electrons or ions on giving e its proper sign and on assigning the appropriate subscripts e or i. The first four moments of f were written as

$$\begin{aligned} n &\equiv \int f d^3c, \\ \mathbf{v} &\equiv \frac{1}{n}\int \mathbf{c} f d^3c, \\ \mathbf{p} &\equiv M\int (\mathbf{c}-\mathbf{v})(\mathbf{c}-\mathbf{v}) f d^3c, \\ \mathbf{Q} &\equiv M\int (\mathbf{c}-\mathbf{v})(\mathbf{c}-\mathbf{v})(\mathbf{c}-\mathbf{v}) f d^3c, \end{aligned} \tag{2}$$

* A development closely paralleling in many respects that given here has been found by Chew, Goldberger, and Low. This treatment is to be published separately.

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[1] K. M. Watson, preceding paper [Phys. Rev. **102**, 12 (1956)]. Equations in Part I will be referred to here as Eq. (I-1), etc. Part I itself will be referred to as I, for brevity. The notation is the same as in I: **E** is the electric field, **B** the magnetic field, **C** the particle velocity, etc. The unit vector $\hat{e}_1$ is in the direction of $\mathbf{B}_0$, $\hat{e}_2$ in the direction of the principal radius of curvature of the $\mathbf{B}_0$-lines. $\hat{e}_3$ is $\hat{e}_1 \times \hat{e}_2$.

where $\mathbf{p}$ and $\mathbf{Q}$ are the pressure and heat flow tensors. Taking the first three moments of Eq. (1), we obtain the familiar differential relations between the four moments (2).

$$\frac{\partial n}{\partial t}+\nabla\cdot(\mathbf{v}n)=0, \quad (3)$$

$$\frac{\partial \mathbf{v}}{\partial t}+\mathbf{v}\cdot\nabla\mathbf{v}=-\frac{1}{Mn}\nabla\cdot\mathbf{p}+\frac{e}{M}\left[\mathbf{E}+\frac{1}{C}\mathbf{c}\times\mathbf{B}\right], \quad (4)$$

$$\frac{\partial \mathbf{p}}{\partial t}+\nabla\cdot\mathbf{Q}+\mathbf{v}\cdot\nabla\mathbf{p}+\mathbf{p}\nabla\cdot\mathbf{v}+[\mathbf{p}\cdot\nabla\mathbf{v}+(\mathbf{p}\cdot\nabla\mathbf{v})^T]$$
$$-\frac{e}{MC}\sum_{i,j=1}^{3}\hat{e}_i\hat{e}_j[\hat{e}_i\cdot\mathbf{p}\cdot(\mathbf{B}\times\hat{e}_j)+\hat{e}_j\cdot\mathbf{p}\cdot(\mathbf{B}\times\hat{e}_i)]=0. \quad (5)$$

(The unit vectors and coordinate system which we shall use are defined in Appendix A of Part I.[1]) Here the symbol $(\mathbf{p}\cdot\nabla\mathbf{v})^T$ represents the transpose of the dyadic $\mathbf{p}\cdot\nabla\mathbf{v}$. We emphasize once again that there will be a duplicate set of Eqs. (1) to (5) for electrons and ions. To these must be added Maxwell's equations (I-3) and (I-4). Equations (3), (4), and (5) do not, of course, determine the four moments (2) unless an assumption is made which relates the heat flow tensor $\mathbf{Q}$ to the lower moments. Consequently, we must solve Eq. (1) and thus determine $\mathbf{Q}$ if we desire only the first few moments of the distribution. Actually, we shall consider solutions to Eq. (I-11) instead, since we desire the solution f which differs very little from the known static distribution f_0.

Before attempting to solve these equations, we recall the four assumptions (A), (B), (C), and (D) which were made in Sec. III of Part I concerning the properties of the gas. In particular, we must use the conditions that $\eta\ll 1$.[2] As an application of the way in which we shall use these assumptions, we observe that to calculate $\mathbf{v}$ from Eq. (4) to second order in reciprocal dimensions of the system, (i.e., to second order in η) we need calculate the pressure tensor to first order only in η if

$$\frac{e}{MC}\mathbf{v}\times\mathbf{B}\simeq\frac{1}{Mn}\nabla\cdot\mathbf{p},$$

is used to give $\mathbf{v}$. Again, to calculate $\mathbf{p}$ to first order in η, the heat flow $\mathbf{Q}$ need be calculated only to order zero if $\mathbf{p}$ is obtained from the term proportional to $\omega=eB/MC$ in Eq. (5). It is thus clear that the smallness of η implies that the higher moments of the f-function are of limited importance for calculating the lower moments by means of Eqs. (3), (4), and (5). There is, however, a very important limitation on this argument. That is, the moment Eqs. (4) and (5) are tensor equations and the Larmor terms appear only in certain of their components. The smallness of η does not simplify the remaining components as effectively. Indeed, to obtain these we must solve the Boltzmann equation. In spite of this limitation, the existence of a strong external magnetic field is of great importance in permitting one to deduce a quasi-fluid dynamics.

[2] We recall that the quantity $\eta=$ (Larmor radius)$\times$(dimensions of system)$^{-1}$. The only assumption (of our four) of essential importance is that $\eta\ll 1$. The others are made only as a matter of convenience in simplifying the discussion at a later stage.

Before continuing, it is perhaps worth while to discuss the significance of Eq. (5). This contains, for instance, the appropriate law of adiabatic compression for our problem. In the static case $\nabla\cdot\mathbf{Q}=0$ (Part I), and to a good approximation in general (the smallness of η), we may write this equation as

$$\hat{e}_i\cdot\mathbf{p}\cdot(\mathbf{B}\times\hat{e}_j+\hat{e}_j\cdot\mathbf{p}\cdot(\mathbf{B}\times\hat{e}_i)=0, \quad (6)$$

for all (i,j). Recalling our definition of the $\hat{e}$'s and writing Eq. (6) out in detail, we obtain from it

$$p_{13}=p_{12}=p_{23}=0, \quad p_{22}=p_{33}. \quad (7)$$

These are just the relations deduced in Part I.[3] The relations (7) are actually more general. Indeed, it is evident that for nonstatic problems whose motions are characterized by frequencies small compared to the Larmor frequency the relations (7) will still be valid to the extent that Eq. (6) is a good approximation. This is a further consequence of our approximation that $\eta\ll 1$.

When the divergence of the heat flux $(\nabla\cdot\mathbf{Q})$ vanishes, Eq. (5) leads also to a set of laws for adiabatic compression. To obtain these we consider Eqs. (7) to be constraints on the compression. We then find

$$\hat{e}_1\cdot\frac{d\mathbf{p}}{dt}\cdot\hat{e}_1=-p_{11}\left[\operatorname{div}\mathbf{v}+2\hat{e}_1\cdot\frac{\partial\mathbf{v}}{\partial x_1}\right], \quad (8)$$
$$\hat{e}_2\cdot\frac{d\mathbf{p}}{dt}\cdot e_2=\hat{e}_3\cdot\frac{d\mathbf{p}}{dt}\cdot\hat{e}_3$$
$$=-p_{33}\left[2\operatorname{div}\mathbf{v}-\hat{e}_1\cdot\frac{\partial\mathbf{v}}{\partial x_1}\right].$$

Chew, Goldberger, and Low[3] have found integrals of these equations.

The compression laws (8) take on a simple form if we suppose the magnetic field lines to be straight enough that $\hat{e}_1$, $\hat{e}_2$, and $\hat{e}_3$ may be treated as constants. The result is

$$\frac{dp_{11}}{dt}=-3p_{11}\frac{\partial v_1}{\partial x_1}-p_{22}\left(\frac{\partial v_2}{\partial x_2}+\frac{\partial v_3}{\partial x_3}\right),$$
$$\frac{dp_{22}}{dt}=\frac{dp_{33}}{dt}=-2p_{33}\left(\frac{\partial v_2}{\partial x_2}+\frac{\partial v_3}{\partial x_3}\right)-p_{33}\frac{\partial v_1}{\partial x_1}. \quad (9)$$

[3] These relations were first derived in collaboration with M. L. Goldberger. See, for instance, the forthcoming paper by Chew, Goldberger, and Low.

These represent simple generalizations of the usual adiabatic law for scalar pressure. Indeed, on imposing the condition that $p_{11}=p_{22}=p_{33}$, we obtain from Eqs. (9) just the usual adiabatic law with $\gamma=5/3$. The form of Eqs. (9) results from the fact that heat exchange is not possible between longitudinal and transverse degrees of freedom, i.e., parallel to and perpendicular to **B**. The longitudinal compression has $\gamma=3$, since it is one dimensional. Transverse compression has $\gamma=2$, since it is two dimensional. The cross terms in Eqs. (9) represent the effect of density changes only on the pressure.

II. NONSTATIC PROBLEM

The static distribution function is

$$f_0=f_0(\mathbf{r},\mathbf{c}),$$

which to zero order in η has the form

$$f_0(c_\perp^2,c_1^2,\mathbf{r}),\tag{10}$$

where c_1 is the component of the velocity along the magnetic field and $c_\perp$ is the perpendicular component, i.e.,

$$c_1=\mathbf{c}\cdot\hat{e}_1,\quad c_\perp^2=c^2-c_1^2.\tag{11}$$

That expression (10) is the form of the static distribution function was shown in Part I.

The nonstatic distribution function is

$$f=f_0(\mathbf{c}-\mathbf{u})+f',\tag{12}$$

where we define **u** to be the solution to

$$\frac{d\mathbf{u}}{dt}=\frac{e}{M}\left[\mathbf{E}'+\frac{1}{C}\mathbf{u}\times\mathbf{B}_0\right],\tag{13}$$

according to Eqs. (I-9) and (I-10). We remember that **u** and f' are considered to be infinitesimal quantities. Equation (I-11) for f' is

$$\mathfrak{D}f'=-\mathbf{u}\cdot\nabla f_0+\mathbf{c}\cdot(\nabla\mathbf{u})\cdot\nabla_c f_0 - (e/MC)(\mathbf{c}\times\mathbf{B}')\cdot\nabla_c f_0,\tag{14}$$

where the operator $\mathfrak{D}$ was defined in I to be $\partial/\partial t+\mathbf{c}\cdot\nabla+(e/MC)(\mathbf{c}\times\mathbf{B}_0)\cdot\nabla_c$.

First of all, we solve Eq. (4) on the assumption that

$$|\partial\mathbf{v}/\partial t|\ll|\omega\mathbf{v}|,$$

Where $\omega=eB_0/MC$. This is valid if we may introduce the assumption:

(E) Neither $\nabla\cdot\mathbf{p}$ nor **E** has a component in the direction of **B**.

This assumption is not arbitrary, since its self-consistency can be checked once $\nabla\cdot\mathbf{p}$ and **E** have actually been calculated. If it should turn out not to be correct, the appropriate modification can be made in the following developments by solving Eq. (4) for the component of **v** parallel to **B**. This is a very plausible initial *ansatz*, however, since the gas is presumably able to adjust itself to maintain equilibrium conditions along field lines. For instance, a component of $\nabla\cdot\mathbf{p}$ or **E** along **B** would give rise to large currents and matter flow. We shall later return to a justification for assumption (E).

Equation (4) is now solved by successive approximation. We note first that the largest part of the electromagnetic interaction, assuming small **E**, is the term in $\mathbf{v}\times\mathbf{B}$. In the quasi-static case, this is approximately balanced by the pressure term. This suggests that we separate **v** into two parts

$$\mathbf{v}=\mathbf{v}_1+\mathbf{v}^{(1)},$$

where $\mathbf{v}_1$ is defined by

$$\frac{e}{MC}\mathbf{v}_1\times\mathbf{B}=\frac{1}{Mn}\nabla\cdot\mathbf{p},\tag{15}$$

and $\mathbf{v}^{(1)}$ will be smaller and related to the departure from equilibrium. We can further solve Eq. (15) for $\mathbf{v}_1$ (recalling that **v** has no component parallel to **B**) to give

$$\mathbf{v}_1=\hat{e}_1\times(\nabla\cdot\mathbf{p})/Mn\omega.\tag{16}$$

This result shows that $\mathbf{v}_1$ is of order $1/\omega$, or of order η; consequently the term $\mathbf{v}_1\cdot\nabla\mathbf{v}_1\simeq\mathbf{v}\cdot\nabla\mathbf{v}$ can be dropped as of order η^2. Equation (4) now is

$$\frac{\partial\mathbf{v}_1}{\partial t}+\frac{\partial\mathbf{v}^{(1)}}{\partial t}=\frac{e}{M}\left(\mathbf{E}+\frac{\mathbf{v}^{(1)}}{C}\times\mathbf{B}\right).\tag{17}$$

Again we notice that in the quasi-static case $\mathbf{E}+(\mathbf{v}^{(1)}/C)\times\mathbf{B}$ tends to be nearly zero, which suggests that we separate $\mathbf{v}^{(1)}$ into

$$\mathbf{v}^{(1)}=\mathbf{v}_2+\mathbf{v}^{(2)},$$

where

$$\mathbf{v}_2=+C\mathbf{E}\times\mathbf{B}/B^2\simeq C\mathbf{E}\times\mathbf{B}_0/B_0^2.\tag{18}$$

The last step follows from the assumed smallness of **E** which allows us in first order to replace **B** by its static value. We proceed to next order by defining

$$\mathbf{v}^{(2)}=\mathbf{v}_3+\mathbf{v}_4+\mathbf{v}^{(4)},$$

where $\mathbf{v}_3$ and $\mathbf{v}_4$ are equated to the time derivatives of $\mathbf{v}_2$ and $\mathbf{v}_1$, respectively. The result is

$$\mathbf{v}_3=-\frac{1}{e}\frac{MC^2}{B_0^2}\frac{\partial\mathbf{E}}{\partial t},\tag{19}$$

$$\frac{e}{MC}\mathbf{v}_4\times\mathbf{B}=-\frac{\partial}{\partial t}\frac{\hat{e}_1\times(\nabla\cdot\mathbf{p})}{Mn\omega}.\tag{20}$$

The order of magnitude of $\mathbf{v}_4$ is obtained by setting

$$\frac{\partial\mathbf{v}_1}{\partial t}\simeq\Omega\frac{\hat{e}_1\times(\nabla\cdot\mathbf{p})}{Mn\omega},\tag{21}$$

where Ω is a frequency characteristic of the motion of the system. Then

$$v_4 \simeq (\Omega/\omega) v_1. \tag{22}$$

Since v_1 is already of first order in η, this quantity is of second order and will be dropped.

We have now the fluid velocity to the required accuracy and from this may calculate the electric current density. This is [see Eq. (I-3)]

$$\mathbf{j} = e(n_i \mathbf{v}_i - n_e \mathbf{v}_e). \tag{23}$$

We shall neglect by assumptions (A) and (D) the convection current which results from $\mathbf{v}_2$ since the electric field drift is the same for electrons and ions, and in the absence of charge separation no current results. Consequently, taking $n_i = n_e = n$, we have

$$\mathbf{j} = \mathbf{j}_1 + \mathbf{j}_3, \tag{24}$$

where

$$\begin{aligned} \mathbf{j}_1 &= en(\mathbf{v}_{1i} - \mathbf{v}_{1e}) = C\hat{e}_1 \times (\nabla \cdot \mathbf{p}_T)/B, \\ \mathbf{j}_3 &= en(\mathbf{v}_{3i} - \mathbf{v}_{3e}) \\ &\simeq en\mathbf{v}_{3i} \\ &= [(\kappa - 1)/4\pi] \partial \mathbf{E}/\partial t. \end{aligned} \tag{25}$$

Here we mean by $\mathbf{p}_T$ the sum of ion and electron pressures and κ is defined by

$$\kappa = 1 + 4\pi\rho C^2/B_0^2, \tag{26}$$

where $\rho = n M_i$ is the gas density. The charge density ϵ [see Eq. (I-3)] is

$$\epsilon = e(n_i - n_e), \tag{27}$$

and is related, of course, to $\mathbf{j}$ by

$$\nabla \cdot \mathbf{j} + \partial\epsilon/\partial t = 0. \tag{28}$$

To solve Maxwell's equations, we first of all separate out the static part of

$$-\frac{1}{C}\frac{\partial \mathbf{B}}{\partial t} = \nabla \times \mathbf{E}, \tag{29}$$

using

$$-\frac{1}{C}\frac{\partial \mathbf{B}_0}{\partial t} = \nabla \times \mathbf{E}_0. \tag{30}$$

Thus

$$-\frac{1}{C}\frac{\partial \mathbf{B}'}{\partial t} = \nabla \times \mathbf{E}'. \tag{31}$$

It is now convenient to work not with $\mathbf{E}'$, but with the drift velocity resulting from $\mathbf{E}'$. We do this by writing the drift velocity as

$$\mathbf{v}_2 = C\frac{\mathbf{E}_0 \times \mathbf{B}_0}{B_0^2} + \frac{\partial \boldsymbol{\xi}}{\partial t}, \tag{32}$$

where

$$\frac{\partial \boldsymbol{\xi}}{\partial t} = C\frac{\mathbf{E}' \times \mathbf{B}_0}{B_0^2} \tag{33}$$

is the nonstatic drift velocity. We can thus express $\mathbf{E}'$ in terms of $\boldsymbol{\xi}$ by

$$\mathbf{E}' = -\frac{1}{C}\frac{\partial \boldsymbol{\xi}}{\partial t} \times \mathbf{B}_0. \tag{34}$$

Substituting into Eq. (31) and integrating with respect to time gives

$$\mathbf{B}' = \nabla \times (\boldsymbol{\xi} \times \mathbf{B}_0). \tag{35}$$

This permits us to eliminate $\mathbf{B}'$ from the remaining Maxwell equations:

$$\begin{aligned} \nabla \times \mathbf{B} &= \frac{4\pi}{C}(\mathbf{j}_1 + \mathbf{j}_3) + \frac{1}{C}\frac{\partial \mathbf{E}}{\partial t} \\ &= \frac{4\pi}{C}\mathbf{j}_1 + \frac{\kappa}{C}\frac{\partial \mathbf{E}}{\partial t}, \end{aligned} \tag{36}$$

using Eq. (25). We remove the static part:

$$\nabla \times \mathbf{B}_0 = \frac{4\pi}{C}\mathbf{j}_1^0, \tag{37}$$

where

$$\mathbf{j}_1 = \mathbf{j}_1^0 + \mathbf{j}', \tag{38}$$

with $\mathbf{j}_1^0$ the static and $\mathbf{j}'$ the perturbed part of $\mathbf{j}_1$. Then Eq. (36) becomes

$$\frac{\kappa}{C}\frac{\partial \mathbf{E}'}{\partial t} = \nabla \times \mathbf{B}' - \frac{4\pi}{C}\mathbf{j}'. \tag{39}$$

Eliminating $\mathbf{E}'$ and $\mathbf{B}'$ using Eqs. (34) and (35) we obtain

$$-\frac{\kappa}{C^2}\frac{\partial^2 \boldsymbol{\xi}}{\partial t^2} \times \mathbf{B}_0 = \nabla \times \{\nabla \times (\boldsymbol{\xi} \times \mathbf{B}_0)\} - \frac{4\pi}{C}\mathbf{j}'. \tag{40}$$

Taking the vector product with $\mathbf{B}_0$, we obtain the equation

$$\rho_0 \frac{\partial^2 \boldsymbol{\xi}}{\partial t^2} = \frac{1}{C}\mathbf{B}_0 \times \mathbf{j}' - \frac{1}{4\pi}\mathbf{B}_0 \times \{\nabla \times [\nabla \times (\boldsymbol{\xi} \times \mathbf{B}_0)]\}, \tag{41}$$

using Eq. (26) for κ and neglecting $B_0^2/4\pi$ relative to $nM_i c^2$. We shall discuss in a moment the determination of $\mathbf{j}'$ from the solution of the perturbed Boltzmann equation (14). When this has been done, Eq. (41) becomes a differential equation for the variable $\boldsymbol{\xi}$. It is evident that $\boldsymbol{\xi}$ is a particularly useful variable to describe both the motion and stability of a plasma.[4] We must emphasize that only motion perpendicular to $\mathbf{B}_0$ is defined by the differential equation (41). To obtain

[4] Frieman, Kruskal, Bernstein, and Kulsrud, Revs. Modern Phys. (to be published), have introduced a very similar variable $\boldsymbol{\xi}$ as a hydrodynamic Lagrangian variable in connection with the theory of stability of hydromagnetic fluids. In our application, $\boldsymbol{\xi}$ replaces $\mathbf{E}'$ as a variable through Eq. (33) and represents only a *part* of the fluid motion. As we shall later see, for an extensive class of problems an adiabatic pressure law holds and Eq. (41) becomes identical with the corresponding equation in hydromagnetics.

motion along the field lines, we must return to Eq. (4) and relax our assumption (4) that $\nabla\cdot\mathbf{p}$ and $\mathbf{E}$ have no $\hat{e}_1$ components. Since $\nabla\cdot\mathbf{p}$ may be calculated from Eq. (16) and $\mathbf{E}$ calculated from $\nabla\cdot\mathbf{E}=4\pi\epsilon$ we may also solve Eq. (4) for the component $\hat{e}_1\cdot\mathbf{v}$. For a wide class of problems this is not of importance.

To proceed with the discussion of Eq. (41), we must evaluate $\mathbf{j}'$. Since $\mathbf{j}'$ is the fluctuation of $\mathbf{j}_1=C\mathbf{B}\times\nabla\cdot\mathbf{p}_T/B^2$ from its static value, we have

$$\mathbf{j}'=C\frac{\hat{e}_1}{\mathbf{B}_0}\times[\nabla\cdot\mathbf{p}_T']+\frac{C}{\mathbf{B}_0^2}\mathbf{B}'\times[\nabla\cdot\mathbf{p}_T^0]-\frac{2C}{\mathbf{B}_0^4}[\mathbf{B}_0\cdot\mathbf{B}']\mathbf{B}_0\times(\nabla\cdot\mathbf{p}_T^0), \tag{42}$$

where

$$\mathbf{p}_T=\mathbf{p}_T^0+\mathbf{p}_T',$$

is separated into static and fluctuation parts. Using Eqs. (35) and (41), we can now simplify Eq. (42) to

$$\rho_0\frac{\partial^2\boldsymbol{\xi}}{\partial t^2}=-\nabla\cdot\mathbf{p}_T'+\frac{1}{\mathbf{B}_0^2}[\mathbf{B}_0\cdot\nabla\times(\boldsymbol{\xi}\times\mathbf{B}_0)](\nabla\cdot\mathbf{p}_T^0)-\frac{1}{4\pi}\mathbf{B}_0\times\{\nabla\times[\nabla\times(\boldsymbol{\xi}\times\mathbf{B}_0)]\}. \tag{43}$$

It remains only to calculate $\mathbf{p}_T'$ from the Boltzmann equation (14). Before doing this we evaluate the virial of Eq. (43). A qualitative discussion of the motion is, indeed, often possible from the virial. Taking the scalar product with $\boldsymbol{\xi}$ and integrating over space, we have (after some partial integrations and discarding surface integrals)

$$\int\rho_0\boldsymbol{\xi}\cdot\frac{\partial^2\boldsymbol{\xi}}{\partial t^2}d\tau=-\int\left\{\frac{1}{4\pi}[\nabla\times(\boldsymbol{\xi}\times\mathbf{B}_0)]^2-\mathbf{j}_1^0\cdot\boldsymbol{\xi}\times[\nabla\times(\boldsymbol{\xi}\times\mathbf{B}_0)]-(\nabla\boldsymbol{\xi}):\mathbf{p}_T'\right\}d\tau. \tag{44}$$

In obtaining the final form, we have assumed that $\boldsymbol{\xi}=0$ on the bounding surface of the integration volume and used the identity $\nabla\cdot\mathbf{p}_T^0=(1/C)\mathbf{j}_1^0\times\mathbf{B}_0$. [$(\nabla\boldsymbol{\xi}):\mathbf{p}_T'$ represents the double scalar product of the two dyadics.]

The importance of the virial for our problem is as follows: If the motion across magnetic field lines is slow compared to thermal velocities, contours of constant density will remain parallel to magnetic field lines. This means that $\boldsymbol{\xi}\cdot\partial^2\boldsymbol{\xi}/\partial t^2$ will have the same sign everywhere along a field line. If

$$\boldsymbol{\xi}\cdot\partial^2\boldsymbol{\xi}/\partial t^2>0,$$

the acceleration is essentially parallel to the "displacement" and the system may be called "unstable." If

$$\boldsymbol{\xi}\cdot\partial^2\boldsymbol{\xi}/\partial t^2<0,$$

the acceleration is opposite to the "displacement" and the system is "stable." This is not quite an absolute criterion for stability since the system may show "overstability" (oscillations of increasing amplitude). In this case it would be necessary to return to the differential equation (43).

For any assumed $\boldsymbol{\xi}$, the frequency Ω of the motion may be estimated from Eq. (44) by setting $\partial^2\boldsymbol{\xi}/\partial t^2=\Omega^2\boldsymbol{\xi}$. On evaluating the integrals, one may solve for Ω^2.

The use of the virial in such problems as this is not uncommon.[5] In the hydrodynamic discussion of Frieman *et al.*,[4] the right-hand side of Eq. (39) was shown to be the negative of twice the "potential energy" of the system. This observation by Frieman *et al.* permitted them to give a variational principle for the motion of the system.[6]

To complete our discussion of Eq. (43), we must now determine the pressure fluctuation $\mathbf{p}_T'$. Thus we turn finally to the discussion of the Boltzmann equation (14). The drift velocity $\mathbf{u}$ is evaluated from Eq. (13) assuming that the time derivative is negligible, as is valid in the low η limit. Then

$$\mathbf{E}'+(1/C)\mathbf{u}\times\mathbf{B}_0=0,$$

determines $\mathbf{u}$. But this is the defining equation for $\partial\boldsymbol{\xi}/\partial t$. Thus

$$\partial\boldsymbol{\xi}/\partial t=\mathbf{u}. \tag{45}$$

With Eq. (32) for $\mathbf{B}'$, Eq. (14) becomes [here $\mathfrak{D}$ can be written as $\mathfrak{D}\equiv\partial/\partial t+\mathbf{c}\cdot\nabla+(e/MC)\mathbf{c}\times\mathbf{B}_0\cdot\nabla_c$]

$$\mathfrak{D}f'=-\frac{\partial\boldsymbol{\xi}}{\partial t}\cdot\nabla f_0+\mathbf{c}\cdot\left(\nabla\frac{\partial\boldsymbol{\xi}}{\partial t}\right)\cdot\nabla_c f_0-\frac{e}{MC}\mathbf{c}\times[\nabla\times(\boldsymbol{\xi}\times\mathbf{B}_0)]\cdot\nabla_c f_0=G\left(\frac{\partial\boldsymbol{\xi}}{\partial t},\mathbf{c}\right). \tag{46}$$

Since

$$\mathbf{p}'=M\int\mathbf{c}\mathbf{c}f'd^3c, \tag{47}$$

we can evaluate $\mathbf{p}'$ in terms of $\boldsymbol{\xi}$ once f' is found from Eq. (46). It is clear that $\mathbf{p}'$ will be a linear function of $\boldsymbol{\xi}$ and its derivatives, which shows that Eq. (38) is an eigenvalue equation for Ω,[2] where

$$\partial\boldsymbol{\xi}/\partial t=\Omega\boldsymbol{\xi}. \tag{48}$$

We wish to solve Eq. (46) to lowest order in η, which will be to first order in reciprocal dimensions of the system. Thus f_0 is assumed to have the form (13), with density gradients in the X_2 direction [except for the X_1 dependence determined by Eq. (I-33)].

Since $\hat{e}_1$ depends upon $\mathbf{r}$ we must evaluate the gradient

[5] See, for instance, S. Chandrasekhar and E. Fermi, Astrophys. J. **118**, 116 (1953).

[6] If Eq. (39) could be replaced by an energy integral, a variational principle would be immediately available. We have not succeeded in obtaining such an integral in the general case.

in Eq. (41) rather carefully. That is,

$$\nabla c_1 = \nabla(\mathbf{c}\cdot\hat{e}_1) = (\nabla\hat{e}_1)\cdot\mathbf{c},$$
$$\nabla c_\perp^2 = -\nabla c_1^2 = -2c_1\nabla(c_1), \tag{49}$$

for instance. The gradient of f_0 is therefore

$$\frac{\partial f_0}{\partial x_1} = \left(\frac{\partial f_0}{\partial x_1}\right)_{c_1} + \frac{c_1 c_2}{R_1}\left[\frac{\partial f_0}{\partial c_1^2} - \frac{\partial f_0}{\partial c_\perp^2}\right],$$
$$\frac{\partial f_0}{\partial x_2} = \left(\frac{\partial f_0}{\partial x_2}\right)_{c_1} + \frac{c_1 c_2}{R_2}\left[\frac{\partial f_0}{\partial c_1^2} - \frac{\partial f_0}{\partial c_\perp^2}\right], \tag{50}$$
$$\frac{\partial f_0}{\partial x_3} = c_1 c_3 \hat{e}_3\cdot\frac{\partial \hat{e}_1}{\partial x_3}\left[\frac{\partial f_0}{\partial c_1^2} - \frac{\partial f_0}{\partial c_\perp^2}\right].$$

Here the symbol $(\partial f_0/\partial x_1)_{c_1}$ means differentiation holding c_1 constant, and R_1, R_2, etc., were defined in Part I. The derivatives of f' can be evaluated in the same manner. It is convenient to first write

$$f' = f'' + f''' + \cdots, \quad f'' = A(c_\perp^2, c_1^2, \mathbf{r}) + c_1 A_2(c_\perp^2, c_1^2, \mathbf{r}). \tag{51}$$

The terms $f''' + \cdots$ will be seen to be of $0(\eta)$ compared to the A_1 and A_2, terms and will thus be neglected in our final result. Inserting Eq. (51) into Eq. (46) and evaluating space derivatives as in Eqs. (50), we obtain [as in Part I, $D^{-1} = -(1/B_0)(\partial B_0/\partial x_1)$]

$$\frac{\partial f''}{\partial t} + \mathbf{c}\cdot\nabla f''\Big|_{c_1} + \frac{c_1 c_\perp^2}{D}\left[\frac{\partial f''}{\partial c_1^2} - \frac{\partial f''}{\partial c_\perp^2}\right]$$
$$+2\left[\frac{\partial f''}{\partial c_1^2} - \frac{\partial f''}{\partial c_\perp^2}\right]\left[\frac{c_1^2 c_2}{R_1} + \frac{c_1}{2}(c_2^2 - c_3^2)\left(\frac{1}{R_2} - \frac{1}{R_3}\right)\right]$$
$$+\mathfrak{D}(f''' + \cdots) = G(\partial\xi/\partial t, \mathbf{c}). \tag{52}$$

To effect an expansion in η, we shall classify the terms in Eq. (52) as "L-type" and "P-type." The P-type terms are those which vanish on averaging over the angles of c_2 and c_3 (that is, averaging over the plane perpendicular to $\mathbf{B}_0$). The L-type are the remaining terms. As an example, f'' in Eq. (51) is an L-type term. We may, for instance, write

$$G = G_1(\partial\xi/\partial t, c_\perp^2, c_1^2) + G_2(\partial\xi/\partial t, c_1, c_2, c_3), \tag{53}$$

where G_1 is L-type and G_2 is P-type. From Eq. (46), we obtain

$$G_1 = -\mathbf{u}\cdot\nabla f_0\Big|_{c_1} + 2c_1^2[\hat{e}_1\cdot(\nabla\mathbf{u})\cdot\hat{e}_1]\frac{\partial f_0}{\partial c_1^2}$$
$$+c_\perp^2[\hat{e}_2\cdot(\nabla\mathbf{u})\cdot\hat{e}_2 + \hat{e}_3\cdot(\nabla\mathbf{u})\cdot\hat{e}_3]\frac{\partial f_0}{\partial c_\perp^2}. \tag{53'}$$

The term G_2 may easily be obtained, also, but we shall not need it.

Recalling that f'' is an L-type term, we set

$$(e/MC)\mathbf{c}\times\mathbf{B}_0\cdot\nabla_c f''' = G_2$$
$$-2\left[\frac{\partial f''}{\partial c_1^2} - \frac{\partial f''}{\partial c_\perp^2}\right]\left[\frac{c_1^2 c_2}{R_1} + \frac{c_1}{2}(c_2^2 - c_3^2)\left(\frac{1}{R_2} - \frac{1}{R_3}\right)\right]$$
$$-c_2\frac{\partial f''}{\partial x_2}\Big|_{c_1} - c_3\frac{\partial f''}{\partial x_3}\Big|_{c_1}. \tag{54}$$

This equation may be solved easily for f''', since the terms on the right-hand side are all P-type terms.[7] But f''' is of $O(1/\omega)$, as is apparent from Eq. (54), so may be neglected elsewhere in Eq. (52). On dropping these terms, Eq. (52) reads

$$\frac{\partial f''}{\partial t} + c_1\frac{\partial f''}{\partial x_1}\Big|_{c_1} + \frac{c_1 c_\perp^2}{D}\left[\frac{\partial f''}{\partial c_1^2} - \frac{\partial f''}{\partial c_\perp^2}\right] = G_1. \tag{55}$$

We next substitute Eq. (51) for f'' into this equation. We obtain some terms which are even in c_1 and some which are odd. Since G_1 is even in c_1, we may equate to zero the sum of all the odd terms. This leads to two coupled equations for A_1 and A_2:

$$\frac{\partial A_2}{\partial t} + \left(\frac{\partial A_1}{\partial x_1}\right)_{c_1} + \frac{c_\perp^2}{D}\left[\frac{\partial A_1}{\partial c_1^2} - \frac{\partial A_1}{\partial c_\perp^2}\right] = 0, \tag{56}$$

and

$$\frac{\partial A_1}{\partial t} + c_1^2\left(\frac{\partial A_2}{\partial x_1}\right)_{c_1} + \frac{c_\perp^2}{D}\left[\frac{1}{2}A_2 + c_1^2\left(\frac{\partial A_2}{\partial c_1^2} - \frac{\partial A_2}{\partial c_\perp^2}\right)\right]$$
$$= G_1\left(\frac{\partial\xi}{\partial t}, c_\perp^2, c_1^2\right). \tag{57}$$

To the required order, the solution to Eq. (14) is now just

$$f' = A_1(c_\perp^2, c_1^2) + c_1 A_2(c_\perp^2, c_1^2). \tag{58}$$

The components of the fluctuating part of the pressure tensor are calculated from

$$p_{11}' = M\int c_1^2 A_1 d^3c, \quad p_{22}' = p_{33}' = \frac{M}{2}\int c_\perp^2 A_1 d^3c, \tag{59}$$

with the off-diagonal elements vanishing. This was already predicted in Eq. (7). The only two components of the heat flow tensor are

$$Q_1 = M\int c_1^4 A_2 d^3c, \quad Q_\perp = \tfrac{1}{2}M\int c_1^2 c_\perp^2 A_2 d^3c. \tag{60}$$

Here we do not have, in general, $\nabla\cdot\mathbf{Q} = 0$ as was true for the static solution of I.

Before discussing some specific examples, a few general remarks are in order. First, having evaluated

[7] It is easily verified by substitution that f''' has the form $f''' = (c_2^2 - c_3^2)\Gamma_{22} + c_2 c_3\Gamma_{23} + c_2\Gamma_2 + c_3\Gamma_3$, where the Γ's are L-type terms. Equation (54) gives a set of *algebraic* equations for the Γ's.

Eqs. (56) and (57), we may substitute the resulting pressure tensor $\mathbf{p}'$ into our differential equation (43) to obtain the equation of motion of $\boldsymbol{\xi}$—or we may use it in the virial integral (44). On the other hand, one may evaluate the heat flow from Eqs. (60) and use the differential equation (5) to obtain $\mathbf{p}'$. This involves solving the extra set of differential equations (5), but has the advantage of giving greater accuracy in general, since $\nabla\cdot\mathbf{Q}$, rather than $\mathbf{Q}$, appears in (5).

The motion is evidently much simpler for situations in which $A_2=0$. In this case all the odd moments of f' vanish, including the heat flux $\mathbf{Q}$. This is just the condition that the compression law for $\mathbf{p}'$ be adiabatic.

For $A_2=0$, Eq. (57) may be integrated to give explicitly

$$A_1=G_1(\xi,c_\perp^2,c_1^2). \tag{61}$$

Equation (56) is then

$$\left.\frac{\partial A_1}{\partial x_1}\right|_{c_1}+\frac{c_\perp^2}{D}\left[\frac{\partial A_1}{\partial c_1^2}-\frac{\partial A_1}{\partial c_\perp^2}\right]=0, \tag{62}$$

which is a condition on the "displacement" $\boldsymbol{\xi}$, when we insert (61) into (62). We recall that Eq. (62) is exactly the same as the equation satisfied by f_0 (see Part I), which was interpreted as the condition for equilibrium "along field lines." The interpretation of Eq. (62) is thus that for $A_2=0$, the gas moves in such a manner that equilibrium is instantaneously maintained along magnetic field lines.

In this case, using Eqs. (59), we may easily evaluate $\mathbf{p}'$:

$$p_{11}'=-\boldsymbol{\xi}\cdot\nabla p_{11}^0-p_{11}^0\left[\nabla\cdot\boldsymbol{\xi}+2\hat{e}_1\cdot\frac{\partial\boldsymbol{\xi}}{\partial x_1}\right]$$
$$p_{22}'=p_{33}'=-\boldsymbol{\xi}\cdot\nabla p_{33}^0-p_{33}^0\left[2\nabla\cdot\boldsymbol{\xi}-\hat{e}_1\cdot\frac{\partial\boldsymbol{\xi}}{\partial x_1}\right]. \tag{63}$$

These values are to be used in Eq. (43) to give a differential equation linear in $\boldsymbol{\xi}$—or in the virial (44), showing that the virial is quadratic in $\boldsymbol{\xi}$. Equations (63) are seen to be identical in form with Eqs. (8).

In general, when Eqs. (56) and (57) must be integrated, it is convenient to change from $c_1^2,c_\perp^2$ to c_1^2,c^2 as variables. Then $[\partial/\partial c_1^2-\partial/\partial c_\perp^2]$ is replaced by $\partial/\partial c_1^2$. Also let us set

$$\partial\boldsymbol{\xi}/\partial t=\Omega\boldsymbol{\xi},\quad \partial A_1/\partial t=\Omega A_1,$$
$$\partial A_2/\partial t=\Omega A_2,\quad A_2\equiv\Omega\Gamma. \tag{64}$$

Then Eqs. (56) and (57) become [with A_1 and Γ functions of c_1^2, c^2, and $\mathbf{r}$]

$$\left.\frac{\partial A_1}{\partial x_1}\right|_{c_1}+\frac{[c^2-c_1^2]}{D}\frac{\partial A_1}{\partial c_1^2}=-\Omega^2\Gamma,$$
$$A_1+c_1^2\left.\frac{\partial\Gamma}{\partial x_1}\right|_{c_1}+\frac{[c^2-c_1^2]}{D}\left[\frac{1}{2}\Gamma+c_1^2\frac{\partial\Gamma}{\partial c_1^2}\right]=G_1(\xi,c_\perp^2,c_1^2). \tag{65}$$

This indicates that motion only in the $\hat{e}_1$ direction is involved in the integration of these equations. The total square velocity, c^2, enters only as a parameter, so these are equations in two independent variables. Upon integrating these, we may evaluate $\mathbf{p}'$ as a linear functional of $\boldsymbol{\xi}$. This, in turn, implies that Eq. (43) is a linear homogeneous equation for $\boldsymbol{\xi}$. We are thus dealing with a system which is holonomic, all quantities expressible in terms of $\boldsymbol{\xi}$.

When the motion is very slow, so Ω is small, the $\Omega^2\Gamma$ term is negligible on the right-hand side of the first of Eqs. (65). This is then equivalent to Eq. (62), which has the obvious interpretation that for very slow motion {i.e., for $\Omega\ll$[thermal velocity]×["length" of plasma]$^{-1}$} the system always moves so as to maintain equilibrium "along field lines."

We must finally say something about our neglect of motion along field lines. When $A_2\neq0$, we have

$$v_1'=\int c_1^2A_2d^3c,$$

since we have set $\xi_1=0$. We may use v_1' to calculate the current j_1' and thus solve Maxwell's equations to give E_1'. From this we may calculate ξ_1 and then re-solve Eq. (65). In this way we can check the consistency of our approximation (E).[8]

Before considering several applications of our theory, we may summarize the method. In a manner analogous to that of Chapman and Enskog, we solve the Boltzmann equations (56) and (57) along with the hydrodynamic equation (43). The variable $\boldsymbol{\xi}$ plays the role of the Lagrangian variable in the conventional hydromagnetic equations for a highly conducting gas, such as

$$\mathbf{E}+\frac{1}{C}\frac{d\boldsymbol{\xi}}{dt}\times\mathbf{B}=0.$$

Of particular interest is the conclusion that when conditions for adiabatic compression obtain, the Boltzmann equation may be solved explicitly, the solution being given by Eq. (61).

III. APPLICATIONS

In this section we consider applications of the theory of the preceding sections to hydromagnetic waves,

[8] Actually, this rather cumbersome method of investigating the $\hat{e}_1$-motion is unnecessary. It will be shown in a subsequent publication that the $\hat{e}_1$-motion may easily be included in the general discussion and that its neglect is justifiable for a well-defined class of problems. A second point to be discussed in this subsequent publications concerns our tacit assumption that B'/B_0 is of order η. This was assumed, because the B'-term in G was of the P-type and thus was included in G_2. Consequently, f''' contains B'/B_0, as is clear from Eq. (54). We discarded this term as being of order η, because it contained B_0^{-1}. For small β, we have [change in magnetic field energy]×[total magnetic field energy]$^{-1}=(B'/B_0)^2\simeq\beta$, justifying our neglect of this term. In the general case a minor modification of our formulas is needed.

electron plasma oscillations, and the Kruskal-Schwarzschild[9] problem of gravitational instability.

Example A. Hydromagnetic Waves

We suppose the plasma to be uniform and $\mathbf{B}_0$ to be constant. We then look for a transverse solution of the form

$$\boldsymbol{\xi}=\mathbf{a}e^{ik_1x_1},\quad a\cdot\hat{e}_1=0, \tag{66}$$

so

$$\nabla\cdot\boldsymbol{\xi}=0. \tag{67}$$

Then $G_1=0$ in Eq. (53), so $\mathbf{p}_T'=0$, as does $\nabla\cdot\mathbf{p}_T^0$. We may then substitute Eq. (66) into Eq. (43) to obtain

$$\frac{\partial^2\boldsymbol{\xi}}{\partial t^2}=-\frac{B_0^2}{4\pi\rho_0}k_1^2\boldsymbol{\xi}.$$

Taking $\partial^2\boldsymbol{\xi}/\partial t^2=\Omega^2\boldsymbol{\xi}$, we obtain the usual dispersion relation for hydromagnetic waves:

$$\Omega^2=-(B_0^2/4\pi\rho_0)k_1^2. \tag{68}$$

Example B. Electron Plasma Oscillations

We take $\mathbf{B}_0=0$,

$$\boldsymbol{\xi}=\hat{e}_1\cdot ae^{ik_1x_1},\quad da/dt=-i\Omega_0 a, \tag{69}$$

and the vector $\hat{e}_1$ to be constant. Then Eq. (4) is (for electrons)

$$-\Omega_0^2\boldsymbol{\xi}=-\frac{e}{M_e}\mathbf{E}'-\frac{\hat{e}_1}{Mn_0}\frac{\partial p_{e,11}'}{\partial x_1}. \tag{70}$$

To calculate $p_{e,11}'$, we shall use the first of Eqs. (9), assuming the compression to be adiabatic (our discussion of the Boltzmann equation requires modification for $B_0=0$). Then

$$p_{e,11}'=-iek_1p_{0,11}\xi. \tag{71}$$

From Eq. (3), we have

$$n_e'=-ikn_0\xi. \tag{72}$$

The charge density is $\epsilon=-en_e'$. Using $\nabla\cdot\mathbf{E}'=\partial E'/\partial x_1=4\pi\epsilon$, we obtain

$$\mathbf{E}'=4\pi n_0e\xi\hat{e}_1. \tag{73}$$

Substituting Eqs. (71) and (73) into Eq. (70), we obtain the Bohm-Gross[10] dispersion relation

$$\Omega_0^2=\frac{4\pi n_0e^2}{M_e}+3k_1^2\frac{p_{0,11}}{M_en_0}. \tag{74}$$

Example C. Gravitational Instability[9]

We consider a plasma with an infinite plane bounding surface, which is the plane $x_2=0$. $\mathbf{B}_0$ is a constant and the plasma density is uniform except at the boundary, which is supposed to be sharp. A constant acceleration $-g\hat{e}_2$ acts in the direction from plasma to vacuum.[11] We shall first discuss the motion using the differential equation (43) and then using the virial (44).

We suppose the variable $\boldsymbol{\xi}$ of Sec. II to have the form

$$\boldsymbol{\xi}=\mathbf{a}e^{i\mathbf{k}\cdot\mathbf{r}}, \tag{75}$$

with

$$\mathbf{k}\cdot\hat{e}_1=0,\quad \mathbf{a}\cdot\hat{e}_1=0,\quad d\mathbf{a}/dt=\Omega\mathbf{a}. \tag{76}$$

We wish to determine the dispersion relation between Ω and the vector $\mathbf{k}$.

The gravitational force causes a current

$$\mathbf{j}_g=\hat{e}_3(\rho C/B)g, \tag{77}$$

to flow in the plasma interior.[12] The infinitesimal change in $\mathbf{j}_g$, that is $\mathbf{j}_g'$, is easily calculated just as was $\mathbf{j}'$ in Eq. (42). $\mathbf{B}'$ is given by Eq. (37). We have also $A_2=0$, $A_1=G_1$. Thus $\mathbf{p}'$ can be evaluated. We may verify from our final solution, however, that only the last term on the right-hand side of Eq. (43) need be kept when $\beta\ll1$, so

$$\rho_0\partial^2\boldsymbol{\xi}/\partial t^2=-(B_0^2/4\pi)\hat{e}_1\times\{\nabla\times[\nabla\times(\boldsymbol{\xi}\times\hat{e}_1)]\}. \tag{78}$$

The electric field $\mathbf{E}'$ is

$$\mathbf{E}'=-(\Omega/C)\boldsymbol{\xi}\times\mathbf{B}_0=-\left(\frac{\Omega B_0}{C}\right)\mathbf{a}\times\hat{e}_1e^{i\mathbf{k}\cdot\mathbf{r}}. \tag{79}$$

For the same reason that $\mathbf{p}'$ will turn out to be small in our final solution, the charge density $e[n_i'-n_e']$ will also turn out to be small (this is easily calculated from our A_{1e} and A_{1i}), so

$$\nabla\cdot\mathbf{E}'=0,$$

in the plasma interior. This implies that

$$k_2a_3=k_3a_2. \tag{80}$$

Using Eqs. (76) and (80) in Eq. (78), we obtain

$$\Omega^2\mathbf{a}=-(B_0^2/4\pi\rho_0)[k_2^2+k_3^2]\mathbf{a}. \tag{81}$$

To obtain a second relation between these quantities, we must satisfy the proper boundary conditions at the plasma surface. The displacement "tilts" the plasma surface, whereas the current $\mathbf{j}_g$ remains horizontal. Thus $\mathbf{j}_g$ will have a component flowing across the plasma boundary causing surface charge density to develop at the rate[13]

$$\epsilon_s=-\mathbf{j}_g\cdot\hat{n},$$

where $\hat{n}$ is the normal to the plasma boundary. The component of $\hat{n}$ in the $\hat{e}_3$-direction is $(\partial/\partial x_3)[\boldsymbol{\xi}\cdot\hat{e}_3]$, so

$$\dot{\epsilon}_s=\Omega\epsilon_s=-ik_3a_2j_g. \tag{82}$$

[9] M. Kruskal and M. Schwarzschild, Proc. Roy. Soc. (London) A223, 348 (1954).

[10] D. Bohm and E. Gross, Phys. Rev. 75, 1851 (1949).

[11] A semi-infinite plasma of uniform density is incompatible with a uniform $\mathbf{B}_0$ in a gravitational field. We may suppose the density to be uniform for a depth greater than other distances encountered, however, as did Kruskal and Schwarzschild (reference 9).

[12] This is calculated by setting $g\hat{e}_2=(e/M_iC)\mathbf{v}_g\times\mathbf{B}$ and $\mathbf{j}_g=en_i\mathbf{v}_g$ for ions. The electrons contribute to $\mathbf{j}_g$ only in the order M_e/M_i.

[13] This method of solving the boundary conditions at the plasma surface was suggested to us by Dr. C. Longmire.

If $\mathbf{E}'(0)$ is the electric field in vacuum, we have the boundary conditions at $x_2=0$:

$$E_2'-\frac{E_2'(0)}{\kappa}=\frac{4\pi}{\kappa}\epsilon_s,\quad E_3'=\frac{E_3'(0)}{\kappa}. \tag{83}$$

Here κ is the "dielectric constant" of Eq. (26). Since κ is assumed to be a large number, we neglect $E_2'(0)$ in Eq. (83). We may now eliminate ϵ_s between Eqs. (82) and (83) and express E_2' in terms of $\mathbf{a}$ using Eq. (79). This, combined with $\nabla\cdot\mathbf{E}'=0$ and Eq. (81) leads to

$$\Omega^2=gk_3, \tag{84}$$

and

$$-ik_2\equiv k\simeq k_3. \tag{85}$$

This is the Kruskal-Schwarzschild instability rate.

To solve the same problem with the virial (44), we note that

$$\mathbf{j}_g'\simeq\hat{e}_3\frac{gC}{B_0}M_i[-\nabla\cdot(\xi n_0)], \tag{86}$$

and that Eqs. (43) and (44) are modified to read

$$\rho_0\frac{\partial^2\xi}{\partial t^2}=-\nabla\cdot\mathbf{p}_T'+\frac{1}{B_0^2}[\mathbf{B}_0\cdot\nabla\times(\xi\times B_0)]\frac{(\mathbf{j}_1^0\times\mathbf{B}_0)}{C} -\frac{1}{4\pi}\mathbf{B}_0\times\{\nabla\times[\nabla\times(\xi\times B_0)]\}t\hat{e}_2g\nabla\cdot(\xi\rho_0), \tag{87}$$

$$\int\rho_0\xi\cdot\frac{\partial^2\xi}{\partial t^2}d\tau=-\int\left\{\frac{1}{4\pi}[\nabla\times(\xi\times\mathbf{B}_0)]^2-\mathbf{j}_1^0\cdot\xi \times[\nabla\times(\xi\times\mathbf{B}_0)]-(\nabla\xi):\mathbf{p}_T'-g\xi_2\nabla\cdot[\xi\rho_0]\right\}d\tau. \tag{88}$$

To make the right-hand side of Eq. (88) as large as possible, we must take $\nabla\times(\xi\times\mathbf{B}_0)\simeq0$. This is true for our solution (84) and (83). Also $\mathbf{p}_T'$ is small when $\beta\ll1$, so Eq. (88) reduces to

$$\int\rho_0\xi\cdot\frac{\partial^2\xi}{\partial t^2}d\tau=\int g\xi_2\nabla\cdot[\xi\rho_0]d\tau\simeq\int g\xi_2^2\rho_0 d\Sigma, \tag{89}$$

where $d\tau=dx_2d\Sigma$, $d\Sigma$ is an element of area on the surface of the plasma, and the contribution to $\int dx_2$ comes from the discontinuity at the boundary. To evaluate the left-hand side, we have $|\xi_2|\simeq|\xi_3|\simeq e^{-kx_2}$. Thus, Eq. (89) becomes

$$\Omega^2\int d\Sigma\frac{\rho_0}{k}\xi_2^2=\int g\xi_2^2\rho_0 d\Sigma. \tag{90}$$

Because of the uniformity of the plasma, we have

$$\Omega^2=gk, \tag{91}$$

which is equivalent to Eq. (84).

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Thermoelectric Powers in Palladium-Silver and Palladium-Rhodium Alloys*

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Thermoelectric powers have been measured in the temperature range 77°K to 273°K in the alloys of palladium with silver. The absolute thermoelectric power is found to be highly sensitive to the presence of unoccupied d-band states, and a marked variation with composition is also found in alloys with more than 90% of either element. This latter behavior is ascribed to a departure of the Fermi surface from an accurately spherical form as the s-electron concentration increases above 0.9 per atom in the silver-rich alloys, and to a contribution to conduction from d-band holes in the palladium-rich alloys. The values of thermoelectric power obtained for the pure metals by extrapolation from the results for alloys where these effects are absent are in good agreement with estimates made on the basis of simple theoretical models. Results of measurements on some palladium-rich palladium-rhodium alloys are also presented and discussed briefly.

INTRODUCTION

In the metallic state palladium has about 0.6 unoccupied states per atom in the band derived from the $4d$ levels of the free atoms.[1] In its alloys with silver, which form a continuous series of solid solutions, these empty states or d-band holes are gradually filled, pure silver having a full d-band and one electron per atom in the $5s$-band. The magnetic properties of the alloys give a clear indication of the general character of the change in electronic structure produced in traversing the system, and the most recent susceptibility measurements[2] suggest that d-band holes are present only when the silver content is less than about 60%.[3] The present

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[1] E. P. Wohlfarth, Proc. Leeds Phil. Lit. Soc., Sci. Sect. 5, 89 (1948).

[2] Hoare, Matthews, and Walling, Proc. Roy. Soc. (London) A216, 502 (1953).

[3] Concentrations are expressed in atomic percentages throughout.