d-band is full but the hole conduction cannot be neglected. The experimental value of R_0 then requires that $\sigma_d/\sigma_s=0.3$. According to existing theory of conductivity[8] for temperatures above the characteristic temperature of the lattice

$$\frac{\sigma_d}{\sigma_s}=\frac{m_s\tau_d}{m_d\tau_s}, \tag{9}$$

where m_s= effective mass of s-band electrons, m_d= effective mass of d-band electrons (absolute value), τ_s= relaxation time of s-band electrons, and τ_d= relaxation time of d-band electrons. The specific heat measurements of Keesom and Clark[10] indicate that $m_s/m_d \cong 1/28$, which would give $\tau_d/\tau_s \cong 8$. No physically significant calculations have yet been given for τ_d/τ_s, but it is plausible that $\tau_d > \tau_s$ because of the low velocity of d-band electrons relative to the velocity of s-band electrons.

[10] W. H. Keesom and C. W. Clark, Physica 2, 513 (1935).

Of the two possible interpretations of the experimental value of R_0, the latter, in which the negative Hall effect from s-band conduction is counteracted by the positive Hall effect due to hole conduction, appears to be the most likely alternative. The result $\sigma_d/\sigma_s=0.3$ does not appear to contradict any existing experimental evidence. It is believed that this is the only quantitative evaluation of the conductivity ratio that has been attempted. Furthermore, it is necessary that $\sigma_d/\sigma_s>1$ for Co and Fe to produce the observed positive Hall effects, unless some agency entirely different from hole conduction is responsible for these positive Hall effects.

It is expected that a clearer understanding of the hole conduction will be achieved with the completion of measurements of R_0 and R_1 that are now being carried out on the Ni−Cu and the Ni−Co series of alloys.

The authors are grateful for helpful discussions with a number of individuals including F. Seitz and G. Wannier and to the International Nickel Company for supplying the nickel samples.

PHYSICAL REVIEW VOLUME 80, NUMBER 4 NOVEMBER 15, 1950

Magneto-Hydrodynamic Shocks*

F. DE HOFFMANN AND E. TELLER
Los Alamos Scientific Laboratory, Los Alamos, New Mexico

(Received July 10, 1950)

A mathematical treatment of the coupled motion of hydrodynamic flow and electromagnetic fields is given. Two simplifying assumptions are introduced: first, the conductivity of the medium is infinite, and second, the motion is described by a plane shock wave. Various orientations of the plane of the shock and the magnetic field are discussed separately, and the extreme relativistic and unrelativistic behavior is examined. Special consideration is given to the behavior of weak shocks, that is, of sound waves. It is interesting to note that the waves degenerate into common sound waves and into common electromagnetic waves in the extreme cases of very weak and very strong magnetic fields.

I. INTRODUCTION

It has been shown recently that the interaction between hydrodynamic motion and magnetic fields in a conducting liquid is of importance in problems of astrophysics, geophysics, and the behavior of interstellar gas masses.[1] The non-linear character of the hydrodynamic equations raises difficulties in the treatment of these problems. So far only the linear problem of sound propagation has been treated[2] and this one only for a transverse wave propagating along the lines of force. It is the purpose of the following investigation to clarify the behavior of plane waves in magneto-hydrodynamics. The non-linear case of shock waves will be of primary interest and the simpler behavior of sound waves will be obtained by considering shocks of small amplitude. Two special cases of magneto-hydrodynamic waves are well known in physics. One is the hydrodynamic shock and the other the pure electromagnetic wave. It will be clear from the formalism which we are going to develop that these can be obtained from the magneto-hydrodynamic shock as limiting cases. In order to permit a treatment of waves which are similar to electromagnetic waves we shall need to discuss shock velocities which are close to the velocity of light. We therefore must include a relativistic treatment.

In order to limit ourselves to the simplest possible case we shall make the assumption that the conductivity is infinite. One consequence of this assumption is that the self-induction will prevent a change in the magnetic fields if the substance carrying the magnetic field is at rest. Actually the conductivity is finite. However, the

* This document is based on work performed at Los Alamos Scientific Laboratory of the University of California under the auspices of the AEC.

[1] H. Alfvén, Arkiv. f. Mat. Astron. Fysik 29A, 12 (1943). E. C. Bullard, Proc. Roy. Soc. A 197, 433 (1949)—See this paper for additional bibliography. E. Fermi, Phys. Rev. 75, 1169 (1949). C. Walen, Arkiv. f. Mat. Astron. Fysik 30A, 15 (1944).

[2] H. Alfvén, Arkiv. f. Mat. Astron. Fysik 29B, 2 (1943).

time in which magnetic fields can change appreciably is proportional to the conductivity times the square of the linear dimensions of the system under consideration. In the applications to cosmic problems the linear dimension is usually so great that changes of the magnetic field in a system at rest are effectively excluded even if the conductivity is moderate. Thus the case of infinite conductivity will furnish us with a good model of the situations of interest.

If the conducting liquid is in motion it can flow freely along the magnetic lines of force without affecting the latter. If, on the other hand, material streams in a direction perpendicular to the lines of force, these will behave as though they were firmly attached to the fluid. This can be seen by the following qualitative argument. Consider an imaginary closed path which moves along with the liquid. Since the conductivity is infinite, the e.m.f. along this path must at all times be zero. It follows that the magnetic flux enclosed by the closed path is time independent. Since this statement holds for any closed path, it follows that no sidewise shift of magnetic lines with respect to the fluid is possible.

It also follows from the above assumption that the electric fields vanish in a coordinate system that is at rest in the liquid. The vector $\mathfrak{D}$ will vanish in the same system and it is unnecessary to introduce the dielectric constant. As a further simplification we shall assume $\mathcal{H}=\mathcal{B}$ so that only the six vector $\mathcal{E}$, $\mathcal{H}$ is needed to describe the electromagnetic field.

While most of the derivations will refer to a general equation of state of matter, we shall be interested in particular in the case in which the fluid is an ideal gas for which γ, the ratio of specific heats at constant volume to constant pressure, is a fixed number. This is a satisfactory approximation for the equation of state within stars and for interstellar gases. In the latter case the effective value of γ may depend on the presence of dust and on the question of whether or not equilibrium with radiation is established.

II. LONGITUDINAL AND TRANSVERSE SHOCKS

In normal shock hydrodynamics it is proved that the motion in a shock proceeds perpendicularly to the shock front. We shall see later that in magneto-hydrodynamics this is no longer necessarily so. When treating shocks whose direction of propagation is parallel or perpendicular to the magnetic force lines, one obtains solutions of the shock equations if one assumes that all motion is parallel to the direction of shock propagation. These shocks we shall call longitudinal shocks. In addition we shall consider discontinuities in which the motion of material is along the surface of discontinuity (and therefore perpendicular to the direction of propagation). These are transverse shocks. These transverse shocks have in normal hydrodynamics a propagation velocity equal to zero and are there called slipstreams rather than shocks. In magneto-hydrodynamics transverse shocks have a finite propagation velocity if the magnetic field is perpendicular to the shock front; the propagation velocity of transverse waves is zero (as in normal hydrodynamics) when the magnetic field is parallel to the shock front. All this will be proved later.

In the case of a transverse shock the magnetic lines of force will have in general a different direction before and after the shock has passed and therefore the expression "transverse shock" does not apply in a strict sense. The only case in which this expression may be applied is that of a weak transverse shock, i.e., in transverse sound propagation, in which case the magnetic lines are only slightly different in direction before and after the passage of the wave. These are in fact the magneto-hydrodynamic waves discussed by Alfvén[2] and they possess the velocity $\mathcal{H}/(4\pi\rho)^{\frac{1}{2}}$. It will be observed that for a vanishing magnetic field this velocity goes to zero. We shall not treat this transverse sound wave in detail here because its properties will be obtained by appropriate specialization from the oblique shock equation.

In the following Sections III and IV we discuss the longitudinal shocks for the cases of shock propagation parallel and perpendicular to $\mathcal{H}$. The discussion of oblique shocks in which the terms transverse and longitudinal motion do not apply directly will be reserved for a final Section V.

III. PARALLEL SHOCKS

We shall assume that the shock propagates in the x direction, i.e., that its plane is the $y-z$ plane. The magnetic fields have components only along the x direction. It is easy to see that in this case the hydrodynamic motion is not coupled to the magnetic field and the shock proceeds as in ordinary hydrodynamics. We give a detailed discussion, however, in order to introduce the proper relativistic terminology. There are two kinds of coordinate systems which are relevant:

(a) *The co-moving systems:* These systems are those in which the observer is at rest relative to the medium on the one or other side of the shock. They will be denoted by a prime. Owing to the infinite value of the conductivity all electric fields vanish in the primed systems.

(b) *The shock system:* This system is one in which the observer is at rest relative to the shock front and will be denoted by unprimed letters. Viewed from this coordinate system the flow will appear time-independent.

We denote the two regions on either side of the shock by 1 and 2, respectively. In the region 1 the flow velocity v_1 will be directed toward the shock plane; in region 2 the velocity v_2 is directed away from that plane.

According to the definition of the parallel shock all components of the magnetic fields vanish with the exception of[3] $\mathcal{H}_{1,x}'$ and $\mathcal{H}_{2,x}'$. By a Lorentz transforma-

[3] The index 1 or 2 will always appear at the bottom of the quantity it qualifies as the first subscript. It is separated from the symbols of the coordinates (x, y, z, t, or in general n) by a comma. If no index 1 or 2 appears the formula applies to both 1 and 2.

tion we find

$$\left.\begin{matrix}\mathcal{E}_{1,x}=0 & \mathcal{H}_{1,x}=\mathcal{H}_{1,x}' \\ \mathcal{E}_{1,y}=0 & \mathcal{H}_{1,y}=0 \\ \mathcal{E}_{1,z}=0 & \mathcal{H}_{1,z}=0\end{matrix}\right\}, \tag{1}$$

and

$$\left.\begin{matrix}\mathcal{E}_{2,x}=0 & \mathcal{H}_{2,x}=\mathcal{H}_{2,x}' \\ \mathcal{E}_{2,y}=0 & \mathcal{H}_{2,y}=0 \\ \mathcal{E}_{2,z}=0 & \mathcal{H}_{2,z}=0\end{matrix}\right\}. \tag{2}$$

Furthermore, $\mathrm{div}\mathcal{H}=0$, so that

$$\mathcal{H}_{1,x}=\mathcal{H}_{2,x}. \tag{3}$$

We see that on the boundary $\mathrm{curl}\,\mathcal{E}$ vanishes and hence $\partial\mathcal{H}/\partial t=0$ as was to be expected.

In order to set up the relativistic shock equations we define the quantity η' as the relativistic energy density for the primed system:

$$\eta' c^2=n'(m_0'c^2+E'), \tag{4}$$

where n' is the density of particles, m_0' the average rest mass, and E' the excess energy per particle over and above the rest energy. We shall restrict the discussion to fluids. For these the energy momentum stress tensor has a particularly simple form in the primed system: only the diagonal components T_{nn}' differ from zero.

$$T_{xx}'=p'-(\mathcal{H}_x')^2/8\pi, \tag{5}$$

$$T_{yy}'=T_{zz}'=p'+(\mathcal{H}_x')^2/8\pi, \tag{6}$$

$$T_{tt}'=\eta'+(\mathcal{H}_x')^2/8\pi c^2, \tag{7}$$

where p' is the pressure in the co-moving system.

If one performs the Lorentz transformation to the shock system which moves with a velocity v relative to the primed system, he obtains for the tensor:

$$T=\begin{vmatrix} \dfrac{\eta' v^2+p'}{1-(v^2/c^2)}-\dfrac{(\mathcal{H}_x')^2}{8\pi} & 0 & 0 & \dfrac{-[v\eta'+(v/c^2)p']}{1-(v^2/c^2)} \\ 0 & p'+\dfrac{(\mathcal{H}_x')^2}{8\pi} & 0 & 0 \\ 0 & 0 & p'+\dfrac{(\mathcal{H}_x')^2}{8\pi} & 0 \\ \dfrac{-[v\eta'+(v/c^2)p']}{1-(v^2/c^2)} & 0 & 0 & \dfrac{\eta'+(v^2/c^4)p'}{1-(v^2/c^2)}+\dfrac{(\mathcal{H}_x')^2}{8\pi c^2} \end{vmatrix}. \tag{8}$$

Equation (8) will hold on both sides of the shock; i.e., all quantities should be given the subscript 1 or all quantities the subscript 2.

One group of shock equations is obtained from the fact that the four-dimensional divergence of T vanishes:

$$\mathrm{div}T=0. \tag{9}$$

Since the flow depends only on the x coordinate, we know that $\partial/\partial y$ and $\partial/\partial z$ are zero and since the shock system is stationary, $\partial/\partial t$ is zero. Thus $\partial T_{xx}/\partial x=0$ and $\partial T_{xt}/\partial x=0$ and in general all components T_{xn} are independent of x. This statement continues to hold across the shock discontinuity, since it must hold in any approximation in which the shock is replaced by an increasingly steep but continuous change of the physical variables. We therefore find that the xx components of the tensors T_1 and T_2 are equal and similarly the xt components of these tensors are equal. Recalling that

$$\mathcal{H}_{1,x}'=\mathcal{H}_{2,x}' \tag{10}$$

we obtain

$$(\eta_1'v_1^2+p_1)/(1-v_1^2/c^2)=(\eta_2'v_2^2+p_2)/(1-v_2^2/c^2) \tag{11}$$

and

$$\left(\eta_1'v_1+\frac{v_1}{c^2}p_1'\right)\Big/\left(1-\frac{v_1^2}{c^2}\right)=\left(\eta_2'v_2+\frac{v_2}{c^2}p_2'\right)\Big/\left(1-\frac{v_2^2}{c^2}\right). \tag{12}$$

An additional shock equation is obtained from the conservation of particles; i.e., the number of particles arriving and leaving per unit time must be equal:

$$n_1v_1=n_2v_2. \tag{13}$$

This may be written in the co-moving systems as

$$n_1'v_1(1-v_1^2/c^2)^{-\frac{1}{2}}=n_2'v_2(1-v_2^2/c^2)^{-\frac{1}{2}}. \tag{14}$$

Equations (11)–(13) do not contain[4] $\mathcal{H}$. It is easy to see that these equations reduce to the Rankine-Hugoniot equations in the non-relativistic approximation. In fact, (14) reduces to the equation of conservation of mass. Neglecting v^2/c^2 and replacing η' by nm_0 in (11) one obtains the unrelativistic equation for the conservation

[4] They are the same equations as have been derived by A. H. Taub, Phys. Rev. **74**, 328 (1948), for relativistic shocks, with a slightly different notation.

of momenta. The leading term in (12) is identical with (14). If one expands (12) in powers of v^2/c^2, subtracts (14), and substitutes from (4), the leading terms in the difference give

$$c^{-2}[\tfrac{1}{2}n_1m_0v_1^3+n_1E_1+p_1v_1] \\ =c^{-2}[\tfrac{1}{2}n_2m_0v_2^3+n_2E_2+p_2v_2]. \quad (15)$$

This is the Rankine-Hugoniot equation expressing conservation of energy.

IV. PERPENDICULAR SHOCK

The shock again propagates in the x direction and owing to the infinite value of the conductivity all electric fields vanish in the primed systems. According to the definition of the perpendicular shock all components of the magnetic fields vanish with the exception of $\mathcal{H}_{1,y}'$.

By a Lorentz transformation we find

$$\left.\begin{array}{ll} \mathcal{E}_{1,x}=0 & \mathcal{H}_{1,x}=0 \\ \mathcal{E}_{1,y}=0 & \mathcal{H}_{1,y}=\dfrac{\mathcal{H}_{1,y}'}{[1-(v_1/c)^2]^{\frac{1}{2}}} \\ \mathcal{E}_{1,z}=\dfrac{(v_1/c)\mathcal{H}_{1,y}'}{[1-(v_1/c)^2]^{\frac{1}{2}}} & \mathcal{H}_{1,z}=0 \end{array}\right\}. \quad (16)$$

$\mathcal{H}_{1,y}$ and $\mathcal{H}_{2,y}$ are related by virtue of the fact that we have assumed the magnetic force lines to be attached to the particles. Thus, the density of magnetic force lines is proportional to the density of particles and we have

$$\mathcal{H}_{1,y}'/n_1'=\mathcal{H}_{2,y}'/n_2' \quad (17)$$

or since

$$n_1'=n_1[1-(v_1/c)^2]^{\frac{1}{2}}, \quad (18)$$

it follows that

$$\mathcal{H}_{1,y}/n_1=\mathcal{H}_{2,y}/n_2. \quad (19)$$

Equation (19) is merely a special case of the general statement that the density of magnetic lines of force divided by the density of particles in a perpendicular shock will retain the same value in all coordinate systems. Furthermore because of Eq. (13)

$$\mathcal{H}_{1,y}v_1=\mathcal{H}_{2,y}v_2. \quad (20)$$

Finally, since

$$\mathcal{E}_{2,z}=(v_2/c)\mathcal{H}_{2,y}, \quad (21)$$

we have

$$\mathcal{E}_{2,z}=\mathcal{E}_{1,z}, \quad (22)$$

so that in the shock system curl$\mathcal{E}=0$ at the boundary, which yields $\partial\mathcal{H}/\partial t=0$ as expected.

The energy-momentum tensor follows from (16) as:

$$T=\left|\begin{array}{cccc} \dfrac{\eta'v^2+p'+[(\mathcal{H}_y')^2/8\pi][1+(v^2/c^2)]}{1-(v^2/c^2)} & 0 & 0 & \dfrac{-[v\eta'+(v/c^2)p'+(v/c^2)(\mathcal{H}_y')^2/4\pi]}{1-(v^2/c^2)} \\ 0 & p'-\dfrac{(\mathcal{H}_y')^2}{8\pi} & 0 & 0 \\ 0 & 0 & p'+\dfrac{(\mathcal{H}_y')^2}{8\pi} & 0 \\ \dfrac{-[v\eta'+(v/c^2)p'+(v/c^2)(\mathcal{H}_y')^2/4\pi]}{1-(v^2/c^2)} & 0 & 0 & \dfrac{\eta'+(v^2/c^4)p'+[(\mathcal{H}_y')^2/8\pi c^2][1+(v^2/c^2)]}{1-(v^2/c^2)} \end{array}\right|. \quad (23)$$

We introduce the energy density η in the unprimed system

$$\eta=\eta'/[1-(v/c)^2] \quad (24)$$

and define the auxiliary quantity

$$p=p'/[1-(v/c)^2]. \quad (25)$$

To simplify the notation for the remainder of this section we set $\mathcal{H}=\mathcal{H}_y$, $\mathcal{H}'=\mathcal{H}_y'$. Making use of (16) we get

$$\mathcal{H}=\mathcal{H}'/[1-(v/c)^2]. \quad (26)$$

Thus, in analogy to the treatment presented for the parallel shock, Eq. (23) with the use of Eqs. (24) through (26) yields

$$\eta_1v_1^2+p_1+(\mathcal{H}_1^2/8\pi)=\eta_2v_2^2+p_2+(\mathcal{H}_2^2/8\pi) \quad (27)$$

$$\eta_1v_1+(v_1p_1/c^2)+(v_1\mathcal{H}_1^2/c^24\pi) \\ =\eta_2v_2+(v_2p_2/c^2)+(v_2\mathcal{H}_2^2/c^24\pi). \quad (28)$$

RELATIVISTIC IDEAL GAS

The first use we shall make of these equations will be to derive the sound velocity for a monoatomic gas in the extreme relativistic case. We must, therefore, find the relevant equation of state. For a relativistic gas we may write

$$\eta'c^2=A\int[P^2c^2+m_0^2c^4]^{\frac{1}{2}}P^2dP \\ \times\exp[-(P^2c^2+m_0^2c^4)^{\frac{1}{2}}/kT], \quad (29)$$

$$p'=\frac{Ac^2}{3}\int\frac{P^4dP}{(P^2c^2+m_0^2c^4)^{\frac{1}{2}}} \\ \times\exp[-(P^2c^2+m_0^2c^4)^{\frac{1}{2}}/kT], \quad (30)$$

where P is a momentum and A a normalization constant. In the extreme relativistic case

$$kT\gg m_0c^2, \quad P^2c^2\gg m_0^2c^4 \quad (31)$$

and Eqs. (29) and (30) are related by

$$\eta' c^2 = 3p' \tag{32}$$

and also using (24) and (25)

$$\eta c^2 = 3p. \tag{33}$$

Since we are interested in the sound velocity, the quantities in Eqs. (27) and (28) carrying subscripts 1 and 2 differ by infinitesimal amounts. If we take the differences of the two sides of these two equations and divide by $v_2 - v_1 = dv$, we get using (33)

$$\frac{d}{dv}\left[\eta v^2 + \frac{\eta c^2}{3} + \frac{\mathcal{H}^2}{8\pi}\right] = 0 \tag{34}$$

$$\frac{d}{dv}\left[\frac{4}{3}\eta v + \frac{v}{c^2}\frac{\mathcal{H}^2}{4\pi}\right] = 0. \tag{35}$$

Similarly, from (20)

$$d(\mathcal{H}v)/dv = 0. \tag{36}$$

Using the last four equations and eliminating η, $d\eta/dv$, $d\mathcal{H}/dv$ as well as re-introducing p' and $\mathcal{H}'$ we find

$$(3v^2 - c^2)/(c^2 - v^2) = r, \tag{37}$$

where

$$r = 9(\mathcal{H}')^2/16\pi\eta' c^2. \tag{38}$$

Thus, when $\mathcal{H}' = 0$, i.e. there is no applied magnetic field, the sound velocity is

$$v = c/\sqrt{3}. \tag{39}$$

When $\mathcal{H}' \rightarrow \infty$, then

$$v \approx c. \tag{40}$$

At intermediate values of $\mathcal{H}$ we find

$$v = c[(1+r)/(3+r)]^{\frac{1}{2}}. \tag{41}$$

Equation (40) shows that the sound velocity becomes equal to the light velocity when the contribution of matter to the energy momentum tension tensor becomes relatively unimportant. This agrees with expectation. One may find it surprising that in the absence of a magnetic field the sound velocity[5] is $c/\sqrt{3}$ while the velocity of the particles in the extreme relativistic case is equal to c. The reason is that in the random thermal motion the particles move obliquely in the propagating wave.

THE LIMITING CASE $v \rightarrow c$

We have seen above that, in the special case of an extremely relativistic equation of state, v is equal to c in the case $\mathcal{H}' \rightarrow \infty$. We shall now investigate the conditions necessary for $v \rightarrow c$.

First we shall assume that $v_1 \approx c$ and $v_2 \approx c$. Making use of the contraction

$$\mu = (1 - v^2/c^2)^{-\frac{1}{2}} \tag{42}$$

[5] A. R. Curtis, Proc. Roy. Soc. **A200**, 248 (1950).

we thus wish to examine the case $\mu_1 \gg 1$; $\mu_2 \gg 1$. In the approximation $v_1 = v_2$ Eq. (14) reduces to

$$n_1'\mu_1 = n_2'\mu_2. \tag{43}$$

Similarly, Eq. (20) becomes $\mathcal{H}_1 = \mathcal{H}_2$ or

$$\mathcal{H}_1'\mu_1 = \mathcal{H}_2'\mu_2. \tag{44}$$

Equations (27) and (28) then reduce to the same statement, namely,

$$\eta_1 c^2 + p_1 = \eta_2 c^2 + p_2. \tag{45}$$

Multiplying (28) by $-c$, adding (27), and retaining terms of the order $(c - v_1)$, $(c - v_2)$, and $(\mathcal{H}_1 - \mathcal{H}_2)$ one obtains

$$\left.\begin{aligned} &-\eta_1 c^2\left(1 - \frac{v_1}{c}\right) + p_1\left(1 - \frac{v_1}{c}\right) + \frac{\mathcal{H}_1^2}{8\pi} - \frac{v_1\mathcal{H}_1^2}{4\pi c} \\ &= -\eta_2 c^2\left(1 - \frac{v_2}{c}\right) + p_2\left(1 - \frac{v_2}{c}\right) + \frac{\mathcal{H}_2^2}{8\pi} - \frac{v_2\mathcal{H}_2^2}{4\pi c} \end{aligned}\right\}. \tag{46}$$

Using the approximation

$$\mu^2 \approx \tfrac{1}{2}(1 - v/c)^{-1} \tag{47}$$

and the equation

$$\frac{H_1^2}{8\pi} - \frac{v_1\mathcal{H}_1^2}{4\pi c} = \left(1 - \frac{v_1}{c}\right)^2 \frac{\mathcal{H}_1^2}{8\pi} - \frac{v_1^2\mathcal{H}_1^2}{8\pi c^2}. \tag{48}$$

Equation (46) can be written as

$$\frac{-\eta_1 c^2}{(\mu_1)^2} + \frac{p_1}{(\mu_1)^2} + \frac{\mathcal{H}_1^2}{16\pi(\mu_1)^4} = \frac{-\eta_2 c^2}{(\mu_2)^2} + \frac{p_2}{(\mu_2)^2} + \frac{\mathcal{H}_2^2}{16\pi(\mu_2)^4}. \tag{49}$$

Equations (45) and (49) can be written as

$$(\mu_1)^2\eta_1' c^2 + (\mu_1)^2 p_1' = (\mu_2)^2\eta_2' c^2 + (\mu_2)^2 p_2', \tag{50}$$

$$\begin{aligned} &-\eta_1' c^2 + p_1' + (\mathcal{H}_1')^2/16\pi(\mu_1)^2 \\ &= -\eta_2' c^2 + p_2' + (\mathcal{H}_2')^2/16\pi(\mu_2)^2. \end{aligned} \tag{51}$$

Examine now Eqs. (50) and (51) for the special case of a weak shock (sound velocity). In particular, we shall investigate whether $v \approx c$ is consistent with the presence of a cold ideal gas (i.e., non-relativistic equation of state of the matter). Then

$$\eta' c^2 = n' m_0 c^2 + (3n'kT/2) \tag{52}$$

$$p' = n'kT. \tag{53}$$

Hence our equations become

$$d(n'\mu)/d\mu = 0 \tag{54}$$

$$d(\mathcal{H}'\mu)/d\mu = 0 \tag{55}$$

$$\frac{d}{d\mu}\mu^2\left(n' m_0 c^2 + \frac{5}{2}n'kT\right) = 0 \tag{56}$$

$$\frac{d}{d\mu}\left[-\left(n' m_0 c^2 + \frac{1}{2}n'kT\right) + \frac{(\mathcal{H}')^2}{16\pi\mu^2}\right] = 0. \tag{57}$$

Since for our unrelativistic equation of state $kT \ll m_0$, we find that (54) and (56) yield

$$n' m_0 c^2 + \frac{5}{2} \mu n' \frac{d(kT)}{d\mu} = 0. \quad (58)$$

From (54), (55), and (57) it follows that

$$n' m_0 c^2 - \tfrac{1}{2} \mu n' [d(kT)/d\mu] - [(\mathcal{H}')^2 / 4\pi\mu^2] = 0. \quad (59)$$

Eliminating $d(kT)/d\mu$ from (58) and (59) one obtains

$$(\mathcal{H}')^2 / 8\pi = 3\mu^2 n' m_0 c^2 / 5. \quad (60)$$

Thus it is seen that for $v \approx c$ there is indeed a possibility that the material stays cold provided the magnetic energy is considerably greater than the rest energy. Equations (60) and (41) show that the sound velocity approaches light velocity for both the unrelativistic and the relativistic ideal gas, provided that the magnetic energy density becomes large compared with the material energy density.

We shall now enquire in greater generality whether the case of a predominant magnetic field will indeed have as a consequence that the propagation velocity is close to the velocity of light. We assume that $(1 - v_1/c)$ is not small compared with unity and we shall show that (except for quite peculiar equations of state) this is incompatible with the assumption that $\mathcal{H}^2/8\pi$ is large compared with the material energy density.

If the magnetic terms dominate comparison of (20), (27) and (28) shows that $v_1 \approx v_2$ and $\mathcal{H}_1 \approx \mathcal{H}_2$. If in addition $(1 - v_1/c) \ll 1$ and also $\mu^2 \gg 1$ do not hold, it follows that

$$\left.\begin{aligned} v &= \tfrac{1}{2}(v_1 + v_2) \approx v_1 \approx v_2 \\ \mu^2 &= \tfrac{1}{2}(\mu_1^2 + \mu_2^2) \approx \mu_1^2 \approx \mu_2^2 \\ \mathcal{H} &= \tfrac{1}{2}(\mathcal{H}_1 + \mathcal{H}_2) \approx \mathcal{H}_1 \approx \mathcal{H}_2. \end{aligned}\right. \quad (61)$$

Equations (13), (20), (27), and (28) can be written in powers of the quantities $\Delta v = v_1 - v_2$, $\Delta\eta = \eta_1 - \eta_2$, $\Delta p = p_1 - p_2$ and $\Delta\mathcal{H} = \mathcal{H}_1 - \mathcal{H}_2$ as follows:

$$\mathcal{H}\Delta v + v\Delta\mathcal{H} = 0 \quad (62)$$

$$v^2\Delta\eta + v(\eta_1 + \eta_2)\Delta v + \tfrac{1}{4}\Delta\eta(\Delta v)^2 + \Delta p + (\mathcal{H}\Delta\mathcal{H}/4\pi) = 0 \quad (63)$$

$$\left.\begin{aligned} v\Delta\eta + \frac{\eta_1+\eta_2}{2}\Delta v + \frac{v}{c^2}\Delta p + \frac{(p_1+p_2)}{2c^2}\Delta v \\ + \frac{v\mathcal{H}\Delta\mathcal{H}}{4\pi c^2} + \frac{\Delta v(\Delta\mathcal{H})^2}{16\pi c^2} = 0 \end{aligned}\right\}. \quad (64)$$

Multiplying (64) by v and subtracting from (63) we get

$$\frac{\Delta p}{\mu^2} + \frac{\mathcal{H}\Delta\mathcal{H}}{4\pi\mu^2} = \frac{v\Delta v}{2c^2}\left\{ -(\eta_1+\eta_2)c^2 + (p_1+p_2) + \frac{(\Delta\mathcal{H})^2}{8\pi} \right\} - \frac{\Delta\eta}{4}(\Delta v)^2. \quad (65)$$

The last term on the right-hand side is of higher order and may be neglected. The first term can be transformed with the help of (62)

$$\frac{v\Delta v}{2c^2}\left\{ -(\eta_1+\eta_2)c^2 + (p_1+p_2) + \frac{(\Delta\mathcal{H})^2}{8\pi} \right\} = -\frac{\mathcal{H}\Delta\mathcal{H}}{2}\left(\frac{v^2}{c^2}\right)\left\{ \frac{-(\eta_1+\eta_2)c^2 + (p_1+p_2) + (\Delta\mathcal{H})^2/8\pi}{\mathcal{H}^2} \right\}. \quad (66)$$

The bracket on the right-hand side of (66) is small compared with unity. Thus all of the right-hand side of (65) is negligible and one obtains

$$\Delta p = -\mathcal{H}\Delta\mathcal{H}/4\pi. \quad (67)$$

Next (63) will be multiplied by (v/c^2) and subtracted from (64)

$$\frac{v\Delta\eta}{\mu^2} = -\frac{\Delta v}{2c^2}\left\{ (\eta_1+\eta_2)(c^2 - 2v^2) + (p_1+p_2) + \frac{(\Delta\mathcal{H})^2}{8\pi} \right\} + \frac{v(\Delta v)^2}{4c^2}\Delta\eta. \quad (68)$$

The last term on the right-hand side of (68) is small compared with the left hand side and will be neglected. Transforming the first term in a manner similar to (66) and using (67) leads to the result

$$c^2\Delta\eta = -\tfrac{1}{2}\mu^2\Delta p + \left\{ \frac{(\eta_1+\eta_2)(c^2-2v^2) + (p_1+p_2) + (\Delta\mathcal{H})^2/8\pi}{(\mathcal{H}^2/4\pi)} \right\}. \quad (69)$$

It is seen from (69) that

$$c^2|\Delta\eta| \ll |\Delta p|. \quad (70)$$

Dividing by μ one has

$$c^2|\Delta\eta'| \ll |\Delta p'|. \quad (71)$$

Condition (71) does not seem to be in direct contradiction to any general law but it requires a somewhat peculiar equation of state. For the extreme relativistic ideal gas Eq. (32) states that $c^2\Delta\eta' = 3\Delta p'$, which contradicts (71). In general, using (4), and introducing

$$\left.\begin{aligned} n' &= \tfrac{1}{2}(n_1' + n_2') \approx n_1' \approx n_2' \\ \Delta n' &= n_1' - n_2' \\ \Delta E' &= E_1' - E_2' \end{aligned}\right\} \quad (72)$$

we get

$$c^2\Delta\eta' = [m_0c^2 + \tfrac{1}{2}(E_1' + E_2')]\Delta n' + n'\Delta E'. \quad (73)$$

With the help of (41), (62), and (67) the quantity $\Delta n'$ can be expressed in terms of $\Delta p'$.

$$c^2\Delta\eta' = -\frac{\mu^2 n'[m_0c^2 + \frac{1}{2}(E_1' + E_2')]}{[(\mathcal{H}')^2/4\pi]}\Delta p' + n'\Delta E'. \quad (74)$$

Here the coefficient of $\Delta p'$ is small compared with unity and it follows from (74) and (71) that

$$n'|\Delta E'| \ll |\Delta p'|. \tag{75}$$

Remembering that $(\Delta n'/n') \ll 1$ and applying (71) to the non-relativistic case one sees that we are indeed led to this strange conclusion: The particle density changes little and the energy density (exclusive of rest energy) changes by a negligible amount compared to the change in pressure. This is not absurd but quite unusual.

Thus for the extreme relativistic ideal gas and for most non-relativistic equations of state one may conclude that $(1-v_1/c) \ll 1$ is not compatible with predominance of magnetic field energies. It seems the rule that the propagation velocity of the shock approaches c as $(\mathcal{H}')^2/8\pi$ becomes the leading term in the energy density.

NON-RELATIVISTIC PERPENDICULAR SHOCK

Equation (27) may be written as

$$\frac{n_1'(m_0'+E_1'/c^2)v_1^2+p_1'+(\mathcal{H}_1')^2/8\pi}{1-(v_1/c)^2} = \frac{n_2'(m_0'+E_2'/c^2)v_2^2+p_2'+(\mathcal{H}_2')^2/8\pi}{1-(v_2/c)^2}. \tag{76}$$

To obtain the non-relativistic approximation we neglect all terms containing c^{-2}. Setting the material density

$$\rho' = n'm_0' \tag{77}$$

and dropping the primes for simplicity in the non-relativistic case we get the equation

$$p_1+\rho_1 v_1^2+(\mathcal{H}_1)^2/8\pi = p_2+\rho_2 v_2^2+(\mathcal{H}_2)^2/8\pi, \tag{78}$$

which expresses conservation of momentum.

If in Eq. (28) all terms containing c^{-2} should be dropped, the result would be identical with the conservation of matter, Eq. (13),

$$\rho_1 v_1 = \rho_2 v_2. \tag{79}$$

Expanding (14) and (28) in powers of v^2/c^2 and subtracting the highest remaining terms gives the non-relativistic approximation to the energy equation

$$E_1\rho_1 v_1+\tfrac{1}{2}\rho_1 v_1^3+p_1 v_1+v_1(\mathcal{H}_1)^2/4\pi = E_2\rho_2 v_2+\tfrac{1}{2}\rho_2 v_2^3+p_2 v_2+v_2(\mathcal{H}_2)^2/4\pi. \tag{80}$$

We may derive the sound velocity in this case from Eqs. (78) and (80). By reasoning analogous to that presented in obtaining Eqs. (34) and (35) we find that

$$(dp/dv)+2\rho_0 v+(v^2 d\rho_0/dv)-(\mathcal{H})^2/4\pi v = 0 \tag{81}$$

$$\rho_0 v+v^2 d\rho_0/dv = 0 \tag{82}$$

and hence

$$v = \left(\frac{dp/dv}{d\rho_0/dv}+\frac{\mathcal{H}^2}{4\pi\rho_0}\right)^{+\frac{1}{2}} \tag{83}$$

which reduces to the familiar expression for the sound velocity in case the magnetic field vanishes. In case the magnetic field dominates, (83) reduces to

$$v = \mathcal{H}/(4\pi\rho_0)^{\frac{1}{2}}. \tag{84}$$

This formula for the sound velocity looks like the Alfvén velocity. This agreement is fortuitous. The derivation of the Alfvén velocity is given later under the section treating the oblique shock.

Equations (78) and (80) can be simplified by the substitutions

$$p_1^* = p_1+(\mathcal{H}_1)^2/8\pi \tag{85}$$

$$E_1^* = E_1+(\mathcal{H}_1)^2/8\pi\rho_1 \tag{86}$$

with analogous equations for p_2^* and E_2^*. In fact, the quantities p^* and E^* are the sums of material and magnetic pressures and energies.

Equations (78) and (80) then become the familiar Rankine-Hugoniot equations with p and E replaced by their respective starred quantities which include the appropriate electromagnetic contribution. A solution of these equations yields the familiar shock equations.

$$v_1-v_2 = \left[(p_2^*-p_1^*)\left(\frac{1}{\rho_1}-\frac{1}{\rho_2}\right)\right]^{\frac{1}{2}} \tag{87}$$

$$E_2^*-E_1^* = \left(\frac{1}{\rho_1}-\frac{1}{\rho_2}\right)\left(\frac{p_1^*+p_2^*}{2}\right). \tag{88}$$

IDEAL GAS

Let us now specialize the previous section for the case of an ideal gas; i.e., let

$$E = p/(\gamma-1)\rho. \tag{89}$$

Here γ satisfies the inequality $1<\gamma\leqslant 3/5$. Using (17) we define the constant

$$\nu = \mathcal{H}_1/\rho_1 = \mathcal{H}_2/\rho_2. \tag{90}$$

Then we may verify that the following relation holds between E^* and p^*:

$$E^* = \frac{p^*}{(\gamma-1)\rho}+\frac{(\gamma-2)}{(\gamma-1)}\frac{\rho\nu^2}{8\pi}. \tag{91}$$

Substituting (91) into (88) we find

$$p_2^*\left[\frac{1}{2\rho_1}-\frac{1}{2\rho_2}-\frac{1}{(\gamma-1)\rho_2}\right]-p_1^*\left[\frac{1}{2\rho_2}-\frac{1}{2\rho_1}-\frac{1}{(\gamma-1)\rho_1}\right] = \frac{\nu^2}{8\pi}\frac{(\gamma-2)}{(\gamma-1)}(\rho_2-\rho_1). \tag{92}$$

In ordinary shock hydrodynamics of an ideal gas for $\rho_2>\rho_1$ it follows that $p_2>p_1$ and $(\rho_2/\rho_1)<(\gamma+1)/(\gamma-1)$. Thus $(\gamma+1)/(\gamma-1)$ is the maximum compression obtainable. Similar relations are true in magneto-hydrodynamic shocks. For this purpose we set

$$\alpha=(\gamma+1)/(\gamma-1) \tag{93}$$

$$\beta=-\left(\frac{\nu^2}{4\pi}\right)\frac{\gamma-2}{\gamma-1}. \tag{94}$$

We note that $\alpha>1$ and that for the case of interest to us, namely, $\gamma<2$, the expression $\beta>0$. Equation (92) may then be written as

$$p_2^*\left[\frac{\alpha}{\rho_2}-\frac{1}{\rho_1}\right]-p_1^*\left[\frac{\alpha}{\rho_1}-\frac{1}{\rho_2}\right]=\beta(\rho_2-\rho_1). \tag{95}$$

Assuming $(\rho_2-\rho_1)>0$ we see that $(\alpha/\rho_1-1/\rho_2)>0$ and therefore from (95), $(\alpha/\rho_2-1/\rho_1)>0$ or

$$(\rho_2/\rho_1)<\alpha=(\gamma+1)/(\gamma-1). \tag{96}$$

Since, furthermore, $(\alpha/\rho_2)-(1/\rho_1)<(\alpha/\rho_1)-(1/\rho_2)$ follows from $\rho_2>\rho_1$, it can be further seen from (95) that $p_2^*>p_1^*$.

STABILITY AND CHANGE OF ENTROPY

We shall now investigate whether a compressive shock $\rho_2>\rho_1$ is thermodynamically stable, i.e., whether the entropy increases in a compressive shock. For this purpose plot p_2^* against $1/\rho_2$ for given p_1^* and ρ_1 as indicated in Fig. 1. We introduce the notation $1/\rho=x$. Point 1 represents the initial condition of the gas. Then we draw the locus of all possible shocks with 1 as the initial point. Examples of such points are k and l. Let these be infinitesimally close. Now consider the difference dS of entropy between points k and l. This is given by

$$dS=dQ/T, \tag{97}$$

where dQ is the heat added to get from k to l:

$$dQ=E_l^*-E_k^*+p_k^*(x_l-x_k) \tag{98}$$

but from Eq. (88)

$$E_l^*-E_k^*=\tfrac{1}{2}(p_l^*-p_k^*)(x_1-x_k)-\tfrac{1}{2}(p_1^*+p_k^*)(x_l-x_k) \tag{99}$$

so that

$$dQ=\tfrac{1}{2}[(p_l^*-p_k^*)(x_1-x_k)+(p_k^*-p_1^*)(x_l-x_k)]. \tag{100}$$

Consider now a move from point k to point l. Then (x_l-x_k) is negative and dp_2^* is positive. Moreover, *if* the curve is everywhere concave then

$$(x_1-x_k)/(p_k^*-p_1^*)>(x_k-x_l)/(p_l^*-p_k^*) \tag{101}$$

so that dQ_2 and hence dS is positive. If now we consider another point j, then we can repeat the argument for this point. Thus eventually we can by differential regression arrive at point 1. Hence point l has higher entropy than point 1 provided that the curve is concave throughout. This argument is analogous to the usual argument in hydrodynamics, except that of course E^* is replaced by E and p^* by p.

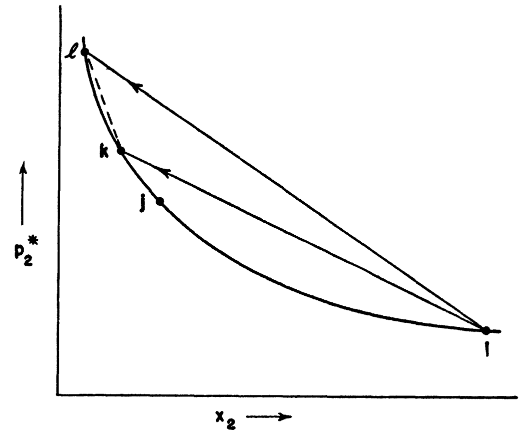

FIG. 1. The pressure p_2^* *versus* $1/\rho_2$ in a compressive shock.

We shall now prove that indeed for an ideal gas a plot of p_2^* *vs.* x_2 is concave throughout. From Eq. (95) we find

$$\frac{d^2p_2^*}{dx_2^2}=\frac{2\alpha(\alpha p_2^*+p_1^*)}{(\alpha x_2-x_1)^2}+\frac{2\beta}{x_2^2}\left\{\frac{\alpha}{(\alpha x_2-x_1)^2}+\frac{1}{x_2(\alpha x_2-x_1)}\right\}. \tag{102}$$

However, Eq. (96) shows that in general $(x_1/x_2)<\alpha$, so that $(\alpha x_2-x_1)>0$. Hence $d^2p_2^*/dx_2^2>0$ at all points and thus the entropy indeed increases in a compressive shock for an ideal gas.

In the limiting case of sound waves the entropy tends to zero. It is of interest to investigate the entropy change for weak shocks.[6] Let the entropy S and the inverse density x be the independent variables and expand E_2^* and p_2^* in powers of $\Delta x=x_2-x_1$, and $\Delta S=S_2-S_1$. Then

$$E_2^*=E_1^*+(\partial E^*/\partial x)_S\Delta x+\tfrac{1}{2}(\partial^2E^*/\partial x^2)_S(\Delta x)^2+\tfrac{1}{6}(\partial^2E^*/\partial x^3)_S(\Delta x)^3+(\partial E^*/\partial S)_{\Delta x}\Delta S+\cdots. \tag{103}$$

Since we have seen that E^* plays the role of the actual total energy and p^* of the pressure, we have from thermodynamics:

$$(\partial E^*/\partial x)_S=-p^* \tag{104}$$

and

$$(\partial E^*/\partial S)_x=T. \tag{105}$$

We can see this in a more formal way by recalling that

$$E^*=E+(\nu^2/8\pi x) \tag{106}$$

$$p^*=p+(\nu^2/8\pi x^2) \tag{107}$$

and carrying out the indicated differentiations in (104) and (105) and using (87) we find

$$\tfrac{1}{6}(\partial^2p^*/\partial x^2)_S(\Delta x)^2-T_1\Delta S/\Delta x=\tfrac{1}{4}(\partial^2p^*/\partial x^2)_S(\Delta x)^2+\tfrac{1}{2}(\partial p^*/\partial S)_x\Delta S. \tag{108}$$

[6] This derivation for the case of conventional hydrodynamics is given in a private communication by R. W. Goranson, entitled "Seminar Notes," dated May 13, 1949.

For small Δx

$$T_1/\Delta x \gg \tfrac{1}{2}(\partial p^*/\partial S)_x \tag{109}$$

holds and we get

$$\Delta S = -[(\Delta x)^3/12T_1](\partial^2 p^*/\partial x^2)_S. \tag{110}$$

Now for a compressive wave $(\Delta x)^3$ is negative and for the special case of the ideal gas we have shown that

$$d^2p_2^*/dx^2 > 0 \tag{111}$$

so that the ΔS is positive, which is in accord with the general proof previously given.

ENERGY DISSIPATION IN A STRONG SHOCK

In ordinary hydrodynamics strong shocks transform a considerable portion of the kinetic energy into heat. It is of interest to determine whether in the presence of a magnetic field this continues to be true or whether the magnetic field can act as a sort of cushion producing elastic recoil where normal hydrodynamics would predict a strongly inelastic collision. For the sake of simplicity only transverse shocks are considered.

Let us therefore investigate whether the following assumptions are consistent. First, we have a strong shock, i.e., the relation $|v_1-v_2| \ll |v_1|$ does not hold, and second,

$$\left.\begin{aligned} E_1 &\ll \rho_1 v_1^2 \\ p_1 &\ll \rho_1^2 v_1^2 \end{aligned}\right\} \tag{112}$$

and

$$\left.\begin{aligned} E_2 &\ll \rho_2 v_2^2 \\ p_2 &\ll \rho_2^2 v_2^2 \end{aligned}\right\}. \tag{113}$$

Now we make use of these assumptions in (78) and (80) and obtain

$$\rho_1 v_1^2 + (\mathcal{H}_1)^2/4\pi = \rho_2 v_2^2 + (\mathcal{H}_2)^2/4\pi \tag{114}$$

$$\tfrac{1}{2}\rho_1 v_1^3 + (\mathcal{H}_1)^2 v_1/8\pi = \tfrac{1}{2}\rho_2 v_2^3 + (\mathcal{H}_2)^2 v_2^2/8\pi. \tag{115}$$

Let us, in accordance with (13), set

$$M = \rho_1 v_1 = \rho_2 v_2 \tag{116}$$

and divide (114) and (115) by M^2. Then, using (90)

$$\frac{1}{M}(v_1-v_2) = \frac{\nu^2}{8\pi}\left(\frac{1}{v_1^2}-\frac{1}{v_2^2}\right) \tag{117}$$

$$\frac{1}{2M}(v_1^2-v_2^2) = \frac{\nu^2}{4\pi}\left(\frac{1}{v_2}-\frac{1}{v_1}\right). \tag{118}$$

We note that in (117) and (118) the pressure and internal energy terms are neglected as compared with terms containing difference of velocities. This is justified because according to the definition of a strong shock (v_1-v_2) is not very much less than v_1 and hence the differences retained in (117) and (118) are not small.

Dividing now (118) into (117) we find

$$\frac{2}{v_1+v_2} = \frac{1}{2}\left(\frac{1}{v_1}+\frac{1}{v_2}\right) \tag{119}$$

or

$$4 = 2 + (v_1/v_2) + (v_2/v_1). \tag{120}$$

Equation (120) is satisfied only if $v_1 = v_2$; i.e., for sound velocity and *not* for a strong shock. Thus the assumptions (112) and (113) require $|v_1-v_2| \ll v_1$ and the magnetic field does not provide the cushioning effect.

V. OBLIQUE SHOCK

In order to simplify the treatment of the oblique shock we shall introduce a convenient coordinate system. Consider first the co-moving system on side 1 of the shock. As usual we shall denote the quantities in this system by primes. Thus $\mathcal{E}_{1,x}' = \mathcal{E}_{1,y}' = \mathcal{E}_{1,z}' = 0$. Further we choose our coordinate system such that the shock front is perpendicular to the x axis and the y axis is so oriented as to make $\mathcal{H}_1'$ lie in the xy plane. In consequence $\mathcal{H}_{1,z}' = 0$. We shall assume obliquity of the shock so that $\mathcal{H}_{1,x}' \neq 0$ and $\mathcal{H}_{1,y}' \neq 0$. Now make a transformation to a system in which the shock is stationary and whose velocity relative to the primed system is parallel to the magnetic lines of force.[7] This "shock system" will be denoted by unprimed symbols. In particular then

$$\mathcal{H}_{1,x}/\mathcal{H}_{1,y} = v_{1,x}/v_{1,y} = \phi_1. \tag{121}$$

Since $\mathbf{v}_1$ and $\mathcal{H}_1$ are parallel, it follows that

$$\mathcal{E}_{1,x} = \mathcal{E}_{1,y} = \mathcal{E}_{1,z} = 0. \tag{122}$$

Because $\mathcal{H}_1$ lies in the xy plane, we find $\mathcal{H}_{1,z} = 0$ and $v_{1,z} = 0$. Since our transformation is such that the translation is along the lines of force, the direction or magnitude of the magnetic field does not change, i.e.,

$$\mathcal{H}_{1,x} = \mathcal{H}_{1,x}' \neq 0, \quad \mathcal{H}_{1,y} = \mathcal{H}_{1,y}' \neq 0. \tag{123}$$

In order to find the electromagnetic fields on the other side of the shock we note that the equation $\mathrm{div}\mathcal{H} = 0$ becomes

$$\mathcal{H}_{1,x} = \mathcal{H}_{2,x} \neq 0. \tag{124}$$

The particular simplicity of our coordinate system lies in the fact that

$$\mathcal{E}_{2,x} = \mathcal{E}_{2,y} = \mathcal{E}_{2,z} = 0. \tag{125}$$

This can be seen as follows. Consider the co-moving system for side 2. In this system $\mathcal{E}_2' = 0$ and hence $\mathcal{E}_2' \cdot \mathcal{H}_2' = 0$. Since the quantity $\mathcal{E} \cdot \mathcal{H}$ is a relativistic invariant, we find $\mathcal{E}_2 \cdot \mathcal{H}_2 = 0$. Thus $\mathcal{E}_2$ is perpendicular to $\mathcal{H}_2$. Furthermore, the shock system is stationary so that $\dot{\mathcal{H}} = 0$ and hence $\mathrm{curl}\,\mathcal{E} = 0$ at the shock front. Since

[7] It should be noted that such a transformation obviously cannot be made for the limiting case of the perpendicular shock. In the latter case the magnetic lines of force are perpendicular to the x direction and thus along the shock front. Hence no motion along the shock front can bring the shock front to rest.

$\mathcal{E}_1=0$, it follows that $\mathcal{E}_2$ is perpendicular to the shock front. Considering $\mathcal{H}_{2,x}\neq 0$, the vector $\mathcal{E}_2$ cannot be perpendicular to both the shock front and the magnetic field. We must therefore (as stated in (125)) set $\mathcal{E}_2=0$. In order that no electric fields arise on side 2 because of the transformation from the primed to the unprimed system, $\mathbf{v}_2$ must be parallel to $\mathcal{H}_2$.

$$\mathcal{H}_{2,x}/\mathcal{H}_{2,y}=v_{2,x}/v_{2,y}=\phi_2, \tag{126}$$

$$\mathcal{H}_{2,z}/\mathcal{H}_{2,y}=v_{2,z}/v_{2,y}=\phi_3. \tag{127}$$

The picture obtained in our particular coordinate system is the following. The shock is stationary. Material streams in from one side and out on the other. The streamlines are oblique to the shock front. One expects that the streamlines are refracted in the shock front. They are everywhere parallel to the magnetic lines of force which, therefore, are similarly refracted. It is known that no oblique shocks are possible in an isotropic medium in the absence of magnetic fields because the shock front can transfer momentum to the streaming material only in a direction perpendicular to the front. The refraction of magnetic lines makes this type of momentum transfer possible.

To obtain the quantitative effect of this momentum transfer we now evaluate the T_{xn} components of the energy momentum tensor. For the system with a velocity $\mathbf{v}$ relative to the co-moving system one finds

$$T_{xx}=p'+(v_x^2/c^2)\mu^2(p'+c^2\eta')+(\mathcal{H}^2/8\pi)-(\mathcal{H}_x)^2/4\pi \tag{128}$$

$$T_{xy}=(v_xv_y/c^2)\mu^2(p'+c^2\eta')-(\mathcal{H}_x\mathcal{H}_y/4\pi) \tag{129}$$

$$T_{xz}=(v_xv_z/c^2)\mu^2(p'+c^2\eta')-(\mathcal{H}_x\mathcal{H}_z/4\pi) \tag{130}$$

$$T_{xt}=(-v_x/c^2)\mu^2(p'+c^2\eta'). \tag{131}$$

In (128) through (130) we have written $\mathcal{H}$ instead of $\mathcal{H}'$, since it has been shown that in the present case the two vectors are equal. Equations (128) through (131) apply to side 1 or 2 of the shock depending on whether all quantities carry subscript 1 or 2. Because of our assumptions, $T_{1,xz}=0$ holds and according to (9) it follows that also $T_{2,xz}=0$.

The most natural way in which to satisfy these equations is to assume $v_{2,z}=\mathcal{H}_{2,z}=0$. This, indeed, seems to follow from symmetry. We shall see, however, in the further discussion that $v_{2,z}=\mathcal{H}_{2,z}=0$ does not hold in a particular degenerate case. We shall exclude at first such a degeneracy by demanding the following conditions.

Case I, $T_{xy}\neq 0$

Equation (127) and comparison of (129) and (130) show that

$$T_{xz}=v_zT_{xy}/v_y. \tag{132}$$

From $v_{1,z}=\mathcal{H}_{1,z}=0$ we get $T_{1,xz}=0$ and from (9) we get $T_{2,xz}=0$. This last statement is compatible with (132) and $T_{xy}\neq 0$ only if $v_{2,z}=0$. It then follows that $\mathcal{H}_{2,z}$ is also zero and the whole phenomenon takes place in the xy plane.

The equations in this case are fairly involved. We limit ourselves to the derivation of formulas for the non-relativistic sound velocity. In the non-relativistic case[8] Eqs. (128), (129), and (131) read as follows:

$$\rho_1v_{1,x}^2+p_1+(\mathcal{H}_{1,y})^2/8\pi=\rho_2v_{2,x}^2+p_2+(\mathcal{H}_{2,y})^2/8\pi \tag{133}$$

$$\rho_1v_{1,x}v_{1,y}-(\mathcal{H}_{1,x}\mathcal{H}_{1,y}/4\pi)=\rho_2v_{2,x}v_{2,y}-(\mathcal{H}_{2,x}\mathcal{H}_{2,y}/4\pi) \tag{134}$$

$$E_1\rho_1v_{1,x}+\tfrac{1}{2}\rho_1(v_{1,x}^2+v_{1,y}^2)v_{1,x}+p_1v_{1,x}=E_2\rho_2v_{2,x}+\tfrac{1}{2}\rho_2(v_{2,x}^2+v_{2,y}^2)v_{2,x}+p_2v_{2,x}. \tag{135}$$

Equating the expressions for mass flow and the magnetic flux on the two sides of the shock and replacing the difference of the two sides of these equations by differentials we find

$$v_xd\rho+\rho dv_x=0 \tag{136}$$

$$d\mathcal{H}_x=0. \tag{137}$$

From (121) and (126)

$$\mathcal{H}_x/\mathcal{H}_y=v_x/v_y=\phi. \tag{138}$$

Equations (136), (137), and (138) in combination with (133), (134), and (135) then yield

$$-v_x^2+\frac{dp}{d\rho}+\frac{(\mathcal{H}_x)^2}{4\pi}\phi\frac{d\phi}{d\rho}=0 \tag{139}$$

$$-\phi v_x^2+\rho v_x^2\frac{d\phi}{d\rho}-\frac{(\mathcal{H}_x)^2}{4\pi}\frac{d\phi}{d\rho}=0 \tag{140}$$

$$\rho v_x\frac{dE}{d\rho}+v_x^3\left(-1+\phi\frac{d\phi}{d\rho}\rho-\phi^2\right)-\frac{v_x}{\rho}p+v_x\frac{dp}{d\rho}=0. \tag{141}$$

Substituting the values of $dp/d\rho$ and $d\phi/d\rho$ obtained from (139) and (140) into (141) we find that (141) reduces merely to the adiabatic condition

$$dE/d\rho=p/\rho^2. \tag{142}$$

Elimination of $d\phi/d\rho$ between Eqs. (139) and (140) yields the desired equation for v_x^2

$$4\pi\rho(v_x^2)^2-\left(4\pi\rho\frac{dp}{d\rho}+\mathcal{H}_x^2+\phi^2\mathcal{H}_x^2\right)v_x^2+\mathcal{H}_x^2\frac{dp}{d\rho}=0. \tag{143}$$

Thus the general expression for v_x^2 is given by

$$v_x^2=\frac{4\pi\rho(dp/d\rho)+\mathcal{H}_x^2(1+\phi^2)\pm[\{4\pi\rho(dp/d\rho)+(\mathcal{H}_x)^2(1+\phi^2)\}^2-16\pi\rho(\mathcal{H}_x)^2(dp/d\rho)]^{\frac{1}{2}}}{8\pi\rho}. \tag{144}$$

[8] Since we are discussing the non-relativistic case, primes can be dropped.

It might be noted that the sum of the solutions (where the two solutions are denoted by a subscript + or −) is given by

$$(v_x^2)_+ + (v_x^2)_- = (dp/d\rho) + (\mathcal{H}^2/4\pi\rho), \tag{145}$$

which is the sum of the squares of the ordinary sound velocity and the Alfvén velocity (84). The product of the solutions of (144) yields

$$(v_x)_+ (v_x)_- = (\mathcal{H}_x^2/4\pi\rho)(dp/d\rho), \tag{146}$$

which is the product of the squares of the ordinary velocity and the Alfvén velocity for $\mathcal{H}_x$.

Two special cases of (144) are of particular interest to us. The first case is $\mathcal{H}_y \to 0$. There (144) reduces to

$$(v_x^2)_+ = dp/d\rho \tag{147}$$

$$(v_x^2)_- = (\mathcal{H}_x)^2/4\pi\rho. \tag{148}$$

The first solution is the familiar velocity for the parallel longitudinal shock in hydrodynamics. The second solution corresponds to the transverse shock and is indeed identical with that derived by Alfvén[2] for this case.

The second case of interest to us is that where $\mathcal{H}_x \to 0$; then (144) reduces to

$$8\pi\rho v_x^2 = 4\pi\rho(dp/d\rho) + \mathcal{H}_y^2 \pm [4\pi\rho(dp/d\rho) + \mathcal{H}_y^2] \tag{149}$$

and

$$(v_x^2)_+ = 0 \tag{150}$$

$$(v_x^2)_- = (dp/d\rho) + (\mathcal{H}_y)^2/4\pi\rho. \tag{151}$$

The first solution is the hydrodynamic transverse velocity which (as in normal hydrodynamics) is equal to zero. The second of these solutions is the magnetohydrodynamic velocity for the perpendicular longitudinal shock previously expressed by Eq. (83).

Case II. $T_{xy} = 0$

In this case (132) shows that $T_{2,xz} = 0$ is compatible with $v_z \neq 0$, $\mathcal{H}_z \neq 0$. The assumption $T_{xy} = 0$ will be used in the form

$$(v_x v_y/c^2)\mu^2(p' + c^2\eta') = \mathcal{H}_x\mathcal{H}_y/4\pi. \tag{152}$$

Using (121), (126), and (127) it follows that

$$(v_x^2/c^2)\mu^2(p' + c^2\eta') = (\mathcal{H}_x)^2/4\pi. \tag{153}$$

Since $\mathcal{H}_{1,x} = \mathcal{H}_{2,x}$, the left side of (153) has the same value for subscripts 1 and 2:

$$[(v_{1,x})^2/c^2]\mu_1^2(p_1' + c^2\eta_1') = [(v_{2,x})^2/c^2]\mu_2^2(p_2' + c^2\eta_2'). \tag{154}$$

The condition $T_{1,xt} = T_{2,xt}$ and (131) leads to

$$(v_{1,x}/c^2)\mu_1^2(p_1' + c^2\eta_1') = (v_{2,x}/c^2)\mu_2^2(p_2' + c^2\eta_2'). \tag{155}$$

Combining (154) and (155) yields

$$v_{1,x} = v_{2,x} \tag{156}$$

and

$$\mu_1^2(p_1' + c^2\eta_1') = \mu_2^2(p_2' + c^2\eta_2'). \tag{157}$$

Thus from (128)

$$p_1' + (1/8\pi)(\mathcal{H}_{1,y})^2 = p_2' + (1/8\pi)[(\mathcal{H}_{2,y})^2 + (\mathcal{H}_{2,z})^2]. \tag{158}$$

Suppose now that we find a solution that is satisfied for a given $\mathcal{H}_{2,z}$ and corresponding $v_{2,z}$. We shall verify below that in this case there exists an entire set of solutions in which the values of $[(\mathcal{H}_{2,z})^2 + (\mathcal{H}_{2,y})^2]$ and $[(v_{2,z})^2 + (v_{2,y})^2]$ are maintained and μ_2, p_2 and η_2 are also unchanged. Physically this means that for each outgoing velocity $\mathbf{v}_2$ there is a whole set of solutions such that the $\mathbf{v}_2$'s describe a cone with the perpendicular to the shock front as the center line. We can easily verify that all the values corresponding to this cone are indeed solutions; $T_{2,xx}$ as expressed by Eq. (158) is unchanged, since $[(\mathcal{H}_{2,z})^2 + (\mathcal{H}_{2,y})^2]$ is maintained; $T_{2,xy}$ is still zero, since v_y and $\mathcal{H}_y$ are changed in proportion; $T_{2,xz}$ is still zero, since v_z and $\mathcal{H}_z$ are changed in proportion, and, lastly, $T_{2,xt}$ is not affected, since it does not depend on y or z components. Thus we see that if the condition $T_{1,xy} = 0$ is satisfied in the incident stream, the solution is degenerate and the direction of the outgoing stream is indeterminate. In this way out-of-the-plane solutions are compatible with the symmetry of the incident stream.

In order to simplify further discussion we shall consider here only the case $\mathcal{H}_{2,z} = v_{2,z} = 0$ and recognize that any such solution really corresponds to a cone of solutions. Then (158) reduces to

$$p_1' + (1/8\pi)[(\mathcal{H}_{1,x})^2 + (\mathcal{H}_{1,y})^2] = p_2' + (1/8\pi)[(\mathcal{H}_{2,x})^2 + (\mathcal{H}_{2,y})^2]. \tag{159}$$

Use of Eqs. (154) and (126) yields

$$p_1' + \tfrac{1}{2}(v_1/c)^2\mu_1^2(p_1' + c^2\eta_1') = p_2' + \tfrac{1}{2}(v_2/c)^2\mu_2^2(p_2' + c^2\eta_2'). \tag{160}$$

Since $(v/c)^2\mu^2 = \mu^2 - 1$, the use of Eq. (157) yields

$$p_1' - c^2\eta_1' = p_2' - c^2\eta_2'. \tag{161}$$

Thus Eqs. (157) and (161) together with the equation of state determine the nature of the solution. Clearly, a particular solution satisfying (157) and (161) is given by

$$\left.\begin{aligned} p_1' &= p_2' \\ \eta_1' &= \eta_2' \\ \mu_1^2 &= \mu_2^2 \end{aligned}\right\}. \tag{162}$$

Equations (162) then imply (for the case of $v_{2,z} = \mathcal{H}_{2,z} = 0$) that

$$v_{1,y} = \pm v_{2,y}, \quad \mathcal{H}_{1,y} = \pm\mathcal{H}_{2,y}. \tag{163}$$

The positive sign corresponds to no shock, the case of the negative sign we shall denote by the term "symmetrical shock." For the symmetrical shock $T_{1,xy}=T_{2,xy}$ implies that both these quantities vanish. Using (129), (121), and the definition of μ^2 one obtains for the velocity of the symmetrical shock[9]

$$v=\left[\frac{1}{c^2}+\frac{4\pi}{\mathcal{H}^2}\left(\frac{p'}{c^2}+\eta'\right)\right]^{-\frac{1}{2}}. \tag{164}$$

Note that Eq. (164) states that if $\mathcal{H}\rightarrow\infty$, then $v=c$.

Examine now whether (162) is the only condition satisfying Eqs. (157) and (161) or whether there are other solutions. Since $v_{1,x}=v_{2,x}$, the equation expressing the conservation of particles is

$$\mu_1 n_1'=\mu_2 n_2'. \tag{165}$$

Hence, use of Eq. (4) permits the rewriting of (161) and (157), respectively, as

$$n_1'm_0c^2+n_1'E_1'-p_1'=n_2'm_0c^2+n_2'E_2'-p_2' \tag{166}$$

$$\frac{m_0c^2}{n_1'}+\frac{n_1'E_1'+p_1'}{(n_1')^2}=\frac{m_0c^2}{n_2'}+\frac{n_2'E_2'+p_2'}{(n_2')^2}. \tag{167}$$

Multiplying Eq. (167) by $n_1'n_2'$, Eqs. (166) and (167) can be written as

$$m_0c^2\Delta n'=-\Delta(n'E')+\Delta p' \tag{168}$$

$$n_0c^2\Delta n'=+\Delta(n'E')+(p_1'n_2'/n_1')-(p_2'n_1'/n_2'). \tag{169}$$

We now assume that the equation of state is nonrelativistic on both sides of the shock. If we divide (168) by nm_0c^2, then we notice that

$$[-\Delta(n'E')+\Delta p']/nm_0c^2\ll 1 \tag{170}$$

so that

$$\Delta n'/n'\ll 1. \tag{171}$$

Hence Eq. (169) may be rewritten as

$$m_0c^2\Delta n'=+\Delta(n'E')+\Delta p'. \tag{172}$$

Combining (170) with (168) we find

$$\Delta p'\gg n'(\Delta E'). \tag{173}$$

Note that Eqs. (172) and (71) are identical. We have seen that the latter equation together with (171) requires a peculiar equation of state. Therefore, the symmetrical shock and the associated cone of solutions are as a general rule the only ones compatible with $T_{xy}=0$ and an unrelativistic equation of state.

If on both sides of the shock the extreme relativistic ideal gas law holds, $3p'=c^2\eta'$. This is evidently inconsistent with (161) except for $p_1'=p_2'$ and $\eta_1'=\eta_2'$. Thus we are brought back to (162).

[9] This agrees with the velocity given by H. Alfvén in his book, *Cosmical Electrodynamics* (Oxford University Press, London, 1950), p. 85, for a somewhat more specialized case.