Electron and Ion Runaway in a Fully Ionized Gas. II*

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The treatment presented in an earlier paper is extended to give a more exact estimate of the particle runaway rate in a fully ionized gas under the action of a weak applied electric field. By analyzing the motion of particles in various regions of velocity space, it is shown that in any weak applied electric field some particles will always run away. The rate at which this occurs is determined by the flow of particles from the collision-dominated to the electric-field-dominated region of velocity space. The probability, $Q(\tau)$, of electron runaway as a function of time is calculated with the help of the Boltzmann-Fokker-Planck equation and can be expressed in the form $Q(\tau)=1-\exp(-\lambda_1\tau)$. The runaway rate, λ_1, is presented as a function of applied electric field, and the plasma temperature and density. It exceeds by several orders of magnitude the rate recently proposed by Harrison. The runaway rate for positive ions is shown to be exceedingly small compared to λ_1, in the circumstances usually encountered.

A brief discussion is devoted to the experimental evidence, the effects of magnetic fields, and the excitation of plasma instabilities. The correction which particle runaway introduces in the equation of pressure balance is presented for the case of a static pinched discharge.

I. INTRODUCTION

IN the first paper of this series[1] (hereafter denoted by R-I) hydrodynamic equations were used to describe the flow of electrons and positive ions of a fully ionized gas under the influence of an applied electric field. In this description, the dynamical friction force which arises through the agency of two-body electron-ion encounters was evaluated for Maxwellian electron and ion velocity distributions. Under the action of an electric field, these distributions move through velocity space with the electron and ion drift velocities, and in this approximation the motion carried out by any particle in the distribution is very similar to that carried out by the average particle moving with the drift velocity. Specifically, if the applied field is weak compared to the critical electric field for runaway, E_c, introduced in R-I, and the average particle does not runaway, then according to the hydrodynamic approximation no other particle in the distribution can run away.

This result is not strictly valid. It ignores the existence of certain groups of energetic particles in the distribution which take part in collisions so infrequently that for these almost any applied field, E, may be considered to be strong (in the sense defined in R-I), even though the same field fulfills the weak-field criterion $E\ll E_c$ when the main body of particles in the distribution are considered.

The primary purpose of this paper is to examine the runaway effect in this weak-field limit. These calculations are presented in Sec. II.[2] In Sec. III a brief discussion is devoted to the experimental evidence, the role of plasma instabilities and magnetic fields, and the corrections which runaway introduces into the equations of pressure balance.

II. WEAK FIELD RUNAWAY

A. Definition of Problem

In this treatment, we consider a uniform fully ionized plasma of electrons and protons (or deuterons). Prior to the instant when the externally applied electric field is switched on, the electron and ion velocities are assumed to be distributed according to the Maxwellian law

$$F_e(\mathbf{c})=n(\beta_e/\pi)^{\frac{3}{2}}\exp(-\beta_e c^2),$$
$$F_i(\mathbf{c})=n(\beta_i/\pi)^{\frac{3}{2}}\exp(-\beta_i c^2),$$

where n is the particle density

$$\beta_e=m/2kT_e,\quad \beta_i=M/2kT_i,$$

and the electron and ion masses and temperatures are given by m, M and T_e, T_i, respectively. For simplicity, the main part of this paper is restricted by the temperature relation $T_e\geq T_i$ which is usually encountered in practice. The opposite limit is briefly discussed in Sec. II, D.

At some instant of time an external electric field, E, is switched on which is weak in the sense defined in R-I, i.e.,

$$E\ll E_c=n(m/e)\Gamma_e\beta_e, \tag{1}$$

where

$$\Gamma_e=4\pi(e^2/4\pi\epsilon_0 m)^2\ln(\lambda/p_0),$$

$\lambda=(\epsilon_0 kT_e/ne^2)^{\frac{1}{2}}$ = Debye radius,

p_0 = average impact parameter for a 90° Coulomb deflection,

$4\pi\epsilon_0=\frac{1}{9}\times10^{-9}$ coulomb-volt^{-1}—meter^{-1}.

This inequality implies that in the mean free electron-

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[1] H. Dreicer, Phys. Rev. **115**, 238 (1959).

[2] These results were first reported by the author at the Tenth Annual Gaseous Electronics Conference, Cambridge, Massachusetts, October, 1957 [H. Dreicer, Bull. Am. Phys. Soc. 3, 86 (1958)].

ion collision time ν^{-1}, where

$$\nu=(e/m)E_c\beta_e^{\frac{1}{2}}, \tag{2}$$

the electric field alters the energy of an average electron in the distribution by an amount which is small compared to kT_e. The collision rate ν is also approximately the rate of encounter between those electrons in a Maxwellian distribution which are moving with average random speeds.

Electron-electron encounters cannot alter the drift energy imparted to the electrons by the field. This can only be brought about by electron-ion encounters, i.e., Joule heating. In weak electric fields this process limits the drift velocity of the electrons in the main body of the Maxwellian distribution (hereafter called body electrons) to approximately the directed velocity imparted to such an electron in a single mean free electron-ion collision time. This behavior is expressed by the relation

$$v_e(t)=\frac{3\sqrt{\pi}}{4}\frac{E}{E_c}\beta_e^{-\frac{1}{2}}\left[1-\exp\left(-\frac{4\nu t}{3\sqrt{\pi}}\right)\right], \tag{3}$$

(derived in R-I) which shows that v_e reaches its terminal value several mean free collision times after the field is turned on. Any subsequent increase in v_e is coupled to the rise in electron temperature. For the body electrons a rise in temperature is a result of electron-ion encounters which randomize the drift energy imparted by the field, and electron-electron encounters which redistribute this energy into a Maxwellian distribution. As long as we concern ourselves with the body electrons, the results of R-I will apply; and with the help of Eqs. (22b) and (29) (R-I), we find that these electrons are Joule heated at the rate

$$\frac{d}{dt}(\tfrac{3}{2}kT_e)=\frac{4}{3\sqrt{\pi}}(eE_c\beta_e^{-\frac{1}{2}})\beta_e v_e^2.$$

With the help of Eqs. (2) and (3) this expression may be rewritten in the form

$$\frac{d}{dt}(\tfrac{3}{2}kT_e)=\pi^{\frac{1}{2}}(\tfrac{3}{2}kT_e)\left(\frac{E}{E_c}\right)^2\nu\left[1-\exp\left(-\frac{4\nu t}{3\sqrt{\pi}}\right)\right]^2. \tag{4}$$

This result shows that the instantaneous collision rate ν exceeds the instantaneous Joule heating rate by the factor $E_c^2/(\pi^{\frac{1}{2}}E^2)$, and the lifetime of the weak-field regime, i.e., the time taken for the body electron drift velocity to reach the value corresponding to $\beta_e v_e^2=1$ starting from $v_e=0$, is roughly $(E_{c0}/E)^2\nu^{-1}$, where E_{c0} is the critical field expressed in terms of the initial electron temperature T_{e0}. Solutions of the more exact equations derived in R-I which take into account the strong variation of the dynamical friction force with drift velocity are tabulated for several electric fields in Table I and are seen to yield somewhat longer lifetimes. We are thus led to the following picture of the behavior of the body electrons in the weak-field regime: After the electric field is switched on, the terminal drift speed is established in several mean free collision times. The subsequent increase in the drift speed follows the rise in temperature for an interval of time equal to about $(E_{c0}/E)^2\nu^{-1}$, after which the strong field regime is reached and runaway of the body electrons occurs. Since the lifetime of the weak-field regime for the body electrons exceeds ν^{-1}, we may expect the body velocity distribution to behave adiabatically and remain very closely Maxwellian throughout this time.

TABLE I. Lifetime of weak-field regime.

E/E_c	Lifetime obtained in R-I (in units of ν^{-1})	$\pi^{-\frac{1}{2}}(E_c/E)^2$
0.1	80	57
0.2	20	14
0.3	9	6.2
0.4	5.5	3.5

We consider next that there must exist fast electrons in the high-energy tail of the velocity distribution which make collisions so infrequently that almost any applied electric field controls their motion and thus may be considered to be a strong field. When this effect is examined we find that velocity space can roughly be divided into a runaway region where the applied field plays the role of a strong field, and into a region dominated by collisions where the same field is weak, and runaway does not take place.

Our first problem is to locate these regions in velocity space in the period immediately following the switching on of the electric field. For this purpose, we consider a single test electron moving with the velocity $\mathbf{c}$ under the influence of a weak electric field through a gas of electrons and singly charged ions. During the early times, the relative drift velocity between the electron and ion gases is very small compared to the average random electron speed, and the ion as well as body-electron velocity distributions are closely Maxwellian. Under these conditions, the Langevin equation[3] for the test electron is

$$\frac{d\mathbf{c}}{dt}+\frac{\mathbf{c}}{c}\frac{eE_c}{m}[2\Psi(V)+\gamma^{-2}\Psi(\gamma^{-1}V)]=\frac{e}{m}\mathbf{E}+\mathbf{A}(t),$$

where

$$V=\beta_e^{\frac{1}{2}}c,\quad \gamma=(\beta_e/\beta_i)^{\frac{1}{2}},$$

$$\Psi(V)=\frac{\mathcal{E}_2(V)-V d\mathcal{E}_2/dV}{V^2},$$

$$\mathcal{E}_2(V)=\frac{2}{\sqrt{\pi}}\int_0^V \exp(-t^2)dt.$$

In this equation, the total instantaneous acceleration due to particle-particle interaction is separated into a

[3] For a review of methods involving the Langevin equation, see S. Chandrasekhar, Revs. Modern Phys. **15**, 1 (1943).

time-average part, the dynamical friction, and a part, $\mathbf{A}(t)$, describing the fluctuations about the average. By making use of Gauss' theorem, the dynamical friction is easily derived and expressed in terms of Ψ functions, one due to electron-electron encounter, the other due to electron-ion encounter. Chandrasekhar has shown[3] that $\mathbf{A}(t)$ arises chiefly from encounters with nearest neighbors. Its exact value is determined at each instant by all particle positions, and its rate of change involves, in addition, the velocity of all particles.

The solution of the Langevin equation is complicated by the strong velocity dependence of the Ψ function. Fortunately, a particularly simple situation exists for electrons which are moving so fast (i.e., $V\gg1$ and $\gamma^{-1}V\gg1$) that we may consider practically all other plasma particles to be at rest. Under these circumstances, use of the asymptotic form of the Ψ function leads to the following simplification:

$$2\Psi(V)+\gamma^{-2}\Psi(V/\gamma)\rightarrow 3/V^2.$$

It also follows that the faster the test electron moves the smaller are the velocity fluctuations produced by the acceleration $\mathbf{A}(t)$. The main frequency component of $\mathbf{A}(t)$ is inversely proportional to the time of flight between nearest neighbors, and because of its inertia the response of the test electron to $\mathbf{A}(t)$ must decrease as V increases. This result holds even though a fast particle can penetrate deeper into the Coulomb fields it encounters.

These remarks are borne out by more detailed kinetic calculations to be presented in Sec. II, B. [See argument leading up to Eq. (33).] There we find that the mean square change in the velocity of the electrons per unit time is primarily the result of energy gained in the applied field, rather than the result of energy exchanged between electrons provided

$$V>(E_c/E)^{\frac{1}{2}}=V_t. \tag{5}$$

Subject to this restriction on the velocity we therefore neglect $\mathbf{A}(t)$ compared to $\mathbf{E}$, and in dimensionless form the high-energy limit of the Langevin equation becomes

$$\frac{d\mathbf{V}}{d\tau}+3\frac{\mathbf{V}}{V^3}=\frac{\mathbf{E}}{E_c}.$$

The dimensionless time variable τ used here is defined with the help of Eq. (2) by $\tau=\nu t$.

The motion of a fast electron in velocity space follows the flow lines of the total force $(\mathbf{E}/E_c)-3\mathbf{V}/V^3$, and in any plane containing E these lines are determined by

$$\frac{Vd\theta}{dV}=\frac{V^2\sin\theta}{3(E_c/E)-V^2\cos\theta}.$$

Here θ is the angle subtended by $\mathbf{V}$ and $\mathbf{E}$. With the help of the identity

$$V^2\sin\theta\cos\theta d\theta+V\sin^2\theta dV=V\sin\theta d(V\sin\theta),$$

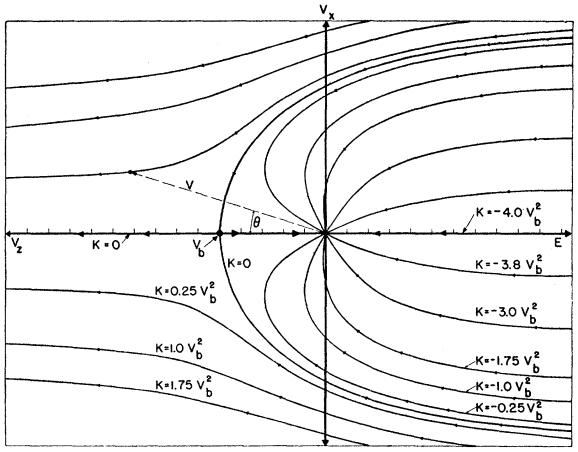

FIG. 1. Flow lines in the V_x, V_z plane for an electron moving under the combined action of the dynamical friction force, $(-3\mathbf{V}/V^3)$, and an electric field pointing along the negative V_z axis. Each K line generates a surface of revolution about the V_z axis.

we find[4]

$$V^2=V_b^2\sec^2(\theta/2)+K\csc^2\theta, \tag{6}$$

where

$$V_b^2=3E_c/E, \tag{7}$$

and K is the arbitrary constant which generates the family of flow lines shown in Fig. 1. For $K>0$ the electric field force is strong compared to the dynamical friction force, and flow lines enter and leave velocity space at infinity. Examination of Eqs. (6) and (7) shows that in this region (we call it the runaway region) the condition given in Eq. (5) is everywhere satisfied and use of the asymptotic form of Ψ is justified. For $K<0$ all flow lines converge on the origin, and the dynamical friction force appears to bring all electrons to rest. This, however, is a result of neglecting $\mathbf{A}(t)$. The motion of the body electrons is collision dominated and in general the condition given in Eq. (5) is violated for $K<0$. In this region $\mathbf{A}(t)$ causes frequent transitions between and along flow lines and such particle motion is best described as a diffusion in velocity space.

The runaway and collision-dominated regions are separated by the surface on which $K=0$. If $\mathbf{A}(t)$ were retained in the Langevin equation, this surface would be smeared out, and the regions would merge gradually. The concept of the $K=0$ surface is nevertheless a useful one. It suggests an approximate method of solution which we enter into next.

After the electric field is switched on, the high-energy electrons which originate in the runaway region with speed V_0 will increase their velocity in the direction of the field by V_0 in a time given roughly by

$$\tau=(E_c/E)V_0,$$

[4] Derived earlier in H. Dreicer, Ph.D. thesis, Massachusetts Institute of Technology, 1955; and published in W. P. Allis' "Motion of Electrons and Ions," *Handbuch der Physik*, edited by S. Flügge (Springer-Verlag, Berlin, 1956), Vol. 21, p. 436.

and subsequently we may consider these to have run away. The motion of these *early* runaways is dominated by the electric field, and is easily followed. Their total particle density is, however, only of the order

$$\frac{2n}{\sqrt{\pi}}\int_0^{\pi}\int_{V_b\sec(\theta/2)}^{\infty} V^2\exp(-V^2)\sin\theta d\theta dV \simeq nV_b\exp(-V_b^2). \quad (8)$$

A more difficult question involves those electrons which originate in the collision dominated region. These consist very largely of body electrons whose velocity distribution is very nearly Maxwellian. In the course of time, these electrons diffuse across the $K=0$ surface and run away. Our problem is to calculate the rate with which this mechanism depletes the collision dominated region, and we turn to the Boltzmann equation for a solution to this problem.

B. Boltzmann Treatment

The calculation proceeds from the Boltzmann and Fokker-Planck equations which have the following forms in our dimensionless units:

$$\frac{\partial F_e}{\partial\tau}-\frac{\mathbf{E}}{E_c}\cdot\nabla_V F_e=\left(\frac{\partial F_e}{\partial\tau}\right)_c, \quad (9a)$$

$$\left(\frac{\partial F_e}{\partial\tau}\right)_c=-\frac{\partial}{\partial V_k}\left[F_e\left(\frac{\partial H_{ee}}{\partial V_k}+\frac{\partial H_{ei}}{\partial V_k}\right)\right] +\frac{1}{2}\frac{\partial^2}{\partial V_k\partial V_j}\left[F_e\left(\frac{\partial^2 G_{ee}}{\partial V_k\partial V_j}+\frac{\partial^2 G_{ei}}{\partial V_k\partial V_j}\right)\right], \quad (9b)$$

where

$$H_{ee}=2\int\frac{F_e(\mathbf{V}',\tau)}{|\mathbf{V}-\mathbf{V}'|}d^3V', \quad (10a)$$

$$H_{ei}=\left(1+\frac{m}{M}\right)\int\frac{F_i(\mathbf{V}',\tau)}{|\mathbf{V}-\mathbf{V}'|}d^3V', \quad (10b)$$

$$G_{ee}=\int F_e(\mathbf{V}',\tau)|\mathbf{V}-\mathbf{V}'|d^3V', \quad (11a)$$

$$G_{ei}=\int F_i(\mathbf{V}',\tau)|\mathbf{V}-\mathbf{V}'|d^3V'. \quad (11b)$$

F_e and F_i are the electron and ion velocity distributions, respectively. Summation over the components of the velocity V_k, V_j is implied wherever these components appear repeatedly. Integrations over the entire velocity space are implied in Eqs. (10a) to (11b).

An exact analytic solution of the Boltzmann equation does not seem feasible, and it is necessary to resort to an approximate treatment. In this paper we make use of an expansion in spherical harmonics[5] and search for a solution of the form

$$F_e(\mathbf{V},\tau)=F^0(V,\tau)+\mu F^1(V,\tau), \quad (12)$$

$$F^1(V,\tau)\ll F^0(V,\tau), \quad (13)$$

where μ is the cosine of the angle subtended by $\mathbf{E}$ and $\mathbf{V}$. As shown in R-I, this expansion is valid for the body electrons in the weak-field limit $E\ll E_c$. The introduction of Eq. (12) into Eqs. (9a) and (9b) leads, after integration over μ, to a set of coupled equations for F^0 and F^1 which take the form

$$\frac{\partial F^0}{\partial\tau}=\frac{1}{V^2}\frac{\partial}{\partial V}(V^2J), \quad (14)$$

$$\frac{\partial F^1}{\partial\tau}=\frac{F^1}{V}\frac{\partial H_{ei}}{\partial V}+\frac{E}{E_c}\frac{\partial F^0}{\partial V}. \quad (15)$$

In this equation the current J and the potential H_{ei} are given by

$$J=-F^0\frac{\partial H_{ee}^0}{\partial V}+\tfrac{1}{2}F^0\frac{\partial^3 G_{ee}^0}{\partial V^3}+\frac{F^0}{V}\frac{\partial^2 G_{ee}^0}{\partial V^2} +\frac{1}{2}\frac{\partial F^0}{\partial V}\frac{\partial^2 G_{ee}^0}{\partial V^2}-\frac{F^0}{V^2}\frac{\partial G_{ee}^0}{\partial V}+\frac{1}{3}\frac{E}{E_c}F^1, \quad (16)$$

and

$$H_{ei}=\int_0^{\infty}\delta(V')\frac{dV'}{|\mathbf{V}-\mathbf{V}'|}=\frac{1}{V}. \quad (17)$$

In Eqs. (15) and (16) all terms which are proportional to m/M have been neglected. The ions are therefore taken to be infinitely massive, and for this reason their velocity distribution is given by a Dirac delta function. Terms proportional to $(F^1)^2$ have also been dropped in this derivation. The H_{ee}^0 and G_{ee}^0 functions in Eq. (16) are generated by F^0, the spherically symmetric part of the distribution.

At this point we introduce two other simplifications. The first of these restricts the solution of Eqs. (14) and (15) to the collision dominated ($K<0$) region by requiring that the electron velocity distribution vanish on the $K=0$ transition surface. This assumption is based on the expectation that runaway reduces the electron population in the neighborhood of this surface below the value, $\pi^{-\frac{3}{2}}\exp(-V_b^2)$, which exists in the absence of the applied electric field. As a consequence of this simplification, all interaction between electrons in the runaway region which results in their reappearance in the $K<0$ region, and all interaction between electrons on opposite sides of the transition surface is neglected. The second simplification replaces the $K=0$ transition surface by a sphere of radius V_b, and thus introduces spherical symmetry into the problem. Justification for this approximation is presented *a posteriori* in Sec. II, C.

Equation (14) is a nonlinear diffusion equation for F^0.

[5] For an example of this method see T. Holstein, Phys. Rev. **70**, 367 (1946).

The current density J is therefore radial in velocity space, and receives contributions only from processes which involve energy exchange. Among these processes, the contribution from electron-ion encounters is smaller than that due to electron-electron encounters by the factor m/M and has therefore been neglected. This accounts for the fact that G_{ei} does not appear in Eq. (16). In Eq. (15), on the other hand, we are dealing with momentum transferred to and from the electrons, and here it is the electron-ion encounters, whose collision rate in dimensionless units is just V^{-3}, which play the dominant role. Directed energy gained from the electric field is randomized in direction by these collisions, and contributes the Joule heating current $(F^1/3)(E/E_c)$ to J. We have neglected the contribution to $\partial F^1/\partial\tau$ from electron-electron encounters since these do not alter the momentum of the electron gas directly.

The solution of Eq. (15) subject to the condition that no current flows prior to the application of E is

$$F^1(V,\tau)=\exp\left(-\frac{\tau}{V^3}\right)\int_0^\tau \exp\left(\frac{s}{V^3}\right)\frac{E}{E_c}\frac{\partial F^0(V,s)}{\partial V}ds. \quad (18)$$

The exponential time factors in this equation express the contribution from particles moving with the speed V to the transient buildup in the asymmetry F^1. The majority of electrons in the collision dominated region move with speeds in the neighborhood of the most probable speed $V=1$, and for these the transient motion dies out in several mean free collision times (i.e., $\tau\approx 2$–3). The subsequent time dependence of F^1 arises from Joule heating, and diffusion of electrons into the runaway region, and is related to the time-dependent behavior of F^0 which we consider next.

Following an approach used by Chandrasekhar[6] in the theory of star clusters, we specify the initial velocity of a test electron and trace its subsequent statistical behavior by means of Eq. (14). Moreover, as we have argued before, it appears reasonable to assume that all other body electrons are distributed according to a Maxwellian distribution since we can expect this region to be depleted slowly compared to the rate at which these electrons exchange energy. The utility of this approximation lies in the fact that we have thus reduced Eq. (14) to a linear partial differential equation, with the interaction terms expressed once and for all in the form

$$H_{ee}^0(V)=2\mathcal{E}_2(V)/V, \quad (19)$$

$$G_{ee}^0(V)=\mathcal{E}_2(V)\left[V+\frac{1}{2V}\right]+\frac{1}{2}\frac{d\mathcal{E}_2}{dV}. \quad (20)$$

A discussion of the limitations which linearization imposes upon the validity of the solution is given in Sec. II, C. However, we may add at this point that the distribution function obtained in this way is very nearly Maxwellian except for the immediate neighborhood of V_b (see curves of w_1 in Figs. 2 and 3). Substitution of Eqs. (19) and (20) into Eq. (14) yields

$$\frac{\partial F^0}{\partial\tau}=\frac{\Psi(V)}{2V}\frac{\partial^2 F^0}{\partial V^2}+\left[\frac{\mathcal{E}_2(V)}{V^2}-\frac{\Psi(V)}{2V^2}\right]\frac{\partial F^0}{\partial V}+2\frac{d\mathcal{E}_2}{dV}F^0+\frac{1}{3}\frac{E}{E_c}\frac{1}{V^2}\frac{\partial}{\partial V}(V^2F^1), \quad (21)$$

which we must solve subject to the initial and boundary conditions

$$F^0(V,0)=\frac{\delta(V-V_0)}{4\pi V_0^2}, \quad V_0<V_b, \quad (22)$$

$$F^0(V_b,\tau)=0, \quad \tau\geq 0. \quad (23)$$

After several mean free collision times, the transient buildup terms in F^1 die out, and under certain conditions Eq. (21) can subsequently be satisfied by a separable solution of the form

$$F^0(V,\tau)=f(V)\exp(-\lambda\tau). \quad (24)$$

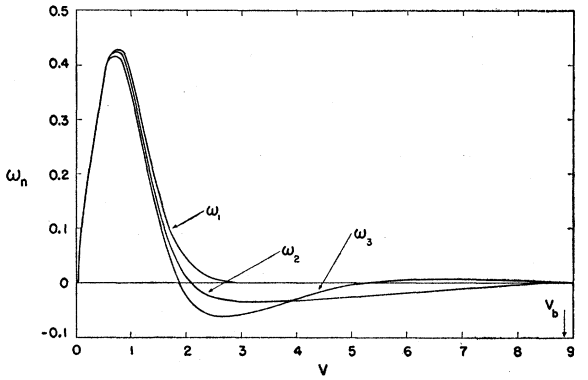

FIG. 2. The variation of the eigenfunctions w_1, w_2, and w_3 with the normalized speed V, for $E/E_c=0.03$ and $V_b=8.85$.

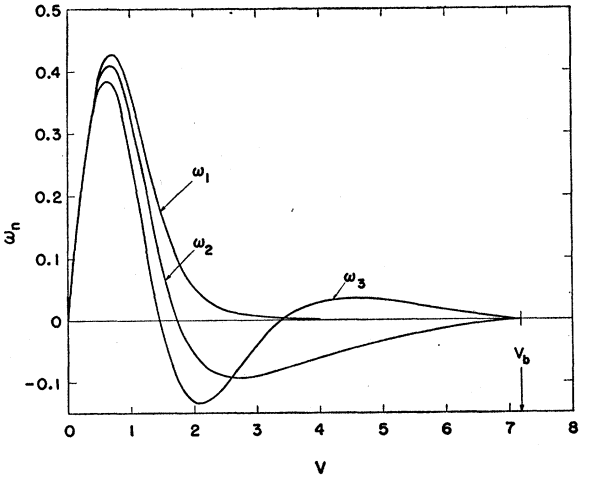

FIG. 3. The variation of the eigenfunctions w_1, w_2, and w_3 with the normalized speed V, for $E/E_c=0.1$ and $V_b=7.20$.

[6] S. Chandrasekhar, Astrophys. J. 98, 54 (1943).

To show this we substitute Eq. (24) into Eq. (18). The result

$$F^1(V,\tau)=\frac{E}{E_c}\frac{\partial f}{\partial V}\frac{\exp(-\lambda\tau)-\exp(-\tau/V^3)}{(1/V^3)-\lambda}$$

is nearly proportional to $\exp(-\lambda\tau)$ provided

$$\lambda<V^{-3}\leq V_b^{-3}. \tag{25}$$

In Sec. II, C, we shall show that except for the early times the diffusion of electrons across the transition sphere is governed primarily by the smallest eigenvalue λ_1 which appears in the solution, and it develops that λ_1 satisfies the above inequality when $E/E_c\ll 1$. Thus the separation of variables is possible provided an accurate solution is sacrificed during the first few mean free collision times.

The approximation can also be described from a different point of view. Neglect of the transients in Eq. (18) is equivalent to the neglect of the time derivative of F^1 in Eq. (15). Since the asymmetry F^1 alone determines the momentum of the electron gas the approximation implies that the motion of the electrons is quasi-static with the inertial term taking up the negligibly small difference between the dynamical friction force and the applied field force acting on the body electrons. This is merely a restatement of the fact that the body electrons do not run away together until Joule heating has lowered the dynamical friction force to the point where it cannot balance the applied field and the inertial term becomes important. In this approximation, the quasi-static form of F^1 is given by

$$F^1=\frac{E}{E_c}V^3\frac{\partial f}{\partial V}\exp(-\lambda\tau), \tag{26}$$

and F^0 is given by Eq. (24).

For convenience in the subsequent numerical analysis, a new function $w(V)$ is defined by

$$w(V)=Vf(V).$$

On the boundary we then have

$$w(V_b)=0. \tag{27}$$

In terms of this new function, the differential equation becomes

$$\frac{d^2w}{dV^2}\left[\frac{\Psi(V)}{2V}+\frac{1}{3}\left(\frac{E}{E_c}\right)^2V^3\right]+\frac{dw}{dV}\left[\left(\frac{E}{E_c}\right)^2V^2+\frac{\mathcal{E}_2(V)}{V^2}-\frac{3\Psi(V)}{2V^2}\right]+w\left[2\frac{d\mathcal{E}_2(V)}{dV}-\frac{\mathcal{E}_2(V)}{V^3}-\left(\frac{E}{E_c}\right)^2V+\frac{3\Psi(V)}{2V^3}+\lambda\right]=0. \tag{28}$$

It must be solved subject to the additional boundary condition

$$w(0)=0. \tag{29}$$

For each choice of V_b this equation yields a distinct set of eigenfunctions $w_1(V)$, $w_2(V)$, $\cdots$, and a corresponding set of eigenvalues λ_1, λ_2, $\cdots$. The latter give the decay rates of the various modes as a result of diffusion across the transition surface.

The probability that an electron, whose initial speed is V_0, be found in the range between V and $V+dV$ after a time τ is given by

$$F^0(V|V_0|\tau)=\sum_{n=1}^{\infty}\frac{w_n(V)w_n(V_0)r(V_0)}{4\pi VV_0N_n}\exp(-\lambda_n\tau), \tag{30}$$

where

$$N_n=\int_0^{V_b}[w_n(V)]^2r(V)dV,$$

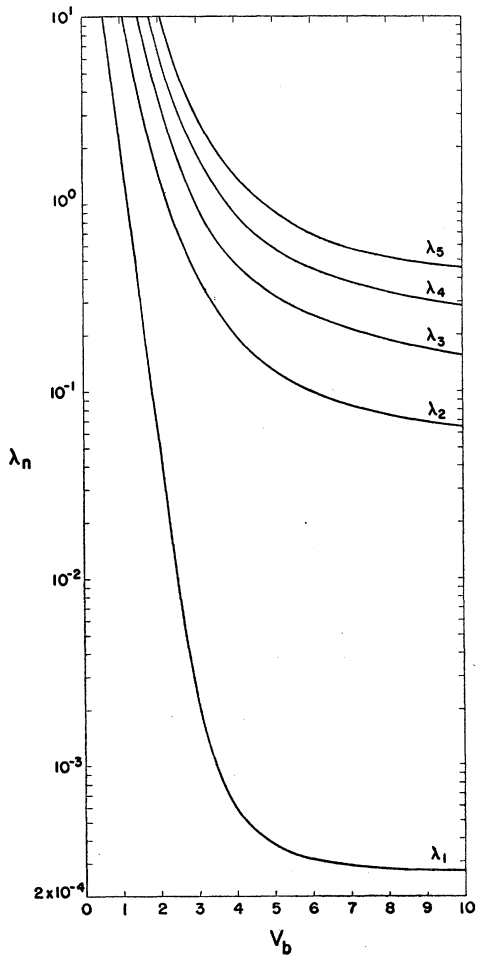

FIG. 4. The variation of the eigenvalues λ_1 to λ_5, expressed in units of the collision rate ν, with the normalized transition sphere radius V_b for $E/E_c=0.06$.

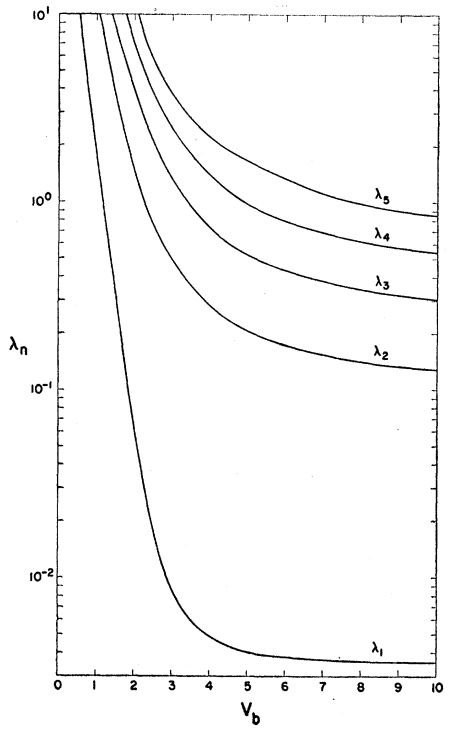

FIG. 5. The variation of the eigenvalues λ_1 to λ_5, expressed in units of the collision rate ν, with the normalized transition sphere radius V_b for $E/E_c=0.10$.

and the weighting function is given by

$$r(V)=p(V)\Big/\left[\frac{1}{3}\left(\frac{E}{E_c}\right)^2V^3+\frac{\Psi(V)}{2V}\right],$$

$$p(V)=\exp\left\{\int_0^V\frac{[\mathcal{E}_2/V^2-3\Psi(V)/2V^2+(E/E_c)^2V^2]}{[\Psi(V)/2V+\frac{1}{3}(E/E_c)^2V^3]}dV\right\}.$$

$F^0(V|V_0|\tau)$ is the conditional velocity distribution which satisfies Eqs. (21), (22), and (23).

The conditional probability $P(V_b|V_0|\tau)$ that an electron whose initial speed is V_0 crosses the transition boundary for the first time between τ and $\tau+d\tau$ is simply the negative of the total current, $4\pi V^2J$, evaluated at $V=V_b$. Use of Eqs. (16), (19), (20), (26), and (30) together with the boundary condition $F^0(V_b,\tau)=0$ yields

$$P(V_b|V_0|\tau)$$
$$=-4\pi V_b^2\left[\frac{1}{3}\left(\frac{E}{E_c}\right)^2V_b^3+\frac{\Psi(V_b)}{2V_b}\right]$$
$$\times\sum_{n=1}^{\infty}\frac{w_n(V_0)r(V_0)}{4\pi V_0V_bN_n}\left(\frac{dw_n}{dV}\right)_{V=V_b}\exp(-\lambda_n\tau).$$

The coefficient in the square bracket is the mean square increment in the speed per unit time of an electron moving with the speed V_b, and it plays the role of a diffusion constant. The total conditional probability,

$$Q(V_b|V_0|\tau)=\int_0^\tau P(V_b|V_0|\tau)d\tau,$$

satisfies the normalization condition $Q(V_b|V_0|\infty)=1$. It may therefore be expressed in the equivalent form

$$Q(V_b|V_0|\tau)$$
$$=\sum_n Q_n(V_b,V_0)[1-\exp(-\lambda_n\tau)]$$
$$=\sum_{n=1}^{\infty}C_n(V_b)w_n(V_0)[1-\exp(-\lambda_n\tau)]\Big/$$
$$\times\sum_{m=1}^{\infty}C_m(V_b)w_m(V_0),$$

where

$$C_n(V_b)=\frac{1}{N_n\lambda_n}\left(\frac{dw_n}{dV}\right)_{V=V_b}.$$

Finally Q is averaged over a Maxwellian distribution of the initial speeds V_0 with the result

$$Q(V_b|\tau)=\sum_n A_n(V_b)[1-\exp(-\lambda_n\tau)], \quad (31)$$

where

$$A_n(V_b)=\int_0^{V_b}V_0^2\exp(-V_0^2)Q_n(V_b,V_0)dV_0\Big/$$
$$\int_0^{V_b}V_0^2\exp(-V_0^2)dV_0.$$

After several mean free collision times, Q is determined almost entirely by the decay of the fundamental mode and we have

$$Q(V_b|\tau)\cong A_1[1-\exp(-\lambda_1\tau)]. \quad (32)$$

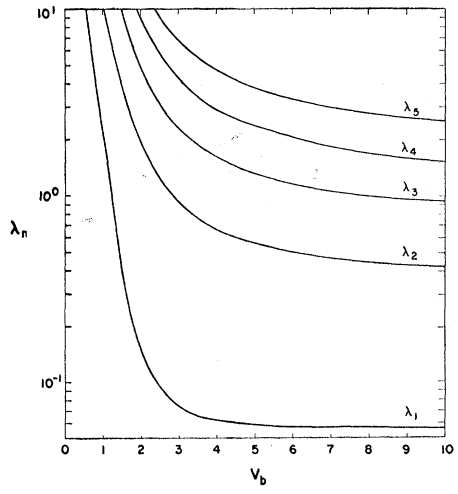

FIG. 6. The variation of the eigenvalues λ_1 to λ_5, expressed in units of the collision rate ν, with the normalized transition sphere radius V_b for $E/E_c=0.223$.

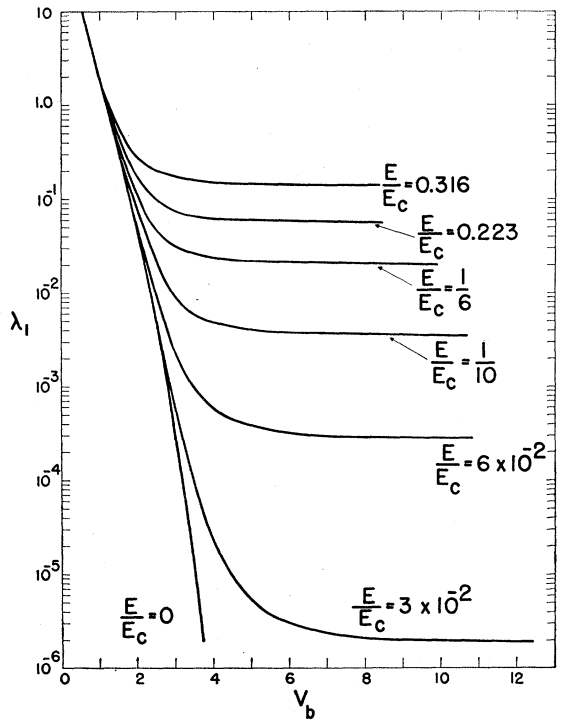

FIG. 7. The variation of the runaway rate, λ_1, expressed in units of the collision rate ν_1 with the normalized transition sphere radius V_b. The normalized electric field E/E_c is treated as a parameter.

Except for the inclusion of electric field effects, Eq. (32) is a result obtained earlier by Chandrasekhar[6] in connection with the escape of stars from clusters.

C. Numerical Results

The eigenvalue problem defined by Eqs. (27), (28), and (29) has been solved on a 704 IBM digital computer by a standard Runge-Kutta integrating technique. Eigenfunctions and eigenvalues were obtained as a function of the boundary coordinate V_b with the electric field E/E_c playing the role of a parameter. Typical numerical results are shown in Figs. 2 to 7.

The most significant result to come out of the calculation is that λ_1 becomes essentially independent of V_b for sufficiently large V_b. In particular, these results show that λ_1 becomes independent of V_b for a sphere radius which agrees well with the value derived in Eq. (7) from the Langevin equation. The values of λ_1 obtained by Chandrasekhar[6] correspond to $E/E_c=0$ in the range $2.3<V_b<2.7$ and agree closely with our results.

The variation of λ_1 with V_b and E/E_c can be explained in terms of the processes which give rise to J. Of these, mutual electron encounters contribute a thermalizing current to J which depends upon the deviation of F^0 from the Maxwellian distribution, but represents no consistent diffusion of electrons to higher energies. The Joule heating current

$$\frac{1}{3}\left(\frac{E}{E_c}\right)F^1=\frac{1}{3}\left(\frac{E}{E_c}\right)^2 V^3\frac{\partial F^0}{\partial V},$$

on the other hand, does represent a consistent diffusion of electrons to higher energies. This is brought about by the fact that electrons exchange very little energy (in our approximate treatment none at all) with the positive ions, and store nearly all of the energy gained from the electric field. Energy exchange between electrons is a result of Coulomb encounters, whereas energy gain from the electric field is hindered by encounters. To see this we note that the Joule heating current given in the above equation is proportional to $(E/E_c)^2$ and thus decreases with an increase in the rate of collision. Since the Rutherford scattering law governs the rate of collision, we must expect to find that at large velocities J is almost completely determined by the electric field. More precisely it can be seen from the coefficient of d^2w/dV^2 in Eq. (28), which is just the mean square increment per unit time in the speed of an electron moving with speed V, that the contribution from the electric field equals the contribution arising from collisions when

$$\frac{1}{3}\left(\frac{E}{E_c}\right)^2 V_t^3=\frac{\Psi(V_t)}{2V_t}\simeq\frac{1}{2V_t^3}, \tag{33}$$

or

$$V_t\simeq(E_c/E)^{\frac{1}{5}}.$$

The inverse of λ_1 is essentially the time required for an average body electron to diffuse from $V\simeq1$ to $V=V_t$ under the action of electron-electron encounters. The slight slope which remains in the curve of λ_1 *versus* V_b when λ_1 saturates above V_b is contributed by the additional time required by an electron to be accelerated from V_t to larger velocities by the action of the electric field. It is this behavior of λ_1 which we invoke to justify our substitution of a sphere for the transition surface defined by Eq. (6). Since λ_1 becomes essentially independent of V_b for large V_b, we can expect that deformation of the transition surface in this region cannot affect the results very strongly. Moreover, this result seems to bear out the validity of both the boundary condition $F^0(V_b,\tau)=0$ and the neglect of the interactions which scatter electrons back into the sphere.

In our treatment, electrons which cross the sphere run away. Actually electrons which enter the region shaded in Fig. 8 do not get very far before they return to the main body of electrons in the $K<0$ region. This error has roughly the effect of halving the true diffusion length in the problem, or increasing λ_1 by a factor of four. The purely angular scattering of electrons from the shaded region into the true runaway region due to encounters with positive ions will tend to reduce this error somewhat.

TABLE II. Probability coefficients for the first five decay modes.

E/E_c	A_1	A_2	A_3	A_4	A_5	$\sum_{n=1}^{5} A_n$	V_b
0.223	1.120	-1.14×10^{-1}	-1.38×10^{-2}	9.2×10^{-3}	5.0×10^{-4}	1.002	4.0
0.166	1.106	-1.23×10^{-1}	1.18×10^{-2}	9.4×10^{-3}	-2.9×10^{-3}	1.001	5.0
0.10	1.049	-7.16×10^{-2}	2.96×10^{-2}	-5.9×10^{-3}	-1.7×10^{-3}	0.999	7.5
0.06	1.007	-1.19×10^{-2}	6.64×10^{-3}	-2.89×10^{-3}	7.07×10^{-4}	1.000	8.0
0.03	1.000	-2.32×10^{-4}	1.91×10^{-4}	-1.6×10^{-4}	1.1×10^{-4}	1.000	9.0

Strictly speaking, the spherical harmonic expansion of the distribution function defined in Eqs. (12) and (13) is a poor approximation for velocities in the range $V_t<V<V_b$. In this velocity range, collisions are already so rare that the electric field must produce a pronounced drift of these electrons in one direction, and the velocity distribution must deviate rather seriously from a spherical distribution. Our treatment does not take this effect into account, but we believe that a more rigorous treatment will preserve the essential features of Fig. 7. Our results seem to indicate that the eigenvalue λ_1 depends primarily upon the net loss of electrons out of the $V<V_t$ portion of the collision dominated region, rather than the direction in which these traverse the region $V_t<V<V_b$ and cross the transition surface. This result follows from the fact that the time an electron spends in the region between V_t and V_b is small compared to λ_1^{-1}, independent of the direction in which it flows across this region. A more precise calculation has not been made to check this point.

In Sec. II, B, the F^1 distribution was simplified by the neglect of transients. This approximation was based upon the inequality $\lambda_1<V_b^{-3}$ [see Eq. (25)], and a comparison of λ_1 and V_b, made with the help of Eq. (7), shows that this inequality is indeed satisfied for $E/E_c \leq 10^{-1}$.

The eigenvalues associated with w_2 to w_5 are shown in Figs. 4 to 6 as a function of V_b with E/E_c treated as a parameter. Examination of these as well as the values of A_1 to A_5 listed in Table II shows that the asymptotic form

$$Q(V_b|\tau)\simeq 1-\exp(-\lambda_1\tau)$$

is a very good approximation after several mean free collision times. The validity of this relation is, however, limited by the linearization which we have imposed upon the Boltzmann equation. This effect shows up in essentially two ways. First the density of electron scatterers decreases with time, and secondly Joule heating of the body electrons may increase their temperature if there are no compensating heat losses. These changes have the effect of decreasing the electron-electron collision rate, thus increasing the mean free collision time to which we normalize λ_1. On the other hand, the ratio E/E_c increases with the rise in temperature, but is only partially affected by electron density changes in the collision dominated region since E_c is a result of electron-ion as well as electron-electron encounters. To estimate the rise in temperature when there are no energy losses we make use of Eq. (4) which states that strong field runaway of the body electrons can be expected after a time given approximately by $(E_{c0}/E)^2$. For each E/E_{c0} studied, this time is either comparable to or much shorter than λ_1^{-1}. If Joule heating is balanced by other energy losses and the electron temperature remains constant, then our results are limited roughly to the e-folding time λ_1^{-1}. Very reasonable corrections can be made by adjusting the mean free collision time to take account of the gradual depletion of electrons in the collision dominated region. Eventually, however, the interaction between electrons in the runaway region must be taken into account.

D. Effect of Random Ion Motion

In this section we will briefly consider the effect of random ion motion on the results we have obtained so far. To extend the treatment we consider the possibility of ion runaway by the method of the Langevin equation. The dynamical friction force exerted upon an ion moving with velocity $\mathbf{c}$ in a gas composed of ions and electrons, whose velocity distributions are Maxwellian, is obtained with the help of R-I [Eqs. (6), (9), and (17a)] in the form

$$\mathbf{f}_i=M(\nabla H_{ii}+\nabla H_{ie})$$
$$=-eE_c\left[2\frac{q_i}{q_i}\frac{T_e}{T_i}\Psi(\beta_i^{\frac{1}{2}}q_i)+\frac{q_e}{q_e}\frac{M+m}{M}\Psi(\beta_e^{\frac{1}{2}}q_e)\right], \quad (34)$$

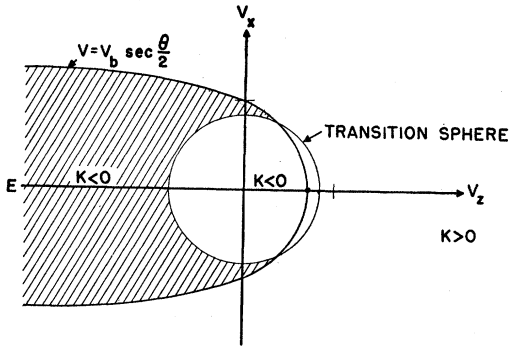

FIG. 8. Two-dimensional view of the actual and approximate transition surfaces. In the approximate treatment, electrons in the $K>0$ region as well as in the shaded portion of the $K<0$ region runaway.

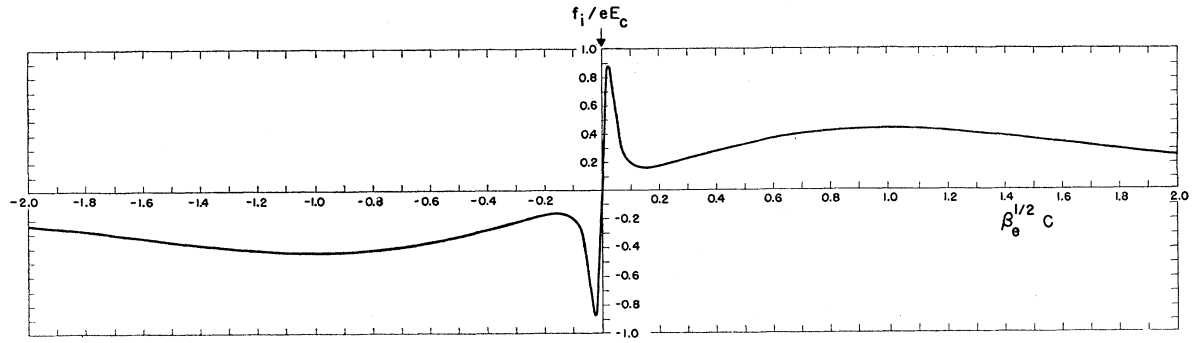

FIG. 9. The dynamical friction force f_i, normalized to eE_c, which acts on an ion moving with the speed c through a Maxwellian distribution of electrons and ions. For this illustration the electron and ion temperatures were taken to be equal, and $v_e=v_i=0$.

where

$$\mathbf{q}_e=\mathbf{c}-\mathbf{v}_e, \quad \mathbf{q}_i=\mathbf{c}-\mathbf{v}_i,$$

and $\mathbf{v}_e$ and $\mathbf{v}_i$ are the electron and ion drift velocities, respectively. The force f_i, normalized to eE_c, is illustrated in Figs. 9 and 10 as a function of $\beta_e^{\frac{1}{2}}c$ for a test ion moving in the direction of $(\mathbf{v}_e-\mathbf{v}_i)$. It is clear from these graphs that under many of the circumstances usually encountered in practice the large majority of the high-energy ions in the tail of the ion velocity distribution are subject to a dynamical friction force (originating with the electrons) which increases with ion velocity.

Equation (34) shows that ion runaway is possible for any electron and ion temperature provided the velocity distributions are well separated, i.e., $|\mathbf{v}_e-\mathbf{v}_i| \geq \beta_e^{-\frac{1}{2}}+\beta_i^{-\frac{1}{2}}$. This situation is best handled by the methods of R-I. However, when the velocity distributions overlap appreciably, then it is necessary to estimate runaway by locating the transition surface. The asymptotic limit of Eq. (34) takes the form

$$M\frac{d\mathbf{c}}{dt}+eE_c\left(2\frac{m}{M}\frac{\mathbf{q}_i}{\beta_e q_i^3}+\frac{\mathbf{q}_e}{\beta_e q_e^3}\right)=e\mathbf{E}.$$

By equating the dynamical friction and the applied electric force, we find the condition for runaway to be

$$\beta_e^{\frac{1}{2}}|\mathbf{c}|>(E_c/E)^{\frac{1}{2}}-\beta_e^{\frac{1}{2}}|\mathbf{v}_e|.$$

Since the present treatment requires $\beta_e^{\frac{1}{2}}|\mathbf{v}_e| \leq 1$ and $E \ll E_c$, we see that ion velocities must exceed the most probable random electron speed by an appreciable factor before runaway becomes possible. To calculate accurately the rate with which ions leave their collision-dominated region, it is necessary to solve the Boltzmann-Fokker-Planck equation for the ions as well as for the electrons. This has not been done. However, in a qualitative manner it is possible to predict that the rate involved must be much smaller than the eigenvalue λ_1 characteristic of electron runaway. For not only does the electron Joule heating rate exceed the ion Joule heating rate by the factor M/m, but the ion self-collision rate which gives rise to a diffusion through velocity space is smaller than the corresponding electron self-collision rate by the factor $(m/M)^2$ when $\beta_e=\beta_i$. Moreover, since we require $T_i \gg T_e$ for ion runaway, we can expect appreciable energy transfer from ions to electrons to take place. As a consequence both energy and momentum transfer to the electrons acts to reduce the ion runaway rate.

E. Comparison with Other Work

Recently Harrison[7] has made Eq. (7) the basis of an estimate on the rate at which electrons appear in the runaway region of velocity space. His treatment proceeds from the continuity equation for runaway

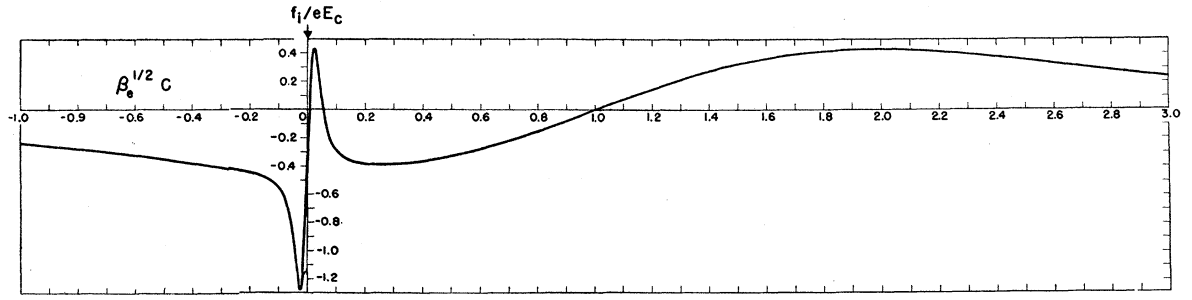

FIG. 10. The dynamical friction force f_i, normalized to eE_c, is shown as a function of $\beta_e^{\frac{1}{2}}c$ for an ion moving parallel to the relative electron-ion drift velocity with the speed c. For this illustration the electron and ion temperatures were taken to be equal, and their drift speeds were chosen to be $v_e=\beta_e^{-\frac{1}{2}}$; $v_i=-v_e m/M$.

[7] E. R. Harrison, Phil. Mag. 3, 1318 (1958).

electrons which he writes in the form

$$\frac{\partial n_r}{\partial t}+\nabla\cdot\mathbf{j}_r=\int\Delta f_r(\mathbf{v})d^3v.$$

Here n_r is the number density of runaways, $\mathbf{j}_r$ is their current density in real space, and $\Delta f_r(\mathbf{v})$ is the rate at which electrons qualify for the runaway state per unit volume of phase space at the velocity $\mathbf{v}$. He states, without presenting the physical reasoning involved, that when the applied electric field vanishes Δf_r is to a first approximation given by

$$\Delta f_r(\mathbf{v})=\frac{n}{2\tau_c}\left(\frac{m}{2\pi kT_e}\right)^{\frac{3}{2}}\exp\left(-\frac{mv_b^2}{2kT_e}\right). \quad (35)$$

The mean free collision time τ_c, appearing in this equation, is determined from the dynamical friction force acting on an electron moving with the speed v, and is given by

$$\tau_c=v^3/3n\Gamma_e.$$

For the total runaway rate induced by the field E, Harrison then finds

$$\frac{1}{2}\int_{v_b}^{\infty}\Delta f_r(v)4\pi v^2dv=\frac{3n^2}{\sqrt{\pi}}\Gamma_e\beta_e^{\frac{3}{2}}\frac{\exp(-V_b^2)}{V_b^2}, \quad (36)$$

where

$$V_b^2=mv_b^2/2kT_e,$$

and the effect of the electric field appears in the size of the runaway region only.

Harrison does not present the derivation of Δf_r, but presumably it is based upon a detailed balancing argument. The number of electrons entering the runaway region must just balance the number which leave this region provided steady state conditions prevail. When the electric field vanishes the velocity distribution is Maxwellian, and the above expression gives approximately the number which leave the runaway region since the main contribution to the integral comes from the neighborhood of v_b. For the moment we must ignore the fact that in the absence of the electric field v_b becomes infinite and the integral in Eq. (36) vanishes. The factor, 0.5, multiplying the integral presumably was introduced to take approximate account of the geometrical shape of the transition surface. This estimate can be said to give the rate of runaway in the presence of an applied electric field, provided (1) the velocity distribution remains Maxwellian right up to the transition surface, (2) the velocity distribution vanishes for $v>v_b$, i.e., no scattering back into the $v<v_b$ region, and (3) Joule heating is neglected. With these restrictions in mind Eq. (36) can be directly compared with our λ_1 provided that we normalize the former with the collision rate defined in Eq. (2). This

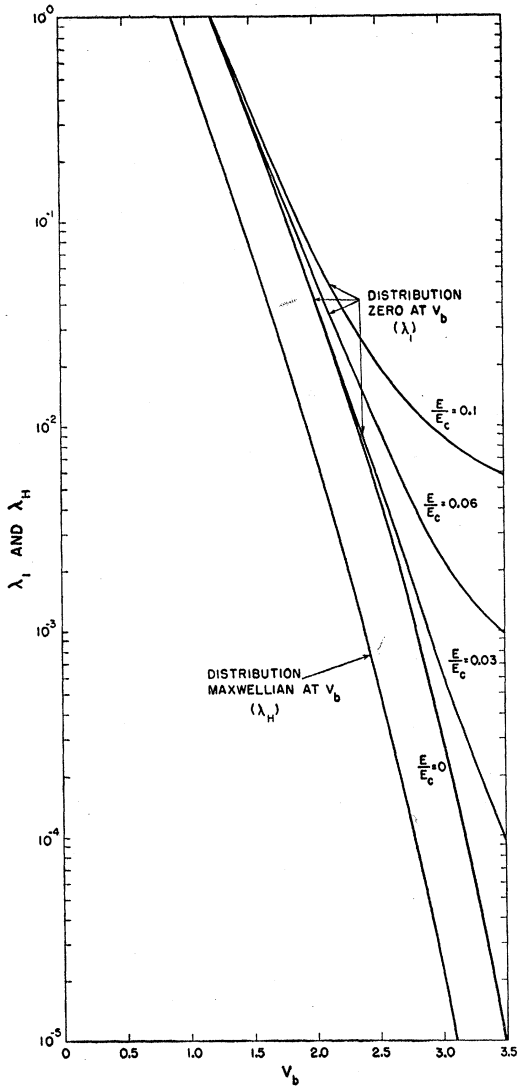

FIG. 11. Comparison, as a function of V_b, of the runaway rate λ_1 with the runaway rate λ_H proposed by Harrison. These curves agree most closely for the case $E/E_c=0$.

transforms Harrison's result to

$$\lambda_H=\frac{3}{\sqrt{\pi}}\frac{\exp(-V_b^2)}{V_b^2}.$$

A comparison of λ_H and λ_1 for $E/E_c=0$ is presented in Fig. 11 as a function of V_b. It shows that for this case λ_H is about one order of magnitude smaller than λ_1. When λ_H is compared with λ_1 for $E/E_c>0$, then the results differ even more. This difference arises partly from the fact that we have calculated λ_1 for a sphere, whereas λ_H involves the factor 0.5 to account for the geometry. It is also connected with the difference in boundary conditions at v_b. In our treatment the distribution is forced to vanish on v_b, and its value for $v<v_b$ is calculated from the Boltzmann equation.

Harrison assumes that the distribution remains unaltered (i.e., Maxwellian) for $v<v_b$, and then he counts the scattering out of the collision-dominated region only.

The crux of the matter is that both methods lead to incorrect results as long as the flow across the transition surface brought about by the electric field is neglected. In the absence of the applied field no runaway occurs at all. Therefore, it is precisely the deviation from the Maxwellian distribution induced by the electric field which is required for an estimate of the runaway rate. In our present treatment no physical significance is attached to λ_1 for finite V_b and $E/E_c=0$. However, this case does serve the purpose of separating the purely collisional diffusion in velocity space from the diffusion brought about by the electric field. Our results show that the inclusion of the electric field results in a runaway rate which exceeds by orders of magnitude the rate proposed by Harrison.

III. EXPERIMENTAL EVIDENCE AND RELATED PHENOMENA

A. Experimental Evidence

Evidence for the existence of the runaway effect is meager, and originates almost exclusively with observations carried out on experimental thermonuclear machines. Several years ago x-rays in the range of hundreds of kev were observed to be emitted from a toroidal pinched plasma[8] ("Perhapsatron" at Los Alamos), and from a figure-eight electrical discharge[9] ("Stellarator" at Princeton). Additional evidence obtained by the Princeton group[10] indicates the existence of long time constant current plateaus in the Stellarator even after the application of external electric fields has ceased. Presumably these plateaus can be attributed to runaway electrons whose mean free collision time for momentum transfer is exceedingly long. Large x-ray yields are frequently measured at and immediately following electrical breakdown of toroidal discharges.[8] This indicates that electron runaway also plays an important role in the breakdown mechanism, and during the early ionization stage of toroidal discharges. Another class of experiments is concerned with the spectral analysis of the neutrons emitted from plasmas as a result of the d-D reaction. For a number of experimental machines[11] the measured spectrum requires the center-of-mass motion of a considerable number of deuterons in the direction of the applied electric field. Since, as we have pointed out, ion runaway is an exceedingly slow process, it is not unlikely that other mechanisms operate to bring ions into the runaway region where they are accelerated to higher energies by the electric field.

The physical phenomena occuring in these experiments are unfortunately large in number and too complex to permit a clear separation of the runaway phenomenon, and so far no true measurement of the velocity distribution or the runaway rate has been obtained.

B. Effects due to Magnetic Fields

We have already pointed out[1] that particle runaway cannot take place when the electric and magnetic fields in the plasma are mutually perpendicular. Our results are therefore to be interpreted strictly in terms of the component of **E** which is parallel to **B**. Magnetic field effects are completely absent only in certain idealized geometries. Into this category fall the purely radial discharges between coaxial cylinders or concentric spheres. In practical geometries charged particle motion may be jointly controlled by the existing electric and magnetic fields to an extent which can make a clear separation of the runaway process difficult.

An extreme example in this category is the cylindrically pinched plasma, confined solely by its own magnetic field, and driven by an axial electric field. In this situation runaway can occur only near the pinch axis where the self-magnetic field vanishes. Certain charged particles in this plasma configuration traverse complicated orbits which repeatedly cross the axis where acceleration in the electric field alone can take place. With the addition of an axial magnetic field, runaway becomes possible at all radii, although the curvature of the resulting helical magnetic field introduces effects which we have not taken into account. Strictly speaking, our treatment applies only insofar as the electrical currents local to a point in space are not strongly affected by the curvature of the magnetic field.

C. Excitation of Plasma Instabilities

The runaway of plasma particles creates asymmetries in their velocity distribution which in turn can promote plasma instabilities. A detailed study of this question requires exact solutions of the Boltzmann-Fokker-Planck equation. Such a program requires the use of high-speed digital computers, and is presently under way at Los Alamos.[12] Under certain simplified condi-

[8] Burkhardt, Sawyer, and Stratton, Conference on Thermonuclear Reactions, Princeton University, October, 1954, Atomic Energy Commission Washington Report 184 (unpublished), p. 68; Burkhardt, Sawyer, Stratton, and Williams, Conference on Thermonuclear Reactions, Berkeley, February, 1955, Atomic Energy Commission Washington Report 289 (unpublished), p. 49.

[9] F. F. Chen, Conference on Thermonuclear Reactions, Berkeley, February, 1955, Atomic Energy Commission Washington Report 289 (unpublished), p. 297.

[10] Bernstein, Chen, Heald, and Kranz, Phys. Fluids **1**, 430 (1958).

[11] Conner, Hagerman, Honsaker, Karr, Mize, Osher, Phillips, and Stovall, *Proceedings of the Second United Nations Conference on the Peaceful Uses of Atomic Energy, Geneva, 1958* (United Nations, Geneva, 1959), Vol. 32, p. 297. Also in the same volume, see Butt, Carruthers, Mitchell, Pease, Thonemann, Bird, Blears, and Hartill, p. 63.

[12] For preliminary results, see H. Dreicer, see reference 11, Vol. 31, p. 57.

tions the instability problem has been solved.[13] The relative drift motion of electrons and ions in an infinite uniform plasma can be shown to cause the growth of electrostatic plasma oscillations provided that the particle velocity distributions are sufficiently peaked and separated. This is the so-called two-beam instability well-known to the research field concerned with traveling wave tubes.[14]

Instabilities of this type can be expected to affect the rate of electron and ion Joule heating. Moreover, under these conditions the relative electron-ion drift velocity is probably controlled by the excitation of plasma oscillations as well as by the electric field. In spite of the existence of these cooperative plasma effects, random two-body encounters can still be expected to carry particles into regions of velocity space where the applied field dominates the particle motion.

D. Effect of Runaway on Pressure Balance

In this section we will derive the correction terms which particle runaway introduces into the pressure balance equation for a static cylindrically symmetric pinched plasma. In addition to its self-magnetic field, B_θ, the plasma is subjected to an axial magnetic field, B_z. All macroscopic properties are assumed to be independent of axial distance. For convenience we introduce a local Cartesian coordinate system whose basis vectors are defined by

$$\boldsymbol{\varepsilon}_1=\mathbf{r}_0, \quad \boldsymbol{\varepsilon}_2=\boldsymbol{\varepsilon}_3\times\boldsymbol{\varepsilon}_1, \quad \boldsymbol{\varepsilon}_3=\mathbf{B}/B,$$

where $\mathbf{B}=\mathbf{k}_0B_z+\boldsymbol{\theta}_0B_\theta$, and the vectors $\mathbf{r}_0$, $\boldsymbol{\theta}_0$, and $\mathbf{k}_0$, form the basis vectors for the cylindrical coordinate system whose z axis coincides with the pinch axis. The particle velocities are assumed to be distributed according to the displaced Maxwellian form

$$F_\alpha(\mathbf{c})=n\left(\frac{m_\alpha}{2\pi kT_{1\alpha}}\right)\left(\frac{m_\alpha}{2\pi kT_{3\alpha}}\right)^{\frac{1}{2}} \times\exp\left[-\sum_j\frac{m_\alpha}{2kT_{j\alpha}}(c_j-v_{j\alpha})^2\right], \quad (37)$$

with $T_{1\alpha}=T_{2\alpha}$. The summation over $\mathbf{j}$ refers to the three orthogonal directions in the local coordinate frame which are determined by $\boldsymbol{\varepsilon}_1$, $\boldsymbol{\varepsilon}_2$, and $\boldsymbol{\varepsilon}_3$. The subscript α refers to the particle type, i.e., electrons or ions. For simplicity the electron and ion temperatures are chosen to be equal.

$$(T_j)_e=(T_j)_i=T_{j\alpha}=T_j.$$

The drift velocity $\mathbf{v}_\alpha$ is given by [15]

$$v_{1\alpha}=\frac{1}{\sigma_\perp B^2}\boldsymbol{\varepsilon}_1\cdot\nabla(nkT_{1\alpha})+\frac{3n}{4\sigma_\perp B^2}\boldsymbol{\varepsilon}_1\cdot\nabla(kT_{1\alpha})+\frac{\boldsymbol{\varepsilon}_1\cdot(\mathbf{E}\times\mathbf{B})}{B^2}$$

$$v_{2\alpha}=\frac{1}{e_\alpha B}\boldsymbol{\varepsilon}_1\cdot\nabla(nkT_{1\alpha}), \quad v_{3\alpha}=\mathbf{v}_\alpha\cdot(\mathbf{B}/B),$$

$$\frac{1}{\sigma_\perp}=\frac{\sqrt{2}m\Gamma_e}{3(\pi)^{\frac{1}{2}}e^2}\left(\frac{m}{kT_1}\right)^{\frac{1}{2}}.$$

These results are valid provided we are dealing with electrons and singly charged ions, and that $m_\alpha v_{3\alpha}/e_\alpha B$, the distance traveled along the magnetic line during a Larmor period, does not exceed the curvature of the magnetic field.

Since pinch conditions are assumed to be static in this analysis, the radial velocity, $v_{1\alpha}$, vanishes. Moreover, if we assume that the variations in the plasma pressure are small over an ion Larmor radius, then ions and electrons diffuse at the same rate,[16] and radial space charge electric fields are absent. The electric field is then purely axial in direction.

Pressure balance is described by the relation

$$\nabla\cdot(F_{ij}+P_{ij})=0,$$

where

$$F_{ij}=\frac{1}{4\pi}\left[\frac{E^2+B^2}{2}\delta_{ij}-(E_iE_j+B_iB_j)\right],$$

and P_{ij} is the momentum flow tensor whose components in the local coordinate frame are given, with the help of Eq. (37) by

$$P_{ij}=\sum_\alpha(nkT_{i\alpha}\delta_{ij}+nm_\alpha v_{i\alpha}v_{j\alpha}).$$

In terms of these components, pressure balance in Gaussian units is expressed by

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{B^2+E_z^2}{8\pi}\right)-\frac{E_z^2+B_z^2-B_\theta^2}{8\pi r}+\sum_\alpha\left\{\frac{1}{r}\frac{\partial}{\partial r}(nkT_{1\alpha}r) -\frac{1}{r}\left[\left(\frac{B_z}{B}\right)^2(nkT_{1\alpha}+nm_\alpha v_{2\alpha}^2)+2nm_\alpha v_{2\alpha}v_{3\alpha}\frac{B_\theta B_z}{B^2}\right] -\frac{1}{r}\left[\left(\frac{B_\theta}{B}\right)^2(nkT_{3\alpha}+nm_\alpha v_{3\alpha}^2)\right]=0. \quad (38)$$

Finally we may neglect $v_{2\alpha}$ compared to $(kT_1/m_\alpha)^{\frac{1}{2}}$ provided again the variations in plasma pressure are small over an ion Larmor radius.[1] Integration of Eq. (38)

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over radius results in

$$\frac{B^2(r)+E_z^2(r)}{8\pi}+\int_R^r \frac{B_\theta^2}{4\pi r}dr+\sum_\alpha \Big\{ nkT_{1\alpha}(r) + \int_R^r \frac{1}{r}\Big[nkT_{1\alpha}\Big(1-\frac{B_z^2}{B^2}\Big) - \Big(\frac{B_\theta}{B}\Big)^2 (nkT_{3\alpha}+nm_\alpha v_{3\alpha}^2)\Big]dr = \frac{B^2(R)+E_z^2(R)}{8\pi}+\sum_\alpha nkT_{1\alpha}(R),$$

where r and R represent any two radii. For the special case $T_{1\alpha}=T_{3\alpha}$, this expression simplifies to

$$\frac{B^2(r)+E_z^2(r)}{8\pi}+\int_R^r \frac{B_\theta^2}{4\pi r}dr + \sum_\alpha\Big[nkT_{1\alpha}(r)-\int_R^r \frac{nm_\alpha v_{3\alpha}^2}{r}\frac{B_\theta^2}{B^2}dr\Big] = \frac{B^2(R)+E_z^2(R)}{8\pi}+\sum_\alpha nkT_{1\alpha}(R).$$

The integral involving $v_{3\alpha}$ represents the centrifugal pressure associated with the directed motion of runaway plasma particles. It varies monotonically with radius since the integrand is always positive. In general, plasma pressures deduced from the measurement of B_θ and B_z as a function of radius[17] will measure only the sum of the random and centrifugal pressures. However, if the inequality

$$nkT_{1\alpha}\Big(1-\frac{B_z^2}{B^2}\Big)<\frac{B_\theta^2}{B^2}(nkT_{3\alpha}-nm_\alpha v_{3\alpha}^2)$$

holds true everywhere in the plasma, then the total pressure will have a maximum on the axis provided that the centrifugal pressure is the dominant part of the total particle pressure. It is clear from these results that the plasma temperature can be seriously overestimated whenever the centrifugal pressure equals or exceeds the random particle pressure, and this is not taken into account.

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[17] Burkhardt, Lovberg, and Phillips, Nature **181**, 224 (1958). L. C. Burkhardt and R. H. Lovberg, see reference 11, Vol. 32, p. 29.