PLASMA DIFFUSION IN A TOROIDAL STELLARATOR*

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Collisional diffusion in long-mean-free-path, geometrically nonsymmetric regimes is characterized by Bohm-like scaling laws. A variational principle is found for calculation of diffusion constants.

In toroidal confinement geometries with rotational transform, the magnetic field can be designed to form surfaces. The trajectories of the particles, in general, form drift surfaces that lie roughly within a gyroradius of the magnetic surfaces. However, in the case of particular toroidal geometries and particles having small velocities parallel to the magnetic field, the displacements from the magnetic surfaces can be much larger. Collisional diffusion can then lead to a considerable increase of the particle and heat transport relative to the classical expectation.

In axisymmetric toruses or in straight stellarators the conservation of canonical angular momentum limits the particle excursions[1] to the gyroradius times a geometric factor that becomes large only for small rotational transform.[2] The largest excursions are made by low-parallel-velocity particles that are trapped ("banana" orbits), or nearly trapped, in regions of weak magnetic field. The enhanced transport coefficients that result in the axisymmetric case have been calculated by Galeev and Sagdeev,[3] and were found to scale with magnetic field strength and collision frequency like the classical transport coefficients, but to exceed them by factors depending on the torus geometry.

In the present analysis we consider particle motion and associated transport in a toroidal stellarator. We treat the limit of small gyroradius r_g, where excursions and diffusion would become negligible in a symmetric geometry. In the toroidal stellarator, however, canonical angular momentum is not conserved, and the trapped particles can make excursions that remain of finite size as r_g becomes small.[4] The resultant collisional transport coefficients[5,6] no longer scale like the classical coefficients, and can even approximate Bohm scaling in certain parameter ranges.

We will be concerned principally with an $l=3$ stellarator model[4] of small inverse aspect ratio $\epsilon_t = r/R$,

$$B_z = B_0[1-\epsilon_h \cos 3(\theta-\alpha z)-\epsilon_t \cos\theta],$$
$$\epsilon_t \ll \epsilon_h \ll 1,\quad d\epsilon_h/dr = 3\epsilon_h/r, \qquad (1)$$

which is valid near the magnetic axis (the z axis, going along the major circumference). The magnetic surfaces are approximately concentric cylinders. We can use $\vec{B} \approx B_z\hat{z}$ in dealing with the trapped-particle drifts. The particle trapping is due to the ϵ_h term (the contribution of the helical stellarator windings), and the existence of finite excursions is due to the ϵ_t term, which spoils the helical symmetry.

The trapped-particle motions are calculated from conservation of the energy W, the first adiabatic invariant μ, and the second adiabatic invariant

$$J = (2/m)^{1/2}\oint dl[W-e\Phi-\mu B_z]^{1/2}$$
$$= \frac{16}{3\alpha}\left(\frac{\mu B_0\epsilon_h}{m}\right)^{1/2}\left[E(k^2)-(1-k^2) + \frac{\epsilon_t}{2\epsilon_h}\cos\theta K(k^2)\right], \qquad (2)$$

where

$$k^2 = \frac{W-e\Phi-\mu B_0(1-\epsilon_h)}{2\mu B_0\epsilon_h}.$$

We include an electrostatic potential $\Phi(r)$. The quantities K and E are the first and second complete elliptic integrals.

The trapped particles bounce between mirrors at a fast rate $\omega_b = [m\partial J(W,\mu,r)/\partial W]^{-1} \sim \alpha^{-1}(\mu B_0\epsilon_h/m)^{1/2}$, and the resultant bananas drift on constant-J orbits along the loci of the mirror points (i.e., $\theta-\alpha z = \text{const}$). In general, the <u>bananas</u> are untrapped (their drifts encompass the minor circumference); and their radial excursions follow from Eq. (2) by expansion to first order in $r_1 = r - r_0$:

$$r_1 = r_0 \frac{\epsilon_t}{3\epsilon_h} \frac{(\cos\theta_0 - \cos\theta)K(k^2)}{2E(k^2) - K(k^2)(1 + er_0\Phi'/\mu B_0\epsilon_h)}. \quad (3)$$

When Φ' is small, we thus have $r_1 = O(\epsilon_t \epsilon_h^{-1})$; at $k^2 = k_0^2 = 0.83$, however, the denominator in Eq. (3) has a null, and we must expand Eq. (2) to second order in r_1. We then obtain a class of trapped bananas whose orbits (superbananas) are localized in θ, and therefore also in z, and whose maximum radial excursions are large: $r_1 = O(\epsilon_t^{1/2}\epsilon_h^{-1/2})$. In the presence of an electrostatic potential such that $er_0\Phi' > \mu B_0 \epsilon_h$, there are only untrapped bananas, making excursions $r_1 = O(\epsilon_t \mu B_0/e\Phi')$. (A potential of this magnitude has a negligible effect on the trapping of particles.)

The time scale of the banana drifts is obtained from

$$\langle \dot{r} \rangle = \frac{c}{eB_0} \frac{\partial J/r\partial\theta}{\partial J/\partial W} \quad (4)$$

using Eq. (2). For the untrapped bananas, which are of principal interest in what follows, the orbits have approximately the form $r = r_1(\theta + \omega t)$, with $\omega = O(\omega_b^2/\omega_c)$, where ω_c is the cyclotron frequency [cf. Eqs. (6) and (7)]. For the trapped bananas, where the orbits are not so simple, it is convenient to use the third adiabatic invariant (the flux $B_0 A$ through the superbanana) and calculate the inverse period $\omega_{rt}/2\pi = (c/eB_0)[\partial A(W, \mu, J)/\partial W]^{-1}$ by manipulation of Eqs. (2) and (3). This precession rate is reduced from that of the untrapped bananas by $(\epsilon_t/\epsilon_h)^{1/2}$.

We now consider the diffusion caused by collisions acting on the drifting banana guiding centers. The governing equation is the Fokker-Planck equation,[7] the important term being that leading to diffusion in the pitch angle $\cos^{-1}\{1 - (\mu B_0/W)\}^{1/2}$. The diffusion in pitch angle leads to a diffusion in the quantity k^2 of Eq. (3), and the correct diffusion equation is obtained by averaging over a banana period ω_b^{-1}. Thus we require the effective collision frequency ν/ϵ_h to be small compared with ω_b. On the other hand, we will require that ν be large compared with the inverse lifetime of the plasma, so that the plasma distribution function is close to Maxwellian and we may use a linearized collision operator; we also treat the scatterers as Maxwellian.

The latter assumption is strictly correct only for the case of plasma diffusion on an independent population of scatterers. However, because of the nonsymmetric nature of the present geometry and the attendant removal of the constraint due to conservation of canonical angular momentum, the assumption of Maxwellian scatterers appears justified also for the more important case of self-diffusion of charged particles, which we will consider here.

We will specialize, finally, to the case where the existence of trapped banana orbits can be neglected, either because ν/ϵ_h exceeds ω_{rt}, or because a banana-untrapping electrostatic potential arises in the course of ambipolar diffusion. This simplifying assumption is appropriate for the parameter ranges of practical interest. (We note merely that in the contrary case the diffusion falls to zero as ν vanishes.)

Under these conditions on the scattering, and using Eqs. (2), (3), and (4), we obtain the banana kinetic equation

$$\langle \dot{r} \rangle \frac{\partial f_0}{\partial r} + \omega(k^2)\frac{\partial f_1}{\partial\theta} = \frac{\nu}{\epsilon_h}\frac{1}{K(k^2)}\frac{\partial}{\partial k^2}\left[\int_0^{k^2} K(s)ds \frac{\partial f_1}{\partial k^2}\right] = \frac{\nu}{\epsilon_h}\mathfrak{L}f_1; \quad (5)$$

$$\langle \dot{r} \rangle = v_r \sin\theta, \quad v_r = \epsilon_t \frac{\mu c}{er_0}, \quad \omega_r = \epsilon_h \frac{ckT}{eBr_0^2}, \quad (6)$$

$$\omega(k^2) = 3\omega_r x\left\{\frac{2E(k^2)}{K(k^2)} - 1\right\} + \frac{c\Phi'}{r_0 B_0}, \quad x = \frac{\mu B_0}{kT}, \quad (7)$$

$$\nu = \hat{\nu}A(x), \quad \hat{\nu} = \frac{2^{1/2}\pi n e^4 \lambda}{m^{1/2}(kT)^{3/2}}, \quad (8)$$

$$A(x) = x^{-\frac{3}{2}}\left\{\eta' + \eta\left(1 - \frac{1}{2x}\right)\right\},$$

$$\eta(x) = 2\pi^{-\frac{1}{2}}\int_0^x e^{-s}s^{\frac{1}{2}}ds,$$

where λ is the usual logarithm of the ratio of Debye length to minimum impact parameter. Here $\partial f_0/\partial r$ is the equilibrium density gradient, and Eq. (5) is to be solved for f_1, subject to the boundary conditions that it be regular at $k^2 = 0$ and vanish at $k^2 = 1$, which is the limit of particle trapping. The diffusion flux F is obtained by integrating $f_1\langle \dot{r} \rangle$ over k^2,

$$F = \frac{(2\epsilon_h)^{1/2}}{2\pi^2}\int_0^{2\pi} d\theta \int_0^1 dk^2 K(k) f_1 \langle \dot{r} \rangle. \quad (9)$$

The problem is solved by writing $f_1 = X\cos\theta + Y\sin\theta$ and setting up a maximal variational principle for the diffusion coefficient $D = -F(\partial f_0/\partial r)^{-1}$,

$$D = \frac{\frac{\epsilon_h}{2}^{1/2} \frac{v_r^{\,2}}{\pi}\left(\frac{\nu}{\epsilon_h}\right)\left[\int_0^1 \frac{1}{\omega}\frac{d}{dk^2}\{X'\int_0^{k^2}K(s)ds\}dk_2\right]^2}{\int_0^1\{\int_0^{k^2}K(s)ds\}\left[\left(\frac{\nu}{\epsilon_h}\right)^2\left\{\frac{d}{dk^2}\left(\frac{1}{\omega K}\frac{d}{dk^2}[X'\int_0^{k^2}K(s)ds]\right)\right\}^2+(X')^2\right]dk^2}, \tag{10}$$

where $X' = dX/dk^2$. At $k^2 = 1$, we require $X = \mathfrak{L}X = (\nu/\epsilon_h)^2\mathfrak{L}\omega^{-1}\mathfrak{L}X - v_r\partial f_0/\partial r = 0$; while at $k^2 = 0$, we require that X and $\mathfrak{L}X$ be regular and that $(\nu/\epsilon_h)^2\mathfrak{L}\omega^{-1}\mathfrak{L}X - v_r\partial f_0/\partial r = 0$. To find the overall diffusion coefficient, D should be maximized for each energy μB_0 and then averaged over μB_0.

The variational principle (10) can now be applied to various cases of interest. If the collision frequency is in the range $\hat{\nu}/\epsilon_h < \omega_r$ (but still $>\omega_{rt}$), and if there is no electric field, then the maximizing trial function X' is peaked around $k^2 = 0.83$, where ω vanishes, and we find

$$D = D_0 = 0.46\frac{\epsilon_t^{\,2}}{\epsilon_h^{\,1/2}}\frac{ckT}{eB} = 7.3\frac{\epsilon_t^{\,2}}{\epsilon_h^{\,1/2}}D_{\text{Bohm}}. \tag{11}$$

This rate is independent of particle species and collision rate, within the limitations assumed above, and for practical parameters does not lie too far below the Bohm value. The limitations on collision frequency, however, are seldom satisfied in practice for both ions and electrons simultaneously.

The usual case for both present-day experiments and fusion applications is that the ion collision rate lies near the range envisaged above, but the electron collision rate is somewhat too high. Equal diffusion is attained only with the appearance of an electrostatic potential $\Phi(r,\theta,z)$, in which the θ and z dependence can be important.[8] For present purposes, we confine ourselves to the much simpler model $\Phi(r)$, assuming a plasma component with sufficient mobility within each magnetic surface to provide the required charge neutralization, and neglecting the contribution to Eq. (9) from the perturbation in the neutralizing component. To solve this problem, we first consider the diffusion coefficient in the case of large collision rate $\hat{\nu}/\epsilon_h \gg \omega_r$ (but still $<\omega_b$). The maximizing trial function is given by $\omega^{-1}\mathfrak{L}X = k^2 - 1$, and we find in this case

$$D = 12D_0\omega_r\epsilon_h/\hat{\nu}. \tag{12}$$

If now we treat the case $\hat{\nu}_e/\epsilon_h > \omega_r > \hat{\nu}_i/\epsilon_h$, then for the electrons Eq. (12) still holds, but for the ions the diffusion is not so large as in Eq. (11) because the electric-field rotation $\omega_E = (c/r_0B_0)\Phi' > \omega_r$ leads to a reduction of particle excursions. The maximizing trial function for the ions is localized near $k^2 = 1$, and leads to a diffusion of order

$$D \sim 1.8D_0(\hat{\nu}_i/\epsilon_h)^{1/2}\omega_r/\omega_E^{\,3/2}. \tag{13}$$

To obtain self-consistency, two cases must be considered. (a) We equate the electron and ion diffusion given by Eqs. (12) and (13), by choosing ω_E appropriately. The particle diffusion and thermal conductivity are then given by Eq. (12) for electron parameters. This regime pertains as long as $e\varphi < kT$, or $\omega_E < \omega_r\epsilon_h^{-1}$. (b) For larger values of calculated ω_E, the potential cannot actually rise above kT, since the ions will go toward electrostatic equilibrium, with the ion flux $F = (n'/n - e\Phi'/kT)D$ correspondingly reduced. In this limit the particle diffusion is given by Eq. (12) for electron parameters, but the thermal diffusion is given by Eq. (13) for ion parameters and $\omega_E = \omega_r\epsilon_h^{-1}$.

We note, finally, that collective plasma modes might act to short out the ambipolar potential Φ, and that diffusion could then proceed at the rate given by Eq. (11).

The present results for purely collisional diffusion are of practical interest in the toroidal reactor regime, where substantial improvements over the Bohm time are required. It appears, however, that collisional diffusion alone will not constitute a barrier to the stellarator-type reactor. These are also the conclusions of a numerical computation carried out by Mason and Gibson.[9]

The enhanced collisional diffusion due to stellarator geometry is unlikely to explain the observations on anomalous losses in present-day torus experiments. Our theoretical loss rates are somewhat too small to account for Bohm-like diffusion in stellarators; and the observation of

Bohm-like loss rates also in certain axisymmetric torus regimes suggests that the symmetry of the magnetic-field geometry cannot be the sole relevant consideration.

In this context, it is suggestive to note that irregularities of the electrostatic potential[10] have precisely the same desymmetrizing effects on particle orbits[11] as has the stellarator geometry. It seems plausible that potential irregularities amounting to a fraction of kT could arise from technical imperfections of plasma generation, or from slow-growing instabilities (e.g., trapped-particle modes), or indeed from the diffusion process itself, and could entail collisional transport rates accounting for the experimentally observed anomalous losses.

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