[6]P. Acloque and C. Guillemet, Compt. Rend. 250, 4328 (1960).
[7]H. Osterberg and L. W. Smith, J. Opt. Soc. Am. 54, 1078 (1964).
[8]F. I. Fedorov, Dokl. Akad. Nauk SSSR 105, 465 (1955).
[9]J. Frenkel, Z. Physik 37, 243 (1926); M. Mathisson, *ibid.* 67, 270 (1931); Math. Ann. 107, 400 (1933); A. Proca, Ann. Phys. (Paris) 20, 347 (1933); O. Costa de Beauregard, Compt. Rend. 214, 904 (1942); J. Math. 22, 85 (1943); J. Weyssenhof and A. Raabe, Acta Phys. Polon. 9, 7 (1947); 9, 9 (1947); J. Weyssenhof, *ibid.* 9, 26 (1947); 9, 34 (1947); 9, 46 (1947).
[10]O. Costa de Beauregard, Phys. Rev. 134, B471 (1964); Ann. Inst. Henri Poincaré 2, A131 (1965).
[11]O. Costa de Beauregard, Cahiers Phys. 18, 471 (1964); Phys. Rev. 139, B1446 (1965).
[12]O. Costa de Beauregard, Compt. Rend. 272, B1116 (1971).
[13]H. Schilling, Ann. Physik 16, 122 (1965).
[14]F. Noether, Ann. Physik 11, 141 (1931).
[15]C. Imbert, Compt. Rend. 267, B1401 (1968).
[16]N. Kristoffel, Scient. Acta Tartu State Univ., 1956 (private communication).
[17]R. H. Renard, J. Opt. Soc. Am. 54, 1190 (1964).
[18]C. Imbert, Compt. Rend. 269, B1227 (1969); 270, B529 (1970); Phys. Letters 31A, 337 (1970).
[19]J. Ricard, Nouv. Revue d'Optique 1, 275 (1970).
[20]H. Wolter, *Handbuch der Physik*, edited by S. Flügge (Springer, Berlin, 1956), Vol. 24, p. 582.
[21]J. Pavageau, Compt. Rend. 263, B276 (1966).
[22]In fact our formula is stationary near the limiting angle.
[23]D. Hestenes and R. Gurtler, Am. J. Phys. 39, 1028 (1971); D. Hestenes, *ibid.* (to be published).
[24]H. Corben, Nuovo Cimento 20, 529 (1961); *Classical and Quantum Theories of Spinning Particles* (Holden-Day, San Francisco, 1968).

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Frequency Dependence of the Speed of Light in Space

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To characterize the possible dispersion of the velocity of light in space (vacuum) a Cauchy-type formula, $n^2=1+A/\nu^2+B\nu^2$, is used. It is shown that relativity only allows a nonzero A term, independent of the nature of the waves or a quantization thereof. Recent experimental data provide upper bounds for A and B, limiting thereby the dispersion in the microwave, infrared, visible, and ultraviolet regions of the spectrum to less than one part in 10^{20}.

Recent observations of radio-wave,[1] visible,[2] and x-ray[3] emissions from pulsars have been interpreted to provide experimental bounds on the dispersion of light in interstellar space.[2,4,5] The dispersion or lack thereof has been discussed using the expression[2,6,7]

$$p=\frac{c}{\Delta c}\frac{\lambda_2}{\lambda_1} \tag{1}$$

to relate measurements in different regions of the spectrum giving different limits of dispersion Δc in the velocity. It was pointed out by Brown[7] that p clearly cannot be a good constant to characterize the variation of velocity with energy, because it would be infinite for $\lambda_1=\lambda_2$. Even though that difficulty could be avoided by the introduction of $(\lambda_2-\lambda_1)/\lambda_1$ in place of λ_2/λ_1 in the definition of p, there remains the more serious objection against the concept of p, in our opinion, that it suggests a linear dependence of c on λ.

It appears preferable to the present authors to represent the dispersion (if there is any) via a modified Cauchy expression of the form[8]

$$n^2=1+\frac{A}{\nu^2}+B\nu^2 \tag{2}$$

or

$$n^2=1+A'\lambda^2+\frac{B'}{\lambda^2}, \tag{3}$$

where n is defined by $c_{\text{phase}}=c_0/n$ and c_0 is the velocity of light in the absence of dispersion. The corresponding group velocities, to be used in the analysis of the experimental data, are readily calculated from the above expressions.

In anticipation of their use in regions of the spectrum remote from resonances, only the leading terms in ν^2 and $(1/\nu)^2$, or λ^2 and $(1/\lambda)^2$, are retained in Eqs. (2) and (3). The absence of odd powers of ν or λ is assured by the presumed symmetry with respect to reversal of the direction of time.[9]

Expressions (2) and (3) describe the frequency dependence of the speed of light in any dispersive medium in frequency domains far from all resonances. It should be noted that in vacuum the only possible dispersion formula compatible with the special theory of relativity is one in which A or A' may be nonzero but $B=B'=0$. This can be seen by applying the relativistic addition theorem for velocities to c_{phase} which leads to the transformation rule for n

$$n'=\frac{n+\beta}{1+n\beta}, \qquad (4)$$

where $\beta=v/c$ and (for simplicity) the relative velocity, v, between the two systems is along the direction of light propagation. Simultaneously the frequency, ν, transforms by the relativistic Doppler formula

$$\frac{\nu'}{\nu}=\frac{1+n\beta}{(1-\beta^2)^{1/2}}. \qquad (5)$$

Elimination of β from Eqs. (4) and (5) leads to

$$[n'(\nu')^2-1]\nu'^2=[n(\nu)^2-1]\nu^2. \qquad (6)$$

Now, if $n(\nu)$ is a dispersion function characteristic for vacuum (space), it should transform into itself, thus $n'(\nu')=n(\nu')$. From this it follows that $(n^2-1)\nu^2=A$, where A is a relativistic invariant and therefore the only possible dispersion formula for vacuum is

$$n^2=1+\frac{A}{\nu^2} \qquad (7)$$

if the special theory of relativity is correct. Causality requires $A<0$. (For $A>0$ the group velocity exceeds c_0 and a signal may appear to propagate backwards in time to some observer.)

It seems to us that the statement contained in Eq. (7) was not recognized before in its full generality. It applies to any wave for which the phase velocity is defined as a function of frequency; it does not require specification of the physical nature of equations of motion of the wave.

As a special case, it is known for electromagnetic waves that the only possible linear generalization of Maxwell's equations which is relativistically invariant[10] leads via Proca equations to the dispersion formula of Eq. (7). The same is true for quantized matter waves of de Broglie or of Klein and Gordon, for which $A=-(mc^2/h)^2$, where m is the rest mass of the particles. Consequently, a dispersion formula of the above type can also be attributed to a finite rest mass of the photons. The general character of Eq. (7) implies that it also should be valid for gravity waves whether or not they should prove to be quantized by de Broglie's rule or in some other way.

Although the principle of relativity rigorously rules out $B\neq 0$ in vacuum, we shall nonetheless retain the B term in Eq. (2) in interpreting data on the frequency dependence of the speed of light, to allow for suggested breakdowns of Einstein's principle of relativity at very short wavelengths.[11,12]

The results of several recent pulsar measurements interpreted according to Eq. (2) are summarized in Table I, together with other evidence concerning the lack of dispersion of the speed of light in vacuum, namely, from measurements of the speed of γ rays[13] and upper bounds determined for the "mass" of a photon.[10,14-17]

It is to be noted that the visible measurements by themselves,[2] while they provide the least stringent bounds on A and B, nonetheless limit the dispersion of c in the visible region to $\sim 10^{-16}$. Radio-wave measurements[1,4,5] provide a tighter bound on A, by about 8 orders of magnitude, while x-ray measurements[3] reduce B by 5 orders. Taken together, the pulsar data interpreted according to

TABLE I. Measurements providing upper bounds on the dispersion of the speed of light in different regions of the electromagnetic spectrum, and values of $|A|$ and $|B|$ to be used in interpolating to other regions of the spectrum via Eq. (2).

Type of measurement	Frequency range (Hz)	$\Delta c/c$ visible	$\lvert A\rvert$ (Hz^2)	$\lvert B\rvert$ (Hz^{-2})	Ref.
Pulsar emissions:					
Radio wave	$1-4\times 10^8$	$<10^{-10}$	$<10^6$		a
Visible	$5-8\times 10^{14}$	$<10^{-16}$	$<10^{14}$	$<10^{-45}$	b
x-ray	$4-24\times 10^{17}$	$<10^{-14}$		$<10^{-50}$	c
γ-ray velocity	10^{24}	$<10^{-3}$		$<10^{-51}$	d
Photon mass limit	Static field		<1		e

[a]References 1, 4, and 5.
[b]Reference 2.
[c]Reference 3.
[d]Reference 13.
[e]Reference 17.

Eq. (2) indicate that the speed of light is constant to $<10^{-20}$ throughout the visible, near infrared, and ultraviolet regions of the spectrum.

A somewhat tighter bound on B is provided by the direct measurements of the speed of 6-GeV γ rays.[13] It may be noted that if the γ-ray or x-ray results are interpreted according to the theory of Pavlopoulos,[12] they put a bound on the fundamental length l_0 in his theory, which causes dispersion at high frequencies, of $l_0 < 10^{-16}$ cm, which is the Compton wavelength of a mass $m_0 > 100$ GeV. This upper bound on l_0 is about 3 orders of magnitude smaller than expected by Pavlopoulos.

By far the most stringent limit on A comes from measurements of the "mass" of the photon. According to de Broglie, if the photon has a mass, m, then the constant A in Eq. (7) becomes $A = -(mc^2/h)^2$. An upper bound of[17] $m < 4\times 10^{-48}$ g (or $<1.15\times 10^{-10}$ cm^{-1} for use in the classical Proca equation) leads to $|A| < 1$ Hz^2, as indicated in the last line of Table I. More accurate bounds on m promise still smaller values for A.[18] Thus if Einstein's principle of relativity holds for all frequencies, then c in vacuum is constant to better than 10^{-20} for all frequencies $>10^{10}$ Hz.

It is worth noting in conclusion that available experimental data interpreted according to Eq. (2) already indicate c is constant to an accuracy which exceeds that of any metrological experiment likely to be performed in the foreseeable future at any frequency between that of short radio waves and that of x rays.

This fact provides additional experimental support for the suggestion reviewed recently[19,20] that c be used in metrology to connect the unit of time (the second) and the unit of length (the meter). The connection is made by assigning an agreed upon value to c in m/sec. The results of this paper show that in the above-mentioned broad spectrum this assignment can be made without reference to frequency.

After making the connection between the two units, it is preferable to consider the unit of time and c as standards rather than the unit of length and c. The preference for time follows from the fact that in any system of inertia under steady conditions the periodicity in time (frequency) is conserved in wave propagation while the periodicity in space (wavelength) depends on the geometry of the wave propagation and changes in general from point to point.

Besides the theoretical appeal and simplicity of the unified time-length measuring system, its advantages are the following:

(1) The accuracies of the two units with respect to their definitions are the same and equal to that of an optical or microwave transition judged to be the best choice for a standard.

(2) In case of future improvements only one unit needs to be redefined. Presumably this will be the unit of time, the second. Simultaneously, the meter will be automatically refined via c for the possible use of improvements in length-measuring techniques.

The practical requirements for the introduction of such a unified system are discussed in Refs. 19 and 20.

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[1] *Pulsating Stars,* with introductions by F. G. Smith and A. Hewish (Plenum, New York, 1968).

[2] B. Warner and R. Edward Nather, Nature 222, 157 (1969).

[3] S. Rappaport, H. Bradt, and W. Mayer, Nature Physical Science 229, 40 (1971).

[4] E. K. Conklin, H. T. Howard, J. S. Miller, and E. J. Wampler, Nature 222, 552 (1969).

[5] G. Feinberg, Science 166, 879 (1969).

[6] B. Lovell, F. L. Whipple, and L. H. Solomon, Nature 202, 377 (1964).

[7] B. C. Brown, Nature 224, 1189 (1969).

[8] M. Born and E. Wolf, *Principles of Optics* (Pergamon, New York, 1959), p. 94.

[9] Manuel Cordona, in *Optical Properties of Solids,* edited by S. Nudelman and S. S. Mitra (Plenum, New York, 1969), Chap. 6, p. 142. For an alternative discussion see J. S. Toll, Phys. Rev. 104, 1760 (1956).

[10] M. A. Gintsburg, Astron. Zh. 40, 703 (1963) [Soviet Astron.–AJ 7, 536 (1964)]; A. S. Goldhaber and M. M. Nieto, Rev. Mod. Phys. 43, 277 (1971).

[11] D. I. Blokhintsev, Phys. Letters 12, 272 (1964).

[12] T. G. Pavlopoulos, Phys. Rev. 159, 1106 (1967).

[13] T. Alväger, J. M. Bailey, F. J. M. Farley, J. Kjellman, and I. Wallin, Arkiv Fysik 31, 145 (1966).

[14] E. Schrödinger, Proc. Roy. Irish Acad. A49, 135 (1943).

[15] L. Bass and E. Schrödinger, Proc. Roy. Soc. (London) A232, 1 (1955).

[16] I. Yu. Kobzarev and L. B. Okun, Usp. Fiz. Nauk 95, 131 (1968) [Soviet Phys. Usp. 11, 338 (1968)].

[17] A. S. Goldhaber and M. M. Nieto, Phys. Rev. Letters 21, 567 (1968).

[18] E. Williams and D. Park, Phys. Rev. Letters 26, 1651 (1971).

[19] *Precision Measurement and Fundamental Constants,* edited by D. N. Langenberg and B. N. Taylor, NBS Special Publication 343 (U.S. GPO, Washington, D. C., 1971). See especially the papers by Z. Bay, p. 59; Z. Bay and G. G. Luther, p. 63; K. M. Evenson, J. S. Wells,

and L. M. Matarrese, p. 67.
[20]Z. Bay, paper presented at the Fourth International Conference on Atomic Masses and Fundamental Constants, Teddington, England, 1971 (to be published).

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Momentum and Angular Momentum in Relativistic Classical Particle Mechanics*

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For a classical-mechanical system of any fixed number of particles it is observed that space-translation invariance and conservation of angular momentum imply conservation of momentum. For three particles it is shown, as previously for two, that Poincaré invariance implies that the total kinematic momentum cannot be a constant of motion unless the accelerations are zero. The equations involved make it appear most likely that this is true for any number of particles.

We have learned only recently how relativistically invariant classical mechanics can describe interactions of a fixed number of particles, without fields, as in ordinary Newtonian equations of motion.[1–8] As yet, not very much is known about these interactions. For two particles it has been shown that their constants of motion do not include the total kinematic particle momentum or angular momentum.[4,9] (These quantities could have the same values before and after a collision by being asymptotic limits of constants of the motion which would depend on the interaction and could correspond to translation and rotation invariance. From the field-theory point of view there is momentum in the fields that propagate the interaction; Newton's third law does not hold because the fields do not propagate the interaction instantaneously.)

Here we observe that for any number of particles the impossibility of kinematic momentum being a constant of motion implies the same for angular momentum. We prove the statement about momentum for three particles. The equations involved make it appear most likely that it is true for any number of particles.

The idea is very simple. Suppose the kinematic momentum is a constant of motion. It is also space-translation invariant. We assume the dynamics is Poincaré invariant. It follows that every Lorentz transform of the kinematic momentum is a constant of motion, that is, the sum of the kinematic momenta of the particles taken at the same time in the transformed frame. It seems the only way every one of these can be a constant of motion is for the individual particle momenta to be constants of motion, which means there is no interaction.[10]

The same idea for space-translation invariance shows that conservation of total kinematic momentum follows from that of angular momentum, as we shall see as soon as we introduce some notation.

Let $\vec{x}^n$ and $\vec{v}^n$ be the position and velocity of the nth particle, m_n its mass, and

$$\vec{u}^n = m_n \vec{v}^n [1 - (\vec{v}^n)^2]^{-1/2}$$

its (kinematic) momentum.

Suppose that the angular momentum

$$\sum_n \vec{x}^n \times \vec{u}^n$$

is a constant of motion. If the dynamics is space-translation invariant, it follows that the translated angular momentum

$$\sum_n (\vec{x}^n + \vec{\epsilon}) \times \vec{u}^n,$$

that is, the angular momentum in a frame translated a distance $\vec{\epsilon}$, is a constant of motion. For this to be true for every $\vec{\epsilon}$ the momentum

$$\sum_n \vec{u}^n$$

must be a constant of motion.

For a Lorentz transformation with velocity $\tanh \epsilon$ in the kth direction, the jth component of the transformed position of the nth particle, that is, the