ZENO'S RELATIVISTIC EFFECT.
A PHYSICAL SOLUTION TO THE APORIA "ACHILLES AND THE TORTOISE"

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The Internet forums on physical, mathematical, philosophical and even religious issues frequently become a stage for discussions on Zeno of Elea, the ancient Greek philosopher. Even though in such discussions an idea of the inanity of Zeno's aporiae and the naivety of this ancient Greek is frequently heard, an interest in Zeno's aporiae is not waning.

No doubt, these days there are very few who seriously accept the conclusions of Eleatic thinkers as to the impossibility of motion; still their logical deductions have not lost their attraction even today. Zeno's name is mentioned not only in the history of ancient Greek philosophy. The slowdown of quantum processes discovered a quarter of a century ago was given the name of the quantum Zeno effect (QZE) [1] and is treated in the most serious scientific journals.

One of the most widely discussed Zeno's aporiae is the "Achilles and the Tortoise". In this relation Leo Tolstoy in his "War and Peace" wrote:

"The widely-known sophism of the ancients contends that Achilles would never outrun the tortoise moving ahead of him though Achilles is running ten times faster than the tortoise: no sooner has Achilles covered the distance separating him from the tortoise, than the latter would have covered one tenth ahead of this distance; Achilles would cover this tenth, and then the tortoise would have done one hundredth and so on to infinity. To the ancients this problem seemed to have no solution...

The new branch of mathematics, having become artful in dealing with infinitely small quantities as well as with other more intricate issues of motion, is now able to give answers to the questions that seemed to have no solution before".

For all this, strictly speaking, "the new branch of mathematics" does not give an answer to the "seemingly unsolvable" question put by Zeno. What is now called the solution to Zeno's aporia "Achilles and the Tortoise" is a mathematical description of the self-evident and seemingly absolutely unquestionable fact concluding that, contrary to Zeno's deductions, Achilles can outrun the tortoise fast enough. But is this fact so unquestionable in the light of modern science? Is it possible that under certain conditions, let it be pure theory, the duration of the contest between Achilles and the tortoise may be if not infinite though arbitrarily large?

The answer is positive.

Who has not heard today of the relativistic time dilation?

According to the special theory of relativity, within the reference frame where a running track is moving at a speed little different from that of light, Achilles' clock, if such existed, would stop ticking, and both Achilles and the tortoise would have stopped in frozen poses. If the speed of the running track were close enough to that of light, Achilles within such a reference frame would not outrun the tortoise in the future that the observer could possibly foresee.

Dealing with the relativistic side of Zeno's aporia "Achilles and the Tortoise", the current article demonstrates that Achilles' inability to start running (him "stopping in a frozen pose" at the start), logically "proven" by Zeno, is fully and directly congruous with the delay effect in relation to physical processes in fast-moving objects and such inability may conditionally be called Zeno's relativistic effect.

1. ACHILLES "STOPPED FROZEN" AT THE START

The aporia "Achilles and the Tortoise" is only one of a number of Zeno's aporiae showing, as Eleates believed, the illusiveness of motion.

The final deduction from the aporia "Achilles and the Tortoise" should not mean that Achilles would never outrun the tortoise at the finish, but that he would never start running. The analysis of Achilles' behaviour at the finish is merely a means to prove Achilles' inability to start running.

Indeed, if between Achilles and the tortoise were another tortoise running away from him, Achilles would not outrun it either, and, consequently, would never cover even half of the way to the first tortoise. If there is a third tortoise between the second one and Achilles, then ... etc., etc. However, if Achilles cannot outrun the tortoise close to him, then he finds himself in a "frozen" state, being unable to cross the start line and start moving.

Can one regard this deduction naive and impossible under any circumstances?

Strange enough, but within the framework of modern physics one may not make such a deduction. We will attempt to give an outline of it below.

Different from the well-known solutions actually presenting a mathematical description of the apparent result of Achilles' and the tortoise's "race" as contradicting Zeno's logical deduction, the solution given below demonstrates a physically correct case when Zeno's logic proves to be true.

At first sight any assumption of the impossibility of the ready-to-go Achilles to start running seems absurd; however, referring to the special theory of relativity, one can see a certain analogy between the seemingly impossible Zeno's deduction and seemingly impossible at the beginning of the 20^{th} century, but nowadays the quite habitual effect of the clock rate slowdown.

According to Einstein's special theory of relativity, the clock going within a certain inertial reference frame at a speed arbitrarily close to that of light shall slow down its rate. Similarly, the speeds of physical and biological processes both in physical objects and living organisms moving within certain inertial reference frames slow down at near-light speed. Thus, a sprinter, running the distance between the start and the finish lines of a cinder track in 10 seconds with reference to the clock mounted on this track, within a reference frame where the cinder track is moving at a speed v arbitrarily close to the speed of light, to make this run would need the time of $(1- v^2/c^2)^{-1/2}$, where c is the speed of light, times exceeding 10 seconds. At the speed v of the cinder track tending to the speed of light c, the time of the sprinter's run tends to infinity. As a result, in any of the intermediate positions the sprinter also remains infinitely or arbitrarily long. Likewise, the sprinter remains arbitrarily long at the start line of the cinder track speeding within a certain reference frame, i.e. the sprinter stops frozen in the start pose at the same time moving together with the cinder track at a near-light speed within this reference frame.

However, the most important thing is not even that for observers within this reference frame the sprinter is found stopped frozen, but that in this case the key condition of Zeno's aporia is met, namely, that the sprinter outrunning a slow creature running away from him can during an arbitrarily long time interval periodically find himself at a point in space where the slow creature found itself some time before.

2. THE POSITIVE SOLUTION TO THE APORIA "ACHILLES AND A TORTOISE"

Let us give a closer look at Achilles' and the tortoise's run from different inertial reference frames.

Let us call the running race between Achilles and the tortoise within a time interval from the race start from the initial positions of Achilles and the tortoise to the moment of the first arrival of Achilles at a place where the tortoise found itself Achilles' and the tortoise's first stage of approach. Accordingly, the movement of Achilles from this position to the next location of the tortoise is called the second stage, etc.

Considering the length of space occupied by Achilles and the tortoise when they are not considered as points or very thin linear objects, the aporia "Achilles and the tortoise" becomes meaningless already at the first stage provided the tortoise is standing still or is crawling so slowly that in the period needed for Achilles to cover the distance between the initial position and the tortoise's location, the latter either does not make any movement at all, or has moved so little that Achilles is unable to reach the place where the tortoise has been and from which it had to go forward meeting the condition of the aporia.

In order for the aporia to comply with Zeno's reasoning, the following conditions should be met:

At each stage of approach, during the time of Achilles' running up to the tortoise, the latter must leave its position and enable Achilles to occupy the place it was located before and from where it has moved forward. This condition is not related to the danger of treading upon the tortoise (Achilles may take the adjacent track), but to the ambiguity of the notion "outrun" in relation to special lengthy objects located at the finish line (when the tip of the tortoise's nose is still outrunning the tip of Achilles' nose, the tortoise's tail end may already be dropping behind).

If a 20 cm long tortoise is crawling at a speed, let's say of 0.1 km per hour, and Achilles is running at a speed of 25 km per hour, and the initial distance between Achilles and the tortoise is equal to 12.5 km, then in half an hour Achilles will complete the first stage of approach, and in 7.2 sec. more the second, following which another stage becomes impossible as the above mentioned condition will not be met.

The number of stages of approach of spatially extended Achilles and the tortoise is determined by the relationship of their movement speeds, their longitudinal dimensions (dimensions in the moving direction) and the initial distance between them.

In order to obtain an arbitrarily large number of stages of approach one may, for example, arbitrarily heavily "flatten" Achilles and the tortoise in their movement direction. In essence, it is by way of treating Achilles and the tortoise as lines possessing no thickness that the tortoise's lead in Zeno's aporia is ensured at each quasi-finish microstage of approach.

Yet there is another way of disintegration of Achilles' and the tortoise's race into an arbitrarily large number of stages, allowing managing without the farfetched and physically invalid "flattening" of Achilles and the tortoise to zero in the direction of their movement. This method consists in the replacement of the tortoise's position on the track with its position in space.

Let us consider Achilles, the tortoise and their running track in relation to different inertial reference frames.

Let us put a question: "What is the speed of the tortoise's movement in space?"

The answer is clear: the speed of the tortoise's movement in space depends on the choice of a reference frame.

If a reference frame were rigidly tied to the track, the tortoise would move within the space of this reference frame at the designated speed of 0.1 km per hour, and then the aporia boils down to the above-mentioned classical interpretation.

Within a reference frame rigidly tied to the track the speed of the tortoise's movement is equal to zero, and Achilles moving to the tortoise it being at rest within this frame at a speed of 24.9 km per hour will already at the first stage overtake the tortoise in 30 min. 7.2 sec.

For all that, there is also a reference frame where the tortoise is moving at a speed of 1000 km per hour, and Achilles, while overtaking it, is moving within this reference frame at a speed of 1024.9 km per hour in the direction of the tortoise's movement.

If we undertand the tortoise's position not as its position on the track, but as its position in the space of the given reference frame, then the number of approach stages in this space is significantly greater.

In this case, only 44 seconds after the start of the race from the initial position Achilles will find himself in such a place of space within the given reference frame where the tortoise found itself these 44 seconds before with the latter having moved forward and finding itself 12.2 km away from Achilles. The next stage of approach will require some less time (around 43 sec). When the distance between Achilles and the tortoise has gone down to 5 meters, another stage of approach would require only 0.017 sec.

The total time of Achilles' running up to the tortoise does not depend on the number of stages - it is the same and is, roughly speaking, equal in all the three reference frames considered (30 min. and 7.2 sec.). We say “roughly speaking” because according to the special theory of relativity one may speak of this "sameness” only at speeds much less than that of light, and that but within bounded accuracy of measurement.

Within the reference frames where the tortoise’s speed is arbitrarily close to that of light, the number of stages tends to infinity and the time of approach of Achilles and the tortoise becomes arbitrarily long. In this case Achilles and the tortoise “freeze” as if in an infinitely prolonged still picture.

Furthermore, within such reference frames, Achilles and the tortoise because of Lorentz contraction flatten in a natural way and their longitudinal dimensions become arbitrarily small without doing them any harm at that.

Thus, within a reference frame fully meeting all Zeno’s conditions, Achilles and the tortoise are moving at a near-light speed, and Achilles will practically never overtake the tortoise; he, flat as a pancake, will freeze in his initial position and practically never be able to start the run to the pancake-like tortoise.

The “race” will take place in the following mode (where A stands for Achilles and T for the tortoise):

1) Initial position:

A-------T

2) Stage 1: The starting Achilles, moving in the tortoise' direction, takes the place where the tortoise was; the tortoise moves to the right. Achilles does not change his pose.

----------A-------T

3) Stage 2: The starting Achilles, without changing his pose, takes the place where the tortoise was; the tortoise moves to the right.

----------A-------T

All subsequent displacement stages are analogous. The pose of starting Achilles remains invariable at all stages. Achilles, frozen in this pose, is moving in space but is not starting the run.

CONCLUSION

This is a remarkable confirmation of the rightness of Zeno’s logical reasoning. Two and a half millennia ago Zeno revealed, though in a naive form, the time dilation of physical processes under the conditions of his aporia, now contained in Einstein's special theory of relativity.

Zeno’s reasoning when not limited by a reference frame tied to the track where Achilles is running after the tortoise turns correct, and Zeno's final deduction, no matter how improbable it could seem, may no longer be regarded as a play of imagination of ancient Greek philosophers.

LITERATURE

1. Misra B., Sudarshan E.C.G. // J. Math. Phys. 1977. Vol. 18. P. 756-763.