Problems in Magneto-Fluid Dynamics

LYMAN SPITZER, JR.

Project Matterhorn, Princeton University, Princeton, New Jersey

SINCE Alfvén's classic work[1] a vast amount of research has been carried out on the properties of a conducting liquid or gas, moving in a magnetic field. This subject, variously known as magnetohydrodynamics, hydromagnetics, magneto-gas dynamics, etc., originated as a branch of astrophysics, but has since received great impetus from its application in controlled fusion and rocket propulsion. The present survey outlines the various major problems in this active field, which we call by its most general designation, magneto-fluid dynamics, and provides a certain background for the more detailed technical papers presented at the IUTAM Symposium.

In the past, there has been some tendency to restrict magneto-fluid dynamics to those problems which could be treated within the framework of a continuum theory, with the term "plasma physics" used to denote those more intricate problems in which the individual particles must be considered. Since the IUTAM Symposium contains a substantial number of papers on shocks and related problems in plasma physics, we consider that magneto-fluid dynamics includes all problems that may arise when a real gas or liquid is interacting with a magnetic field.

In the organization of the subject adopted here, the different problems are grouped together according to the magnitude of the mean macroscopic velocity **v**, defined as the mean velocity of all the particles which are within a small volume element at a particular time. After a first section, in which the basic processes of this field are surveyed, a second section deals with the situation in which **v** vanishes, and a magnetostatic equilibrium is present. In a subsequent section the problems of infinitesimal **v** are surveyed, including both waves and instabilities. Next, problems of finite **v** are considered, first those in which the system is in a steady state, and next the more general situation in which time derivatives are not zero. A final section includes a brief discussion of the most important magneto-fluids found in nature and in the laboratory.

References are given to one or more basic papers in each major area discussed. The references chosen are for the most part those readily available, and no attempt has been made to list all the more important works. As a result, the references do not adequately represent the extensive work in this new field by foreign scientists, particularly in the Soviet Union.

[1] H. Alfvén, *Cosmical Electrodynamics* (Clarendon Press, Oxford, England, 1950).

BASIC PROCESSES

In the most general magneto-fluid, three types of basic processes may be considered: (1) trajectory of an individual particle in a given electric and magnetic field; (2) interaction of a particle with radiation; (3) encounters of a particle with another particle. We consider each of these three processes in turn. Only charged particles are considered. The trajectory of a neutral particle is trival; interactions of neutral particles with radiation and with each other may be important in many situations, but the basic analysis of such encounters is not properly a part of magneto-fluid dynamics.

Since the motion of a charged particle in electric and magnetic fields has been studied for many years, it is perhaps surprising that new general results in this subject can still be obtained. The recent study of adiabatic invariants, however, has yielded information of substantial value. The most important such invariant is the magnetic moment μ of a gyrating electron. It is well known[2] that μ is constant to first order in the magnetic field derivatives. It has now been demonstrated[3] that in this asymptotic expansion μ is constant to all orders of the magnetic field derivatives, although the definition of μ must be somewhat modified in each successive order. These results have been generalized[4] to other adiabatic invariants. The existence of these invariants is a powerful tool in the analysis of many situations. The exact way in which these invariants change with changing conditions is a problem of considerable interest.[5]

The interaction of a free electron with the radiation field is, in principle, simple. The rate of radiation by a gyrating electron has been considered recently[6]; the absorption coefficient can be determined from Kirchhoff's law, provided the electron velocity distribution is Maxwellian. The problem becomes complicated if the propagation of the emitted radiation is substantially modified by the presence of a plasma.

Collisions of electrons and ions with other particles have been studied in some detail. Collisions with walls and with neutral atoms may, in fact, have a dominating

[2] L. Spitzer, Jr., *Physics of Fully Ionized Gases* (Interscience Publishers, Inc., New York, 1956).

[3] M. D. Kruskal, *Rendiconti del Terzo Congresso Internazionale sui Fenomeni d'Ionizzazione nei Gas* (Societa Italiani di Fisica, Milan, 1957); U. S. Atomic Energy Commission Rept. NYO-7903 (PM-S-33, 1958).

[4] C. S. Gardner, Phys. Rev. **115**, 791 (1959).

[5] F. Hertweck and A. Schlüter, Z. Naturforsch. **12a**, 844 (1957); D. Northrop and E. Teller, Phys. Rev. **117**, 215 (1960).

[6] B. A. Trubnikov and V. S. Kudryavtsev, in *Proceedings of the Second United Nations International Conference on the Peaceful Uses of Atomic Energy* (United Nations, New York, 1958), Vol. 31, p. 93.

influence in many laboratory experiments, but are ignored here. Coulomb collisions between charged particles alter the velocity distribution functions, in accordance with the Fokker-Planck equation.[7] Analysis of such collisions yields values for the electric resistivity and other transport coefficients.[8] While the essential features are clear, many uncertainties of detail regarding such collisions remain to be explored. A chief problem is associated with the long-range nature of Coulomb forces. With these forces it is uncertain, in principle, how to distinguish between a collision among several electrons and an interaction between an electron and a plasma wave. The Debye shielding distance h provides a rough criterion; electric fields produced by particles within a distance h away from a particular ion produce an effect describable through the Fokker-Planck equation (provided that the relatively very close encounters, producing large deflections, may be ignored), while the effect of particles further away than h may be described in terms of cooperative phenomena such as plasma waves.[9] In principle, so sharp a dividing line is a fiction, albeit a useful one. Moreover, the shielding of a charge varies with the velocity of the charge and also with the magnetic field. Evidently there are many intricate effects still to be considered in connection with collisions among charged particles. Fortunately, these additional effects are probably minor for the bulk of particles in a plasma.

The mean free path λ of a particle in a plasma has a particular significance in magneto-fluid dynamics. If this quantity, which we need not define precisely, is much shorter than the "characteristic dimension of the fluid," defined as the distance over which macroscopic quantities change significantly (including, for example, the wavelengths of any disturbances being considered), then the familiar continuum theory is applicable. In detail, the velocity distribution function is very nearly isotropic and Maxwellian, and the stress tensor can be described by a scalar pressure. If, on the other hand, λ is much longer than the characteristic dimension of the fluid, then the velocity distribution function f may be very complex. In this situation the Boltzmann equation must be used to determine f, and the mathematical problem, like the physical situation, becomes very complicated. We refer to this type of description as the "particle picture," which must be used when the continuum picture becomes invalid.

One great simplification occurs if a magnetic field is present in sufficient strength so that the radius of gyration a is less than the characteristic dimension of the plasma, and if also all temporal changes are negligible during a period of gyration around a line of force. In this situation particles cannot rapidly cross the lines of force, and the velocity distribution function at a point depends only on conditions along the particular tube of force passing through the point. Under the additional special assumption that conditions are uniform along each line of force, the particle picture is almost identical with the continuum picture (except for a change of γ, the specific heats).

VELOCITY ZERO

When the macroscopic velocity $\mathbf{v}$ is zero at all times, the equations of magneto-fluid dynamics undergo a great simplification. If λ is small, conditions are uniform along a line of force, the stress tensor is nearly isotropic, and a single equation of magnetostatic equilibrium results. Under some situations (as, for example, in the toroidal "pinched" discharge) this same result obtains even if λ is very large, provided that a is small. A very large variety of solutions[10] to this equation are possible under different conditions. For λ large, some collisions are required to yield the uniform conditions assumed, and the resistivity associated with such collisions ultimately reduces to zero any current in the fluid. Thus, the solutions of the magnetostatic equation which possess volume currents must be regarded as quasi-equilibrium solutions.

Similarly, confinement of a gas in a magnetic-mirror geometry, where conditions are not uniform along a line of force, is again possible only in a quasi-equilibrium. In this case, the quasi-equilibrium is destroyed not by the resistivity but by the more rapid deflection of the individual particles into the escape cone of the mirror.

A problem of equilibrium which is relatively unexplored is that in which both a and λ are comparable with or greater than the characteristic dimension of the fluid. Solutions have been obtained[11] in the simplest two-dimensional systems, when the trajectory of a particle is fully described by the energy and generalized momentum in one direction. More general situations remain to be treated.

VELOCITY INFINITESIMAL

Given any equilibrium situation, the investigation of infinitesimal disturbances is, in principle, straightforward. Since the analysis is linear, the solutions may be divided into two groups—waves in which the disturbances oscillate, without change of amplitude, and disturbances whose amplitude can grow exponentially, with or without oscillations, and which we call "instabilities."

Analyses of possible waves have mostly been carried through only for infinite uniform plasmas. Moreover, the analysis is normally simplified by use of the con-

[7] S. Chandrasekhar, Revs. Modern Phys. **15**, 1 (1943); M. N. Rosenbluth, W. M. MacDonald, and D. L. Judd, Phys. Rev. **107**, 1 (1959).

[8] L. Spitzer, Jr., and R. Härm, Phys. Rev. **89**, 977 (1953); M. N. Rosenbluth and A. N. Kaufman, Phys. Rev. **109**, 1 (1958).

[9] D. Pines and D. Bohm, Phys. Rev. **85**, 338 (1952).

[10] H. Grad and H. Rubin, in *Proceedings of the Second United Nations International Conference on the Peaceful Uses of Atomic Energy* (United Nations, New York, 1958), Vol. 31, p. 190; M. D. Kruskal and R. M. Kulsrud, Phys. Fluids **1**, 265 (1958); R. Kippenhahn, Z. Naturforsch. **13a**, 260 (1958).

[11] E. G. Harris, Naval Research Lab. Rept. No. 4944 (1957).

tinuum picure; for a sufficiently low gas temperature the continuum treatment and the particle treatment become identical. It is well known[2] that there are three basically simple types of modes; electromagnetic, hydromagnetic (or magnetohydrodynamic), and electrostatic. Different types of each mode can occur; thus an incompressible hydromagnetic transverse oscillation of low frequency is called an Alfvén wave, and its velocity, $B/(4\pi\rho)^{\frac{1}{2}}$, the Alfvén velocity; the compressional wave traveling perpendicular to the magnetic field **B** is called a magnetosonic wave. At higher frequencies, hydromagnetic waves can show electrostatic characteristics as, for example, ion-cyclotron resonance waves.[12] A general analysis of infinitesimal waves in a uniform magneto-fluid, in the absence of collisions, has been carried out[13] by means of Laplace transforms. A detailed inventory of all such waves has also appeared.[14]

There are two outstanding areas for investigation in this field. The first of these is the effect of finite pressure. In the limit of small mean free path the analysis is not difficult, and existing treatments[15] appear adequate. When the mean free path is very long, however, the existence of particles moving at about the wave velocity complicates the analysis. In electrostatic waves such particles give rise to Landau damping,[16] whose significance and probable importance in actual oscillations is still not entirely clear. Similar effects for hydromagnetic waves have also been explored.[17] Extension of the particle picture for all waves is an important problem. If the velocity distribution in the unperturbed state is arbitrary, the range of possibilities is very great, but even weak collisions tend to establish a Maxwellian velocity distribution, limiting the problem substantially.

A second area is the problem of wave disturbances in bounded nonuniform media. In the presence of a density gradient there exists coupling between the various simple modes described in the foregoing, and the conversion of electrostatic oscillations into electromagnetic waves at a density discontinuity has been computed.[18] However, there is not as yet any general theory of the normal modes of oscillation of a hot gas confined by a magnetic field.

Instabilities in a uniform infinite medium have also been investigated. These instabilities are all due to departures from the Maxwellian velocity distribution, and cannot be obtained from a continuum picture. Electrostatic instabilities can arise either if the electron velocity distribution is somewhat shifted with respect to the positive ions or if the electron velocity distribution has at least two maxima, with sufficient separation between them. While the theory of such instabilities has been known for some time,[19] it is not yet known in detail when such instability can occur or what the frequency of the fastest growing instability will be under different conditions. In addition, hydromagnetic instabilities can arise if the mean-square velocity dispersion perpendicular to the lines of force differs from the parallel value.[20]

Perhaps the most thoroughly explored subject in this particular area is the hydromagnetic instability of a plasma confined by a magnetic field. Since this problem is of obvious importance in the controlled fusion program, much effort has been devoted to it. In an exhaustive treatment based on the continuum picture, methods have been developed[21] for determining if the energy change can be negative for any arbitrary deformation. Corresponding analyses,[22] based on the particle picture, have been developed based on the condition that the radius of gyration is very small. If the stress tensor is isotropic in the unperturbed state, then a plasma which is stable on the continuum picture is also stable on the particle picture.

The outstanding general problem remaining in the analysis of hydromagnetic instabilities is the effect of finite radius of gyration a. Since some of the instabilities predicted by the present theories correspond to displacements changing sharply over a distance equal to a, the effect of finite a could be an important one. The problem is difficult and no published results along this line have as yet appeared.

VELOCITY FINITE BUT STEADY

When the macroscopic velocity is finite substantial progress can still be made if in some reference frame the flow is steady, with all time derivatives vanishing. The two main areas in which this type of flow has been investigated so far are the motion of gas around a solid object, such as an airfoil, and the propagation of a finite amplitude wave through an infinite medium.

Motion of gas around an airfoil has been investigated[23] on the familiar aerodynamic assumption of compressible but linearized flow. On this picture the transverse velocity imparted to the flow by the obstacle is small, and can be treated by the linearized theories referred to in the preceding section. Analysis has shown that to satisfy

[12] T. H. Stix, Phys. Rev. **106**, 1146 (1957).
[13] I. B. Bernstein, Phys. Rev. **109**, 10 (1958).
[14] W. P. Allis, Mass. Inst. Technol. Research Lab. of Electronics, Quart. Rept. No. 54, p. 5 (1959).
[15] A. Banos, Jr., Phys. Rev. **97**, 1435 (1955); Proc. Roy. Soc. (London) **A233**, 350 (1955); J. Bazer and O. Fleischman, Phys. Fluids **2**, 366 (1959).
[16] L. Landau, J. Phys. U. S. S. R. **10**, 25 (1946); N. G. van Kampen, Physica **21**, 949 (1955); **23**, 641 (1957).
[17] B. N. Gershman, Zhur. Eksptl. i Teoret. Fiz. **24**, 453 (1953).
[18] G. B. Field, Astrophys. J. **124**, 555 (1956).
[19] J. R. Pierce, J. Appl. Phys. **15**, 721 (1944); **19**, 231 (1948); D. Bohm and E. P. Gross, Phys. Rev. **75**, 1864 (1949).
[20] H. K. Sen, Phys. Rev. **88**, 816 (1952); R. S. Sagdeyev, B. B. Kadomtsev, L. I. Rudakov, and A. A. Vedyanov, in *Proceedings of the Second United Nations International Conference on the Peaceful Uses of Atomic Energy* (United Nations, New York, 1958), Vol. 31, p. 151; E. G. Harris, Phys. Rev. Letters **2**, 34 (1959).
[21] I. B. Bernstein, E. A. Frieman, M. D. Kruskal, and R. M. Kulsrud, Proc. Roy Soc. (London) **A244**, 17 (1958); K. Hain, R. Lüst, and A. Schlüter, Z. Naturforsch. **12a**, 833 (1957).
[22] M. D. Kruskal and C. R. Oberman, Phys. Fluids **1**, 275 (1958); M. N. Rosenbluth and N. Rostoker, *ibid.* **2**, 23 (1959).
[23] W. R. Sears and E. L. Resler, Jr., J. Fluid Mech. **5**, 257 (1959); J. E McCune and E. L. Resler, Jr., J. Aero/Space Sci. **27**, 493 (1960).

the boundary conditions at the obstacle requires hydromagnetic waves of various types radiating out. The velocity pattern depends qualitatively on the relationship between the fluid velocity relative to the obstacle and the group velocity of these various waves. Under some conditions "wakes" appear upstream, with undisturbed flow downstream. The physical significance of these solutions still remains to be explored.

The propagation of finite amplitude disturbances has been investigated both for purely electrostatic waves and for hydromagnetic disturbances. In each case, the disturbance can be regarded as a standing wave, with the fluid moving by. The solution[24] for electrostatic waves takes into account the dispersion of particle velocities, and shows the importance of trapped particles moving along with the wave. The analysis is of particular interest in that the solutions show no Landau damping; the significance of this result to electrostatic oscillations generally is not clear. The analyses for hydromagnetic waves, both of magnetosonic[25] and of Alfvén[26] type, assume that the gas is cold. Extension of these analyses to include the dispersion of particle velocities constitutes an important problem.

A significant subject for investigation is the general stability of a magneto-fluid in a state of steady flow. It has been demonstrated[27] that any solution in which $\mathbf{v}$ is parallel to $\mathbf{B}$ and equals the local Alfvén velocity is hydromagnetically stable. Conditions for convective instability of rotating magneto-fluids, with thermal gradients, have also been obtained.[28] However, only limited results are available for other situations; in some cases, confined plasmas appear stabilized[29] by addition of fluid velocities. The stability of the finite amplitude waves discussed before would also be an interesting problem.

VELOCITY FINITE AND NONSTEADY

When the velocity of a magneto-fluid is finite and varies with time the analysis can become extremely complicated. This situation is of very great scientific interest, however, since chaotically variable velocities and magnetic fields play the same important role in magneto-fluid dynamics as ordinary turbulence does in conventional fluid dynamics.

One problem in this area is capable of relatively precise analysis, the breaking of finite electron electrostatic oscillations. The exact solution for standing waves of finite amplitude in a cold gas is valid only up to a certain limiting amplitude. For greater amplitudes the trajectories of different electrons cross, and the motion becomes chaotic. The course of this development has been followed[30] by means of machine calculations, showing the decay of wave energy into disturbances of shorter wavelength. The development of a general theory for this decay under wider conditions is a primary goal of magneto-fluid dynamics.

An important type of motion which presumably involves a similar process is a hydromagnetic shock. If the mean free path λ is much less than the radius of gyration a the shock is of ordinary type, with collisions producing the increase of entropy required across the shock front. If a is much less than λ, however, the thickness of a shock traveling across the field is observed[31] to be comparable with a, and some irreversible process other than collisions must be operating. This process has not yet been identified, but it might well be the crossing of trajectories observed in the one-dimensional waves. Quite apart from the detailed structure, a knowledge of the physical mechanisms which are important in a hydromagnetic shock constitutes a major objective of magneto-fluid dynamics.

A general understanding of turbulence in magneto-fluids is probably the most difficult aspect of the entire subject. Turbulence in ordinary fluids is still very imperfectly understood, and the addition of electromagnetic effects complicates the problem substantially. If the conductivity is assumed very high, two different limiting cases may be distinguished. In the first, the magnetic energy is very small compared with the kinetic energy. Here the magnetic lines will be carried along with the fluid, and the small-scale magnetic field amplified[32,33] as the lines of force are stretched and come closer together. The amplification may be limited either by finite conductivity or by increase of the magnetic stresses until they oppose the small-scale turbulent motions responsible for the amplification. This latter case is frequently referred to as a state of equipartition between magnetic and kinetic energy, but since only the disturbances of relatively short scales are apparently involved it is not clear how great a root-mean-square magnetic field may be expected. Further work on this problem is evidently required, and may help to clarify how the magnetic fields of the earth, sun, and stars can be maintained[33,34] by fluid motions.

The second limiting case, again for high conductivity, is that in which the magnetic energy density much exceeds the material pressure. In this situation the lines of force are essentially unaffected by the fluid, except perhaps for small oscillations. The diffusion across the lines of force which has apparently been observed[35] in this situation presumably results from a type of turbu-

[24] I. B. Bernstein, J. M. Greene, and M. D. Kruskal, Phys. Rev. **108**, 546 (1957).
[25] L. Davis, R. Lüst, and A. Schlüter, Z. Naturforsch. **13a**, 916 (1958).
[26] D. Montgomery, Phys. Fluids **2**, 585 (1959).
[27] S. Chandrasekhar, Proc. Natl. Acad. Sci. U. S. **42**, 273 (1956).
[28] S. Chandrasekhar, Proc. Roy. Soc. (London) **A225**, 173 (1954); **A237**, 476 (1956).
[29] S. K. Trehan, Astrophys. J. **129**, 475 (1959).
[30] O. Bunemann, Phys. Rev. **115**, 503 (1959).
[31] R. M. Patrick, Phys. Fluids **2**, 589 (1959).
[32] G. K. Batchelor, Proc. Roy. Soc. (London) **A201**, 405 (1950); A. Schlüter and L. Biermann, Z. Naturforsch. **5a**, 237 (1950).
[33] T. G. Cowling, *Magnetohydrodynamics* (Interscience Publishers, Inc., New York, 1957).
[34] W. M. Elsasser, Revs. Modern Phys. **28**, 135 (1956).
[35] R. A. Ellis, L. P. Goldberg, and J. G. Gorman, Phys. Fluids **3**, 468 (1960).

lence not previously encountered, and still to be analyzed.

OBSERVED MAGNETO-FLUIDS

The analytical approach adopted in the previous sections omits an important branch of magneto-fluid dynamics, the study of particular examples of magneto-fluids observed in nature and in the laboratory. We touch very briefly here on some of the topics included in this study.

Since most of the universe is a plasma, and since magnetic fields are believed to exist almost everywhere, magneto-fluid dynamics is an important part of astronomy.[36] The study of magnetic storms[37] is one of the first problems to be investigated involving the motion of an ionized gas in a magnetic field. Apparently these storms arise when a cloud of gas ejected from the sun sweeps by the earth. The particles which form the auroras probably result from the same process on a smaller scale. The magnetic activity on the sun, and in particular the appearance and disappearance of the great magnetized areas called sunspots, is also a major subject of magneto-fluid research.

Looking further afield, the motions of gas clouds in interstellar space[38] are presumably much affected by the magnetic field that is believed to be present. The origin and collisions of these clouds, and their condensation to form groups of new stars, surely involves magnetic effects. The confinement of cosmic-ray particles, and possibly also their acceleration to the observed high energies, may be attributed to a galactic magnetic field.

The interior of the earth is also a conducting field, and hence magneto-fluid dynamics is also of importance in geophysics. The problem of maintaining the earth's magnetic field, by means of fluid motions in the core (the "self-sustaining dynamo" problem), has already been referred to in the foregoing.

Finally, the investigation of the plasmas in various thermonuclear devices[39] is of independent interest. In some situations, when certain phenomena assume a dominant importance, these devices can be used to provide an important observational check on the theory. Thus, the theory of hydromagnetic instability has received some confirmation of the major effects predicted. More generally, an analysis of these devices, as part of a broad research program in magneto-fluid dynamics, should increase our understanding of this new and challenging field.

[36] *Electromagnetic Phenomena in Cosmical Physics*, I. A. U. Symposium No. 6 (Cambridge University Press, New York, 1958).

[37] S. Chapman and V. C. A. Ferraro, Terrestrial Magnetism and Atmospheric Elec. **37**, 147 (1932); A. J. Dessler and E. N. Parker, J. Geophys. Research **64**, 2239 (1959).

[38] "Proceedings of the Third Symposium on Cosmical Gas Dynamics, I. A. U. Symposium No. 8," Revs. Modern Phys. **30**, 905 (1958).

[39] *Proceedings of the Second United Nations International Conference on the Peaceful Uses of Atomic Energy*, "Controlled fusion devices" (United Nations, New York, 1958), Vol. 32.

DISCUSSION

Session Reporter: G. Kuerti

V. C. A. Ferraro, *Queens College, University of London, London, England:* I should like to hear a comment on the fact that results derived from particle trajectories, if applied to plasma motions in a magnetic field, can lead to erroneous conclusions. In particular, in an inhomogeneous magnetic field ions and electrons acquire inhomogeneity drifts in opposite senses. It is sometimes stated simply that the plasma thereby develops an electric current, without considering the circumstances involved. Clearly, self-induction always tends to inhibit the growth of such a current.

L. Spitzer, Jr.: I have not solved this rather intricate problem; however, I should like to point out that a gradient in magnetic field does not generally produce a current in a confined plasma. Density gradients may produce an electric current, but not gradients in magnetic field strength. A detailed solution of the magnetic-storm problem will certainly take such effects into account.

J. M. Burgers, *University of Maryland, College Park, Maryland:* When you investigate the motion of a single particle, do you assume curl **B** to vanish or not! If curl $\mathbf{B}\neq 0$, electric currents, that is, streams of charged particles, must be in the field, hence the field has a large number of singular points. Usually one averages over these singularities and assumes a more or less smooth distribution of curl **B**. Can this always be done, or may difficulties arise?

L. Spitzer, Jr.: The analyses of free particle trajectories, to which I referred, do not assume an irrotational **B**, and the adiabatic invariants are constant whether or not curl **B** vanishes. Generally, however, one must consider fields with a nonvanishing curl; for example, a confined plasma is characterized by "diamagnetic currents" which weaken the impressed magnetic field. In a self-consistent solution one is interested in examining the trajectories of an individual electron in this confined gas.