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The Scattering of Neutrons by Protons

H. A. Bethe and R. Peierls

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very long time so that one can now carry out measurements in this temperature region quite easily.

(3) Iron ammonium alum, manganese ammonium sulphate, chromium potassium alum, and gadolinium sulphate have been investigated as working substances. Of these salts iron ammonium alum is the most nearly ideal, *i.e.*, the interaction of the magnetic ions is the smallest, so that the salt is the most suitable for this method. With this substance, starting at 1·2° and 14 kilogauss, a temperature of 0·038° was reached.

(4) Experiments with a substance in which the magnetic ions were diluted by forming a mixed crystal indicated that the greater part of the interaction is due to the crystalline field and not to direct interaction.

The Scattering of Neutrons by Protons

By H. A. Bethe and R. Peierls, University of Manchester

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1. The mass defects of the lightest nuclei, particularly the ratio between that of the diplon and the α-particle, make it very probable that the range of the interaction force between proton and neutron is very small, of the order of about 10^{-13} cm.* Therefore, in all experiments in which scattering of neutrons by protons has been observed, their wave-length is larger than the range of the interaction force. In these circumstances it is well known that the scattering intensity will be independent of angle for that co-ordinate system in which the centre of gravity is at rest.† For much higher energies, of course, this will no longer be true and one will expect then an anisotropy in the scattered intensity. This effect will become appreciable for energies for which the wave-length is of the same order as the range of the forces. Exact measurements of the angular distribution of scattered fast neutrons would therefore afford a direct check of the assumption of a short range and an estimate of this range. The existing experiments‡ show an isotropic scattering within

* Wigner, 'Phys. Rev.,' vol. 43, p. 252 (1933).

† Wigner, 'Z. Physik,' vol. 83, p. 253 (1933); Chadwick, 'Proc. Roy. Soc.,' A, vol. 142, p. 1 (1933).

‡ Chadwick, *loc. cit.*; Auger and Monod-Herzen, 'C. R. Acad. Sci., Paris,' vol. 196, p. 1102 (1933); Kurie, 'Phys. Rev.,' vol. 43, p. 672 (1933).

the limits of error, but are not accurate enough to allow very definite conclusions.

2. The importance of such experiments is further increased by the fact that, as was pointed out by Wick,* the sign of the asymmetry in the scattering depends on whether the interaction is of the ordinary type or an exchange force as proposed by Heisenberg† and Majorana,‡ and observations of the asymmetry could therefore decide this question.

For very high energies of the scattered particles (wave-length small compared with range) for " ordinary " forces§ this asymmetry is always such that most of the particles are scattered through small angles (*i.e.*, most of the observed protons move at right angles to the incident neutron beam). For exchange forces one gets the opposite behaviour, *i.e.*, the neutrons will preferentially be scattered backwards in the co-ordinate system of relative motion (in the ordinary co-ordinate system the protons will then mostly go forward). This behaviour can be interpreted as the incident particle going on with a small deflection, but in the process of scattering it has changed role with the scattering particle and has become a proton.

It has been tacitly assumed that this will be quite generally true and that a forward maximum in the scattering will always indicate ordinary forces, while a backward maximum would require exchange.

We shall show, however, that just for the proton-neutron interaction this assumption fails for not very high energies. For these we just get a backward maximum with ordinary and a forward maximum with exchange forces, whereas the effect reverses its sign at a certain rather high energy.‖ This behaviour is closely connected with the fact that the interaction forces admit a state in which the particles are bound together, as we know from the existence of the diplon.

3. In the following we denote by r the length of the radius vector between proton and neutron and by θ its angle with the incident beam, by E, v, k the energy, velocity and wave number of the incident neutrons in the relative co-ordinate system and by E_0, v_0, k_0 the same quantities for the system where the proton is initially at rest.

* 'Z. Physik,' vol. 84, p. 799 (1933).

† 'Z. Physik,' vol. 77, p. 1 (1932).

‡ 'Z. Physik,' vol. 82, p. 137 (1933).

§ We use here and subsequently the word " ordinary " forces as distinct from exchange forces, for interactions corresponding to wave mechanical operators that do not interchange the co-ordinates of protons and neutrons.

‖ The calculations of Wigner ('Z. Physik,' *loc. cit.*) led to different results, but they contained an error in the rather complicated analysis.

M is the mass of the proton or the neutron, $V(r)$ the interaction potential, which for definiteness we first suppose to be an "ordinary" one, and ε the binding energy of the diplon.

The Schrödinger equation then separates in polar co-ordinates; we write

$$r\psi = \sum_l u_l(r) P_l(\theta), \tag{1}$$

where P_l is the spherical harmonic of order l and u_l has to satisfy the equation*

$$\frac{\hbar^2}{M}\left(\frac{d^2u_l}{dr^2} - \frac{l(l+1)}{r^2}u_l\right) + (E - V(r))u_l = 0, \tag{2_l}$$

and asymptotically for large r has the form

$$u_l = \text{const.} \sin(kr - \tfrac{1}{2}l\pi + \delta_l). \tag{3_l}$$

The phases δ_l have to be determined by integration of (2_l). Then the effective cross-section for scattering through an angle θ becomes†

$$d\sigma = \frac{\pi}{2k^2} \left| \sum_l (2l+1) P_l(\theta)(e^{2i\delta_l} - 1) \right|^2 \sin\theta\, d\theta. \tag{4}$$

We now assume V to have a range of order a which is small compared with the wave-length $\lambda = 1/k$. In this case all phases δ_l will be small except for $l = 0$, because the centrifugal force $\hbar^2 l(l+1)/Mr^2$ already makes the eigen-function very small for radii $r < l\lambda$.

4. We begin with the discussion of δ_0. We know of the potential V that the equation (2_0) admits a solution with a negative energy, viz., $-\varepsilon$. There will be only one negative eigen-value, because the difference between the levels must be of the order $\hbar^2/Ma^2$ which with the assumed value of $a \backsim 10^{-13}$ cm becomes of the order 10^8 volts. The eigen-function u_0 belonging to this bound state will increase quickly for $r < a$, and for large r will show a slow exponential drop,‡ of the form

$$\left.\begin{aligned} u_0 &= \text{const.}\, e^{-\alpha r} \\ \alpha &= \sqrt{M\varepsilon}/\hbar \sim 2\cdot 3 \, . \, 10^{12}\ \text{cm}^{-1} \end{aligned}\right\}. \tag{5}$$

In the transitional region $r \sim a$ the expression $\dfrac{1}{u_0}\dfrac{du_0}{dr}$ must therefore be slightly negative and of the order $-\alpha$.

* $\frac{1}{2}$M = reduced mass.

† Mott and Massey, "Atomic Collisions," Oxford Press, 1933, p. 24.

‡ Bethe and Peierls, 'Proc. Roy. Soc.,' A, vol. 148, p. 146 (1935).

If we now consider a small positive energy, the value of $\frac{1}{u_0}\frac{du_0}{dr}$ for $r = a$ will be practically the same as for the small negative energy $-\varepsilon$. This can be shown in the following way: by writing (2_0) for two different values of E one can easily derive the following relation for the corresponding wave functions u_0 and u'_0:

$$\left[u_0 \frac{du'_0}{dr} - u'_0 \frac{du^0}{dr}\right]_{r=a} = \frac{M}{\hbar^2}(E - E') \int_0^a u_0 u'_0 \, dr. \tag{6}$$

If the difference $E - E'$ is small, we have

$$\frac{d}{dE}\left(\frac{1}{u_0}\frac{du_0}{dr}\right)_{r=a} = -\frac{M}{\hbar^2}\frac{1}{u_0^2(a)}\int_0^a u_0^2(r)\, dr. \tag{7}$$

The integral on the right-hand side is of the order $au_0^2(a)$. It actually is somewhat smaller, since $u_0^2(r)$ vanishes at $r = 0$ and its value at $r = a$ is practically equal to its maximum value. Therefore

$$\left(\frac{1}{u_0}\frac{du_0}{dr}\right)_{r=a} \backsim -\alpha - (E + \varepsilon)\frac{\gamma M a}{\hbar^2}; \quad 0 < \gamma < 1. \tag{8}$$

For the special case of a potential hole of rectangular shape, we find $\gamma = \frac{1}{2}$. The right-hand side of (8) differs appreciably from α only if

$$E \backsim \frac{\hbar^2 \alpha}{Ma} = \sqrt{\varepsilon \cdot \frac{\hbar^2}{Ma^2}}, \tag{9}$$

which is of the order 1×10^7 volts (corresponding to a neutron energy $E_0 = 2 \times 10^7$ volts).

If, therefore, the energy is smaller than (9) we have essentially

$$\left(\frac{1}{u_0}\frac{du_0}{dr}\right)_a = -\alpha,$$

and since u_0 will already have the form (3_0) for $r = a$, we see that

$$k\frac{\cos(ka + \delta_0)}{\sin(ka + \delta_0)} = -\alpha, \tag{10}$$

which yields

$$\begin{aligned}\delta_0 &= -ka + tg^{-1}(-k/\alpha)\\ &= \pi/2 + tg^{-1}(\alpha/k) - ka.\end{aligned}$$

According to (4) the differential and total cross-sections then become

$$\left.\begin{aligned} d\sigma &= \frac{2\pi}{\alpha^2 + k^2} \sin\theta\, d\theta \\ \sigma &= \frac{4\pi}{\alpha^2 + k^2} = 24 \times 10^{-25} \frac{\varepsilon}{\varepsilon + \frac{1}{2}\mathrm{E}_0} \mathrm{cm}^2 \end{aligned}\right\}, \tag{11}$$

where $\mathrm{E}_0 = 2\mathrm{E}$ is the energy of the incident neutron and $\varepsilon = 2 \cdot 2 \,.\, 10^6$ volts.

The cross-section (11) is rather larger than the experimental values of Chadwick. For $\mathrm{E}_0 = 4 \cdot 3 \,.\, 10^6$ volts ($v = 2 \cdot 9 \,.\, 10^9$ cm/sec) Chadwick finds a cross-section between 5 and 8 $\times 10^{-25}$ cm^2 (radius 4 to 5 $\times 10^{-13}$), whereas (11) gives 12×10^{-25}; for $\mathrm{E}_0 = 2 \cdot 1 \times 10^5$ volts ($v = 2 \,.\, 10^9$ cm/sec) the corresponding figures are 11 to 15 $\times 10^{-25}$ (experimental) and 16×10^{-25} (theoretical). Considering the very indirect experimental method the agreement can, however, be considered as fair.

The assumption of a short range for the potential can thus be checked both by the isotropy and by the absolute magnitude of the cross-section (11).

For energies that are small compared with (9), δ_0 is larger than $\pi/2$ since $k^2 \ll \alpha/a$, but above a certain energy of the order (9), δ_0 will become smaller than $\pi/2$. In order to see this we remark that $\delta_0 = \pi/2$ is equivalent to

$$\left(\frac{1}{u_0}\frac{du_0}{dr}\right)_a = -k \operatorname{tg} ka \sim -k^2 a,$$

and according to (8)

$$-\alpha - (\mathrm{E} + \varepsilon)\, \gamma \mathrm{M} a/\hbar^2 = -\alpha - \gamma a\, (k^2 + \alpha^2) = -k^2 a,$$

i.e.,

$$k^2 (1 - \gamma) = \alpha/a - \alpha^2 \gamma$$

$$\mathrm{E}_s = \frac{1}{1-\gamma} \sqrt{\varepsilon \frac{\hbar^2}{\mathrm{M}a^2}}. \tag{12}$$

Since $1 - \gamma$ is positive and of order unity this energy is of the order (9).

5. For the evaluation of δ_1, we compare the behaviour of the function u_1 and the function w_1 that is a solution of (2_1) when V is put equal to zero. One easily derives that

$$\left(w_1 \frac{du_1}{dr} - u_1 \frac{dw_1}{dr}\right)_{r=\mathrm{R}} = \frac{\mathrm{M}}{\hbar^2} \int_0^{\mathrm{R}} \mathrm{V}(r)\, u_1(r)\, w_1(r)\, dr,$$

where R is an arbitrary large distance. Inserting (3_1), we find

$$-\operatorname{tg}(k\mathrm{R} + \delta_1) + \operatorname{tg} k\mathrm{R} = \frac{\mathrm{M}}{\hbar^2 k} \frac{1}{u_1(\mathrm{R})\, w_1(\mathrm{R})} \int_0^\infty \mathrm{V} u_1 w_1\, dr.$$

It is convenient to choose such a distance R that $tg\, k\mathrm{R} = 0$, then $w_1(\mathrm{R})$ has just its relative maximum value and

$$tg\,\delta_1 = -\frac{\mathrm{M}}{\hbar^2 k\, u_1(\mathrm{R})\, w_1(\mathrm{R})}\int_0^\infty \mathrm{V}(r)\, u_1 w_1\, dr. \tag{13}$$

The right-hand side is positive since $\mathrm{V} < 0$, and u_1 and w_1 have equal sign.

The expression (13) is small, since V is only appreciable in regions where both the functions u and w_1 are much smaller than outside because of the centrifugal force. For an estimate of (13) we may identify u_1 with

$$w_1 = \frac{\sin kr}{kr} - \cos kr,$$

and obtain

$$\delta_1 = -\frac{\mathrm{M}}{\hbar^2 k}\int_0^\infty \mathrm{V}(r)\,(\tfrac{1}{3}k^2 r^2)^2\, dr.$$

From the existence of a small negative eigen-value of the equation (2_0) one can conclude that

$$-\int \mathrm{V}(r)\, r^4\, dr \backsim \tfrac{1}{2}\frac{\hbar^2}{\mathrm{M}}\, a^3$$

$$\delta_1 \backsim \tfrac{1}{18}(ka)^3.$$

Actually the form of u_1 will differ from that of w_1 and therefore the numerical factor will be even less certain; we may write

$$\delta_1 = \frac{\beta}{18}(ka)^3, \tag{14}$$

where β is of order unity. For the special case of a rectangular potential, it may be shown that

$$\beta = \frac{72}{\pi^2} - 6 = 1\cdot 32.$$

6. We see that δ_1 is very small and positive, δ_2, δ_3, etc., are proportional to still higher powers of ka, and therefore in calculating the deviations from the isotropic distribution we may confine ourselves to the first two terms in (4), and also neglect higher powers of δ_1 than the first:

$$d\sigma = \frac{\pi}{2k^2}\,|\,(\cos 2\delta_0 - 1) + i(\sin 2\delta_0 + 3\delta_1 \cos\theta)\,|^2 \sin\theta\, d\,\theta$$

$$= \frac{\pi}{2k^2}(4\sin^2\delta_0 + 6\delta_1 \sin 2\delta_0 \cos\theta)\sin\theta\, d\,\theta.$$

We therefore see that the sign of the asymmetry depends on the product $\delta_1 \sin 2\delta_0$. According to the properties of δ_0 and δ_1 derived above, we get a maximum backward ($\cos\theta = -1$) for energies smaller than (12), since for these $\delta_0 > \pi/2$, *i.e.*, $\sin 2\delta_0 < 0$. At higher energies we have a maximum forward, and the asymmetry vanishes for the energy E_s given by (12).

The highest absolute value of the asymmetry for energies below (12) is obtained for $E = \frac{1}{2}E_s$; then the difference between forward and backward scattering, divided by the average scattering, becomes

$$\beta\,(a\alpha)^2/12\,(1-\gamma),$$

which is approximately 1%. Such a small asymmetry is difficult to observe and the anisotropy would become easily observable* only for neutron energies above (12), *i.e.*, about $4 \,.\, 10^7$ volts, which at the present moment seem impossible to obtain.

It would, however, be very important to measure the angular distribution with neutron energies up to the highest available, in order to ascertain that our original assumptions are at least approximately correct.

If the interaction force is not an " ordinary " one but of the exchange type, the calculation of δ_0 remains unchanged. In the differential equation for δ_1, however, the sign of the potential will be reversed.† To that accuracy with which we considered this potential as a perturbation on u we simply have to reverse the sign of δ_1. Actually also the coefficient β will be smaller than it would be with an " ordinary " force. (With a rectangular potential, $\beta = \frac{36}{\pi}\frac{1+e^{-\pi}}{1-e^{-\pi}} - 6\left(1+\frac{12}{\pi^2}\right) = -0\cdot89$). Then we shall have a forward maximum for $E < E_s$ and a backward maximum for $E > E_s$.

Summary

The cross-section and the angular distribution are calculated for the scattering of neutrons by protons. The result is practically independent of the special law of force assumed between neutron and proton, it depends only on the known binding energy of the diplon. The cross-section obtained is about 50% larger than the rather uncertain experi-

* For energies comparable with V, *i.e.*, of the order 10^8 volts, the asymmetry would become of the order unity.

† Bethe and Peierls, *loc. cit.*

mental value. The scattering is almost isotropic (in the relative co-ordinate system) for all neutron energies up to about 40 million volts. Only for still higher energies, which are at present unavailable, an experimental determination of the sign of the anisotropy would decide whether the force between neutron and proton is of the exchange type or an ordinary force.

On the Direction of Approach of Microseismic Waves

By A. W. LEE, M.Sc., D.I.C., Kew Observatory

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1—INTRODUCTION

It has been thought for some time that an examination of the relation between the phases of the horizontal and vertical displacements in microseisms would be of interest in showing how closely the oscillations compare with Rayleigh waves, but a practicable scheme for making the observations has only recently been developed. In the earlier attempts the turning points of consecutive oscillations were timed during several minutes, but the accuracy attained by interpolation between the minute breaks was not high enough for reliable comparisons between the components. A solution of this difficulty has now been found in a modification of the method adopted by Leet,* who has examined the relation between the horizontal and vertical phases of the microseisms recorded at Harvard Observatory, using comparisons of the movements *exactly at the minute breaks.* The application of this new method to the seisograms of Kew Observatory is described in the present paper.

2—TABULATION OF THE PHASES OF THE MICROSEISMS

Fig. 1 shows portions of the records obtained from the Galitzin seismographs at Kew on January 11, 1930, when the microseisms were very large. Upward movements on the seismograms correspond with ground

* ' Gerl. Beitr. Geophys.,' vol. 42, p. 232 (1934).