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Double Scattering of Electrons with Magnetic Interaction. II. Depolarization*

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The depolarization occurring in attempts to measure the electron's gyromagnetic ratio by double-scattering experiments in a magnetic field is discussed. Formulas are found for the effects expected to be most significant. These are applied to estimate the possible accuracy obtainable. One part in 10^5 seems to be quite feasible. In addition, formulas are appended which permit rapid estimation of other depolarization effects.

I. INTRODUCTION

IN a previous paper[1] the change in asymmetry produced by a magnetic field acting between the two targets in a double scattering of electrons was discussed. The magnetic field was idealized as being perfectly homogeneous. It is, however, necessary to consider the effects of small inhomogenities in the constant magnetic field. These will tend to depolarize the beam of electrons coming from the first target and thus decrease the asymmetry to be found after a second scattering. If all particles take the same path, variations in the magnetic field will not cause depolarization. The polarization vector will merely be rotated. Depolarization occurs only when particles take different paths through the field.

In the following, the state of polarization will be discussed in terms of the field and orbit parameters. For simplicity we restrict ourselves to the case where the constant magnetic field is oriented along the plane determined by the incident beam and first scattered beam but perpendicular to the direction of the first scattered beam. Thus, the beam executes cyclotron-like motion between the two targets. This corresponds to case (2) discussed in I. The other case in I corresponds to the experiment of Louisell, Pidd, and Crane[2]; for those experimental conditions the time the particles spend in the field is short[3] so that depolarization is negligible.

In the situation to be considered there is a magnetic field that is essentially constant in the z-direction. The small radial decrease of the field is described by the index $n(0<n<1)$. In such a field a charged particle executes small oscillations about an equilibrium cyclotron orbit in the plane perpendicular to the z-direction. The frequencies of the radial and axial oscillations about the equilibrium orbit are[4]

$$\omega_r=\omega_L(1-n)^{\frac{1}{2}}, \tag{1}$$

and

$$\omega_a=\omega_L n^{\frac{1}{2}}, \tag{2}$$

where ω_r, ω_a are the radial and axial frequencies respectively. ω_L is the Larmor frequency

$$\omega_L=e\mathcal{H}_0(1-\beta^2)^{\frac{1}{2}}/(m_0c). \tag{3}$$

($\mathcal{H}_0$ is the magnitude of the field along the orbit axis.)

II. DEPOLARIZATION

In considering the depolarization of the beam due to small oscillations about the equilibrium orbit, the particles will be considered as Dirac particles and the effects of the anomalous moment will be neglected. The depolarization effects of the radial oscillations and oscillations perpendicular to the plane of the equilibrium orbit will be considered separately. It is demonstrated that the depolarization arising from the radial oscillations is small and independent of the number of revolutions executed by the particle. However, the depolarization due to the oscillations perpendicular to the plane of

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[1] H. Mendlowitz and K. M. Case, Phys. Rev. 97, 33 (1955) (hereafter referred to as I).

[2] Louisell, Pidd, and Crane, Phys. Rev. 94, 7 (1954).

[3] See discussion Eq. (30), reference 1.

[4] D. W. Kerst and R. Serber, Phys. Rev. 60, 53 (1941); O. Klemperer, *Electron Optics* (Cambridge University Press, Cambridge, 1953); H. Mendlowitz, thesis, University of Michigan, 1954 (unpublished).

the orbit is not negligible, and places an upper limit on the number of revolutions between the two targets.

A. Radial Oscillation

We first consider the effect of the small radial oscillations. It has been shown,[1,5] that in the Foldy-Wouthuysen representation, the equation of motion for the spin wave function for a Dirac particle is given by

$$i\hbar\dot{\Phi}=-\mu\boldsymbol{\sigma}\cdot\boldsymbol{\mathcal{H}}\Phi, \tag{4}$$

where $\mu=(e\hbar/2m_0c)(1-\beta^2)^{\frac{1}{2}}=(e\hbar/2mc)$. This equation describes the positive energy solutions in the case where the anomalous moment has been neglected.

It is found that the field that the particle sees when executing radial oscillations in the plane of the equilibrium orbit (xy-plane) is

$$\boldsymbol{\mathcal{H}}=\mathcal{H}_z=\mathcal{H}_0+\mathcal{H}_r\sin\omega_r t, \tag{5}$$

and

$$\frac{\mathcal{H}_r}{\mathcal{H}_0}=-\frac{v_r(i)}{v_\theta(i)}\cdot\frac{n}{(1-n)^{\frac{1}{2}}}, \tag{6}$$

where $v_r(i)$ is the initial radial velocity and $v_\theta(i)$ is the initial orbital velocity.

Now let

$$\Phi(t)=U(t)\Phi_0, \tag{7}$$

where U is a unitary transformation matrix operating on the spin components of Φ, and Φ_0 is the spin wave function at time $t=0$. Then

$$U(t)=Q_P(t)\exp\left[\frac{i\omega_L}{\omega_r}\sigma_z\frac{\mathcal{H}_r}{\mathcal{H}_0}\sin^2\omega_r t\right]=Q_P(t)W(t), \tag{8}$$

where

$$Q_P(t)=\exp[i\omega_L\sigma_z t/2]. \tag{9}$$

The transformation matrix $U(t)$ for any given particle in the beam depends upon the amplitude of the oscillating field $\mathcal{H}_r$. The relavant parameter being $\mathcal{H}_r/\mathcal{H}_0$ which is seen to be a function of the initial orbit parameters as given in (6).

If the initial spin density matrix is given by

$$\rho^{(s)}=\tfrac{1}{2}[\mathbf{1}+\mathbf{a}\cdot\boldsymbol{\sigma}], \tag{10}$$

where $\mathbf{1}$ is a two-by-two unit matrix and $\boldsymbol{\sigma}$ is the Pauli spin vector, the degree of polarization can be defined as[6]

$$P=[\nu/(\nu-1)][\mathrm{trace}(\rho^{(s)})^2-1/\nu]=|a|^2. \tag{11}$$

ν being the number of accessible polarization states for the system. The degree of depolarization is then defined as

$$\mathrm{D.P.}=\frac{P(\mathrm{initial})-P(\mathrm{final})}{P(\mathrm{initial})}. \tag{12}$$

Since the Hamiltonian is Hermitian, $U(t)$ is unitary. Therefore, if all the particles travelled in the same orbit, they would obey the same Hamiltonian, and so

$$\rho^{(s)}(t)=U(t)\rho^{(s)}(0)U^+(t), \tag{13}$$

where $\rho(t)$ is the density matrix at time t and $\rho(0)$ is the density matrix at $t=0$. From this follows that

$$\mathrm{trace}[\rho^{(s)}(t)]^2=\mathrm{trace}[\rho^{(s)}(0)]^2, \tag{14}$$

and so the final degree of polarization is the same as the initial polarization. The depolarization vanishes. We see that even though the spin states of the individual particles precess about the field, the beam keeps the same polarization.

The case of interest is the one where the particles take different paths. This is caused by variations in the initial velocities. In this case, the initial density matrix does not describe a pure momentum state. We assume that we can average over the momentum states. We obtain from Eq. (8)

$$\begin{aligned}\rho^{(s)}(t)&=\langle U(t)\rho^{(s)}(0)U^+(t)\rangle\\&=Q_P(t)\langle W(t)\rho^{(s)}(0)W^+(t)\rangle Q_P{}^+(t),\end{aligned} \tag{15}$$

where the angular braces denote an average over the momentum states of the ensemble. Although the particles of the beam spend different times in the magnetic field (depending upon the amplitude of the small oscillation about the equilibrium orbit) the experiment is designed so that the number of revolutions is the same for all the particles of the beam. Therefore $\omega_L t$ is the same for each particle of the beam. $Q_P(t)$ is then the same for each electron in the ensemble. Therefore

$$\underset{\mathrm{spin}}{\mathrm{trace}}[\rho^{(s)}(t)]^2=\underset{\mathrm{spin}}{\mathrm{trace}}[\langle W\rho^{(s)}(0)W^+\rangle\langle W\rho^{(s)}(0)W^+\rangle]. \tag{16}$$

Let $W(t)$ be expressed as

$$W(t)=\exp\left[i\frac{\omega_L}{\omega_r}\sigma_z\frac{\mathcal{H}_r}{\mathcal{H}_0}\sin^2\omega_r t\right]=\exp[(i/2)\gamma\boldsymbol{\sigma}\cdot\mathbf{h}], \tag{17}$$

where $\mathbf{h}$ is the unit vector in the direction of $\mathcal{H}_0$, the main component of the field. Then

$$\begin{aligned}\langle W\rho^{(s)}(0)W^+\rangle=\tfrac{1}{2}\{&\mathbf{1}+\boldsymbol{\sigma}\cdot\mathbf{a}\langle\cos\gamma\rangle+\boldsymbol{\sigma}\cdot\mathbf{h}(\mathbf{h}\cdot\mathbf{a})\\&\times[1-\langle\cos\gamma\rangle]+\boldsymbol{\sigma}\cdot(\mathbf{h}\times\mathbf{a})\langle\sin\gamma\rangle\},\end{aligned} \tag{18}$$

and

$$\begin{aligned}\underset{\mathrm{spin}}{\mathrm{trace}}[\rho^{(s)}(t)]^2=\tfrac{1}{2}+\{&(\mathbf{h}\cdot\mathbf{a})^2+[a^2+(\mathbf{h}\cdot\mathbf{a})^2]\\&\times[\langle\cos\gamma\rangle^2+\langle\sin\gamma\rangle^2]\}.\end{aligned} \tag{19}$$

The depolarization is

$$\mathrm{D.P.}=\frac{[a^2-(\mathbf{h}\cdot\mathbf{a})^2]}{a^2}\{1-[\langle\cos\gamma\rangle^2+\langle\sin\gamma\rangle^2]\}. \tag{20}$$

If the initial polarization were such that the nonvanishing component of $\mathbf{a}$ was along the direction of $\mathbf{h}$, the depolarization given by (20) would be identically zero. However, in general, the other components do not usually vanish. Therefore, it is necessary to study the

[5] K. M. Case, Phys. Rev. **95**, 1323 (1954).
[6] U. Fano, Phys. Rev. **93**, 621 (1954).

behavior of the trigonometric functions as a function of time (number of revolutions) to know how the depolarization will vary with time. The argument of the function γ is given by (17) as

$$|\gamma| = \left|\frac{\omega_L}{\omega_r}\frac{\mathcal{H}_r}{\mathcal{H}_0}\sin^2\omega_r t\right| \leq \left|\frac{\omega_L}{\omega_r}\frac{\mathcal{H}_r}{\mathcal{H}_0}\right|, \qquad (21)$$

so that, independent of time, the upper limit of γ is

$$|\gamma| \leq \left|\frac{\omega_L}{\omega_r}\frac{\mathcal{H}_r}{\mathcal{H}_0}\right| = \left|\frac{n}{1-n}\frac{v_r(i)}{v_\theta(i)}\right|. \qquad (22)$$

The fall off index of the field, n, is much less than unity in the proposed experiment. The ratio of the initial radial velocity to the initial orbital velocity is much smaller than unity. We can therefore consider

$$\langle \sin\gamma\rangle \sim \langle\gamma\rangle, \qquad (23a)$$

$$\langle \cos\gamma\rangle \sim 1-\langle\gamma^2\rangle/2, \qquad (23b)$$

and

$$[\langle\cos\gamma\rangle^2+\langle\sin\gamma\rangle^2] \cong 1-\langle\gamma^2\rangle+\langle\gamma\rangle^2 = [1-\langle(\gamma-\langle\gamma\rangle)^2\rangle]. \qquad (23)$$

If this is substituted in into (20), we see that the depolarization is extremely small, of the order of small quantities squared. It is seen that no matter how much time the particles spend in the field, the depolarization arising from the radial oscillations is negligible. This is independent of the initial polarization of the beam.

B. Transverse Oscillations

The depolarization of the ensemble arising from small oscillations of parts of the beam in a direction transverse to the plane of the equilibrium orbits will now be considered. The particles oscillate in the z-direction and "see" an oscillating magnetic field in the radial direction given by

$$(\boldsymbol{\mathcal{H}})_a = \mathcal{H}_a \sin\omega_a t, \qquad (24)$$

where

$$(\mathcal{H}_a/\mathcal{H}_0) = -[v_z(i)n^{\frac{1}{2}}/v_\theta(i)]. \qquad (25)$$

Here $v_z(i)$ is the initial velocity in the z-direction, $v_\theta(i)$ is the initial orbital velocity, and n is the fall off index of the "constant" magnetic field discussed above.

Each electron is considered as traveling in a cyclotron-like orbit but each "sees" a radial oscillating field of magnitude depending upon the amplitude of oscillations in the z-direction (initial z-velocity). The direction of the oscillating field that the electrons "see" is always in the same direction as the electron goes about in its orbit. Let us consider this the x-direction. This can be decomposed into two circular fields[7]

$$\mathcal{H}_x = \tfrac{1}{2}(\mathcal{H}_x, \mathcal{H}_y, 0) + \tfrac{1}{2}(\mathcal{H}_x, -\mathcal{H}_y, 0), \qquad (26)$$

where

$$\mathcal{H}_x = \mathcal{H}_a \sin\omega_a t, \qquad (27)$$

$$\mathcal{H}_y = \mathcal{H}_a \cos\omega_a t. \qquad (28)$$

The depolarization arising from one of the circular fields in (26) will be considered. This will probably tend to depolarize the beam more than the superposition of both fields of (26) where one would tend to cancel the effects of the other. We now consider the following field $\boldsymbol{\mathcal{H}}$ acting on the electron.

$$\mathcal{H}_z = \mathcal{H}_0, \qquad (29a)$$

$$\mathcal{H}_x = \tfrac{1}{2}\mathcal{H}_a \sin\omega_a t, \qquad (29b)$$

$$\mathcal{H}_y = \tfrac{1}{2}\mathcal{H}_a \cos\omega_a t, \qquad (29c)$$

where $\mathcal{H}_a$ is different for different parts of the beam.

The equation of motion of the spin wave function is given by (4) and a solution in the form of (7) is assumed. This gives us

$$\frac{\partial U}{\partial t} = \frac{i\omega_L}{2}\left[\sigma_z + \frac{\mathcal{H}_a}{2\mathcal{H}_0}(\sigma_x \sin\omega_a t + \sigma_y \cos\omega_a t)\right]U. \qquad (30)$$

The solution of (30) is given by (Appendix A)

$$U(t) = Q_P(t)W(t), \qquad (31)$$

where $Q_P(t)$ is given by (9) and

$$W(t) = \exp[-\tfrac{1}{2}(\omega_L-\omega_a)\sigma_z t] \times \exp(\tfrac{1}{2}i\tau\sigma_x)\exp(\tfrac{1}{2}i\lambda\sigma_z t). \qquad (32)$$

λ and τ are defined by

$$\lambda \cos\tau = (\omega_L - \omega_a), \qquad (33a)$$

$$\lambda \sin\tau = \tfrac{1}{2}\omega_L \mathcal{H}_a/\mathcal{H}_0. \qquad (33b)$$

For the case of interest, the fields will be held as uniform as possible so that

$$n \ll 1. \qquad (34)$$

The initial velocity in the z direction will be very small compared to the orbital velocity because it is desired that the electrons execute a very large number of revolutions in a comparatively small distance δz, so that

$$v_z(i)/v_\theta(i) \ll 1. \qquad (35)$$

From (2), (25), and (33), we have

$$\tan\tau \approx \tau \approx \mathcal{H}_a/(2\mathcal{H}_0) \ll 1. \qquad (36)$$

Therefore, the term in $W(t)$ depending on σ_x can be expressed as

$$\exp(\tfrac{1}{2}i\tau\sigma_x) \approx 1 + \tfrac{1}{2}i\sigma_x\tau.$$

The contributions of this term to $[\rho^{(s)}(t)]^2$ differ from unity by small quantities of the order of magnitude of τ, which is small for any part of the beam. It should also be noted that this quantity is independent of the time. In further considerations this will therefore be replaced

[7] Rabi, Ramsey, and Schwinger, Revs. Modern Phys. 26, 167 (1954).

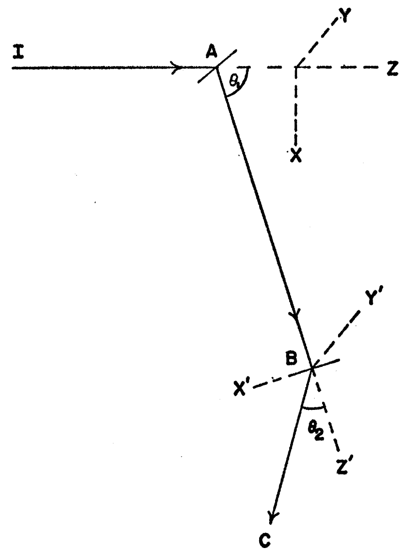

FIG. 1. Schematic diagram of a double-scattering experiment. In the case of interest $\theta_1=90°$.

by the unit matrix. $W(t)$ can now be given by

$$W(t)\approx\exp\{-\tfrac{1}{2}i[\lambda-(\omega_L-\omega_a)\sigma_z t]\}. \tag{37}$$

From (33), we can write

$$\lambda\approx(\omega_L-\omega_a)+\frac{1}{2}\frac{\omega_L^2}{(\omega_L-\omega_a)}\left(\frac{\mathcal{H}_a}{2\mathcal{H}_0}\right)^2,$$

so that

$$[\lambda-(\omega_L-\omega_a)]t\approx\tfrac{1}{2}(\mathcal{H}_a/2\mathcal{H}_0)^2\omega_L t=\gamma'. \tag{38}$$

Therefore

$$W(t)=\exp(\tfrac{1}{2}i\sigma_z\gamma')=\exp(\tfrac{1}{2}i\gamma'\boldsymbol{\sigma}\cdot\mathbf{h}), \tag{39}$$

where $\mathbf{h}$ is the unit vector in the direction of the main field $\mathcal{H}_0$.

If the initial spin density matrix is given by (10), then the depolarization is given by

$$\text{D.P.}=\frac{[a^2-(\mathbf{h}\cdot\mathbf{a})^2]}{a^2}\{1-[\langle\cos\gamma'\rangle^2+\langle\sin\gamma'\rangle^2]\}, \tag{40}$$

where the angular braces denote the averaging over the initial momentum states of the particles in the beam. Although all the particles of the beam do not spend the same amount of time in the field, the experiment is designed so that all the particles make the same number of revolutions between the two scatterings, and therefore, the factor in (38) $\omega_L t$ is the same constant for all electrons. The only variation in γ' among the various electrons in the beam that is to be considered arises from the different initial velocities (radial fields).

If the particles spend a very long time in the field, we have (Appendix B)

$$\lim_{\omega_L t\to\infty}\langle\cos\gamma'\rangle=\lim_{\omega_L t\to\infty}\langle\sin\gamma'\rangle=0, \tag{41}$$

so that from (40) we see that if $\mathbf{a}$ is not parallel to $\mathbf{h}$ the depolarization can be very large. We could then expect the asymmetry in the double scattering experiment to decrease. The oscillations transverse to the equilibrium orbit cause depolarization of the beam and this places an upper limit on the number of revolutions that the electrons can execute between the two scattering events without eliminating the asymmetry.

III. DOUBLE SCATTERING WITH DEPOLARIZATION

We have shown that the major depolarization effects arise from the small oscillations about the equilibrium orbit in the direction of the main component of the magnetic field. Here we will show how the formula describing the double-scattering cross section is modified by the depolarization. The notation of I will be employed.

The most interesting case is the following. The initial incident beam on the first target IA (Fig. 1) is scattered into the direction AB at an angle θ_1 with the initial beam. The magnetic field is perpendicular to the plane IAB. If $\theta_1=90°$, then the magnetic field would be along IA. In the primed coordinate system, the magnetic field is along the x'-axis (Fig. 1).

If there is no depolarization, the density matrix describing the beam incident on the second target is given by Eq. (19) of I as

$$\rho''=P(t)Q(t)\rho' Q^{-1}(t)P^{-1}(t), \tag{42}$$

where $P(t)$ and $Q(t)$ are the transformation matrices related to the rotation of the space and spin states respectively. In this case, Q is given by Eq. (21) of I as

$$Q(t)=Q_P(t)Q_\epsilon(t), \tag{43}$$

where $Q_P(t)$ corresponds to the same angle of rotation as $P(t)$ and $Q_\epsilon(t)$ arises from the rotation due to the anomalous moment.

When the depolarization effects are included, $Q(t)$ is replaced by

$$Q(t)\to Q_P(t)Q_\epsilon(t)W(t)=R(t). \tag{44}$$

Equation (42) describing the beam (without depolarization) incident on the second target is now replaced by

$$\rho^{\text{II}}=P(t)R(t)\rho' R^+P^{-1}(t), \tag{45}$$

where ρ' does not describe a pure momentum state. In (44) only $W(t)$ depends upon the momentum distribution after the first scattering.

Instead of I Eq. (23) describing the beam after the second scattering, we have

$$\langle\rho^{\text{II}}\rangle=P(t)Q_P(t)VQ_\epsilon(t)\times\langle W(t)\rho' W^+(t)\rangle Q_\epsilon^{-1}(t)V^+Q_P^{-1}(t)P^{-1}(t). \tag{46}$$

We have made the following approximation. We considered ρ' as describing a pure momentum state which is an average of the various momentum states, but the averaging over the momentum states has been considered when calculating the spin density matrix of the beam incident on the second target [denoted by the angular braces in (46)]. The scattering cross section is obtained by taking the trace over the spin states in (46). If $P(t)$ is a rotation corresponding to an integral number of revolutions, the cross section in the primed coordinate system is given by

$$d\sigma\sim 1-\delta\cos\varphi_2\langle\cos(\epsilon+\gamma')\rangle, \tag{47}$$

or

$$d\sigma\sim 1-\delta\cos\varphi_2[\cos\epsilon\langle\cos\gamma'\rangle-\sin\epsilon\langle\sin\gamma'\rangle]. \tag{48}$$

Without depolarization, we had from I, Eq. (32),

$$d\sigma\sim 1-\delta\cos\varphi_2\cos\epsilon. \tag{49}$$

The effect of the depolarization is given by the trigonometric functions of γ' which are functions of the number of revolutions executed between the two scatterers. From (41) we see that if the number of revolutions between scatterings is very large, the bracket in (48) tends to vanish. Thus the asymmetry will tend to vanish because of the depolarization of the beam.

IV. APPLICATION TO MEASURING THE GYROMAGNETIC RATIO g

In order that the amount of depolarization be small, we will limit the number of revolutions by requiring that

$$\langle\cos\gamma'\rangle\approx 1, \tag{50a}$$

$$\langle\sin\gamma'\rangle\sim\gamma'\ll 1, \tag{50b}$$

or

$$\tfrac{1}{2}(\mathcal{H}_a/2\mathcal{H}_0)^2\omega_L t\ll 1. \tag{50c}$$

As an upper limit on $\langle\gamma'\rangle$ let us consider

$$\langle\gamma'\rangle\leq\pi/20,$$

so that

$$\frac{n}{8}\langle[v_z(i)/v_\theta(i)]^2\rangle\omega_L t\leq\frac{\pi}{20}. \tag{51}$$

The time for N revolutions is

$$t=2\pi N/\omega_L, \tag{52}$$

so that (51) becomes

$$\frac{2\pi Nn}{8}\langle v_z(i)/v_\theta(i)\rangle\leq\frac{\pi}{20}, \tag{53}$$

or

$$N\leq\langle[v_\theta(i)/v_z(i)]^2\rangle\frac{1}{5n}. \tag{54}$$

Thus we have an upper limit on the number of revolutions in terms of the experimental parameters, the initial velocities and the falloff index n.

Let us consider the case where the maximum velocity in the z direction (x' direction) is given by

$$v_z(i)=v_\theta(i)\sin(1°),$$

or

$$v_z(i)/v_\theta(i)\approx 0.017.$$

If this is the value taken as the average over the beam, then

$$N\lesssim 700/n. \tag{55}$$

For these values of N, (48) is effectively equal to (49), the double scattering cross section without depolarization.

Let us assume that we can measure ϵ of (49) to within ± 0.3 radian and we would like to measure g, the gyromagnetic ratio of the free electron, to one part in one hundred thousand. Let

$$\epsilon=\epsilon_0\pm\Delta\epsilon, \tag{56}$$

and the number of revolutions be given by

$$N=N_0\pm\Delta N. \tag{57}$$

Now

$$\tfrac{1}{2}g\omega_L t=\tfrac{1}{2}g2\pi N=2\pi N+\epsilon,$$

so that

$$\tfrac{1}{2}g=1+\frac{\epsilon_0}{2\pi N_0}\pm\frac{\Delta\epsilon}{2\pi N_0}\pm\frac{\epsilon_0}{2\pi N_0}\frac{\Delta N}{N_0}. \tag{58}$$

We therefore require that

$$\frac{\Delta\epsilon}{2\pi N_0}\leq 10^{-5}, \tag{59a}$$

$$\frac{\epsilon_0}{2\pi N_0}\frac{\Delta N}{N}\leq 10^{-5}. \tag{59b}$$

We assume $\Delta\epsilon\sim 0.3$, which means that we must have

$$N\gtrsim 5000. \tag{60}$$

It was shown in I, Eq. (38), that

$$\epsilon=a\omega_L t/(1-\beta^2)^{\frac{1}{2}},$$
$$a=\alpha/2\pi,$$

α being the fine structure constant. Therefore, from (59b) we have

$$\frac{1}{(1-\beta^2)^{\frac{1}{2}}}\frac{\alpha}{2\pi}\frac{\Delta N}{N_0}\leq 10^{-5},$$

which, for one hundred kilovolt electrons, can be approximated by

$$\frac{\alpha}{2\pi}\frac{\Delta N}{N_0}\leq 10^{-5},$$

so that

$$\Delta N/N_0\lesssim 10^{-2}. \tag{61}$$

Thus, one of the requirements on such an experiment is that the number of revolutions be of the order of 5000. This limits the fall-off index to be of the order of 0.1. Also it is necessary that the number of revolutions be known to within 1%.

V. CONCLUSION

Depolarization effects place certain restrictions on the accuracy with which the gyromagnetic ratio of a free electron can be determined by double scattering experiments in a magnetic field. However, with sufficient care it does seem possible to make the measurement to at least one part in 10^5. This is then another feasible method of checking the radiative corrections.

APPENDIX A

Here we show how we can obtain Eq. (32) from Eq. (30) in the text. We are given Eq. (30)

$$\frac{\partial U}{\partial t}=\tfrac{1}{2}i\omega_L\left[\sigma_z+\frac{\mathcal{H}_a}{2\mathcal{H}_0}(\sigma_x\sin\omega_a t+\sigma_y\cos\omega_a t)\right]U. \quad \text{(A-1)}$$

Let

$$U=\exp[\tfrac{1}{2}i\omega_L\sigma_z t]W=Q_PW, \quad \text{(A-2)}$$

so that

$$\frac{\partial W}{\partial t}=\frac{i\omega_L}{4}\frac{\mathcal{H}_a}{\mathcal{H}_0}\exp\left[-\frac{i}{2}(\omega_L-\omega_a)\sigma_z t\right]\sigma_y \times\exp\left[\frac{i}{2}(\omega_L-\omega_a)\sigma_z t\right]W. \quad \text{(A-3)}$$

We now express W as

$$W=\exp[-\tfrac{1}{2}i(\omega_L-\omega_a)\sigma_z]X, \quad \text{(A-4)}$$

and

$$\frac{\partial X}{\partial t}=\frac{i}{2}\left[\frac{\omega_L}{2}\frac{\mathcal{H}_a}{\mathcal{H}_0}\sigma_y+(\omega_L-\omega_a)\sigma_z\right]X. \quad \text{(A-5)}$$

Now let

$$\lambda\cos\tau=(\omega_L-\omega_a), \quad \text{(A-6)}$$

$$\lambda\sin\tau=\omega_L\mathcal{H}_a/(2\mathcal{H}_0), \quad \text{(A-7)}$$

then

$$\partial X/\partial t=\tfrac{1}{2}i\lambda\exp(\tfrac{1}{2}i\tau\sigma_x)\sigma_z\exp(-\tfrac{1}{2}i\tau\sigma_x)X. \quad \text{(A-8)}$$

If we define Y such that

$$X=\exp(\tfrac{1}{2}i\tau\sigma_x)Y, \quad \text{(A-9)}$$

then

$$\partial Y/\partial t=\tfrac{1}{2}i\lambda\sigma_z Y, \quad \text{(A-10)}$$

so that

$$Y=\exp(\tfrac{1}{2}i\sigma_z t)Y_0. \quad \text{(A-11)}$$

Where Y_0 may be taken as a unit matrix. Therefore

$$U=Q_P(t)W(t), \quad \text{(A-12)}$$

where

$$W(t)=\exp[-\tfrac{1}{2}i(\omega_L-\omega_a)\sigma_z t]\times\exp[\tfrac{1}{2}i\sigma_x\tau]\exp[\tfrac{1}{2}i\lambda\sigma_z t], \quad \text{(A-13)}$$

which is Eq. (32).

APPENDIX B

We are interested in determining how the number of revolutions that the particles execute in the field will affect the average of the trigonometric functions given in Eq. (40) in the text. It is shown that as the number of revolutions becomes very large so that $\omega_L t\to\infty$ the average values of both the sine and cosine functions tend to vanish. Consider

$$\lim_{\omega_L t\to\infty}\langle\cos\gamma'\rangle, \quad \text{(B-1)}$$

$$\lim_{\omega_L t\to\infty}\langle\sin\gamma'\rangle, \quad \text{(B-2)}$$

where γ' is given by

$$\gamma'=\tfrac{1}{2}(\mathcal{H}_a/\mathcal{H}_0)^2\omega_L t=\tfrac{1}{2}\alpha^2\omega_L t, \quad \text{(B-3)}$$

and the maximum of

$$\alpha=\tfrac{1}{2}\mathcal{H}_a/\mathcal{H}_0\ll 1. \quad \text{(B-4)}$$

The average value of one of the trigonometric function is

$$\langle\cos\gamma'\rangle=\int_0^\infty P(\alpha^2)\cos(\tfrac{1}{2}\alpha^2\omega_L t)d(\alpha^2), \quad \text{(B-5)}$$

where $P(\alpha^2)$ is the probability over the beam that a given particle will "see" the fields denoted by the parameter α^2.

Let

$$\tfrac{1}{2}\alpha^2\omega_L t=\lambda, \quad \text{(B-6)}$$

so that

$$\langle\cos\gamma'\rangle=\frac{2}{\omega_L t}\int_0^\infty P\left(\frac{\lambda}{\omega_L t}\right)\cos\lambda d\lambda, \quad \text{(B-7)}$$

and

$$\langle\sin\gamma'\rangle=\frac{2}{\omega_L t}\int_0^\infty P\left(\frac{\lambda}{\omega_L t}\right)\sin\lambda d\lambda. \quad \text{(B-8)}$$

It is assumed that $P(\alpha^2)$ is a reasonable type probability function which is not too peaked at the origin and falls off rapidly for large α. Then

$$\int_0^\infty P\left(\frac{\lambda}{\omega_L t}\right)\begin{Bmatrix}\cos\lambda\\ \sin\lambda\end{Bmatrix}d\lambda,$$

is a bounded function, so that

$$\lim_{\omega_L t\to\infty}\begin{Bmatrix}\langle\cos\gamma'\rangle\\ \langle\sin\gamma'\rangle\end{Bmatrix}=0. \quad \text{(B-9)}$$

This tells us that the beam will tend to depolarize if it stays too long in these fields. This is expressed in the Eq. (40) of the text.

APPENDIX C

The fields causing depolarization considered in the main body of the text are analytically so simple that the integration problems could be handled very easily. For many questions, however, it is convenient to have the approximate formulas for more general situations obtained below.

Let us assume the Schrödinger equation for the spin function to be

$$i\hbar\partial\Phi/\partial t=[H_0(t)+H_1(t,\alpha)]\Phi, \quad \text{(C1)}$$

where $H_0(t)$ is independent of the orbital parameters α and H_1 is small.

If

$$\Phi(t)=U(t)\Phi(0), \quad \text{(C2)}$$

we obtain

$$U=Q(t)W(t,\alpha), \quad \text{(C3)}$$

where

$$i\hbar\partial Q/\partial t=H_0Q, \quad \text{(C4)}$$

and

$$i\hbar\partial W/\partial t=Q^{-1}H_1QW. \quad \text{(C5)}$$

Assuming $Q(t)$ known for Eq. (C4), $W(t,\alpha)$ is, correct to first order in H_1;

$$W(t,\alpha)=\mathbf{1}+W_1(t,\alpha), \quad \text{(C6)}$$

where

$$W_1(t,\alpha)=\frac{1}{i\hbar}\int_0^t Q^{-1}(t')H_1(t',\alpha)Q(t')dt'. \quad \text{(C7)}$$

Using (13), the definition (12), and the approximate formula for U we obtain

$$\text{D.P.}=\frac{-2\,\text{trace}\{[\rho(0)]^2[\langle W_1^2(t)\rangle-\langle W_1(t)\rangle^2]\}}{\text{trace}\rho^2(0)-\frac{1}{2}} + \frac{2\,\text{trace}\{\langle\rho(0)W_1Q^{-1}\rho(0)QW\rangle-\langle\rho(0)W_1Q^{-1}\rangle\langle\rho(0)QW_1\rangle\}}{\text{trace}\rho^2(0)-\frac{1}{2}}. \quad \text{(C8)}$$

$\rho(0)$ is the initial spin density matrix. The angular braces denote averages with respect to α. Equation (C8) is correct to second order in H_1. (This follows somewhat indirectly from an assumed Hermiticity of H_1.) It may be remarked, that since the limitations on experiments are obtained by requiring the depolarization to be small, these could be derived directly using (C8) and

$$H_0=f(t)\boldsymbol{\sigma}\cdot\mathbf{h}, \quad \text{(C9)}$$

$$H_1(t,\alpha)=\mathbf{b}(t,\alpha)\cdot\boldsymbol{\sigma}, \quad \text{(C10)}$$

with $\mathbf{h}$ a constant unit vector, $f(t)$ a given function of time, and b a small, variable vector.

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Diffusion of Like Particles Across a Magnetic Field

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It is shown that the diffusion rate across a magnetic field due to collision of like charged particles is derivable from the macroscopic equations of the plasma. However, it is necessary to include the off-diagonal terms in the stress tensor. The resultant diffusion rate does not obey Fick's law and is proportional to the inverse fourth power of magnetic field strength. This diffusion rate is usually smaller than that due to unlike particle collisions, but may sometimes dominate.

I. INTRODUCTION

MANY of the gross properties of a plasma may be obtained from a consideration of the hydrodynamical (or macroscopic) equations of the plasma.[1] Thus, for example, for a gas consisting of ions and electrons and assuming an isotropic stress tensor, one has the following momentum equation in the steady state

$$\nabla P=\mathbf{j}\times\mathbf{H}+\epsilon\mathbf{E}. \quad (1)$$

Here P is the gas pressure, H and E are the magnetic and electric field strengths respectively, j is the current and ϵ the charge density in the plasma. Note that a nonlinear term in the velocity is neglected. In addition to this equation, we have another expression representing the generalized Ohm's law:

$$\mathbf{E}+(\mathbf{v}\times\mathbf{H})/c=\mathbf{j}/\sigma+\nabla P_i/en. \quad (2)$$

Again, steady state has been assumed and nonlinear terms neglected. The mass velocity of the plasma is denoted by $\mathbf{v}$, the ion partial pressure by P_i and the conductivity of the plasma by σ. The conductivity is defined as

$$\sigma\cong ne^2/mc\nu, \quad (3)$$

where n is the density of electrons, m the electron mass, and ν is the collision frequency for electron-ion impact.

As Spitzer has shown,[1] an expression for the diffusion rate across a magnetic field may be readily derived from Eqs. (1) and (2). Assuming that the density gradient and the electron field are in a single direction (say x) and the magnetic field is in the z-direction, one may eliminate $\mathbf{j}$ between Eqs. (1) and (2). The result is

$$v_x=-(c/\sigma H^2)(\nabla P-\epsilon E). \quad (4)$$

This has the usual form for diffusion in that Fick's law

[1] Lyman Spitzer, Jr., *Physics of Fully Ionized Gases* (Interscience Publishers, New York, to be published).