Violations of Einstein's time dilation formula in particle decays

Eugene V. Stefanovich

2255 Showers Dr., Apt. 153, Mountain View, CA 94040, USA

eugene_stefanovich@usa.net

March 7, 2006

Abstract

A rigorous quantum relativistic approach has been used to calculate the relationship between the decay laws of an unstable particle seen from two inertial frames moving with respect to each other. In agreement with experiment, it is found that the usual Einstein's time dilation formula is rather accurate in this case. However, small corrections to this formula were also obtained. Although the observation of these corrections is beyond the resolution of modern experiments, their presence indicates that special relativistic time dilation is not rigorously applicable to particle decays.

1 Introduction

Unstable particles are a perfect testing ground for theories that try to unify the principle of relativity with quantum mechanics for (at least) two reasons. First, an unstable particle provides a very simple example of a non-trivial interacting quantum system. Due to the specific initial condition characteristic to this problem (there is just one particle in the initial state), a rigorous description of the decay is possible in a small Hilbert space that contains only states of the particle and its decay products, so the solution can be obtained in a closed form. Second, unstable particles played an important role in confirming predictions of Einstein's special relativity. It was demonstrated experimentally that the decay of moving particles slows down in a good agreement with Einstein's time dilation formula [1, 2].

For a long time it was believed that relativistic quantum mechanics and quantum field theory must reproduce exactly the Einstein's time dilation formula for unstable particles [3]. However, a detailed quantum relativistic calculation in [4] has shown that there are small corrections to this formula in the case of particles with definite momentum. This result was later confirmed in other studies [5, 6]. Although surprising, these findings did not challenge directly the applicability of special relativity to unstable systems. This is because relativistic transformations connect results of measurements of two inertial observers moving with constant velocities with respect to each other. Then a fair comparison with the time dilation formula requires consideration of unstable states having definite values of velocity for both observers. Such states do not have a definite momentum for, at least, one observer. Their decay laws were considered in recent article [7] which alleges that the decay accelerates when particle's velocity increases, instead of being slowed down as experiment shows. In the present article we resolve this controversy by performing a detailed calculation of the decay laws of particles with narrow distributions of velocities observed from moving reference frames. We found that the result of [7] is based on an incorrect identification of the subspace of states of the unstable particle in the full Hilbert space of the system. In sections 5 - 7 we present a rigorous quantum relativistic framework required for the description of decays. The exact formula for the time dependence of the non-decay probability in a moving inertial frame of reference is derived in section 8. Particular cases of this formula relevant to unstable particles with sharply defined momenta or velocities are considered in sections 9 and 11, respectively. Section 12 is devoted to numerical calculations of the differences between the accurate quantum mechanical result and the standard time dilation formula (9). Although these differences are much smaller than the resolution of modern experiments, their presence is in sharp contradiction with special relativity which does not tolerate even small deviations from the Einstein's time dilation.

2 Formulation of the problem

The decay of unstable particles is described mathematically by the *non-decay probability* which has the following definition. Suppose that we have a piece of radioactive material with N unstable nuclei prepared simultaneously at time $t = 0$ and denote $N_u(t)$ the number of nuclei that remain undecayed at time $t > 0$. Then the non-decay probability[1] $\omega(0, t)$ (also called the *non-decay law* in this paper) is defined as the fraction of nuclei that survived the decay in the limit of large N

$$\omega(0, t) = \lim_{N \to \infty} N_u(t)/N \tag{1}$$

So, at each time point the piece of radioactive material can be characterized by its *composition* $(N_u(t)/N; (1 - N_u(t)/N)$, where $N_u(t)/N$ is the share of undecayed nuclei

[1]The first argument in $\omega(0, t)$ indicates the rapidity of the observer that measures the non-decay probability, as described below.

in the sample and $1 - N_u(t)/N$ is the share of nuclei that were transformed to the *decay products*. In this paper, in the spirit of quantum mechanics, we will treat N unstable particles as an ensemble of identically prepared systems and consider $\omega(0,t)$ as a property of a single particle (nucleus), i.e., the probability of finding this particle in the undecayed state.

In this paper we are concerned with comparison of decay observations made by two observers O and O' that move with respect to each other. Without loss of generality we will assume that observer O' moves with respect to O with velocity $(0, 0, v)$ along the z-axis and that at time $t = 0$ measured by both observer's clocks the origins of their coordinate systems coincide and all three pairs of coordinate axes are parallel ($x \parallel x'$, etc.). For discussion purposes we will say that O is at rest while O' is moving. To simplify formulas we will use the *rapidity* parameter $\theta = \tanh^{-1}(v/c)$ instead of velocity v. For example, in this notation the famous relativistic factor takes the form $\gamma \equiv (1 - v^2/c^2)^{-1/2} = \cosh\theta$. If from the point of view of O the decay is described by the function $\omega(0,t)$, then the non-decay law of the same particle seen by the moving observer O' will be denoted by $\omega(\theta,t)$.[2] Calculation of this function and, in particular, the relationship between $\omega(\theta,t)$ and $\omega(0,t)$ is the major goal of this work.

3 Postulates of relativity

The purpose of any relativistic theory is to describe the relationships between measurements made by different *inertial observers* or from different *inertial reference frames*, i.e., reference frames moving in space with constant velocities along straight lines and without rotation. The special *principle of relativity* tells us that these reference frames are exactly equivalent

- **Postulate I.** Experiments identically arranged and performed in two different inertial reference frames O and Q always yield the same results.

For any two observers O and Q there is an *inertial transformation* that connects Q to O, i.e., the set of rules that allows one to change from the reference frame O to the reference frame Q. Each inertial transformation is a combination of space and time translations, rotations and boosts. A composition of two inertial transformations is again a valid inertial transformation. This composition obeys the associativity law. The inverse of an inertial transformation is also an inertial transformation. Therefore, they form a 10-parameter Lie group. The structure of this group is specified in the second postulate

- **Postulate II.** Inertial transformations (space and time translations, rotations, and boosts) form the *Poincaré group.*

[2]Note that here parameter t denotes time measured by the clock belonging to the observer O'.

Obviously, two different observers obtain different results by measuring observables of the same physical system. Most problems in physics can be understood as translations of descriptions of the physical system between different reference frames. For example, if we have a full description of the system from the point of view of observer O, then the time evolution is obtained by answering the question "what is the description of the same system from the point of view of observer O'' that is shifted in time with respect to O?" It is important to realize that inertial transformations between observers can be divided into two groups: *kinematical* and *dynamical*. Kinematical transformations are those whose action on observables does not depend on the interaction. For example, if two stationary observers look at the unstable particle from different points in space, they would assign the same non-decay probability to the particle. The same is true for stationary observers having different orientations in space. From this follows[3]

- **Postulate III.** Space translations and rotations are kinematical.

On the other hand, time translations produce non-trivial changes in the system. The composition of the unstable system looks different for observers O and O'' shifted in time with respect to each other. The exact action of time translations on observables of the physical system should be obtained as a result of solution of dynamical equations which depend on the interaction acting in the system. Hence the following postulate is true for all types of isolated interacting systems.

- **Postulate IV.** Time translations are dynamical, i.e., interaction-dependent.

The above postulates I - IV have overwhelming experimental support. They are assumed to be valid throughout this paper. However they are not sufficient for a full description of the unstable particle in a moving reference frame. Such a description requires also knowledge of the nature of boosts. Here we have a choice between two paths forward. One path is to postulate certain properties of boosts. This path was taken by Einstein. It leads to special relativity and to the time dilation formula (9) as explained in the next section. In this paper we will argue in favor of choosing another path: keep postulates I - IV, add to them well-established postulates of quantum mechanics, and see what are the implications for the transformations of observables (in particular, the non-decay probability) with respect to boosts. This approach is employed starting from section 5 throughout the paper.

[3] Our adoption of this postulate means that we are working in Dirac's *instant form* of relativistic dynamics [8]. Other forms of dynamics, e.g., the *point form* and the *front form*, were also introduced by Dirac and are frequently used for description of relativistic interactions. As discussed in section 13, these forms are not appropriate for the description of non-decay laws of unstable particles.

4 Particle decay in special relativity

In addition to postulates I - IV, Einstein's special relativity makes two more assumptions regarding the nature of boosts

- **Assumption V.** Boosts are kinematical.

By postulating the kinematical character of boosts, special relativity insists that the internal composition of a compound system does not depend on the velocity of the observer. For example, if O and O' are two observers moving with respect to each other, then, according to special relativity, both observer will measure the same composition of the unstable system at time $t = 0$. These statements are often considered as self-evident in discussions of special relativity. For example, R. Polishchuk writes in [9] *"Any event that is "seen" in one inertial system is "seen" in all others. For example if observer in one system "sees" an explosion on a rocket then so do all other observers."* In addition to the universality and interaction-independence of boost transformations, special relativity also postulates the exact transformation laws of physical observables with respect to boosts. They are referred to as *Lorentz transformations*. The most fundamental are Lorentz transformations for space-time coordinates of events.[4]

- **Assumption VI.** If from the point of view of observer O an event is localized in a space point (x, y, z) at time t, then from the point of view of observer O' the same event has space-time coordinates (t', x', y', z') given by Lorentz formulas

$$t' = t\cosh\theta - \frac{z}{c}\sinh\theta, \tag{2}$$

$$x' = x, \tag{3}$$

$$y' = y, \tag{4}$$

$$z' = z\cosh\theta - ct\sinh\theta, \tag{5}$$

Let us now demonstrate that in special relativity the Postulates I - IV and Assumptions V - VI are sufficient to unambiguously describe particle decay in different reference frames without invoking any information about the interaction governing the decay. Suppose that from the point of observer O the unstable system is prepared in

[4]By *event* we understand a measurable physical process occurring at a certain point in space at one time instant. An intersection of trajectories of two point-like classical particles is an example of such an event. Note that in quantum mechanics the definition of event is problematic as particles do not have well-defined trajectories. Moreover, it is known that a particle sharply localized from the point of view of the observer O loses its localization from the point of view of the moving observer O' [10]. So, Assumption VI can be applied only in the classical limit.

the state with composition $(1.0; 0.0)$ at rest in the origin $x = y = z = 0$ at time $t = 0$.[5] Then observer O may associate the space-time point

$$(t, x, y, z)_{prep} = (0, 0, 0, 0) \tag{6}$$

with the event of preparation. In accordance with the dynamical character of time translations, the non-decay probability $\omega(0, t)$ decreases with time. From experiment and quantum mechanical calculations (see, e.g., section 12) it is known that the non-decay law has an (almost) exponential shape[6]

$$\omega(0, t) \approx \exp(-\frac{t}{\tau_0}) \tag{7}$$

where τ_0 is the *lifetime* of the unstable particle. At time $t = \tau_0$ the non-decay probability is exactly e^{-1}, so that the composition is $(0.368; 0.632)$. This "one lifetime" event has coordinates

$$(t, x, y, z)_{life} = (\tau_0, 0, 0, 0) \tag{8}$$

according to the observer O.

Let us now take the point of view of the moving observer O'. Due to the Assumption V, both O and O' agree that at the "preparation" and "lifetime" events the non-decay probabilities are 1 and e^{-1}, respectively. However, observer O' may not agree with O about the space-time coordinates of these events. Substituting (6) and (8) in (2) - (5) we see that from the point of view of O', the "preparation" event has coordinates $(0, 0, 0, 0)$, and the "lifetime" event has coordinates $(\tau_0 \cosh\theta, 0, 0, -c\tau_0 \sinh\theta)$. Therefore, the time elapsed between these two events is $\cosh\theta$ times longer than in the reference frame O. This also means that the decay law of the particle is exactly $\cosh\theta$ slower from the point of view of the moving observer O'. This is reflected in the famous "time dilation" formula

$$\omega(\theta, t) = \omega(0, \frac{t}{\cosh\theta}) \tag{9}$$

This formula was confirmed in numerous experiments [1], most accurately for muons accelerated to relativistic speeds in a cyclotron [2]. These experimental findings were

[5] As indicated in the previous footnote, here we use the classical limit. In quantum mechanics the notions "at rest" and "in the origin" cannot be used simultaneously.

[6] The exact form of the non-decay law is not important for our derivation of eq. (9) here.

certainly a triumph of Einstein's theory. However, as we see from the above discussion, in special relativity eq. (9) can be derived only under two assumptions V and VI, which lack proper justification. Therefore, a question remains whether eq. (9) is a fundamental exact result or simply an approximation that can be disproved by more accurate measurements?

Our goal in the rest of this paper is to demonstrate that this classical result of special relativity is not exact. In section 12 we will calculate corrections to the formula (9).

5 Quantum mechanics of particle decays

Let us now turn to the description of an isolated unstable system from the point of view of relativistic quantum theory. In our approach we will keep postulates I - IV. However, we are not going to use Assumptions V and VI.

We will consider a model theory with particles a, b, and c, so that particle a is massive spinless and unstable, while its decay products b and c are stable and their masses satisfy the inequality

$$m_a > m_b + m_c \tag{10}$$

which makes the decay $a \to b+c$ energetically possible. In order to simplify calculations and avoid being distracted by issues that are not relevant to the problem at hand we neglect the spin of the particle a and assume that there is only one decay mode of this particle, i.e., into two decay products b and c. For our discussion, the nature of the particle a is not that important. For example, this could be a muon or a radioactive nucleus or an atom in an excited state.

When measuring the non-decay law, experimentalists simply count the number of particles and determine their types. For example, observations of a muon may result in only two outcomes. One can find either a non-decayed muon or its decay products (an electron, a neutrino, and an antineutrino). Thus the number of particles is a legitimate observable, and for our model system $a \to b + c$ we can introduce Hermitian operators for the number of particles N_a, N_b, and N_c. Since these observables can be measured simultaneously, we can assume that these three operators commute with each other. Two combinations of their common eigenvalues are allowed in our system:

$$n_a = 1, n_b = 0, n_c = 0 \tag{11}$$

$$n_a = 0, n_b = 1, n_c = 1 \tag{12}$$

Hence the Hilbert space of the unstable system should be represented as a direct sum of two orthogonal subspaces[7]

$$\mathcal{H} = \mathcal{H}_a \oplus \mathcal{H}_{bc} \tag{13}$$

where $\mathcal{H}_a$ is the subspace of states of the unstable particle a which corresponds to the set of eigenvalues (11), and $\mathcal{H}_{bc} \equiv \mathcal{H}_b \otimes \mathcal{H}_c$ is the orthogonal subspace of the decay products which corresponds to the set of eigenvalues (12).

We can now introduce a Hermitian operator T (also known as "yes-no experiment") that corresponds to the observable "particle a exists". The operator T can be fully defined by its eigensubspaces and eigenvalues. When a measurement performed on the unstable system finds it in a state corresponding to the particle a (i.e., the state vector is within $\mathcal{H}_a$), the value of T is 1. When the decay products $b + c$ are observed (the state vector lies in $\mathcal{H}_{bc}$), the value of T is 0. Apparently, T is a projection operator on the subspace $\mathcal{H}_a$. For each normalized state vector $|\Psi\rangle \in \mathcal{H}$ the probability of finding the unstable particle a is given by the expectation value of the observable T [3]

$$\omega_{|\Psi\rangle} = \langle \Psi | T | \Psi \rangle$$

In relativistic quantum mechanics, the dynamics of the system is described by a unitary representation U_g of the Poincaré group in the Hilbert space $\mathcal{H}$ [11]. It is convenient to express representatives U_g as exponential functions of *generators* $\mathbf{P}, \mathbf{J}, \mathbf{K}$, and H. Then a general unitary operator from the set U_g can be always written as a product

$$U_g = e^{-\frac{i}{\hbar}Ht} e^{\frac{i}{\hbar}\mathbf{P}\mathbf{a}} e^{\frac{ic}{\hbar}\mathbf{K}\vec{\theta}} e^{\frac{i}{\hbar}\mathbf{J}\vec{\phi}} \tag{14}$$

where $\vec{\phi}$ is the rotation vector, $\mathbf{v} = c\frac{\vec{\theta}}{\theta}\tanh\theta$ is the velocity of the boost, $\mathbf{a}$ is the vector of space translation, and t is the amount of time translation. Generators are Hermitian operators and correspond to certain *total* observables of the system. Space translations are generated by the vector of the total linear momentum $\mathbf{P}$. The generator of rotations $\mathbf{J}$ is the operator of the total angular momentum. The generator of time translations H is the Hamiltonian (total energy). The generator of boosts $\mathbf{K}$ is called the *boost operator*. These generators must satisfy the commutation relations of the Poincaré group Lie algebra. In this paper we will need, in particular, the following commutators

[7] In principle, a full description of systems involving these three types of particles must be formulated in the *Fock space* where integer eigenvalues n_a, n_b, and n_c are allowed to take any values from zero to infinity. However, for most unstable particles the interaction between the decay products in the final state can be ignored, and considering the subspace (13) of the full Fock space is a reasonable approximation.

$$[K_i, P_j] = -i\frac{\hbar}{c^2}H\delta_{ij} \tag{15}$$

$$[K_i, H] = -i\hbar P_i \tag{16}$$

where $i, j = x, y, z$. They imply the following useful relationships[8]

$$e^{\frac{i}{\hbar}\mathbf{K}c\vec{\theta}}\mathbf{P}e^{-\frac{i}{\hbar}\mathbf{K}c\vec{\theta}} = \mathbf{P} + \frac{\vec{\theta}}{\theta}[(\mathbf{P}\cdot\frac{\vec{\theta}}{\theta})(\cosh\theta - 1) + \frac{1}{c}H\sinh\theta] \tag{17}$$

$$e^{\frac{i}{\hbar}\mathbf{K}c\vec{\theta}}He^{-\frac{i}{\hbar}\mathbf{K}c\vec{\theta}} = H\cosh\theta + c(\mathbf{P}\cdot\frac{\vec{\theta}}{\theta})\sinh\theta \tag{18}$$

The representation (14) allows us to relate results of measurements in different reference frames. Let us first take the point of view of the observer O and consider a vector $|\Psi\rangle \equiv |\Psi(0,0)\rangle \in \mathcal{H}_a$[9] that describes a state in which the unstable particle a is found with 100% certainty.

$$\begin{aligned} \omega_{|\Psi\rangle}(0,0) &= \langle\Psi|T|\Psi\rangle \\ &= 1 \end{aligned}$$

Then the time evolution of the state vector $|\Psi\rangle$ in the reference frame O is given by

$$|\Psi(0,t)\rangle = e^{-\frac{i}{\hbar}Ht}|\Psi\rangle$$

and the non-decay law is given by

$$\omega_{|\Psi\rangle}(0,t) = \langle\Psi|e^{\frac{i}{\hbar}Ht}Te^{-\frac{i}{\hbar}Ht}|\Psi\rangle$$

From this equation it is clear that the Hamiltonian H describing the unstable system should not commute with the projection T

$$[H, T] \neq 0 \tag{19}$$

Otherwise, the subspace $\mathcal{H}_a$ of states of the particle a would be invariant with respect to time translations and the particle a would be stable ($\omega_{|\Psi\rangle}(0,t) = 1$ for all t).

[8] For derivation see section 2.2 in [12].

[9] In the notation $|\Psi(0,0)\rangle$ the first argument is the rapidity parameter of the reference frame from which the state $|\Psi\rangle$ is observed and the second argument is the time of observation. This is consistent with the convention adopted for $\omega(\theta,t)$ in section 1.

The moving observer O' describes the initial state (at $t=0$) by the vector

$$|\Psi(\theta,0)\rangle = e^{\frac{ic}{\hbar}K_z\theta}|\Psi\rangle$$

The time dependence of this state is

$$|\Psi(\theta,t)\rangle = e^{-\frac{i}{\hbar}Ht}e^{\frac{ic}{\hbar}K_z\theta}|\Psi\rangle \quad (20)$$

Then the non-decay law from the point of view of O' is

$$\omega_{|\Psi\rangle}(\theta,t) = \langle\Psi(\theta,t)|T|\Psi(\theta,t)\rangle \quad (21)$$

$$= \|T|\Psi(\theta,t)\rangle\|^2 \quad (22)$$

where the last equation follows from the property $T^2 = T$ of the operator T.

6 Non-interacting representation of the Poincaré group

Before calculating the non-decay law in the moving reference frame (21), let us first consider a simpler case when the interaction responsible for the decay is “turned off”. Then the representation of the Poincaré group U_g^0 acting in $\mathcal{H}$ is *non-interacting*. This representation is constructed in accordance with the structure of the Hilbert space (13) as

$$U_g^0 \equiv U_g^a \oplus U_g^b \otimes U_g^c \quad (23)$$

where $U_g^{(a,b,c)}$ are unitary irreducible representations of the Poincaré group corresponding to particles a, b, and c, respectively. The generators of the representation (23) are denoted by $\mathbf{P}_0$, $\mathbf{J}_0$, H_0, and $\mathbf{K}_0$. The operator of non-interacting *mass*

$$M_0 = +\frac{1}{c^2}\sqrt{H_0^2 - P_0^2c^2}$$

commutes with $\mathbf{P}_0$, $\mathbf{J}_0$, H_0, and $\mathbf{K}_0$. According to (10), the operator M_0 has a continuous spectrum in the interval $[m_b + m_c, \infty)$ and a discrete point m_a embedded in this interval.

From definition (23) it is clear that the subspaces $\mathcal{H}_a$ and $\mathcal{H}_{bc}$ are invariant with respect to U_g^0. Moreover, the particle number operator N_a and the projection operator T commute with the non-interacting generators

$$[T, \mathbf{P}_0] = [T, \mathbf{J}_0] = [T, \mathbf{K}_0] = [T, H_0] = 0 \tag{24}$$

This implies that the particle a is stable with respect to time translations and boosts, as expected

$$\begin{aligned}
\omega_{|\Psi\rangle}(\theta, t) &= \langle \Psi | e^{-\frac{ic}{\hbar}(K_z)_0\theta} e^{\frac{i}{\hbar}H_0 t} T e^{-\frac{i}{\hbar}H_0 t} e^{\frac{ic}{\hbar}(K_z)_0\theta} |\Psi\rangle \\
&= \langle \Psi | T | \Psi \rangle \\
&= 1
\end{aligned}$$

The primary reason for considering the representation U_g^0 is that it allows us to build a convenient basis in the subspace $\mathcal{H}_a$. Let us denote $|\mathbf{0}\rangle$ a vector in $\mathcal{H}_a$ that corresponds to the particle a with zero momentum.

$$\begin{aligned}
\mathbf{P}_0|\mathbf{0}\rangle &= \mathbf{0} \\
H_0|\mathbf{0}\rangle &= m_a c^2 |\mathbf{0}\rangle
\end{aligned}$$

Then we find that the vector

$$|\mathbf{p}\rangle = e^{-\frac{ic}{\hbar}\mathbf{K}_0\vec{\theta}} |\mathbf{0}\rangle \tag{25}$$

describes the particle a with definite momentum $\mathbf{p} = \frac{\vec{\theta}}{\theta} m_a c \sinh\theta$. Indeed, using (17) we obtain

$$\begin{aligned}
\mathbf{P}_0|\mathbf{p}\rangle &= \mathbf{P}_0 e^{-\frac{ic}{\hbar}\mathbf{K}_0\vec{\theta}}|\mathbf{0}\rangle \\
&= e^{-\frac{ic}{\hbar}\mathbf{K}_0\vec{\theta}} e^{\frac{ic}{\hbar}\mathbf{K}_0\vec{\theta}} \mathbf{P}_0 e^{-\frac{ic}{\hbar}\mathbf{K}_0\vec{\theta}}|\mathbf{0}\rangle \\
&= e^{-\frac{ic}{\hbar}\mathbf{K}_0\vec{\theta}} (\mathbf{P}_0 + \frac{\vec{\theta}}{\theta}[(\mathbf{P}_0 \cdot \frac{\vec{\theta}}{\theta})(\cosh\theta - 1) + \frac{1}{c} H_0 \sinh\theta])|\mathbf{0}\rangle \\
&= \frac{\vec{\theta}}{\theta} m_a c \sinh\theta e^{-\frac{ic}{\hbar}\mathbf{K}_0\vec{\theta}}|\mathbf{0}\rangle \\
&= \frac{\vec{\theta}}{\theta} m_a c \sinh\theta |\mathbf{p}\rangle
\end{aligned}$$

Since particle a is spinless by our assumption, the eigenvectors $|\mathbf{p}\rangle$ of the total momentum operator $\mathbf{P}_0$ form a full basis in the subspace $\mathcal{H}_a$, so that

$$\langle \mathbf{p}|\mathbf{p}'\rangle = \delta(\mathbf{p}-\mathbf{p}') \tag{26}$$

$$T = \int d\mathbf{p}|\mathbf{p}\rangle\langle\mathbf{p}| \tag{27}$$

Then any state $|\Psi\rangle \in \mathcal{H}_a$ of the particle a can be represented by a linear combination of these basis vectors

$$\begin{aligned} |\Psi\rangle &= T|\Psi\rangle \\ &= \int d\mathbf{p}|\mathbf{p}\rangle\langle\mathbf{p}|\Psi\rangle \\ &= \int d\mathbf{p}\psi(\mathbf{p})|\mathbf{p}\rangle \end{aligned} \tag{28}$$

where $\psi(\mathbf{p}) = \langle\mathbf{p}|\Psi\rangle$ is the wave function in the momentum representation. In order to ensure the normalization $\langle\Psi|\Psi\rangle = 1$, the function $\psi(\mathbf{p})$ must satisfy

$$\int d\mathbf{p}|\psi(\mathbf{p})|^2 = 1 \tag{29}$$

Apparently, vectors $|\mathbf{p}\rangle$ themselves are not normalized. If we want to study the non-decay law of a state with definite momentum $\mathbf{p}_0$, we should use a state vector (which we denote by $|\mathbf{p}_0)$ to distinguish it from $|\mathbf{p}_0\rangle$) that has a normalized momentum-space wave function sharply localized near $\mathbf{p}_0$. In order to satisfy eq. (29) such a wave function may be formally represented as a square root of the Dirac's delta function $\psi(\mathbf{p}) = \sqrt{\delta(\mathbf{p}-\mathbf{p}_0)}$

The action of boosts on the basis vectors $|\mathbf{p}\rangle$ and $|\mathbf{p})$ is known from Wigner's theory of irreducible unitary representations of the Poincaré group (see, e.g., [13, 11])

$$e^{-\frac{ic}{\hbar}(K_z)_0\theta}|\mathbf{p}\rangle = \sqrt{\frac{\Omega_{L\mathbf{p}}}{\Omega_{\mathbf{p}}}}|L\mathbf{p}\rangle \tag{30}$$

where

$$\begin{aligned} L\mathbf{p} &= (p_x, p_y, p_z\cosh\theta + \frac{\Omega_{\mathbf{p}}}{c}\sinh\theta) \\ \Omega_{\mathbf{p}} &= \sqrt{m_a^2c^4 + c^2\mathbf{p}^2} \end{aligned}$$

Using eq. (30) and the property

$$\frac{d\mathbf{p}}{\Omega_{\mathbf{p}}} = \frac{d(L\mathbf{p})}{\Omega_{L\mathbf{p}}} \tag{31}$$

we can find boost transformations for an arbitrary state vector of the form (28)

$$\begin{aligned}
e^{-\frac{ic}{\hbar}(K_z)_0\theta}|\Psi\rangle &= \int d\mathbf{p}\psi(\mathbf{p})e^{-\frac{ic}{\hbar}(K_z)_0\theta}|\mathbf{p}\rangle \\
&= \int d\mathbf{p}\psi(\mathbf{p})\sqrt{\frac{\Omega_{L\mathbf{p}}}{\Omega_{\mathbf{p}}}}|L\mathbf{p}\rangle \\
&= \int d\mathbf{p}\sqrt{\frac{\Omega_{L^{-1}\mathbf{p}}}{\Omega_{\mathbf{p}}}}\psi(L^{-1}\mathbf{p})|\mathbf{p}\rangle
\end{aligned}$$

These transformations can be viewed as transformations of the corresponding momentum-space wave function, e.g.,

$$e^{-\frac{ic}{\hbar}(K_z)_0\theta}\psi(\mathbf{p}) \equiv \sqrt{\frac{\Omega_{L^{-1}\mathbf{p}}}{\Omega_{\mathbf{p}}}}\psi(L^{-1}\mathbf{p}) \tag{32}$$

so that the boost operator can be represented as a differential operator in the momentum space

$$\begin{aligned}
(K_z)_0\psi(\mathbf{p}) &= \frac{i\hbar}{c}\lim_{\theta\to 0}\frac{d}{d\theta}e^{-\frac{i}{\hbar}(K_z)_0 c\theta}\psi(\mathbf{p}) \\
&= \frac{i\hbar}{c}\lim_{\theta\to 0}\frac{d}{d\theta}\sqrt{\frac{\Omega_{L^{-1}\mathbf{p}}}{\Omega_{\mathbf{p}}}}\psi(p_x, p_y, p_z\cosh\theta - \frac{\Omega_{\mathbf{p}}}{c}\sinh\theta) \\
&= -i\hbar(\frac{\Omega_{\mathbf{p}}}{c^2}\frac{d}{dp_z} + \frac{p_z}{2\Omega_{\mathbf{p}}})\psi(\mathbf{p}) \qquad (33)
\end{aligned}$$

For further calculations we will need to define the *Newton-Wigner position* operator [10] in $\mathcal{H}$

$$\mathbf{R}_0 \equiv -\frac{c^2}{2}(H_0^{-1}\mathbf{K}_0 + \mathbf{K}_0 H_0^{-1})$$

which has the property

$$[(R_i)_0, (P_j)_0] = i\hbar\delta_{ij} \tag{34}$$

that can be verified by direct substitution. According to eq. (33)

$$\begin{aligned}
(R_z)_0\psi(\mathbf{p}) &= -\frac{c^2}{2}(H_0^{-1}(K_z)_0 + (K_z)_0 H_0^{-1})\psi(\mathbf{p}) \\
&= \frac{i\hbar}{2}(\Omega_{\mathbf{p}}^{-1}\Omega_{\mathbf{p}}\frac{d}{dp_x} + \Omega_{\mathbf{p}}\frac{d}{dp_x}\Omega_{\mathbf{p}}^{-1} + \frac{p_x c^2}{\Omega_{\mathbf{p}}^2})\psi(\mathbf{p}) \\
&= i\hbar\frac{d}{dp_x}\psi(\mathbf{p})
\end{aligned}$$

Therefore operator $e^{\frac{ic}{\hbar}(R_0)_z b}$ acts as a translation operator in the momentum space. In particular, we can write

$$e^{\frac{ic}{\hbar}(R_0)_z m_a c \sinh\theta}\sqrt{\delta(\mathbf{p})} = \sqrt{\delta(\mathbf{p}-\mathbf{p}_0)}$$

where $\mathbf{p}_0 = (0, 0, m_a c \sinh\theta)$. On the other hand, applying the boost transformation (32) to the momentum eigenfunction we obtain[10]

$$\begin{aligned}
e^{-\frac{ic}{\hbar}(K_0)_z\theta}\sqrt{\delta(\mathbf{p})} &= \sqrt{\frac{\Omega_{L^{-1}\mathbf{p}}}{\Omega_{\mathbf{p}}}}\sqrt{\delta(L^{-1}\mathbf{p})} \\
&= \sqrt{\frac{\Omega_{L^{-1}\mathbf{p}}}{\Omega_{\mathbf{p}}}}\sqrt{\frac{1}{|J|}\delta(\mathbf{p})} \\
&= \sqrt{\delta(\mathbf{p}-\mathbf{p}_0)}
\end{aligned}$$

This suggests that momentum eigenvector (25) has another useful representation

$$|\mathbf{p}\rangle = e^{\frac{i}{\hbar}\mathbf{R}_0\cdot\mathbf{p}}|\mathbf{0}\rangle \tag{35}$$

7 Interacting representation of the Poincaré group

Let us now "turn on" the interaction responsible for the decay and discuss the interacting representation U_g of the Poincaré group in $\mathcal{H}$ with generators $\mathbf{P}$, $\mathbf{J}$, $\mathbf{K}$, and H. According to our postulates III and IV, the generators of space translations and rotations are interaction-free,

[10] where $|J| = \frac{\Omega_{L^{-1}\mathbf{p}}}{\Omega_{\mathbf{p}}}$ is the Jacobian of the transformation $\mathbf{p} \to L^{-1}\mathbf{p}$.

$$\begin{aligned} \mathbf{P} &= \mathbf{P}_0 \\ \mathbf{J} &= \mathbf{J}_0 \end{aligned}$$

while the generator of time translations (the Hamiltonian H) contains an interaction-dependent term V.

$$H = H_0 + V$$

From the commutator (15) it then follows [8] that the generators of boosts must be interaction-dependent as well

$$\mathbf{K} = \mathbf{K}_0 + \mathbf{W}$$

where $\mathbf{W} \neq 0$. This means that we are working in the so-called Dirac's *instant form of dynamics*. We will further assume that the interacting representation U_g belongs to the Bakamjian-Thomas form of dynamics [14], which is characterized by the property that the interacting operator of mass $M \equiv c^{-2}\sqrt{H^2 - \mathbf{P}_0^2 c^2}$ commutes with the Newton-Wigner position operator[11]

$$[\mathbf{R}_0, M] = 0 \tag{36}$$

Our next goal is to define the basis of common eigenvectors of commuting operators $\mathbf{P}_0$ and M in $\mathcal{H}$.[12] These eigenvectors must satisfy conditions

$$\mathbf{P}_0|\mathbf{p}, m\rangle = \mathbf{p}|\mathbf{p}, m\rangle \tag{37}$$

$$M|\mathbf{p}, m\rangle = m|\mathbf{p}, m\rangle \tag{38}$$

They are also eigenvectors of the interacting Hamiltonian $H = \sqrt{M^2c^4 + \mathbf{P}_0^2 c^2}$

[11] The possibilities for the interaction to be not in the Bakamjian-Thomas instant form are discussed in section 13.

[12] In addition to these two operators, whose eigenvalues are used for labeling eigenvectors $|\mathbf{p}, m\rangle$, there are other independent operators in the mutually commuting set containing $\mathbf{P}_0$ and M. These are, for example, the operators of the square of the total angular momentum $\mathbf{J}_0^2$ and the projection of the total angular momentum on the z-axis $(J_0)_z$. Therefore a unique characterization of any basis vector requires specification of all corresponding quantum numbers as $|\mathbf{p}, m, j^2, j_z, \ldots\rangle$. However these quantum numbers are not relevant for our discussion and we omit them.

$$H|\mathbf{p}, m\rangle = \omega_{\mathbf{p}}|\mathbf{p}, m\rangle$$

where $\omega_{\mathbf{p}} \equiv \sqrt{m^2c^4 + c^2\mathbf{p}^2}$.[13] In the zero-momentum eigensubspace of the momentum operator $\mathbf{P}_0$ we can introduce a basis $|\mathbf{0}, m\rangle$ of eigenvectors of the interacting mass M

$$\begin{aligned} \mathbf{P}_0|\mathbf{0}, m\rangle &= \mathbf{0} \\ M|\mathbf{0}, m\rangle &= m|\mathbf{0}, m\rangle \end{aligned}$$

Then the basis $|\mathbf{p}, m\rangle$ in the entire Hilbert space $\mathcal{H}$ can be built by formula (cf. eq. (25))

$$|\mathbf{p}, m\rangle = e^{-\frac{ic}{\hbar}\mathbf{K}\vec{\theta}}|\mathbf{0}, m\rangle$$

where $\mathbf{p} = mc\vec{\theta}\theta^{-1}\sinh\theta$. These eigenvectors are normalized to delta functions

$$\langle\mathbf{q}, m|\mathbf{p}, m'\rangle = \delta(\mathbf{q} - \mathbf{p})\delta(m - m') \tag{39}$$

The actions of inertial transformations on these states are well-known [11]. In particular, for boosts along the z-axis (cf. eq. (30)) and time translations we obtain

$$e^{-\frac{ic}{\hbar}K_z\theta}|\mathbf{p}, m\rangle = \sqrt{\frac{\omega_{\Lambda\mathbf{p}}}{\omega_{\mathbf{p}}}}|\Lambda\mathbf{p}, m\rangle \tag{40}$$

$$e^{\frac{i}{\hbar}Ht}|\mathbf{p}, m\rangle = e^{\frac{i}{\hbar}\omega_{\mathbf{p}}t}|\mathbf{p}, m\rangle \tag{41}$$

where

$$\Lambda\mathbf{p} = (p_x, p_y, p_z\cosh\theta + \frac{\omega_{\mathbf{p}}}{c}\sinh\theta)$$

Next we notice that due to eqs. (34) and (36) vectors $e^{\frac{i}{\hbar}\mathbf{R}_0\cdot\mathbf{p}}|\mathbf{0}, m\rangle$ also satisfy eigenvector equations (37) - (38), so they must be proportional to the basis vectors $|\mathbf{p}, m\rangle$

$$|\mathbf{p}, m\rangle = \gamma(\mathbf{p}, m)e^{\frac{i}{\hbar}\mathbf{R}_0\cdot\mathbf{p}}|\mathbf{0}, m\rangle$$

[13] Note the difference between $\omega_{\mathbf{p}}$ that depends on the eigenvalue m of the interacting mass operator and $\Omega_{\mathbf{p}}$ in eq. (31) that depends on the fixed value of mass m_a of the particle a.

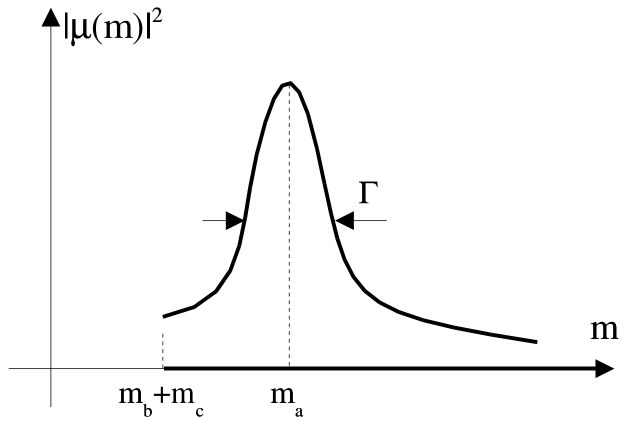

Figure 1: Mass distribution of a typical unstable particle.

where $\gamma(\mathbf{p}, m)$ is an unimodular factor. Unlike in (35), we cannot conclude that $\gamma(\mathbf{p}, m) = 1$, because, generally, the action of $e^{\frac{i}{\hbar}\mathbf{R}_0 \cdot \mathbf{P}}$ on eigenvectors $|\mathbf{q}, m\rangle$ involves multiplication by a unimodular scalar in addition to the shift of momentum. However, if the interaction is not pathological we can assume that the factor $\gamma(\mathbf{p}, m)$ is smooth, i.e., without rapid oscillations. This property will be used in derivation of eq. (59).

Obviously, vector $|\mathbf{0}\rangle$ can be expressed as a linear combination of zero-momentum basis vectors $|\mathbf{0}, m\rangle$, so we can write[14]

$$|\mathbf{0}\rangle = \int_{m_b+m_c}^{\infty} dm \mu(m) |\mathbf{0}, m\rangle \tag{42}$$

where function $|\mu(m)|^2$ describes the *mass distribution* of the unstable particle. For most realistic unstable systems the mass distribution $|\mu(m)|^2$ has the Breit-Wigner form (see Fig. 1)[15]

$$|\mu(m)|^2 \approx \frac{\alpha\Gamma/2\pi}{\Gamma^2/4 + (m - m_a)^2}, \; if \; m \geq m_b + m_c \tag{43}$$

[14]We will assume that interaction responsible for the decay does not change the spectrum of mass. In particular, we will neglect the possibility of existence of bound states of particles b and c, i.e., discrete eigenvalues of M below $m_b + m_c$. Then the spectrum of M (similar to the spectrum of M_0) is continuous in the interval $[m_b + m_c, \infty)$, and integration in (42) should be performed from $m_b + m_c$ to infinity.

[15]Strictly speaking, the center of the resonance (43) could be different from the mass m_a that particle a has in the absence of decay interaction. However, we will disregard this possibility here.

$$|\mu(m)|^2 = 0, \; if \; m < m_b + m_c \tag{44}$$

where parameter α is a normalization factor required to ensure that the mass distribution (43) - (44) is normalized to unity.

We now use eqs. (35) and (42) to expand the vector $|\mathbf{p}\rangle$ in the basis $|\mathbf{p}, m\rangle$

$$\begin{aligned} |\mathbf{p}\rangle &= e^{\frac{i}{\hbar}\mathbf{R}_0\mathbf{p}} \int_{m_b+m_c}^{\infty} dm\mu(m)|\mathbf{0}, m\rangle \\ &= \int_{m_b+m_c}^{\infty} dm\mu(m)\gamma(\mathbf{p}, m)|\mathbf{p}, m\rangle \end{aligned} \tag{45}$$

Then, from (39) we obtain a useful formula

$$\begin{aligned} \langle \mathbf{q}|\mathbf{p}, m\rangle &= \int_{m_b+m_c}^{\infty} dm'\mu^*(m')\gamma^*(\mathbf{q}, m')\langle \mathbf{q}, m'|\mathbf{p}, m\rangle \\ &= \gamma^*(\mathbf{p}, m)\mu^*(m)\delta(\mathbf{q}-\mathbf{p}) \end{aligned} \tag{46}$$

8 General formula for the non-decay law

Suppose that vector $|\Psi\rangle$ in (28) describes a state of the unstable particle a from the point of view of the observer O. The time dependence of this state seen from the moving reference frame O' is obtained by applying eqs (20), (40), (41), and (45)

$$\begin{aligned} |\Psi(\theta, t)\rangle &= \int d\mathbf{p}\psi(\mathbf{p})e^{-\frac{i}{\hbar}Ht}e^{\frac{ic}{\hbar}K_z\theta}|\mathbf{p}\rangle \\ &= \int d\mathbf{p}\psi(\mathbf{p}) \int_{m_b+m_c}^{\infty} dm\mu(m)\gamma(\mathbf{p}, m)e^{-\frac{i}{\hbar}Ht}e^{\frac{ic}{\hbar}K_z\theta}|\mathbf{p}, m\rangle \\ &= \int d\mathbf{p}\psi(\mathbf{p}) \int_{m_b+m_c}^{\infty} dm\mu(m)\gamma(\mathbf{p}, m)e^{-\frac{i}{\hbar}\omega_{\Lambda^{-1}\mathbf{p}}t}\sqrt{\frac{\omega_{\Lambda^{-1}\mathbf{p}}}{\omega_{\mathbf{p}}}}|\Lambda^{-1}\mathbf{p}, m\rangle \end{aligned}$$

The inner product of this vector with $|\mathbf{q}\rangle$ is found by using (46)

$$\begin{aligned} &\langle \mathbf{q}|\Psi(\theta, t)\rangle \\ &= \int d\mathbf{p}\psi(\mathbf{p}) \int_{m_b+m_c}^{\infty} dm\mu(m)\gamma(\mathbf{p}, m)e^{-\frac{i}{\hbar}\omega_{\Lambda^{-1}\mathbf{p}}t}\langle \mathbf{q}|\Lambda^{-1}\mathbf{p}, m\rangle\sqrt{\frac{\omega_{\Lambda^{-1}\mathbf{p}}}{\omega_{\mathbf{p}}}} \end{aligned}$$

$$= \int d\mathbf{p}\psi(\mathbf{p}) \int_{m_b+m_c}^{\infty} dm|\mu(m)|^2\gamma(\mathbf{p},m)\gamma^*(\Lambda^{-1}\mathbf{p},m)e^{-\frac{i}{\hbar}\omega_{\Lambda^{-1}\mathbf{p}}t}\delta(\mathbf{q}-\Lambda^{-1}\mathbf{p})\sqrt{\frac{\omega_{\Lambda^{-1}\mathbf{p}}}{\omega_{\mathbf{p}}}}$$

Introducing new integration variables $\mathbf{r} = \Lambda^{-1}\mathbf{p}$ and taking into account (31), this equation can be rewritten as

$$\begin{aligned}
&\langle \mathbf{q}|\Psi(\theta,t)\rangle \\
&= \int_{m_b+m_c}^{\infty} dm \int d\mathbf{r}\frac{\omega_{\Lambda\mathbf{r}}}{\omega_{\mathbf{r}}}\sqrt{\frac{\omega_{\mathbf{r}}}{\omega_{\Lambda\mathbf{r}}}}\psi(\Lambda\mathbf{r})\gamma(\Lambda\mathbf{r})\gamma^*(\mathbf{r})|\mu(m)|^2e^{-\frac{i}{\hbar}\omega_{\mathbf{r}}t}\delta(\mathbf{q}-\mathbf{r}) \\
&= \int_{m_b+m_c}^{\infty} dm\sqrt{\frac{\omega_{\Lambda\mathbf{q}}}{\omega_{\mathbf{q}}}}\psi(\Lambda\mathbf{q})\gamma(\Lambda\mathbf{q},m)\gamma^*(\mathbf{q},m)|\mu(m)|^2e^{-\frac{i}{\hbar}\omega_{\mathbf{q}}t}
\end{aligned}$$

The non-decay probability in the reference frame O' is then found by substituting (27) in eq. (21)

$$\begin{aligned}
\omega_{|\Psi\rangle}(\theta,t) &= \int d\mathbf{q}\langle\Psi(\theta,t)|\mathbf{q}\rangle\langle\mathbf{q}|\Psi(\theta,t)\rangle \\
&= \int d\mathbf{q}|\langle\mathbf{q}|\Psi(\theta,t)\rangle\ |^2 \\
&= \int d\mathbf{q}|\int_{m_b+m_c}^{\infty} dm\sqrt{\frac{\omega_{\Lambda\mathbf{q}}}{\omega_{\mathbf{q}}}}\psi(\Lambda\mathbf{q})\gamma(\Lambda\mathbf{q},m)\gamma^*(\mathbf{q},m)|\mu(m)|^2e^{-\frac{i}{\hbar}\omega_{\mathbf{q}}t}|^2 \quad (47)
\end{aligned}$$

which is an exact formula valid for all values of θ and t.

9 Decay law in the reference frame at rest

In the reference frame at rest ($\theta = 0$), formula (47) simplifies

$$\omega_{|\Psi\rangle}(0,t) = \int d\mathbf{q}|\psi(\mathbf{q})|^2|\int_{m_b+m_c}^{\infty} dm|\mu(m)|^2e^{-\frac{i}{\hbar}\omega_{\mathbf{q}}t}|^2$$

Let us consider a particular case of this expression when the unstable particle has a well-defined momentum, i.e., described by the state vector $|\mathbf{p})$. If the particle a is in the state $|\mathbf{0})$ with zero momentum then $\psi(\mathbf{q}) = \sqrt{\delta(\mathbf{q})}$ and from the above equation we obtain

$$\omega_{|\mathbf{0})}(0,t) \quad = \quad |\int\limits_{m_b+m_c}^{\infty} dm|\mu(m)|^2 e^{-\frac{i}{\hbar}mc^2t}|^2 \tag{48}$$

which is the standard formula for the non-decay law of a particle at rest.[16] In particular, if we substitute (43) - (44) for the mass distribution $|\mu(m)|^2$ we obtain approximate exponential decay (7) where the lifetime is given by

$$\tau_0 = \hbar/(\Gamma c^2) \tag{49}$$

If the particle a has a definite non-zero momentum $\mathbf{p}$, then

$$\psi(\mathbf{q}) = \sqrt{\delta(\mathbf{q}-\mathbf{p})} \tag{50}$$

and

$$\omega_{|\mathbf{p})}(0,t) \quad = \quad |\int\limits_{m_b+m_c}^{\infty} dm|\mu(m)|^2 e^{-\frac{i}{\hbar}\omega_{\mathbf{p}}t}|^2 \tag{51}$$

In a number of works [4, 5, 6] it was noticed that if one interprets the state $|\mathbf{p})$ (where $|\mathbf{p}| = m_a c \sinh\theta$) as a state of unstable particle moving with definite speed $|\mathbf{v}| = c\tanh\theta$, then the decay law of the moving particle (51) cannot be connected with the non-decay law of the particle at rest (48) by Einstein's formula (9), i.e.,

$$\omega_{|\mathbf{p})}(0,t) \quad \neq \quad \omega_{|\mathbf{0})}(0,t/\cosh\theta) \tag{52}$$

This observation prompted authors of [4, 5, 6] to question the applicability of Einstein's special relativity to particle decays. However, at a closer inspection it appears that this result does not contradict Einstein's time dilation formula (9) directly. Formula (9) refers to observations made on the same particle from two frames of reference moving with respect to each other. If from the point of view of observer O the particle is described by the state vector $|\mathbf{0})$ which has zero momentum and zero velocity, then from the point of view of O' this particle is described by the state

$$|\mathbf{p}) = e^{\frac{ic}{\hbar}\mathbf{K}\vec{\theta}}|\mathbf{0}) \tag{53}$$

[16]By noting that for the particle at rest its energy is identified with mc^2, this formula can be compared, for example, with eq. (3.8) in [16].

which is an eigenstate of the velocity operator[17]

$$\mathbf{V} = c^2\mathbf{P}_0/H \tag{54}$$

but is not an eigenstate of the momentum operator $\mathbf{P}_0$. Therefore, strictly speaking, its non-decay law is not described by $\omega_{|\mathbf{p})}(0,t)$. In order to compare with Einstein's formula (9), we need to calculate the decay law in the moving frame of reference $\omega_{|\Psi\rangle}(\theta,t)$. This is done in section 11.

10 Decays caused by boosts

Let us now discuss the non-decay probability $\omega_{|\Psi\rangle}(\theta,0)$ at $t=0$ in the moving frame of reference. Instead of eq. (47), it is more convenient to use the general definition (22) which expresses $\omega_{|\Psi\rangle}(\theta,0)$ as the square of the norm of the projection of $|\Psi(\theta,0)\rangle$ on the subspace $\mathcal{H}_a$. We are going to prove that for a normalized $|\Psi\rangle$ this probability cannot be equal to 1 for all non-zero θ. Suppose that this statement is wrong, so that (in Agreement with Assumption V) for any $|\Psi\rangle \in \mathcal{H}_a$ and any $\theta > 0$, the vector $e^{\frac{ic}{\hbar}K_z\theta}|\Psi\rangle$ belongs to $\mathcal{H}_a$. Then the subspace $\mathcal{H}_a$ is invariant under action of boosts $e^{\frac{ic}{\hbar}K_z\theta}$ and operator K_z commutes with the projection T. Then from the Poincaré commutator (15) and $[T,(P_0)_z]=0$ it follows by Jacobi identity that

$$\begin{aligned}
[T,H] &= \frac{ic^2}{\hbar}[T,[K_z,(P_0)_z]] \\
&= \frac{ic^2}{\hbar}[K_z,[T,(P_0)_z]] - \frac{ic^2}{\hbar}[(P_0)_z,[T,K_z]] \\
&= 0
\end{aligned}$$

which contradicts eq. (19). This contradiction implies that the state $e^{\frac{ic}{\hbar}K_z\theta}|\Psi\rangle$ does not correspond to the particle a with 100% probability; this state must contain contributions from the decay products even at $t=0$

[17]Indeed, taking into account $V_z|\mathbf{0}) \approx \mathbf{0}$ and eqs. (17) - (18), we obtain

$$\begin{aligned}
V_z e^{\frac{ic}{\hbar}K_z\theta}|\mathbf{0}) &= e^{\frac{ic}{\hbar}K_z\theta}e^{-\frac{ic}{\hbar}K_z\theta}V_z e^{\frac{ic}{\hbar}K_z\theta}|\mathbf{0}) \\
&= e^{\frac{ic}{\hbar}K_z\theta}\frac{V_z - c\tanh\theta}{1-\frac{V_z\tanh\theta}{c}}|\mathbf{0}) \\
&\approx -c\tanh\theta e^{\frac{ic}{\hbar}K_z\theta}|\mathbf{0})
\end{aligned}$$

$$e^{\frac{ic}{\hbar}K_z\theta}|\Psi\rangle \notin \mathcal{H}_a \quad (55)$$

$$\omega_{|\Psi\rangle}(\theta, 0) < 1, \ for \ \theta \neq 0 \quad (56)$$

This is the "decay caused by boost". Thus we found that Assumption V from section 4 is not correct. Sometimes this assumption is formulated as [15] *"Flavor is the quantum number that distinguishes the different types of quarks and leptons. It is a Lorentz invariant quantity. For example, an electron is seen as an electron by any observer, never as a muon."* Although the above statement about the electron is correct (because the electron is a stable particle), this is not true about the muon which, according to (55) can be seen as a single particle by the observer at rest and as a group of three decay products (an electron, a neutrino, and an antineutrino) by the moving observer.

11 Decay law in the moving reference frame

Unfortunately exact evaluation of eq. (47) for $\theta \neq 0$ is not possible, so we need to make approximations. Let us discuss properties of the initial state $|\Psi\rangle$ in more detail. First, in all realistic cases this state is not an eigenstate of the total momentum operator, so the wave function is not localized at one point in the momentum space (as assumed in (50)) but has a spread (or uncertainty) of momentum $|\Delta \mathbf{p}|$ and, correspondingly an uncertainty of position $|\Delta \mathbf{r}| \approx \hbar/|\Delta \mathbf{p}|$. On the other hand, the state $|\Psi\rangle \in \mathcal{H}_a$ is not an eigenstate of the mass operator M, and $|\Psi\rangle$ is characterized by the uncertainty of mass Γ (see Fig. 1) that is related to the lifetime of the particle τ_0 by formula (49). It is important to note that in all cases of practical interest the mentioned quantities are related by inequalities

$$|\Delta \mathbf{p}| \gg \Gamma c \quad (57)$$

$$|\Delta \mathbf{r}| \ll c\tau_0 \quad (58)$$

In particular, the latter inequality means that the uncertainty of position is mush less than the distance passed by light during the lifetime of the particle. For example, in the case of muon $\tau_0 \approx 2.2 \cdot 10^{-6}$s and, according to (58), the spread of the wave function in the position space must be much less than 600m, which is a reasonable assumption. Therefore, we can safely assume that the factor $|\mu(m)|^2$ in (47) has a sharp peak near the value $m = m_a$. Then we can move the value of the smooth function $\sqrt{\frac{\omega_{\Lambda\mathbf{q}}}{\omega_{\mathbf{q}}}}\psi(\Lambda\mathbf{q})\gamma(\Lambda\mathbf{q}, m)\gamma^*(\mathbf{q}, m)$ at $m = m_a$ outside the integral by m.[18]

[18] Note that if we assumed instead of (57) that the spread of the momentum-space wave function is much less than Γc we could have moved the value of the smooth function $|\mu(m_0)|^2$ outside the integral

$$
\begin{aligned}
\omega_{|\Psi\rangle}(\theta,t) &\approx \int d\mathbf{q} \Big| \sqrt{\frac{\Omega_{\Lambda\mathbf{q}}}{\Omega_{\mathbf{q}}}} \psi(\Lambda\mathbf{q})\gamma(\Lambda\mathbf{q},m_a)\gamma^*(\mathbf{q},m_a)|^2 \Big| \int_{m_b+m_c}^{\infty} dm|\mu(m)|^2 e^{-\frac{i}{\hbar}\omega_{\mathbf{q}}t}\Big|^2 \\
&= \int d\mathbf{q} \frac{\Omega_{L\mathbf{q}}}{\Omega_{\mathbf{q}}} |\psi(L\mathbf{q})|^2 \Big| \int_{m_b+m_c}^{\infty} dm|\mu(m)|^2 e^{-\frac{i}{\hbar}\omega_{\mathbf{q}}t}\Big|^2 \\
&= \int d\mathbf{p}|\psi(\mathbf{p})|^2 \Big| \int_{m_b+m_c}^{\infty} dm|\mu(m)|^2 e^{-\frac{i}{\hbar}\omega_{L^{-1}\mathbf{p}}t}\Big|^2
\end{aligned} \tag{59}
$$

Now we are going to use formula (59) to calculate the decay law for an approximate eigenstate of momentum in $\mathcal{H}_a$ whose wave function $\psi(\mathbf{p})$ is localized near zero momentum $\mathbf{p} = \mathbf{0}$, though inequality (57) still remains valid. We denote this state vector by symbol $|\mathbf{0}]$. The state $e^{\frac{ic}{\hbar}K_z\theta}|\mathbf{0}]$ is an approximate eigenstate of the velocity operator (54) for all values of θ.[19] Therefore $\omega_{|\mathbf{0}]}(\theta,t)$ in eq. (60) can be understood as the non-decay law of a particle with definite speed $v = c\tanh\theta$.

Note that the second factor in the integrand in (59) is a slowly varying function of $\mathbf{p}$. Therefore, we can set $|\psi(\mathbf{p})|^2 \approx \delta(\mathbf{p})$ in eq. (59) and obtain

$$
\omega_{|\mathbf{0}]}(\theta,t) \approx \Big| \int_{m_b+m_c}^{\infty} dm|\mu(m)|^2 e^{-\frac{it}{\hbar}\sqrt{m^2c^4+m_a^2c^4\sinh^2\theta}}\Big|^2 \tag{60}
$$

If we approximately[20] identify $m_a c\sinh\theta$ with the momentum $|\mathbf{p}|$ of the particle a from the point of view of the moving observer O' then

$$
\omega_{|\mathbf{0}]}(\theta,t) \approx \Big| \int_{m_b+m_c}^{\infty} dm|\mu(m)|^2 e^{-\frac{i}{\hbar}\omega_{\mathbf{p}}t}\Big|^2 \tag{61}
$$

So, in this approximation the non-decay law (61) in the frame of reference O' moving with the speed $c\tanh\theta$ takes the same form as the non-decay law (51) of a particle moving with momentum $m_a c\sinh\theta$ with respect to the stationary observer O.

by m. This would mean that the non-decay law in the moving frame of reference is controlled by the spread of the particle wavefunction in the momentum space rather than by its mass uncertainty which disagrees with experimental observations.

[19]See footnote 17.

[20]See remark after eq. (53).

12 Numerical results

In this section we are going to perform numerical calculations of the differences between the classical Einstein's formula (9)

$$\omega^{class}_{|\mathbf{0}]}(\theta, t) = \omega_{|\mathbf{0}]}(0, \frac{t}{\cosh\theta}) \tag{62}$$

and the actual non-decay law (60) of a moving particle having either definite momentum or sharply defined speed. In these calculations we assumed that the mass distribution $|\mu(m)|^2$ of the unstable system (see eqs. (43) - (44) and Fig. 1) is centered at the value of mass $m_a = 1000$ MeV/c^2, the total mass of the decay products was $m_b + m_c = 900$ MeV/c^2, and the width of the mass distribution was $\Gamma = 20$ MeV/c^2. These values do not correspond to any real particle, but they are typical for strongly decaying baryons.

It is convenient to measure time in units of the classical lifetime $\tau_0 \cosh\theta$. Denoting $\chi \equiv t/(\tau_0 \cosh\theta)$, we find that Einsteinian non-decay laws (62) for any value of θ are given by the same universal function $\omega^{class}(\chi)$. This function was calculated for values of χ in the interval from 0 to 6 with the step of 0.1 The calculations were performed by direct numerical integration of eq. (48) using the *Mathematica* program shown below

```
gamma = 20

mass = 1000

theta = 0.0

Do[Print[(1/0.9375349) Abs[NIntegrate [gamma/(2 Pi) / (gamma^2/4
+(x - mass)^2) Exp[ I t Sqrt [x^2 + mass^2 (Sinh [theta])^2]
Cosh [theta] / gamma], {x, 900, 1010, 1100, 300000}, MinRecursion -> 3,
MaxRecursion -> 16, PrecisionGoal -> 8, WorkingPrecision -> 18]]^2],
{t, 0.0, 6.0, 0.1}]
```

As expected, function $\omega^{class}(\chi)$ is very close to the exponent $e^{-\chi}$.[21] Next we used eq. (60) and the above *Mathematica* program to calculate the non-decay laws $\omega_{|\mathbf{0}]}(\theta, \chi)$ in moving reference frames for three values of the parameter θ (=`theta`), namely 0.2, 1.4, and 10.0, that correspond to velocities of 0.197c, 0.885c, and 0.999999995c, respectively. These calculations qualitatively confirmed the time dilation formula (62)

[21] This function is represented by a thick solid line in Fig. 2. The magnitudes of $\omega^{class}(\chi)$ for small values of the argument χ are too large to be shown on the scale of Fig. 2.

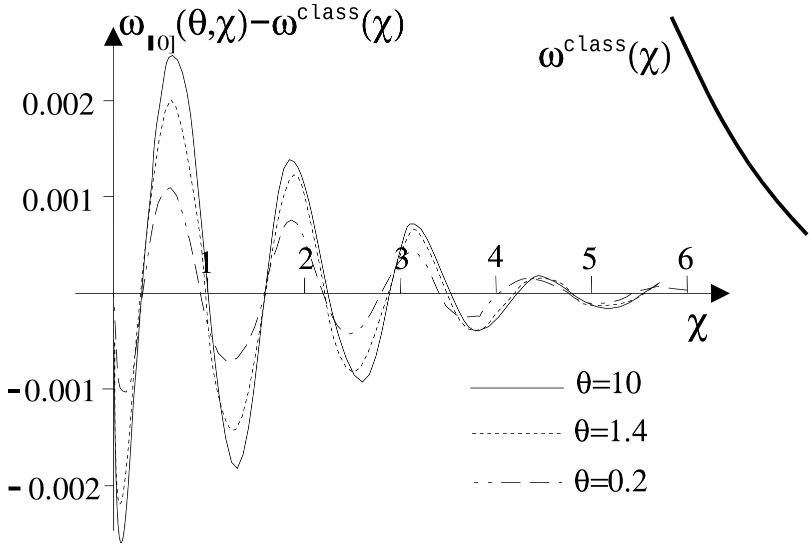

Figure 2: Corrections to the Einstein's "time dilation" formula (9) for the non-decay law of unstable particle moving with the speed $v = c \tanh \theta$. Parameter χ is time measured in units of $\tau_0 / \cosh \theta$.

to the accuracy of better than 0.3%. However, they also revealed important differences $\omega_{|\mathbf{0}]}(\theta, \chi) - \omega^{class}(\chi)$ which are plotted as thin lines in Fig. 2.[22]

The lifetime of the particle a considered in our example ($\tau_0 \approx 2 \times 10^{-22}$ s) is too short to be observed experimentally. So, calculated corrections to the Einstein's time dilation law have only illustrative value. However, from these data we can estimate the magnitude of corrections for particles whose time-dependent non-decay laws can be measured in a laboratory, e.g., for muons ($\tau_0 \approx 2 \times 10^{-6}$s, $\Gamma \approx 2 \times 10^{-9} eV/c^2$, $m_a \approx 105 MeV/c^2$). Taking into account that the magnitude of corrections is roughly proportional to the ratio Γ/m_a [4, 6], we can expect that for muons the maximum correction should be about 2×10^{-18} which is much smaller than the accuracy of modern experiments (about 10^{-3} [2]). So, experimental observation of the predicted corrections requires significant improvements of existing experimental techniques.

Note that formulas (60) and (61) for the particle seen from the moving frame are approximate in the sense that they ignore the "decay caused by boost" expressed by eq. (56). Nevertheless, our major approximation (57) is well justified and it cannot explain the discrepancy of our results (Einsteinian time dilation of the decay plus small interaction-dependent corrections) with the conclusion of ref. [7] about the acceleration of the decay of moving particles. In our view this conclusion does not refer to the experimentally measured non-decay law that is defined as the probability of finding

[22]Results presented in Fig. 2 are different from those in Fig. 1 in ref. [4] due to the more accurate calculation procedure employed in the present work.

one unstable particle. Instead of eq. (21), the non-decay probability was defined in [7] by formula

$$\omega_{|\Psi\rangle}(\theta, t) \quad = \quad |\langle \Psi(\theta, 0)| e^{-\frac{i}{\hbar}Ht} |\Psi(\theta, 0)\rangle|^2$$

whose physical meaning remains unclear.

13 Particle decays in different Dirac's forms of dynamics

In this article we assumed that the interaction responsible for the particle decay has the Bakamjian-Thomas instant form of dynamics. This assumption was used to simplify calculations, but there is no good reason to believe that real interactions have this form. Then, naturally, one may ask a question: "Is there another form of dynamics in which Einstein's time dilation formula (9) is exactly true?" The answer is *No.* In any instant form of dynamics (including non-Bakamjian-Thomas instant forms of dynamics) the boost operators contain interaction terms, so the results (55) - (56) are still valid, and eq. (9) holds only approximately. In the point form dynamics[23] the subspace $\mathcal{H}_a$ of the unstable particle is invariant with respect to boosts, so there can be no boost-induced decays (55). However, due to the interaction-dependence of the total momentum operator $\mathbf{P}$, one should expect decays induced by space translations

$$e^{\frac{i}{\hbar}P_z a}|\Psi\rangle \quad \notin \quad \mathcal{H}_a, \ for \ \ a \neq 0 \tag{63}$$

This prediction is not confirmed by experiments which show that the composition of an unstable particle is not affected by space translations of the observer.[24] Therefore, the point form of dynamics is not acceptable for the description of decays. Similarly, never observed translation- and/or rotation-induced decays are characteristic for all non-instant forms of dynamics, e.g., the front form. Therefore only the instant form of dynamics is appropriate for the description of particle decays. Note, however, that the decay slowdown and the absence of translation-induced decays are firmly established only for unstable systems whose lifetimes are long enough to be measured experimentally, i.e., those governed by electromagnetic and weak interactions. For strongly interacting resonances these properties are beyond experimental resolution, and the non-instant forms of dynamics cannot be ruled out.

[23] where generators of boosts $\mathbf{K}_0$ and rotations $\mathbf{J}_0$ are kinematical, while generators of space translations $\mathbf{P}$ and time translations H contain interaction terms.

[24] An even more striking contradiction between predictions of the point form dynamics and observations is that the decay of moving particles *accelerates* $\cosh\theta$ times instead of experimentally observed slowdown [4].

14 Summary

In this paper we analyzed the relationships between the non-decay laws in the moving reference frame and in the reference frame at rest. We used a rigorous quantum relativistic approach that is applicable to any unstable system independent on the nature of interaction governing the decay. A complete description of dynamics in different reference frames was obtained by using relativistic Postulates I - VI and rules of quantum mechanics only. We found that Assumptions V and VI of special relativity are not needed. Moreover, their consequences (the universal and exact time dilation of the decay of moving particles) are in contradiction with rigorous calculations. Although the time dilation (9) of special relativity was qualitatively confirmed by our results, we also found small corrections to this formula that depend on the strength of interaction. In a broader sense our results indicate that clocks viewed from the moving reference frame do not go exactly $\cosh \theta$ slower. The exact amount of time dilation depends on the physical makeup of the clock and on interactions responsible for the operation of the clock.

I am grateful to Dr. M. I. Shirokov for reading the manuscript, valuable comments, and discussions.

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