CHAPTER VII.

STATICS OF SOLIDS AND FLUIDS.

Rigid body. **551.** WE commence with the case of a *rigid body* or system, that is, an ideal substance continuously occupying a given solid figure, admitting no change of shape, but free to move translationally and rotationally. It is sometimes convenient to regard a rigid body as a group of material particles maintained by mutual forces in definite positions relatively to each other, but free to move relatively to other bodies. The condition of perfect rigidity is approximately fulfilled in natural solid bodies, so long as the applied forces are not sufficiently powerful to break them or to distort them, or to condense or rarefy them to a sensible extent. To find the conditions of equilibrium of a rigid body under the influence of any number of forces, we follow the example of Lagrange in using the principle of work (§ 289) and take advantage of our kinematic preliminary (§ 197).

Equilibrium of free rigid body. **552.** First supposing the body to be perfectly free to take any motion possible to a rigid body:—Give it an infinitesimal translation in any direction, and an infinitesimal rotation round any line.

I. In respect to the translational displacement, the work done by the applied forces is equal to the product of the amount of the displacement (being the same for all the points of application) into the algebraic sum of the components of the forces in its direction. Hence for equilibrium (§ 289) the sum of these components must be zero.

II. In respect to rotational displacement the work done by the forces is (§ 240) equal to the product of the infinitesimal angle of rotation into the sum of the moments (§ 231) of the forces round the axis of rotation. Hence for equilibrium (§ 289) the sum of these moments must be zero. Equilibrium of free rigid body.

Since (§ 197) every possible motion of a rigid body may be compounded of infinitesimal translations in any directions, and rotations round any lines, it follows that the conditions necessary and sufficient for equilibrium are that the sum of the components of the forces in any direction whatever must be zero, and the sum of the moments of the forces round any axis whatever must be zero.

Let X_1, Y_1, Z_1 be the components of one of the forces, and x_1, y_1, z_1 the co-ordinates of its point of application relatively to three rectangular axes. Taking successively these axes for directions of the infinitesimal translations, and axes of the infinitesimal rotations, we find, as *necessary* for equilibrium, the following equations:—

$$\Sigma(X_1)=0,\quad \Sigma(Y_1)=0,\quad \Sigma(Z_1)=0\ldots\ldots\ldots\ldots(1),$$

$$\Sigma(Z_1y_1-Y_1z_1)=0,\ \Sigma(X_1z_1-Z_1x_1)=0,\ \Sigma(Y_1x_1-X_1y_1)=0\ \ldots(2).$$

Of the latter three equations the first members are respectively the sums of the moments round the three axes of co-ordinates, of the given forces or of the components X_1, Y_1, Z_1, &c., which we take for them.

Important proposition: **553.** It is interesting and important to remark that the evanescence of the sum of components in any direction whatever is secured if it is ascertained that the sums of the components in the directions of any three lines not in one plane are each nil; and that the evanescence of the sum of moments round any axis whatever is secured if it is ascertained that the sums of the moments round any three axes not in one plane are each nil.

Let (l, m, n), (l', m', n'), (l'', m'', n'') be the direction cosines of three lines not in one plane, a condition equivalent to non-evanescence of the determinant $l\,m'\,n''-$&c. Let F, F', F'' be the sums of components of forces along these lines. We have proved.

$$\left.\begin{aligned} F\ &=l\ \Sigma(X_1)+m\ \Sigma(Y_1)+n\ \Sigma(Z_1)\\ F'\ &=l'\ \Sigma(X_1)+m'\ \Sigma(Y_1)+n'\ \Sigma(Z_1)\\ F''&=l''\Sigma(X_1)+m''\Sigma(Y_1)+n''\Sigma(Z_1)\end{aligned}\right\}\ldots\ldots\ldots\ldots(3).$$

Equilibrium of free rigid body.

If each of these is zero, each of the components ΣX, ΣY, ΣZ must be zero, as the determinant is not zero. The corresponding proposition is similarly proved for the moments, because (§ 233) moments of forces round different axes follow the same laws of composition and resolution as forces in different directions.

Equilibrium of constrained rigid body.

554. For equilibrium when the body is subjected to one, two, three, four, or five degrees of constraint, equations to be fulfilled by the applied forces, to ensure equilibrium, correspondingly reduced in number to five, four, three, two or one, are found with the greatest ease by giving direct analytical expression to (§ 289), the principle of work in equilibrium.

Let $\dot{x}$, $\dot{y}$, $\dot{z}$, ϖ, ρ, σ be components of the translational velocity of a point O of the body, and of the angular velocity of the body; and (§ 201) let

$$\left.\begin{array}{c} A\dot{x}+B\dot{y}+C\dot{z}+G\varpi+H\rho+I\sigma=0 \\ A'\dot{x}+B'\dot{y}+C'\dot{z}+G'\varpi+H'\rho+I'\sigma=0 \\ \&c., \qquad \&c., \end{array}\right\} \text{.............}(4),$$

be one, two, three, four, or five equations, representing the constraints. The work done by the applied forces per unit of time is

$$\left.\begin{array}{l} \dot{x}\Sigma(X_1)+\dot{y}\Sigma(Y_1)+\dot{z}\Sigma(Z_1) \\ \quad +\varpi\Sigma(Z_1y_1-Y_1z_1)+\rho\Sigma(X_1z_1-Z_1x_1)+\sigma\Sigma(Y_1x_1-X_1y_1) \end{array}\right\}\text{...}(5),$$

or
$$X\dot{x}+Y\dot{y}+Z\dot{z}+L\varpi+M\rho+N\sigma\text{..............}(5'),$$

where X, Y, Z, L, M, N denote the sums that appear in (5), that is to say, the sums of the components of the given forces parallel to the axes of co-ordinates, and the sum of their moments round these lines.

This amount of work, (5), must be zero for all values of $\dot{x}$, $\dot{y}$, $\dot{z}$, ϖ, ρ, σ which satisfy equation or equations (4). Hence, by Lagrange's method of indeterminate multipliers, we find

$$\left.\begin{array}{ll} \Sigma(X_1)+\lambda A+\lambda' A'+\dots & =0 \\ \Sigma(Y_1)+\lambda B+\lambda' B'+\dots & =0 \\ \Sigma(Z_1)+\lambda C+\lambda' C'+\dots & =0 \\ \Sigma(Z_1y_1-Y_1z_1)+\lambda G+\lambda' G'+\dots & =0 \\ \Sigma(X_1z_1-Z_1x_1)+\lambda H+\lambda' H'+\dots & =0 \\ \Sigma(Y_1x_1-X_1y_1)+\lambda I+\lambda' I'+\dots & =0 \end{array}\right\}\text{............}(6);$$

and the elimination of λ, λ',... from these six equations gives the correspondingly reduced number of equations of equilibrium among the applied forces.

Equilibrium of constrained rigid body.

Example. Two constraints;—the four equations of equilibrium found;

To illustrate the use of these equations suppose, for example, the number of constraints to be two, and all except four of the applied forces be given: the six equations (5) determine these four forces, and allow us if we desire it to calculate the two indeterminate multipliers λ, λ'. The use of finding the values of these multipliers is that

$$\lambda A,\ \lambda B,\ \lambda C,\ \lambda G,\ \lambda H,\ \lambda I$$

and the two factors determining the amounts of the constraining forces called into action.

are the components and the moments of the reactions of the first constraining body or system on the given body, and

$$\lambda' A',\ \lambda' B',\ \lambda' C',\ \lambda' G',\ \lambda' H',\ \lambda I'$$

are those of the second.

Equations of equilibrium without expression of constraining reactions.

555. When it is desired only to find the equations of equilibrium, not the constraining reactions, the easiest and most direct way to the object is, to first express any possible motion of the body in terms of the five, four, three, two or one freedoms (§§ 197, 200) left to it by the one, two, three, four or five constraints to which it is subjected. The description in § 102 of the most general motion of a rigid body shows that the most general result of five constraints, or the most general way of allowing just one freedom, to a rigid body, is to give it guidance equivalent to that of a nut on a fixed screw shaft. If we unfix this shaft and give it similar guidance to allow it one freedom, the primary rigid body has two freedoms of the most general kind. Its double freedom may be resolved in an infinite number of ways (besides the one way in which it is thus compounded) into two single freedoms. Triple, quadruple, and quintuple freedom may be similarly arranged mechanically.

556. The conditions of equilibrium of a rigid body with single, double, triple, quadruple or quintuple freedom, when each of the constituent freedoms is given in the manner specified in § 555, are found by writing down the equation or equations expressing that the applied forces do no work when the body moves simply according to any one alone of the given freedoms. We shall take first the case of a single freedom of the most general kind.

Equilibrium of forces applied to a nut on a frictionless fixed screw.

Let s* be the axial motion per radian of rotation; so that $q = s\omega$ expresses the relation between axial translational velocity, and angular velocity in the possible motion. Let HK be the axis of the screw, and N_1 the nearest point to it in $L_1 M_1$, the line of P_1, a first of the applied forces. Let i_1 be the inclination of $L_1 M_1$ to HK, and a_1 the distance of N_1 from HK. At any point in $L_1 M_1$, most conveniently at the point N_1, resolve P_1 into two components, $P_1 \cos i_1$, parallel to the axis of freedom, and $P_1 \sin i_1$ perpendicular to it. The former component does work only on the axial component of the motion, the latter on the rotational; and the rate of work done by the two together is

Work done by a single force on a nut, turning on a fixed screw.

$$s\omega\, P_1 \cos i_1 + a\omega\, P_1 \sin i_1.$$

Hence, if Σ denotes summation for all the given forces, the equation of equilibrium to prevent them from taking advantage of the first freedom is

Equation of equilibrium of forces applied to a nut on a frictionless screw.

$$s\Sigma P_1 \cos i_1 + \Sigma a_1 P_1 \sin i_1 = 0 \quad\ldots\ldots\ldots\ldots\ldots(7);$$

or, in words, *the step of the screw multiplied into the sum of the axial components must be equal to the sum of the moments of the force round the axis of the screw.*

The direction taken as positive for the moments in the preceding statement is the direction opposite to the rotation which the nut would have if it had axial motion in the direction taken as positive for those axial components.

557. The equations of equilibrium when there are two or more freedoms, are merely (7) repeated with accents to denote the elements corresponding to the several guide-screws other than the first. Thus if s, s', s'', &c., denote the screw-steps; a_1, a_1', a_1'', &c., the shortest distances between the axes of the screws and the line of P_1; i_1, i_1', i_1'', &c., the inclinations of this line to the axes; and a_2, a_2', &c., and i_2, i_2', &c., corresponding elements

* The quantity s thus defined we shall, for brevity, henceforth call the screw-step.

for the line of the second force, and so on; we have, for the equations of equilibrium,

$$\left.\begin{array}{l} s\Sigma P_1 \cos i_1 \;\; + \Sigma a_1 P_1 \sin i_1 \;\;\; = 0 \\ s'\Sigma P_1 \cos i_1' \; + \Sigma a_1' P_1 \sin i_1' \;\; = 0 \\ s''\Sigma P_1 \cos i_1'' + \Sigma a_1'' P_1 \sin i_1'' = 0 \\ \quad\&c., \qquad \&c., \end{array}\right\} \ldots\ldots\ldots\ldots\ldots(8).$$

The same analytically and in terms of rectangular co-ordinates.

The equations of constraint being, as in § 553, (4),

$$\left.\begin{array}{l} A\dot{x} + B\dot{y} + C\dot{z} + G\varpi + H\rho + I\sigma = 0 \\ A'\dot{x} + B'\dot{y} + C'\dot{z} + G'\varpi + H'\rho + I'\sigma = 0 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{array}\right\} \ldots\ldots\ldots(9),$$

suppose, for example, these equations to be four in number. Take two more equations

$$\left.\begin{array}{l} a\dot{x} + b\dot{y} + c\dot{z} + g\varpi + h\rho + i\sigma = \omega \\ a'\dot{x} + b'\dot{y} + c'\dot{z} + g'\varpi + h'\rho + i'\sigma = \omega' \end{array}\right\} \ldots\ldots\ldots(10),$$

where $a, b, \ldots$ and $a', b', \ldots$ are any arbitrarily assumed quantities: and from the six equations (9) and (10) deduce the following:

$$\left.\begin{array}{lll} \dot{x} = \mathfrak{A}\omega + \mathfrak{A}'\omega', & \dot{y} = \mathfrak{B}\omega + \mathfrak{B}'\omega', & \dot{z} = \mathfrak{C}\omega + \mathfrak{C}'\omega', \\ \varpi = \mathfrak{G}\omega + \mathfrak{G}'\omega', & \rho = \mathfrak{H}\omega + \mathfrak{H}'\omega', & \sigma = \mathfrak{I}\omega + \mathfrak{I}'\omega', \end{array}\right\} \ldots\ldots(11);$$

Two generalized component velocities corresponding to two freedoms.

where $\mathfrak{A}, \mathfrak{B}, \ldots$ and $\mathfrak{A}', \mathfrak{B}', \ldots$ are known, being the determinantal ratios found in solving (9) and (10). Thus the *six* rectangular component velocities are expressed in terms of *two* generalized component velocities ω, ω', which, in virtue of the four equations of constraint (9), suffice for the complete specification of whatever motion the constraints leave permissible. In terms of this notation we have, for the rate of working of the applied forces,

$$\left.\begin{array}{l} X\dot{x} + Y\dot{y} + Z\dot{z} + L\varpi + M\rho + N\sigma \\ \quad = (\mathfrak{A}X + \mathfrak{B}Y + \mathfrak{C}Z + \mathfrak{G}L + \mathfrak{H}M + \mathfrak{I}N)\,\omega \\ \quad + (\mathfrak{A}'X + \mathfrak{B}'Y + \mathfrak{C}'Z + \mathfrak{G}'L + \mathfrak{H}'M + \mathfrak{I}'N)\,\omega' \end{array}\right\} \ldots\ldots\ldots\ldots(12).$$

This must be nil for every permitted motion in order that the forces may balance. Hence the equations of equilibrium are

$$\left.\begin{array}{l} \mathfrak{A}X + \mathfrak{B}Y + \mathfrak{C}Z + \mathfrak{G}L + \mathfrak{H}M + \mathfrak{I}N = 0 \\ \text{and } \mathfrak{A}'X + \mathfrak{B}'Y + \mathfrak{C}'Z + \mathfrak{G}'L + \mathfrak{H}'M + \mathfrak{I}'N = 0 \end{array}\right\} \ldots\ldots\ldots(13).$$

Two generalized component velocities corresponding to two freedoms.

Similarly with one, or two, or three, or five (instead of our example of four) constraining equations (9), we find five, or four, or three, or one equation of equilibrium (13). These equations express obviously the same conditions as those expressed by (8); the first of (13) is identical with the first of (8), the second of (13) with the second of (8), and so on, provided ω, ω',... correspond to the same components of freedom as the several screws of (8) respectively. The equations though identical in substance are very different in form. The purely analytical transformation from either form to the other is a simple enough piece of analytical geometry which may be worked as an exercise by the student, to be done separately for the first of (8) and the first of (13), just as if there were but one freedom.

Equilibrant and resultant.

558. Any system of forces which if applied to a rigid body would balance a given system of forces acting on it, is called an equilibrant of the given system. The system of forces equal and opposite to the equilibrant may be called a resultant of the given system. It is only, however, when the resultant system is less numerous, or in some respect simpler, than the given system that the term resultant is convenient or suitable. It is used with great advantage with respect to the resultant force and couple (§ 559 *g*, below) to which Poinsot's method leads, or to the two resultant forces which mathematicians before Poinsot had shown to be the simplest system to which any system of forces acting on a rigid body can in general be reduced. It is only when the system is reducible to a single force that the term "resultant" pure and simple is usually applied.

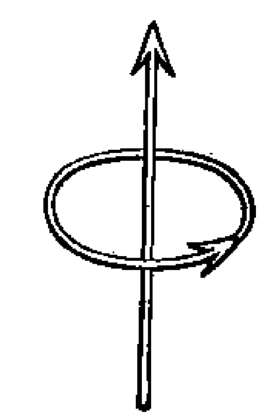
559. As a most useful commentary on and illustration of the general theory of the equilibrium of a rigid body, which we have completed in §§ 552—557, and particularly for the purpose of finding practically convenient resultants in a very simple and clear manner, we may now with advantage introduce the beautiful method of *Couples*, invented by Poinsot.

Couples.

In § 234 we have already defined a couple, and shown that the sum of the moments of its forces is the same about all axes perpendicular to its plane. It may therefore be shifted to any new position in its own plane, or in any parallel plane,

without alteration of its effect on the rigid body to which it is applied. Its arm may be turned through any angle in the plane of the forces, and the length of the arm and the magnitudes of the forces may be altered at pleasure, without changing its effect—provided the *moment* remain unchanged. Hence a couple is conveniently specified by the line defined as its "axis" in § 234. According to the convention of § 234 the axis of a couple which tends to produce rotation in the direction contrary to the motion of the hands of a watch, must be drawn through the *front* of the watch and *vice versâ*. This may easily be remembered by the help of a simple diagram such as we give, in which the arrow-heads indicate the directions of rotation, and of the axis, respectively.

Couples.



Composition of couples.

559 *b*. It follows from §§ 233, 234, that couples are to be compounded or resolved by treating their axes by the law of the parallelogram, in a manner identical with that which we have seen must be employed for linear and angular velocities, and forces.

Hence a couple G, the direction cosines of whose axis are λ, μ, ν, is equivalent to the three couples $G\lambda$, $G\mu$, $G\nu$ about the axes of x, y, z respectively.

Force resolved into force and couple.

559 *c*. If a force, F, act at any point, A, of a body, it may be transferred to any other point, B. Thus: by the principle of superposition of forces, introduce at B, in the line through it parallel to the given force F, a pair of equal and opposite forces F and $-F$. Then F at A, and $-F$ at B, form a couple, and there remains F at B.

Application to equilibrium of rigid body.

From this we have, at once, the conditions of equilibrium of a rigid body already investigated in § 552. For, each force may be transferred to any assumed point as origin, if we introduce the corresponding couple. And the forces, which now act at one point, must equilibrate according to the principles of Chap. VI.; while the resultant couple, and therefore its components about any three lines at right angles to each other, must vanish.

Forces represented by the sides of a polygon.

559 *d*. Hence forces represented, not merely in magnitude and direction, but in lines of action, by the sides of any closed polygon whether plane or not plane, are equivalent to a single couple. For when transferred to any origin, they equilibrate, by the Polygon of Forces (§§ 27, 256). When the polygon is plane, twice its area is the moment of the couple; when not plane, the component of the couple about any axis is twice the area of the projection on a plane perpendicular to that axis. The resultant couple has its axis perpendicular to the plane (§ 236) on which the projected area is a maximum.

Forces proportional and perpendicular to the sides of a triangle.

559 *e*. Lines, perpendicular to the sides of a triangle, and passing through their middle points, meet; and their mutual inclinations are equal to the changes of direction at the corners, in travelling round the triangle. Hence, if at the middle points of the sides of a triangle, and in its plane, forces be applied all inwards or all outwards; and if their magnitudes be proportional to the sides of the triangle, they are in equilibrium. The same is true of any plane polygon, as we readily see by dividing it into triangles. And if forces equal to the areas of the faces be applied perpendicularly to the faces of any closed polyhedron, at their centres of inertia, all inwards or all outwards, these also will form an equilibrating system; as we see by considering the evanescence of (i) the algebraic sum of the projections of the areas of the faces on any plane, and of (ii) the algebraic sum of the volumes of the rings described by the faces when the solid figure is made to rotate round any axis, these volumes being reckoned by aid of Pappus' theorem (§ 569, below).

Composition of force and couple.

559 *f*. A couple and a force in a given line inclined to its plane may be reduced to a smaller couple in a plane perpendicular to the force, and a force equal and parallel to the given force. For the couple may be resolved into two, one in a plane containing the direction of the force, and the other in a plane perpendicular to the force. The force and the component couple in the same plane with it are equivalent to an equal force acting in a parallel line, according to the converse of § 559 *c*.

Composition of any set of forces acting on a rigid body.

559 *g*. We have seen that any set of forces acting on a rigid body may be reduced to a force at any point and a couple. Now (§ 559 *f*) these may be reduced to an equal force acting in a definite line in the body, and a couple whose plane is perpendicular to the force, and which is the least couple which, with a single force, can constitute a resultant of the given set of forces. The definite line thus found for the force is called the *Central Axis*. It is the line about which the sum of the moments of the given forces is least.

Central axis.

With the notation of §§ 552, 553, let us suppose the origin to be changed to any point x', y', z'. The resultant force has still the components $\Sigma(X)$, $\Sigma(Y)$, $\Sigma(Z)$, or Rl, Rm, Rn, parallel to the axes. But the couples now are

$$\Sigma[Z(y-y')-Y(z-z')],\quad \Sigma[X(z-z')-Z(x-x')],\quad \Sigma[Y(x-x')-X(y-y')];$$

or

$$G\lambda - R(ny'-mz'),\quad G\mu - R(lz'-nx'),\quad G\nu - R(mx'-ly').$$

The conditions that the resultant force shall be perpendicular to the plane of the resultant couple are

$$\frac{G\lambda - R(ny'-mz')}{l} = \frac{G\mu - R(lz'-nx')}{m} = \frac{G\nu - R(mx'-ly')}{n}.$$

These two equations among x', y', z' are the equations of the central axis.

We find the same two equations by investigating the conditions that the resultant couple

$$\sqrt{[G\lambda-R(ny'-mz')]^2+[G\mu-R(lz'-nx')]^2+[G\nu-R(mx'-ly')]^2}$$

may be a minimum subject to independent variations of x', y', z'.

Reduction to two forces.

560. By combining the resultant force with one of the forces of the resultant couple, we have obviously an infinite number of ways of reducing any set of forces acting on a rigid body to *two* forces whose directions do not meet. But there is one case in which the result is symmetrical, and which is therefore worthy of special notice.

Symmetrical case.

Supposing the central axis of the system has been found, draw a line, AA', at right angles to it through any point C of

Symmetrical case.

it, and make CA equal to CA'. For R, acting along the central axis, substitute (by § 561) $\frac{1}{2}R$ at each end of AA'. Then, choosing this line AA' as the arm of the couple, and calling it a, we have at one extremity of it, two forces, $\frac{G}{a}$ perpendicular to the central axis, and $\frac{1}{2}R$ parallel to the central axis. Compounding these we get two forces, each equal to $\left(\frac{1}{4}R^2+\frac{G^2}{a^2}\right)^{\frac{1}{2}}$, through A and A' respectively, perpendicular to AA', and inclined to the plane through AA' and the central axis, at angles on the two sides of it each equal to $\tan^{-1}\frac{2G}{Ra}$.

Composition of parallel forces.

561. A very simple, but important, case, is that of any number of *parallel* forces acting at different points of a rigid body.

Here, for equilibrium, obviously it is necessary and sufficient that the algebraic sum of the forces be nil; and that the sum of their moments about any two axes perpendicular to the common direction of the forces be also nil.

This clearly implies (§ 553) that the sum of their moments about any axis whatever is nil.

To express the condition in rectangular coordinates, let P_1, P_2, &c. be the forces; (x_1, y_1, z_1), (x_2, y_2, z_2), &c. points in their lines of action; and l, m, n the direction cosines of a line parallel to them all. The general equations [§ 552 (1), (2)] of equilibrium of a rigid body become in this case,

$$l\Sigma P=0,\quad m\Sigma P=0,\quad n\Sigma P=0\,;$$

$$n\Sigma Py-m\Sigma Pz=0,\quad l\Sigma Pz-n\Sigma Px=0,\quad m\Sigma Px-l\Sigma Py=0.$$

These equations are equivalent to but three independent equations, which may be written as follows:

$$\Sigma P=0,\quad \frac{\Sigma Px}{l}=\frac{\Sigma Py}{m}=\frac{\Sigma Pz}{n}\ \ldots\ldots\ldots\ldots\ldots(1).$$

If the given forces are not in equilibrium a single force may be found which shall be their resultant. To prove this let, if possible, a force $-R$, in the direction (l, m, n), at a point

$(\bar{x}, \bar{y}, \bar{z})$ equilibrate the given forces. By (1) we have, for the conditions of equilibrium of $-R$, P_1, P_2, &c.,

Composition of parallel forces.

$$R=\Sigma P\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2),$$

and

$$\frac{\Sigma Px-R\bar{x}}{l}=\frac{\Sigma Py-R\bar{y}}{m}=\frac{\Sigma Pz-R\bar{z}}{n}\ \ldots\ldots\ldots(3).$$

Equation (2) determines R, and equations (3) are the equations of a straight line at any point of which a force equal to $-R$, applied in the direction (l, m, n), will balance the given system.

Suppose now the direction (l, m, n) of the given forces to be varied while the magnitude P_1, and one point (x_1, y_1, z_1) in the line of application, of each force is kept unchanged. We see by (3) that one point $(\bar{x}, \bar{y}, \bar{z})$ given by the equations

$$\bar{x}=\frac{\Sigma Px}{R},\quad \bar{y}=\frac{\Sigma Py}{R},\quad \bar{z}=\frac{\Sigma Pz}{R}\ \ldots\ldots\ldots(4),$$

is common to the lines of the resultants.

The point $(\bar{x}, \bar{y}, \bar{z})$ given by equations (4) is what is called the centre of the system of parallel forces P_1 at (x_1, y_1, z_1), P_2 at (x_2, y_2, z_2), &c.: and we have the proposition that a force in the line through this point parallel to the lines of the given forces, equal to their sum, is their resultant. This proposition is easily proved synthetically by taking the forces in any order and finding the resultant of the first two, then the resultant of this and the third, then of this second force, and so on. The line of the first subsidiary resultant, for all varied directions of the given forces, passes through one and the same point (that is the point dividing the line joining the points of application of the first two forces, into parts inversely as their magnitudes). Similarly we see that the second subsidiary resultant passes always through one determinate point: and so for the third, and so on for any number of forces.

Centre of gravity.

562. It is obvious, from the formulas of § 230, that if masses proportional to the forces be placed at the several points of application of these forces, the centre of inertia of these masses will be the same point in the body as the centre of parallel

Centre of gravity.

forces. Hence the reactions of the different parts of a rigid body against acceleration in parallel lines are rigorously reducible to one force, acting at the centre of inertia. The same is true approximately of the action of gravity on a rigid body of small dimensions relatively to the earth, and hence the centre of inertia is sometimes (§ 230) called the *Centre of Gravity*. But, except on a centrobaric body (§ 534), gravity is in general reducible not to a single force but to a force and couple (§ 559 *g*); and the force does not pass through a point fixed relatively to the body in all the positions for which the couple vanishes.

Parallel forces whose algebraic sum is zero.

563. In one case the proposition of § 561, that the system has a single resultant force, must be modified: that is the case in which the algebraic sum of the given forces vanishes. In this case the resultant is a couple whose plane is parallel to the common direction of the forces. A good example of this case is furnished by a magnetized mass of steel, of moderate dimensions, subject to the influence of the earth's magnetism. The amounts of the so-called north and south magnetisms in each element of the mass are equal, and are therefore subject to equal and opposite forces, parallel in a rigorously uniform field of force. Thus a compass-needle experiences from the earth's magnetism sensibly a couple (or *directive* action), and is not sensibly attracted or repelled as a whole.

Conditions of equilibrium of three forces.

564. If three forces, acting on a rigid body, produce equilibrium, their directions must lie in one plane; and must all meet in one point, or be parallel. For the proof we may introduce a consideration which will be very useful to us in investigations connected with the statics of flexible bodies and fluids.

Physical axiom.

If any forces, acting on a solid, or fluid body, produce equilibrium, we may suppose any portions of the body to become fixed, or rigid, or rigid and fixed, without destroying the equilibrium.

Applying this principle to the case above, suppose any two points of the body, respectively in the lines of action of two of the forces, to be fixed. The third force must have no moment

Physical axiom.

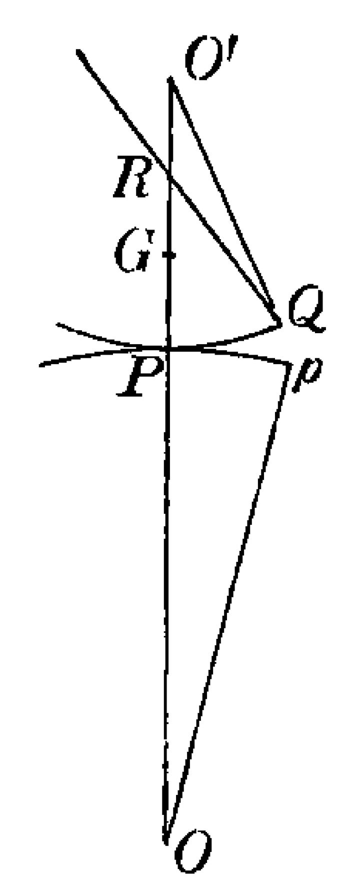
about the line joining these points; in other words, its direction must pass through that line. As any two points in the lines of action may be taken, it follows that the three forces are coplanar. And three forces, in one plane, cannot equilibrate unless their directions are parallel, or pass through a point.

Equilibrium under the action of gravity.

565. It is easy, and useful, to consider various cases of equilibrium when no forces act on a rigid body but gravity and the pressures, normal or tangential, between it and fixed supports. Thus if one given point only of the body be fixed, it is evident that the centre of inertia must be in the vertical line through this point. For *stable* equilibrium the centre of inertia need not be *below* the point of support (§ 566).

Rocking Stones.

566. An interesting case of equilibrium is suggested by what are called Rocking Stones, where, whether by natural or by artificial processes, the lower surface of a loose mass of rock is worn into a convex or concave, or anticlastic form, while the bed of rock on which it rests in equilibrium may be convex or concave, or of an anticlastic form. A loaded sphere resting on a spherical surface is a particular case.



Let O, O' be the centres of curvature of the fixed, and rocking, bodies respectively, when in the position of equilibrium. Take any two infinitely small, equal arcs PQ, Pp; and at Q make the angle $O'QR$ equal to POp. When, by displacement, Q and p become the points in contact, QR will evidently be vertical; and, if the centre of inertia G, which must be in OPO' when the movable body is in its position of equilibrium, be to the left of QR, the equilibrium will obviously be stable. Hence, if it be below R, the equilibrium is stable, and not unless.

Now if ρ and σ be the radii of curvature OP, $O'P$ of the two surfaces, and θ the angle POp, the angle $QO'R$ will be equal to $\frac{\rho\theta}{\sigma}$; and we have in the triangle $QO'R$ (§ 112)

$$RO' : \sigma :: \sin\theta : \sin\left(\theta + \frac{\rho\theta}{\sigma}\right)$$

$$:: \sigma : \sigma + \rho \text{ (approximately).}$$

Rocking stones.

Hence $$PR = \sigma - \frac{\sigma^2}{\sigma+\rho} = \frac{\rho\sigma}{\rho+\sigma};$$

and therefore, for stable equilibrium,

$$PG < \frac{\rho\sigma}{\rho+\sigma}.$$

If the lower surface be plane, ρ is infinite, and the condition becomes (as in § 291)

$$PG < \sigma.$$

If the lower surface be concave the sign of ρ must be changed, and the condition becomes

$$PG < \frac{\rho\sigma}{\rho-\sigma},$$

which cannot be negative, since ρ *must* be numerically greater than σ in this case.

Equilibrium, about an axis,

567. If two points be fixed, the only motion of which the system is capable is one of rotation about a fixed axis. The centre of inertia must then be in the vertical plane passing through those points. For stability it is necessary (§ 566) that the centre of inertia be *below* the line joining them.

on a fixed surface.

568. If a rigid body rest on a frictional fixed surface there will in general be only *three* points of contact; and the body will be in stable equilibrium if the vertical line drawn from its centre of inertia cuts the plane of these three points *within* the triangle of which they form the corners. For if one of these supports be removed, the body will obviously tend to fall towards that support. Hence each of the three prevents the body from rotating about the line joining the other two. Thus, for instance, a body stands stably on an inclined plane (if the friction be sufficient to prevent it from sliding down) when the vertical line drawn through its centre of inertia falls within the base, or area bounded by the shortest line which can be drawn round the portion in contact with the plane. Hence a body, which cannot stand on a horizontal plane, may stand on an inclined plane.

Pappus' theorem.

569. A curious theorem, due to Pappus, but commonly attributed to Guldinus, may be mentioned here, as it is employed with advantage in some cases in finding the centre of gravity (or centre of inertia) of a body. It is obvious from § 230. *If a plane closed curve revolve through any angle about an axis in its plane, the solid content of the surface generated is equal to the product of the area of the curve into the length of the path described by the centre of inertia of the area of the curve; and the area of the curved surface is equal to the product of the length of the curve into the length of the path described by the centre of inertia of the curve.*

570. The general principles upon which forces of constraint and friction are to be treated have been stated above (§§ 293, 329, 452). We add here a few examples for the sake of illustrating the application of these principles to the equilibrium of a rigid body in some of the more important practical cases of constraint.

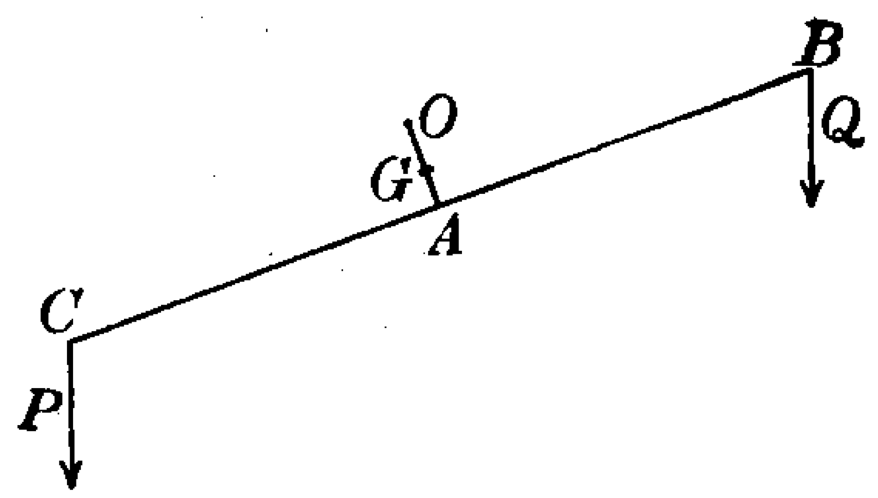
Mechanical powers.

571. The application of statical principles to the *Mechanical Powers*, or elementary machines, and to their combinations, however complex, requires merely a statement of their kinematical relations (as in §§ 79, 85, 102, &c.) and an immediate translation into Dynamics by Newton's principle (§ 269); or by Lagrange's Virtual Velocities (§§ 289, 290), with special attention to the introduction of forces of friction as in § 452. In no case can this process involve further difficulties than are implied in seeking the geometrical circumstances of any infinitely small disturbance, and in the subsequent solution of the equations to which the translation into dynamics leads us. We will not, therefore, stop to discuss any of these questions; but will take a few examples of no very great difficulty, before quitting for a time this part of the subject. The principles already developed will be of constant use to us in the remainder of the work, which will furnish us with ever-recurring opportunities of exemplifying their use and mode of application.

Let us begin with the case of the Balance, of which we promised (§ 431) to give an investigation.

Examples. Balance.

572. *Ex.* I. The centre of gravity of the beam must not coincide with the knife-edge, or else the beam would rest indifferently in any position. We shall suppose, in the first place, that the arms are not of equal length.



Let O be the fulcrum, G the centre of gravity of the beam, M its mass; and suppose that with loads P and Q in the pans the beam rests (as drawn) in a position making an angle θ with the horizontal line.

Sensibility.

Taking moments about O, and, for convenience (see § 220), using gravitation measurement of the forces, we have

$$Q(AB\cos\theta + OA\sin\theta) + M\,.\,OG\sin\theta = P(AC\cos\theta - OA\sin\theta).$$

From this we find

$$\tan\theta = \frac{P\,.\,AC - Q\,.\,AB}{(P+Q)\,OA + M\,.\,OG}.$$

If the arms be equal we have

$$\tan\theta = \frac{(P-Q)\,AB}{(P+Q)\,OA + M\,.\,OG}.$$

Hence the Sensibility (§ 431) is greater, (1) as the arms are longer, (2) as the mass of the beam is less, (3) as the fulcrum is nearer to the line joining the points of attachment of the pans, (4) as the fulcrum is nearer to the centre of gravity of the beam. If the fulcrum be *in* the line joining the points of attachment of the pans, the sensibility is the same for the same *difference* of loads in the pans.

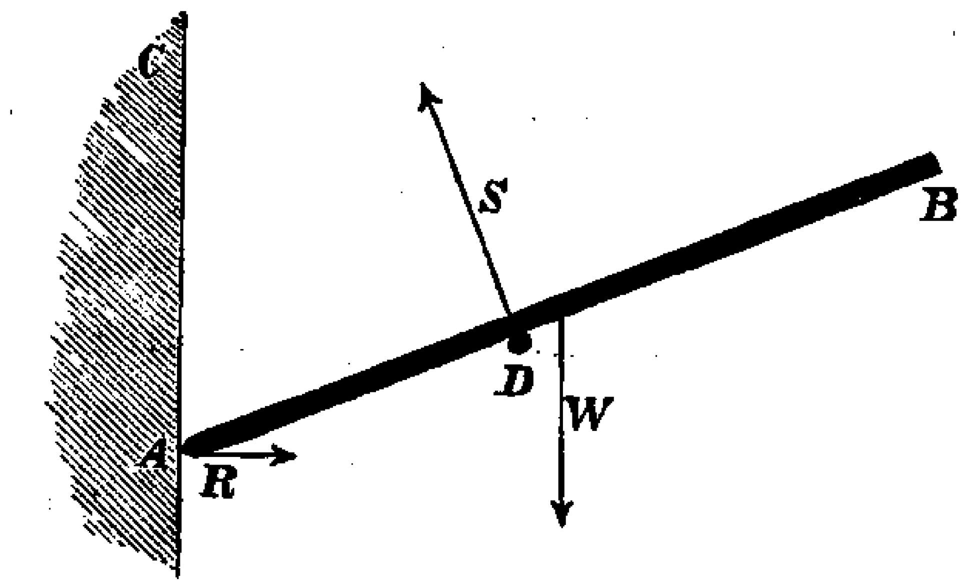
Examples. Rod with frictionless constraint.

Ex. II. Find the position of equilibrium of a rod AB resting on a frictionless horizontal rail D, its lower end pressing against a frictionless vertical wall AC parallel to the rail.

The figure represents a vertical section through the rod, which must evidently be in a plane perpendicular to the wall and rail. The equilibrium is obviously unstable.

Examples. Rod with frictionless constraint.

The only forces acting are three, R the pressure of the wall on the rod, horizontal; S that of the rail on the rod, perpendicular to the rod; W the weight of the rod, acting vertically downwards at its centre of gravity. If the half-length of the rod be a, and the distance of the rail from the wall b, these are given—and all that is wanted to fix the position of equilibrium is the angle, CAB, which the rod makes with the wall. If we call it θ we have $AD = \dfrac{b}{\sin\theta}$.



Resolving horizontally, $R - S\cos\theta = 0$ ....(1),

vertically, $W - S\sin\theta = 0$ ....(2).

Taking moments about A

$$S\,.\,AD - W\,.\,a\sin\theta = 0,$$

or

$$S\,.\,b - W\,.\,a\sin^2\theta = 0 \text{ ....(3).}$$

As there are only three unknown quantities R, S, and θ, these three equations contain the complete solution of the problem. By (2) and (3)

$$\sin^3\theta = \frac{b}{a},\ \text{which gives } \theta.$$

And by (2) $$S = \frac{W}{\sin\theta},$$

and by (1) $$R = S\cos\theta = W\cot\theta.$$

Rod constrained by frictional surfaces.

Ex. III. As an additional example, suppose the wall and rail to be frictional, and let μ be the coefficient of statical friction for both. If the rod be placed in the position of equilibrium just investigated for the case of no friction, none will be called into play, for there will be no tendency to motion to be overcome. If the end A be brought lower and lower, more

Examples. Rod constrained by frictional surfaces.

and more friction will be called into play to overcome the tendency of the rod to fall between the wall and the rail, until we come to a limiting position in which motion is about to commence. In that position the friction at A is μ times the pressure on the wall, and acts *upwards*. That at D is μ times the pressure on the rod, and acts in the direction DB. Putting $CAD = \theta_1$ in this case, our three equations become

$$R_1 + \mu S_1 \sin\theta_1 - S_1 \cos\theta_1 = 0 \quad\ldots\ldots\ldots(1_1),$$
$$W - \mu R_1 - S_1 \sin\theta_1 - \mu S_1 \cos\theta_1 = 0 \quad\ldots\ldots\ldots(2_1),$$
$$S_1 b - Wa \sin^2\theta_1 = 0 \quad\ldots\ldots\ldots(3_1).$$

The directions of both the friction-forces passing through A, neither appears in (3_1). This is why A is preferable to any other point about which to take moments.

By eliminating R_1 and S_1 from these equations we get

$$1 - \frac{a}{b}\sin^3\theta_1 = \mu\frac{a}{b}\sin^2\theta_1 (2\cos\theta_1 - \mu\sin\theta_1)\ldots\ldots(4_1),$$

from which θ_1 is to be found. Then S_1 is known from (3_1), and R_1 from either of the others.

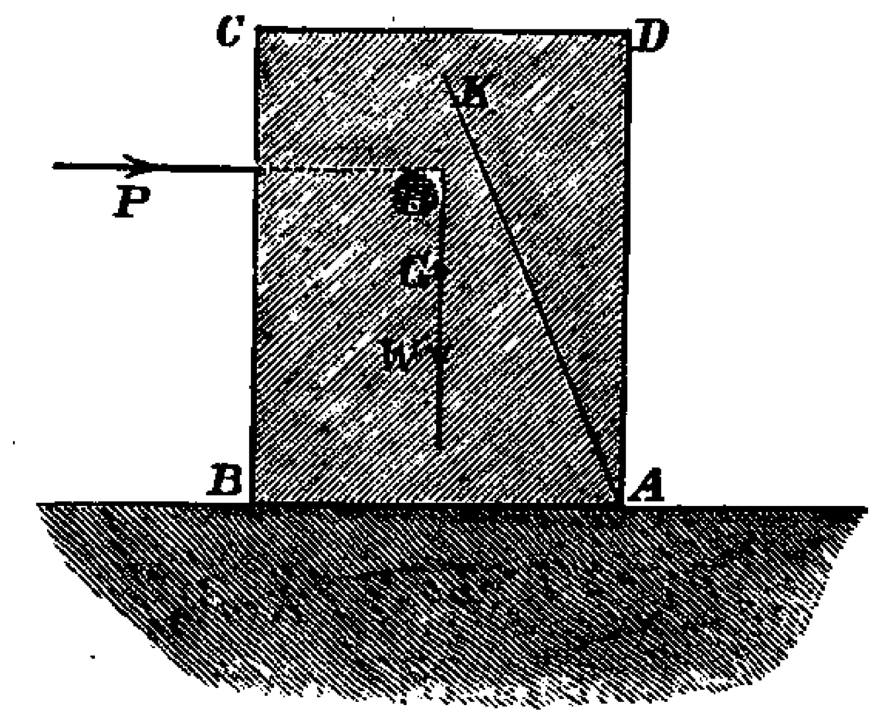
If the end A be raised above the position of equilibrium without friction, the tendency is for the rod to fall *outside* the rail; more and more friction will be called into play, till the position of the rod (θ_2) is such that the friction reaches its greatest value, μ times the pressure. We may thus find another *limiting* position for stability; and in any position between these the rod is in equilibrium.

It is useful to observe that in this second case the direction of each friction is the opposite to that in the former. Hence equations of the first case, with the sign of μ changed, serve for the second case. Thus for θ_2, by (4_1),

$$1 - \frac{a}{b}\sin^3\theta_2 = -\mu\frac{a}{b}\sin^2\theta_2 (2\cos\theta_2 + \mu\sin\theta_2).$$

Examples. Block on frictional plane.

Ex. IV. A rectangular block lies on a frictional horizontal plane, and is acted on by a horizontal force whose line of action is midway between two of the vertical sides. Find the magnitude of the force when just sufficient to produce motion, and whether the motion will be of the nature of *sliding* or *overturning*.



If the force P is on the point of overturning the body, it is evident that it will turn about the edge A, and therefore the pressure, R, of the plane and the friction, S, act at that edge. Our statical conditions are, of course,

$$R = W,$$
$$S = P,$$
$$Wb = Pa,$$

where b is half the length of the solid, and a the distance of P from the plane. From these we have $S = \frac{b}{a} W$.

Now S cannot exceed μR, whence we must not have $\frac{b}{a}$ greater than μ, if it is to be possible to upset the body by a horizontal force in the line given for P.

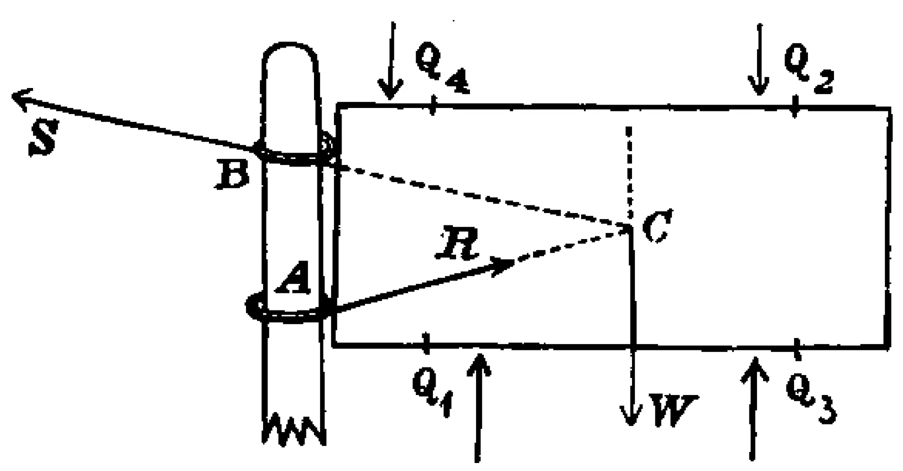
A simple geometrical construction enables us to solve this and similar problems, and will be seen at once to be merely a graphic representation of the above process. Thus if we produce the directions of the applied force, and of the weight, to meet in H, and make at A the angle BAK whose co-tangent is the coefficient of friction: there will be a tendency to upset, or not, according as H is above, or below, AK.

Mass supported by rings passing round a rough post.

Ex. V. A mass, such as a gate, is supported by two rings, A and B, which pass loosely round a vertical post. In equilibrium, it is obvious that at A the part of the ring nearest the

Examples. Mass supported by rings passing round a rough post.

mass, and at B the part farthest from it, will be in contact with



the post. The pressures exerted on the rings, R and S, will evidently be in the directions AC, CB, indicated in the diagram, which, if no other force besides gravity act on the mass, must meet in the vertical through its centre of inertia. And it is obvious that, however small be the coefficient of friction, provided there be any force of friction at all, equilibrium is always possible if the distance of the centre of inertia from the post be great enough compared with the distance between the rings.

When the mass is just about to slide down, the full amount of friction is called into play, and the angles which R and S make with the horizon are each equal to the sliding angle. If the centre of inertia of the gate be farther from the post than the intersection of two lines drawn from A, B, at the sliding angles, it will hang stably held up by friction; not unless. A force pushing upwards at Q_1, or downwards at Q_2, will remove the tendency to fall; but a force upwards at Q_3, or downwards at Q_4, will produce sliding.

A similar investigation is easily applied to the jamming of a sliding piece or drawer, and to the determination of the proper point of application of a force to move it.

573. Having thus briefly considered the equilibrium of a rigid body, we propose, before entering upon the subject of the deformation of elastic solids, to consider certain intermediate cases, in each of which we make a particular assumption the basis of the investigation, and thereby avoid a very considerable amount of analytical difficulty.

Equilibrium of a flexible and inextensible cord.

574. Very excellent examples of this kind are furnished by the statics of a flexible and inextensible cord or chain, fixed at both ends, and subject to the action of any forces. The curve in which the chain hangs in any case may be called a *Catenary*, although the term is usually restricted to the case of a uniform chain acted on by gravity only.

Catenary.

Three methods of investigation.

575. We may consider separately the conditions of equilibrium of each element; or we may apply the general condition (§ 292) that the whole potential energy is a minimum, in the case of any conservative system of forces; or, especially when gravity is the only external force, we may consider the equilibrium of a *finite* portion of the chain treated for the time as a rigid body (§ 564).

Equations of equilibrium with reference to tangent and osculating plane.

576. The first of these methods gives immediately the three following equations of equilibrium, for the catenary in general:—

(1) The rate of variation of the tension per unit of length along the cord is equal to the tangential component of the applied force, per unit of length.

(2) The plane of curvature of the cord contains the normal component of the applied force, and the centre of curvature is on the opposite side of the arc from that towards which this force acts.

(3) The amount of the curvature is equal to the normal component of the applied force per unit of length at any point divided by the tension of the cord at the same point.

The first of these is simply the equation of equilibrium of an infinitely small element of the cord relatively to tangential motion. The second and third express that the component of the resultant of the tensions at the two ends of an infinitely small arc, along the normal through its middle point is directly opposed and is equal to the normal applied force, and is equal to the whole amount of it on the arc. For the plane of the tangent lines in which those tensions act is (§ 8) the plane of curvature. And if θ be the angle between them (or the infinitely small angle by which the angle between their positive directions falls short of π), and T the arithmetical mean of their magnitudes, the component of their resultant along the line bisecting the angle between their positive directions is $2T \sin \frac{1}{2}\theta$, rigorously: or $T\theta$, since θ is infinitely small. Hence $T\theta = N\delta s$, if δs be the length of the arc, and $N\delta s$ the whole

Equations of equilibrium with reference to tangent and osculating plane.

amount of normal force applied to it. But (§ 9) $\theta = \frac{\delta s}{\rho}$ if ρ be the radius of curvature; and therefore

$$\frac{1}{\rho} = \frac{N}{T},$$

which is the equation stated in words (3) above.

Integral for tension.

577. From (1) of § 576, we see that if the applied forces on each particle of the cord constitute a conservative system, and if the cord be homogeneous, the difference of the tensions of the cord at any two points of it when hanging in equilibrium, is equal to the difference of the potential (§ 485) of the forces between the positions occupied by these points. Hence, whatever be the position where the potential is reckoned zero, the tension of the string at any point is equal to the potential at the position occupied by it, with a constant added.

Cartesian equations of equilibrium.

578. Instead of considering forces along and perpendicular to the tangent, we may resolve all parallel to any fixed direction: and we thus see that the component of applied force per unit of length of the chain at any point of it, must be equal to the rate of diminution per unit of length of the cord, of the component of its tension parallel to the fixed line of this component. By choosing any three fixed rectangular directions we thus have the three differential equations convenient for the analytical treatment of catenaries by the method of rectangular co-ordinates.

These equations are

$$\left.\begin{aligned} \frac{d}{ds}\left(T\frac{dx}{ds}\right) &= -\sigma X \\ \frac{d}{ds}\left(T\frac{dy}{ds}\right) &= -\sigma Y \\ \frac{d}{ds}\left(T\frac{dz}{ds}\right) &= -\sigma Z \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

if s denote the length of the cord from any point of it, to a point P; x, y, z the rectangular co-ordinates of P; X, Y, Z the components of the applied forces at P, per unit mass of the cord; σ the mass of the cord per unit length at P; and T its tension at this point.

Cartesian equations of equilibrium.

These equations afford analytical proofs of § 576, (1), (2), and (3) thus:—Multiplying the first by dx, the second by dy, and the third by dz, adding and observing that

$$\frac{dx}{ds}d\frac{dx}{ds} + \frac{dy}{ds}d\frac{dy}{ds} + \frac{dz}{ds}d\frac{dz}{ds} = \tfrac{1}{2}d\frac{dx^2+dy^2+dz^2}{ds^2} = 0,$$

we have

$$dT = -\sigma(Xdx + Ydy + Zdz) = -\sigma\left(X\frac{dx}{ds} + Y\frac{dy}{ds} + Z\frac{dz}{ds}\right)ds \ldots (2),$$

which is (1) of § 576. Again, eliminating dT and T, we have

$$X\left(\frac{dy}{ds}d\frac{dz}{ds} - \frac{dz}{ds}d\frac{dy}{ds}\right) + Y\left(\frac{dz}{ds}d\frac{dx}{ds} - \frac{dx}{ds}d\frac{dz}{ds}\right) + Z\left(\frac{dx}{ds}d\frac{dy}{ds} - \frac{dy}{ds}d\frac{dx}{ds}\right) = 0 \ldots\ldots\ldots (3),$$

which (§§ 9, 26) shows that the resultant of X, Y, Z is in the osculating plane, and therefore is the analytical expression of § 576 (2). Lastly, multiplying the first by $d\frac{dx}{ds}$, the second by $d\frac{dy}{ds}$, and the third by $d\frac{dz}{ds}$, and adding, we find

$$T = -\sigma\frac{\left(Xd\frac{dx}{ds} + Yd\frac{dy}{ds} + Zd\frac{dz}{ds}\right)ds}{\left(d\frac{dx}{ds}\right)^2 + \left(d\frac{dy}{ds}\right)^2 + \left(d\frac{dz}{ds}\right)^2} \ldots\ldots\ldots\ldots (4),$$

which is the analytical expression of § 576 (3).

Method of energy.

579. The same equations of equilibrium may be derived from the energy condition of equilibrium; analytically with ease by the methods of the calculus of variations.

Catenary

Let V be the potential at (x, y, z) of the applied forces per unit mass of the cord. The potential energy of any given length of the cord, in any actual position between two given fixed points, will be

$$\int V\sigma ds.$$

This integral, extended through the given length of the cord between the given points, must be a minimum; while the indefinite integral, s, from one end up to the point (x, y, z) remains unchanged by the variations in the positions of this point. Hence, by the calculus of variations,

$$\delta\int V\sigma ds + \int\lambda\delta ds = 0,$$

where λ is a function of x, y, z to be eliminated.

Catenary.

Now σ is a function of s, and therefore as s does not vary when x, y, z are changed into $x+\delta x$, $y+\delta y$, $z+\delta z$, the co-ordinates of the same particle of the chain in another position, we have

$$\delta(\sigma V) = \sigma\delta V = -\sigma(X\delta x + Y\delta y + Z\delta z).$$

Using this, and

$$\delta ds = \frac{dx d\delta x + dy d\delta y + dz d\delta z}{ds},$$

in the variational equation; and integrating the last term by parts according to the usual rule; we have

$$\int ds\left\{\left[\sigma X + \frac{d}{ds}\left(\overline{V\sigma+\lambda}\frac{dx}{ds}\right)\right]\delta x + \left[\sigma Y + \frac{d}{ds}\left(\overline{V\sigma+\lambda}\frac{dy}{ds}\right)\right]\delta y + \left[\sigma Z + \frac{d}{ds}\left(\overline{V\sigma+\lambda}\frac{dz}{ds}\right)\right]\delta z\right\} = 0:$$

Energy equation of equilibrium.

whence finally

$$\frac{d}{ds}\left\{(V\sigma+\lambda)\frac{dx}{ds}\right\} + X\sigma = 0,$$

$$\frac{d}{ds}\left\{(V\sigma+\lambda)\frac{dy}{ds}\right\} + Y\sigma = 0,$$

$$\frac{d}{ds}\left\{(V\sigma+\lambda)\frac{dz}{ds}\right\} + Z\sigma = 0,$$

which, if T be put for $V\sigma+\lambda$, are the same as the equations (1) of § 578.

Common catenary.

580. The form of the common catenary (§ 574) may be of course investigated from the differential equations (§ 578) of the catenary in general. It is convenient and instructive, however, to work it out *ab initio* as an illustration of the third method explained in § 575.

Third method.—The chain being in equilibrium, *any* arc of it may be supposed to become rigid without disturbing the equilibrium. The only forces acting on this rigid body are the tensions at its ends, and its weight. These forces being three in number, must be in one plane (§ 564), and hence, since one of them is vertical, the whole curve lies in a vertical plane. In this plane let $x_0, z_0, s_0, x_1, z_1, s_1$, belong to the two ends of the arc which is supposed rigid, and T_0, T_1, the tensions at those points. Resolving horizontally we have

$$T_0\left(\frac{dx}{ds}\right)_0 = T_1\left(\frac{dx}{ds}\right)_1.$$

Catenary; common.

Hence $T\dfrac{dx}{ds}$ is constant throughout the curve. Resolving vertically we have

$$T_1\left(\frac{dz}{ds}\right)_1 - T_0\left(\frac{dz}{ds}\right)_0 = \sigma(s_1 - s_0),$$

the weight of unit of mass being now taken as the unit of force.

Hence if T_0 be the tension at the lowest point, where $\dfrac{dz}{ds}=0$, $s=0$, and T the tension at any point (x, z) of the curve, we have

$$T = T_0\frac{ds}{dx} = \sigma s\frac{ds}{dz} \quad\text{.............................(1).}$$

Hence

$$T_0\frac{d}{ds}\left(\frac{dz}{dx}\right) = \sigma,$$

or

$$T_0\frac{d^2z}{dx^2} = \sigma\frac{ds}{dx} = \sigma\sqrt{1+\left(\frac{dz}{dx}\right)^2} \quad\text{.................(2).}$$

Integrating we have

$$\log\left\{\frac{dz}{dx} + \sqrt{1+\left(\frac{dz}{dx}\right)^2}\right\} = \frac{\sigma}{T_0}x + C',$$

and the constant is zero if we take the origin so that $x=0$, when $\dfrac{dz}{dx}=0$, *i.e.*, where the chain is horizontal.

Hence

$$\frac{dz}{dx} + \sqrt{1+\left(\frac{dz}{dx}\right)^2} = \epsilon^{\frac{\sigma}{T_0}x} \quad\text{......................... (3),}$$

whence

$$\frac{dz}{dx} = \tfrac{1}{2}\left(\epsilon^{\frac{\sigma}{T_0}x} - \epsilon^{-\frac{\sigma}{T_0}x}\right);$$

and by integrating again

$$z + C'' = \frac{T_0}{2\sigma}\left(\epsilon^{\frac{\sigma}{T_0}x} + \epsilon^{-\frac{\sigma}{T_0}x}\right).$$

This may be written

$$z = \tfrac{1}{2}a\left(\epsilon^{\frac{x}{a}} + \epsilon^{-\frac{x}{a}}\right) \quad\text{......................... (4),}$$

the ordinary equation of the catenary, the axis of x being taken at a distance a or $\dfrac{T_0}{\sigma}$ below the horizontal element of the chain.

Catenary: common.

The co-ordinates of that element are therefore $x=0$, $z=\frac{T_0}{\sigma}=a$. The latter shows that

$$T_0=\sigma a,$$

or the tension at the lowest point of the chain (and therefore also the horizontal component of the tension throughout) is the weight of a length a of the chain.

Now, by (1), $T=T_0\frac{ds}{dx}=\sigma z$, by (4), and therefore the tension at any point is equal to the weight of a portion of the chain equal to the vertical ordinate at that point.

Relative kinetic problem

581. From § 576 it follows immediately that if a material particle of unit mass be carried along any catenary with a velocity, $\dot{s}$, equal to T, the numerical measure of the tension at any point, the force upon it by which this is done is in the same direction as the resultant of the applied force on the catenary at this point, and is equal to the amount of this force per unit of length, multiplied by T. For, denoting by S the tangential and (as before) by N the normal component of the applied force per unit of length at any point P of the catenary, we have, by § 576 (1), S for the rate of variation of $\dot{s}$ per unit length, and therefore $S\dot{s}$ for its variation per unit of time. That is to say,

Kinetic question relative to catenary.

$$\ddot{s}=S\dot{s}=ST,$$

or (§ 259) the tangential component force on the moving particle is equal to ST. Again, by § 576 (3),

$$NT=\frac{T^2}{\rho}=\frac{\dot{s}^2}{\rho},$$

or the centrifugal force of the moving particle in the circle of curvature of its path, that is to say, the normal component of the force on it, is equal to NT. And lastly, by (2) this force is in the same direction as N. We see therefore that the direction of the whole force on the moving particle is the same as that of the resultant of S and N; and its magnitude is T times the magnitude of this resultant.

Kinetic question relative to catenary.

Or, by taking

$$\frac{ds}{T}=dt,$$

in the differential equation of § 578, we have

$$\frac{d^2x}{dt^2}=-T\sigma X,\quad \frac{d^2y}{dt^2}=-T\sigma Y,\quad \frac{d^2z}{dt^2}=-T\sigma Z,$$

which proves the same conclusion.

When σ is constant, and the forces belong to a conservative system, if V be the potential at any point of the cord, we have, by § 578 (2), $\qquad T=\sigma V+C.$

Hence, if $U=\frac{1}{2}(\sigma V+C)^2$, these equations become

$$\frac{d^2x}{dt^2}=-\frac{dU}{dx},\quad \frac{d^2y}{dt^2}=-\frac{dU}{dy},\quad \frac{d^2z}{dt^2}=-\frac{dU}{dz}.$$

The integrals of these equations which agree with the catenary, are those only for which the energy constant is such that $\dot{s}^2=2U$.

582. Thus we see how, from the more familiar problems of the kinetics of a particle, we may immediately derive curious cases of catenaries. For instance: a particle under the influence of a constant force in parallel lines moves (Chap. VIII.) in a parabola with its axis vertical, with velocity at each point equal to that generated by the force acting through a space equal to its distance from the directrix. Hence, if z denote this distance, and f the constant force, Examples.

$$T=\sqrt{2fz}$$

in the allied parabolic catenary; and the force on the catenary is parallel to the axis, and is equal in amount per unit of length, to

$$\frac{f}{\sqrt{2fz}} \text{ or } \sqrt{\frac{f}{2z}}.$$

Hence if the force on the catenary be that of gravity, it must have its axis vertical (its vertex downwards of course for stable equilibrium) and its mass per unit length at any point must be inversely as the square root of the distance of this point above the directrix. From this it follows that the whole weight of any arc of it is proportional to its horizontal projection. Or, Examples. again, as will be proved later with reference to the motions of comets, a particle moves in a parabola under the influence of a force towards a fixed point varying inversely as the square of the distance from this point, if its velocity be that due to falling from rest at an infinite distance. This velocity being $\sqrt{\frac{2\mu}{r}}$, at distance r, it follows, according to § 581, that a cord will hang in the same parabola, under the influence of a force towards the same centre, and equal to

$$\frac{\mu}{r^2} \div \sqrt{\frac{2\mu}{r}}, \text{ or } \sqrt{\frac{\mu}{2r^3}}.$$

If, however, the length of the cord be varied between two fixed points, the central force still following the same law, the altered catenary will no longer be parabolic: but it will be the path of a particle under the influence of a central force equal to

$$\left(C + \sqrt{\frac{2\mu}{r}}\right)\sqrt{\frac{\mu}{2r^3}},$$

since (§ 581) we should have,

$$T = \sigma V + C = -\sigma\int\sqrt{\frac{\mu}{2r^3}}\,dr + C = \sigma\sqrt{\frac{2\mu}{r}} + C,$$

instead of $\sqrt{\frac{2\mu}{r}}$.

Catenary. Inverse problem. **583.** Or if the question be, to find what force towards a given fixed point, will cause a cord to hang in any given plane curve with this point in its plane; it may be answered immediately from the solution of the corresponding problem in "central forces."

But the general equations, § 578, are always easily applicable; as, for instance, to the following curious and interesting, but not practically useful, inverse case of the gravitation catenary :—

Catenary of uniform strength. *Find the section, at each point, of a chain of uniform material, so that when its ends are fixed the tension at each point may be proportional to its section at that point. Find also the form of the Curve, called the Catenary of Uniform Strength, in which it will hang.*

Here, as the only external force is gravity, the chain is in a vertical plane—in which we may assume the horizontal axis of x to lie. If μ be the weight of the chain at the point (x, z) reckoned per unit of length; our equations [§ 578 (1)] become Catenary of uniform strength.

$$\frac{d}{ds}\left(T\frac{dx}{ds}\right) = 0, \quad \frac{d}{ds}\left(T\frac{dz}{ds}\right) = \mu.$$

But, by hypothesis $T \propto \mu$. Let it be $b\mu$. Hence, by the first equation, if μ_0 be the value of μ at the lowest point

$$\mu = \mu_0\frac{ds}{dx};$$

whence, by the second equation,

$$\frac{d}{ds}\left(\frac{dz}{dx}\right) = \frac{1}{b}\,\frac{ds}{dx},$$

or

$$\frac{d^2z}{dx^2} = \frac{1}{b}\left[1 + \left(\frac{dz}{dx}\right)^2\right].$$

Integrating we find

$$\tan^{-1}\frac{dz}{dx} = \frac{x}{b},$$

no constant being required if we take the axis of x so as to touch the curve at its lowest point. Integrating again we have

$$\frac{z}{b} = -\log\cos\frac{x}{b},$$

no constant being added, if the origin be taken at the lowest point. We may write the equation in the form

$$\sec\frac{x}{b} = \epsilon^{\frac{z}{b}}.$$

From this form of the equation we see that the curve has vertical asymptotes at a horizontal distance πb from each other. Hence πb is the greatest possible span, if the ends are on the same level, or the horizontal projection of the greatest possible span if they be not on the same level; b denoting the length of a uniform rod or wire of the material equal in weight to the tension of the catenary at any point, and equal in sectional area to the sectional area of the catenary at the same point. The greatest possible value of b is the "length modulus of rupture" (§§ 687, 688 below).

Flexible string on smooth surface.

584. When a perfectly flexible string is stretched over a smooth surface, and acted on by no other force throughout its length than the resistance of this surface, it will, when in stable equilibrium, lie along a line of minimum length on the surface, between any two of its points. For (§ 564) its equilibrium can be neither disturbed nor rendered unstable by placing staples over it, through which it is free to slip, at any two points where it rests on the surface: and for the intermediate part the energy criterion of stable equilibrium is that just stated.

There being no tangential force on the string in this case, and the normal force upon it being along the normal to the surface, its osculating plane (§ 576) must cut the surface everywhere at right angles. These considerations, easily translated into pure geometry, establish the fundamental property of the geodetic lines on any surface. The analytical investigations of §§ 578, 579, when adapted to the case of a chain of *not* given length, stretched between two given points on a given smooth surface, constitute the direct analytical demonstration of this property.

In this case it is obvious that the tension of the string is the same at every point, and the pressure of the surface upon it is [§ 576 (3)] at each point proportional to the curvature of the string.

On rough surface.

585. No real surface being perfectly smooth, a cord or chain may rest upon it when stretched over so great a length of a geodetic on a convex rigid body as to be not of minimum length between its extreme points: but practically, as in tying a cord round a ball, for permanent security it is necessary, by staples or otherwise, to constrain it from lateral slipping at successive points near enough to one another to make each free portion a true minimum on the surface.

Rope coiled about rough cylinder.

586. A very important practical case is supplied by the consideration of a rope wound round a rough cylinder. We may suppose it to lie in a plane perpendicular to the axis, as we thus simplify the question very considerably without sensibly

injuring the utility of the solution. To simplify still further, we shall suppose that no forces act on the rope, but tensions and the reaction of the cylinder. In practice this is equivalent to the supposition that the tensions and reactions are very large compared with the weight of the rope or chain; which, however, is inadmissible in some important cases; especially such as occur in the application of the principle to brakes for laying submarine cables, to dynamometers, and to windlasses (or capstans with horizontal axes).

Rope coiled about rough cylinder.

If R be the normal reaction of the cylinder per unit of length of the cord, at any point; T and $T+\delta T$ the tensions at the extremities of an arc δs; $\delta\theta$ the inclination of these lines; we have, as in § 576,

$$T\delta\theta = R\delta s.$$

And the friction called into play is evidently equal to δT. When the rope is about to slip, the friction has its greatest value, and then

$$\delta T = \mu R\delta s = \mu T\delta\theta.$$

This gives, by integration,

$$T = T_0 \epsilon^{\mu\theta},$$

showing that, for equal successive amounts of integral curvature (§ 10), the tension of the rope augments in *geometrical* progression. To give an idea of the magnitudes involved, suppose $\mu = 0{\cdot}25$, $\theta = 2\pi$, then

$$T = T_0 \epsilon^{\cdot 5\pi} = 4{\cdot}81 T_0 \text{ approximately.}$$

Hence if the rope be wound three times round the post or cylinder the ratio of the tensions of its ends, when motion is about to commence, is

$$(4{\cdot}81)^3 : 1 \text{ or about } 111 : 1.$$

Thus we see how, by the aid of friction, one man may easily check the motion of a large ship, by the simple expedient of coiling a rope a few times round a post. This application of friction is of great importance in many other uses, especially for dynamometers.

Rope coiled about rough cylinder.

587. With the aid of the preceding investigations, the student may easily work out for himself the formulæ expressing the solution of the general problem of a cord under the action of any forces, and constrained by a rough surface; they are not of sufficient importance or interest to find a place here.

Elastic wire.

Elastic wire, fibre, bar, rod, lamina, or beam.

588. An elongated body of elastic material, which for brevity we shall generally call a *Wire*, bent or twisted to any degree, subject only to the condition that the radius of curvature and the reciprocal of the twist (§ 119) are everywhere very great in comparison with the greatest transverse dimension, presents a case in which, as we shall see, the solution of the general equations for the equilibrium of an elastic solid is either obtainable in finite terms, or is reducible to comparatively easy questions agreeing in mathematical conditions with some of the most elementary problems of hydrokinetics, electricity, and thermal conduction. And it is only for the determination of certain constants depending on the section of the wire and the elastic quality of its substance, which measure its flexural and torsional rigidity, that the solutions of these problems are required. When the constants of flexure and torsion are known, as we shall now suppose them to be, whether from theoretical calculation or experiment, the investigation of the form and twist of any length of the wire, under the influence of any forces which do not produce a violation of the condition stated above, becomes a subject of mathematical analysis involving only such principles and formulæ as those that constitute the theory of curvature (§§ 5—13) and twist (§§ 119—123) in geometry or kinematics.

589. Before entering on the general theory of elastic solids, we shall therefore, according to the plan proposed in § 573, examine the dynamic properties and investigate the conditions of equilibrium of a perfectly elastic wire, without admitting any other condition or limitation of the circumstances than what is stated in § 588, and without assuming any special quality of isotropy, or of crystalline, fibrous or laminated structure in the substance. The following short geometrical digression is a convenient preliminary :—

Composition and resolution of curvatures in a curved line.

590. The geometrical composition of curvatures with one another, or with rates of twist, is obvious from the definition and principles regarding curvature given above in §§ 5—13 and twist in §§ 119—123, and from the composition of angular velocities explained in § 96. Thus if one line, $\mathfrak{OT}$, of a rigid body be always held parallel to the tangent, PT, at a point P moving with unit velocity along a curve, whether plane or tortuous, it will have, round an axis perpendicular to $\mathfrak{OT}$ and to the radius of curvature (that is to say, perpendicular to the osculating plane), an angular velocity numerically equal to the curvature. The body may besides be made to rotate with any angular velocity round $\mathfrak{OT}$. Thus, for instance, if a line of it, $\mathfrak{OA}$, be kept always parallel to a transverse (§ 120) PA, the component angular velocity of the rigid body round $\mathfrak{OT}$ will at every instant be equal to the "rate of twist" (§ 120) of the transverse round the tangent to the curve. Again, the angular velocity round $\mathfrak{OA}$ may be resolved into components round two lines $\mathfrak{OK}$, $\mathfrak{OL}$, perpendicular to one another and to $\mathfrak{OT}$; and the whole curvature of the curve may be resolved accordingly into two component curvatures in planes perpendicular to those two lines respectively. The amounts of these component curvatures are of course equal to the whole curvature multiplied by the cosines of the respective inclinations of the osculating plane to these planes. And it is clear that each component curvature is simply the curvature of the projection of the actual curve on its plane*.

591. Besides showing how the constants of flexural and torsional rigidity are to be determined theoretically from the form of the transverse section of the wire, and the proper data as to the elastic qualities of its substance, the complete theory simply indicates that, provided the conditional limit (§ 588) of deformation is not exceeded, the following laws will be obeyed by the wire under stress :—

* The curvature of the projection of a curve on a plane inclined at an angle α to the osculating plane, is $(1/\rho)\cos\alpha$ if the plane be parallel to the tangent; and $1/\rho\cos^2\alpha$ if it be parallel to the principal normal (or radius of absolute curvature). There is no difficulty in proving either of these expressions.

Laws of flexure and torsion.

Let the whole mutual action between the parts of the wire on the two sides of the cross section at any point (being of course the action of the matter infinitely near this plane on one side, upon the matter infinitely near it on the other side), be reduced to a single force through any point of the section and a single couple. Then—

I. The twist and curvature of the wire in the neighbourhood of this section are independent of the force, and depend solely on the couple.

II. The curvatures and rates of twist producible by any several couples separately, constitute, if geometrically compounded, the curvature and rate of twist which are actually produced by a mutual action equal to the resultant of those couples.

592. It may be added, although not necessary for our present purpose, that there is one determinate point in the cross section such that if it be chosen as the point to which the forces are transferred, a higher order of approximation is obtained for the fulfilment of these laws than if any other point of the section be taken. That point, which in the case of a wire of substance uniform through its cross section is the centre of inertia of the area of the section, we shall generally call the elastic centre, or the centre of elasticity, of the section. It has also the following important property:—The line of elastic centres, or, as we shall call it, the elastic central line, remains sensibly unchanged in length to whatever stress within our conditional limits (§ 588) the wire be subjected. The elongation or contraction produced by the neglected resultant force, if this is in such a direction as to produce any, will cause the line of *rigorously no elongation* to deviate only infinitesimally from the elastic central line, in any part of the wire finitely curved. It will, however, clearly cause there to be no line of *rigorously unchanged length*, in any straight part of the wire: but as the whole elongation would be infinitesimal in comparision with the effective actions with which we are concerned, this case constitutes no exception to the preceding statement.

Warping of normal section by torsion and flexure, infinitesimal.

593. Considering now a wire of uniform constitution and figure throughout, and naturally straight; let any two planes of reference perpendicular to one another through its elastic central line when straight, cut the normal section through P in the lines PK and PL. These two lines (supposed to belong to the substance, and move with it) will remain infinitely nearly at right angles to one another, and to the tangent, PT, to the central line, however the wire may be bent or twisted within the conditional limits. Let κ and λ be the component curvatures (§ 590) in the two planes perpendicular to PK and PL through PT, and let τ be the twist (§ 120) of the wire at P. We have just seen (§ 590) that if P be moved at a unit rate along the curve, a rigid body with three rectangular axes of reference $\mathfrak{OK}$, $\mathfrak{OL}$, $\mathfrak{OT}$ kept always parallel to PK, PL, PT, will have angular velocities κ, λ, τ round those axes respectively. Hence if the point P and the lines PT, PK, PL be at rest while the wire is bent and twisted from its unstrained to its actual condition, the lines of reference $P'K'$, $P'L'$, $P'T'$ through any point P' infinitely near P, will experience a rotation compounded of $\kappa . PP'$ round $P'K'$, $\lambda . PP'$ round $P'L'$, and $\tau . PP'$ round $P'T'$.

Rotations corresponding to flexure and torsion.

Potential energy of elastic force in bent and twisted wire.

594. Considering now the elastic forces called into action, we see that if these constitute a conservative system, the work required to bend and twist any part of the wire from its unstrained to its actual condition, depends solely on its figure in these two conditions. Hence if $w . PP'$ denote the amount of this work, for the infinitely small length PP' of the rod, w must be a function of κ, λ, τ; and therefore if K, L, T denote the components of the couple-resultant of all the forces which must act on the section through P' to hold the part PP' in its strained state, it follows, from §§ 240, 272, 274, that

$$K\delta\kappa = \delta_\kappa w,\ L\delta\lambda = \delta_\lambda w,\ T\delta\tau = \delta_\tau w \ \ldots\ldots\ldots\ldots (1),$$

where $\delta_\kappa w$, $\delta_\lambda w$, $\delta_\tau w$ denote the augmentations of w due respectively to infinitely small augmentations $\delta\kappa$, $\delta\lambda$, $\delta\tau$, of κ, λ, τ.

595. Now however much the shape of any finite length of the wire may be changed, the condition of § 588 requires

Potential energy of elastic force in bent and twisted wire.

clearly that the changes of shape in each infinitely small part, that is to say, the strain (§ 154) of the substance, shall be everywhere very small (infinitely small in order that the theory may be rigorously applicable). Hence the principle of superposition [§ 591, II.] shows that if κ, λ, τ be each increased or diminished in one ratio, K, L, T will be each increased or diminished in the same ratio: and consequently w in the duplicate ratio, since the angle through which each couple acts is altered in the same ratio as the amount of the couple; or, in algebraic language, w is a homogeneous quadratic function of κ, λ, τ.

Thus if A, B, C, a, b, c denote six constants, we have

$$w = \tfrac{1}{2}(A\kappa^2 + B\lambda^2 + C\tau^2 + 2a\lambda\tau + 2b\tau\kappa + 2c\kappa\lambda) \;\ldots\ldots\ldots\ldots(2).$$

Hence, by § 594 (1),

Components of restituent couple.

$$\left.\begin{aligned} K &= A\kappa + c\lambda + b\tau \\ L &= c\kappa + B\lambda + a\tau \\ T &= b\kappa + a\lambda + C\tau \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots(3).$$

By the known reduction of the homogeneous quadratic function, these expressions may of course be reduced to the following simple forms:—

$$\left.\begin{aligned} &w = \tfrac{1}{2}(A_1\vartheta_1^2 + A_2\vartheta_2^2 + A_3\vartheta_3^2) \\ &L_1 = A_1\vartheta_1, \quad L_2 = A_2\vartheta_2, \quad L_3 = A_3\vartheta_3 \end{aligned}\right\} \ldots\ldots\ldots(4),$$

where ϑ_1, ϑ_2, ϑ_3 are linear functions of κ, λ, τ. And if these functions are restricted to being the expressions for the components round three rectangular axes, of the rotations κ, λ, τ viewed as angular velocities round the axes PK, PL, PT, the positions of the new axes, PQ_1, PQ_2, PQ_3, and the values of A_1, A_2, A_3 are determinate; the latter being the roots of the determinant cubic [§ 181 (11)] founded on (A, B, C, a, b, c). Hence we conclude that

Three principal or normal axes of torsion and flexure.

596. There are in general three determinate rectangular directions, PQ_1, PQ_2, PQ_3, through any point P of the middle line of a wire, such that if opposite couples be applied to any two parts of the wire in planes perpendicular to any one of them, every intermediate part will experience rotation in a plane parallel to those of the balanced couples. The moments

Three principal torsion-flexure rigidities.

of the couples required to produce unit rate of rotation round these three axes are called the *principal torsion-flexure* rigidities of the wire. They are the elements denoted by A_1, A_2, A_3 in the preceding analysis.

597. If the rigid body imagined in § 593 have moments of inertia equal to A_1, A_2, A_3 round three principal axes through ⊕ kept always parallel to the principal torsion-flexure axes through P, while P moves at unit rate along the wire, its moment of momentum round any axis (§§ 281, 236) will be equal to the moment of the component torsion-flexure couple round the parallel axis through P.

Three principal or normal spirals.

598. The form assumed by the wire when balanced under the influence of couples round one of the three principal axes is of course a uniform helix having a line parallel to it for axis, and lying on a cylinder whose radius is determined by the condition that the whole rotation of one end of the wire from its unstrained position, the other end being held fixed, is equal to the amount due to the couple applied.

Let l be the length of the wire from one end, E, held fixed, to the other end, E', where a couple, L, is applied in a plane perpendicular to the principal axis PQ_1 through any point of the wire. The rotation being [§ 595 (4)] at the rate $\dfrac{L}{A_1}$, per unit of length, amounts on the whole to $l\dfrac{L}{A_1}$. This therefore is the angular space occupied by the helix on the cylinder on which it lies. Hence if r denote the radius of this cylinder, and i_1 the inclination of the helix to its axis (being the inclination of PQ_1 to the length of the wire), we have

$$r\frac{Ll}{A_1} = l \sin i_1;$$

whence

$$r = \frac{A_1 \sin i_1}{L} \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5)$$

Case in which elastic central line is a normal axis of torsion.

599. In the most important practical cases, as we shall see later, those namely in which the substance is either "isotropic," as is the case sensibly with common metallic wires, or, as in rods or beams of fibrous or crystalline structure, with an axis of elastic symmetry along the length of the piece, one of the three normal axes of torsion and flexure coincides with the length of the wire, and the two others are perpendicular to it; the first being an axis of pure torsion, and the two others axes of pure flexure. Thus opposing couples round the axis of the wire twist it simply without bending it; and opposing couples in either of the two principal planes of flexure, bend it into a circle. The unbent straight line of the wire, and the circular arcs into which it is bent by couples in the two principal planes of flexure, are what the three principal spirals of the general problem become in this case.

A simple proof that the twist must be uniform (§ 123) is found by supposing the whole wire to turn round its curved axis; and remarking that the work done by a couple at one end must be equal to that undone at the other.

Case of equal flexibility in all directions.

600. In the more particular case in which two principal rigidities against flexure are equal, every plane through the length of the wire is a principal plane of flexure, and the rigidity against flexure is equal in all. This is clearly the case with a common round wire, or rod: or with one of square section. It will be shown later to be the case for a rod of isotropic material and of any form of normal section which is "kinetically symmetrical," § 285, round all axes in its plane through its centre of inertia.

601. In this case, if one end of the rod or wire be held fixed, and a couple be applied in any plane to the other end, a uniform spiral (or helical) form will be produced round an axis perpendicular to the plane of the couple. The lines of the substance parallel to the axis of the spiral are not, however, parallel to their original positions, as (§ 598) in each of the three principal spirals of the general problem: and lines traced along the surface of the wire parallel to its length when straight, become as it were secondary spirals, circling

round the main spiral formed by the central line of the deformed wire; instead of being all spirals of equal step, as in each one of the principal spirals of the general problem. Lastly, in the present case, if we suppose the normal section of the wire to be circular, and trace uniform spirals along its surface when deformed in the manner supposed (two of which, for instance, are the lines along which it is touched by the inscribed and the circumscribed cylinder), these lines do not become straight, but become spirals laid on as it were round the wire, when it is allowed to take its natural straight and untwisted condition.

Case of equal flexibility in all directions.

Let, in § 595, PQ_1 coincide with the central line of the wire, and let $A_1 = A$, and $A_2 = A_3 = B$; so that A measures the rigidity of torsion and B that of flexure. One end of the wire being held fixed, let a couple G be applied to the other end, round an axis inclined at an angle θ to the length. The rates of twist and of flexure each per unit of length, according to (4) of § 595, will be

$$\frac{G\cos\theta}{A}, \text{ and } \frac{G\sin\theta}{B},$$

respectively. The latter being (§ 9) the same thing as the curvature, and the inclination of the spiral to its axis being θ, it follows (§ 126, or § 590, footnote) that $\frac{B\sin\theta}{G}$ is the radius of curvature of its projection on a plane perpendicular to this line, that is to say, the radius of the cylinder on which the spiral lies.

Wire strained to any given spiral and twist.

602. A wire of equal flexibility in all directions may clearly be held in any specified spiral form, and twisted to any stated degree, by a determinate force and couple applied at one end, the other end being held fixed. The direction of the force must be parallel to the axis of the spiral, and, with the couple, must constitute a system of which this line is (§ 559) the *central axis:* since otherwise there could not be the same system of balancing forces in every normal section of the spiral. All this may be seen clearly by supposing the wire to be first brought by any means to the specified condition of strain; then to have rigid planes rigidly attached to its two ends perpendicular to its axis, and these planes to be rigidly

Wire strained to any given spiral and twist.

connected by a bar lying in this line. The spiral wire now left to itself cannot but be in equilibrium: although if it be too long (according to its form and degree of twist) the equilibrium may be unstable. The force along the central axis, and the couple, are to be determined by the condition that, when the force is transferred after Poinsot's manner to the elastic centre of any normal section, they give two couples together equivalent to the elastic couples of flexure and torsion.

Let α be the inclination of the spiral to the plane perpendicular to its axis; r the radius of the cylinder on which it lies; τ the rate of twist given to the wire in its spiral form. The curvature is (§ 126) equal to $\frac{\cos^2 \alpha}{r}$; and its plane, at any point of the spiral, being the plane of the tangent to the spiral and the diameter of the cylinder through that point, is inclined at the angle α to the plane perpendicular to the axis. Hence the components in this plane, and in the plane through the axis of the cylinder of the flexural couple, are respectively

$$\frac{B\cos^2\alpha}{r}\cos\alpha, \quad\text{and}\quad \frac{B\cos^2\alpha}{r}\sin\alpha.$$

Also, the components of the torsional couple, in the same planes, are $\qquad A\tau\sin\alpha,\ $ and $\ -A\tau\cos\alpha.$

Hence, for equilibrium,

$$\left.\begin{aligned} G &= \frac{B\cos^2\alpha}{r}\cos\alpha + A\tau\sin\alpha \\ -Rr &= \frac{B\cos^2\alpha}{r}\sin\alpha - A\tau\cos\alpha \end{aligned}\right\}\ldots\ldots\ldots(6),$$

which give explicitly the values, G and R, of the couple and force required, the latter being reckoned as positive when its direction is such as to pull *out* the spiral, or when the ends of the rigid bar supposed above are pressed *inwards* by the plates attached to the ends of the spiral.

If we make $R=0$, we fall back on the case considered previously (§ 601). If, on the other hand, we make $G=0$, we have

$$\tau = -\frac{1}{r}\frac{B}{A}\frac{\cos^3\alpha}{\sin\alpha},$$

and
$$R = -\frac{B}{r^2}\frac{\cos^2\alpha}{\sin\alpha} = \frac{A\tau}{r\cos\alpha},$$

from which we conclude that

603. A wire of equal flexibility in all directions may be held in any stated spiral form by a simple force along its axis between rigid pieces rigidly attached to its two ends, provided that, along with its spiral form, a certain degree of twist be given to it. The force is determined by the condition that its moment round the perpendicular through any point of the spiral to its osculating plane at that point, must be equal and opposite to the elastic unbending couple. The degree of twist is that due (by the simple equation of torsion) to the moment of the force thus determined, round the tangent at any point of the spiral. The direction of the force being, according to the preceding condition, such as to press together the ends of the spiral, the direction of the twist in the wire is opposite to that of the tortuosity (§ 9) of its central curve.

Twist determined for reducing the action to a single force.

604. The principles and formulæ (§§ 598, 603) with which we have just been occupied are immediately applicable to the theory of spiral springs; and we shall therefore make a short digression on this curious and important practical subject before completing our investigation of elastic curves.

Spiral springs.

A common spiral spring consists of a uniform wire shaped permanently to have, when unstrained, the form of a regular helix, with the principal axes of flexure and torsion everywhere similarly situated relatively to the curve. When used in the proper manner, it is acted on, through arms or plates rigidly attached to its ends, by forces such that its form as altered by them is still a regular helix. This condition is obviously fulfilled if (one terminal being held fixed) an infinitely small force and infinitely small couple be applied to the other terminal along the axis and in a plane perpendicular to it, and if the force and couple be increased to any degree, and always kept along and in the plane perpendicular to the axis of the altered spiral. It would, however, introduce useless complication to work out the details of the problem except for the case (§ 599) in which one of the principal axes coincides with the tangent to the central line, and is therefore an axis of pure torsion; as spiral springs in practice always belong to this case. On the other hand, a very interesting complication occurs if we suppose (a thing easily

Spiral springs.

realized in practice, though to be avoided if merely a good spring is desired) the normal section of the wire to be of such a figure, and so situated relatively to the spiral, that the planes of greatest and least flexural rigidity are oblique to the tangent plane of the cylinder. Such a spring when acted on in the regular manner at its ends must experience a certain degree of turning through its whole length round its elastic central curve in order that the flexural couple developed may be, as we shall immediately see it must be, precisely in the osculating plane of the altered spiral. But all that is interesting in this very curious effect will be illustrated later (§ 624) in full detail in the case of an open circular arc altered by a couple in its own plane, into a circular arc of greater or less radius; and for brevity and simplicity we shall confine the detailed investigation of spiral springs on which we now enter, to the cases in which either the wire is of equal flexural rigidity in all directions, or the two principal planes of (greatest and least or least and greatest) flexural rigidity coincide respectively with the tangent plane to the cylinder, and the normal plane touching the central curve of the wire, at any point.

605. The axial force, on the moveable terminal of the spring, transferred according to Poinsot's method (§ 555) to any point in the elastic central curve, gives a couple in the plane through that point and the axis of the spiral. The resultant of this and the couple which we suppose applied to the terminal in the plane perpendicular to the axis of the spiral is the effective bending and twisting couple: and as it is in a plane perpendicular to the tangent plane to the cylinder, the component of it to which bending is due must be also perpendicular to this plane, and therefore is in the osculating plane of the spiral. This component couple therefore simply maintains a curvature different from the natural curvature of the wire, and the other, that is, the couple in the plane normal to the central curve, pure torsion. The equations of equilibrium merely express this in mathematical language.

Resolving as before (§ 602) the flexural and the torsional couples each into components in the planes through the axis of

Spiral springs.

the spiral, and perpendicular to it, we have

$$\left.\begin{aligned} G &= B\left(\frac{\cos^2\alpha}{r} - \frac{\cos^2\alpha_0}{r_0}\right)\cos\alpha' + A\tau\sin\alpha', \\ -Rr &= B\left(\frac{\cos^2\alpha}{r} - \frac{\cos^2\alpha_0}{r_0}\right)\sin\alpha' - A\tau\cos\alpha', \\ \text{and, by § 126, } \tau &= \frac{\cos\alpha\sin\alpha}{r} - \frac{\cos\alpha_0\sin\alpha_0}{r_0}, \end{aligned}\right\}\ldots(7),$$

where A denotes the torsional rigidity of the wire, and B its flexural rigidity in the osculating plane of the spiral; α_0 the inclination, and r_0 the radius of the cylinder, of the spiral when unstrained; α and r the same parameters of the spiral when under the influence of the axial force R and couple G; and τ the degree of twist in the change from the unstrained to the strained condition.

These equations give explicitly the force and couple required to produce any stated change in the spiral; or if the force and couple are given they determine α', r' the parameters of the altered curve.

As it is chiefly the external action of the spring that we are concerned with in practical applications, let the parameters α, r of the spiral be eliminated by the following assumptions :—

$$\left.\begin{aligned} x &= l\sin\alpha, \quad \phi = \frac{l\cos\alpha}{r} \\ x_0 &= l\sin\alpha_0, \quad \phi_0 = \frac{l\cos\alpha_0}{r_0} \end{aligned}\right\}\ldots\ldots\ldots\ldots\ (8),$$

where l denotes the length of the wire, ϕ the angle between planes through the two ends of the spiral, and its axis, and x the distance between planes through the ends and perpendicular to the axis in the strained condition; and, similarly, ϕ_0, x_0 for the unstrained condition; so that we may regard (ϕ, x) and (ϕ_0, x_0) as the co-ordinates of the movable terminal relatively to the fixed in the two conditions of the spring. Thus the preceding equations become

$$\left.\begin{aligned} L &= \frac{B}{l^3}\{\sqrt{(l^2-x^2)}\,\phi - \sqrt{(l^2-x_0^2)}\,\phi_0\}\sqrt{(l^2-x^2)} + \frac{A}{l^3}(x\phi - x_0\phi_0)\,x \\ R &= -\frac{B}{l^3}\{\sqrt{(l^2-x^2)}\,\phi - \sqrt{(l^2-x_0^2)}\,\phi_0\}\frac{x\phi}{\sqrt{(l^2-x^2)}} + \frac{A}{l^3}(x\phi - x_0\phi_0)\,\phi \end{aligned}\right\}(9).$$

Spiral springs.

Here we see that $Ld\phi + Rdx$ is the differential of a function of the two independent variables, x, ϕ. Thus if we denote this function by E, we have

$$\left.\begin{array}{c} E = \frac{1}{2}\frac{B}{l^3}\{\sqrt{(l^2-x^2)}\,\phi - \sqrt{(l^2-x_0^{\,2})}\,\phi_0\}^2 + \frac{1}{2}\frac{A}{l^3}(x\phi - x_0\phi_0)^2 \\ L = \frac{dE}{d\phi}, \quad R = \frac{dE}{dx} \end{array}\right\} \quad (10),$$

a conclusion which might have been inferred at once from the general principle of energy, thus :—

606. The potential energy of the strained spring is easily seen from § 595 (4), above, to be

$$\tfrac{1}{2}[B(\varpi - \varpi_0)^2 + A\tau^2]l,$$

if A denote the torsional rigidity, B the flexural rigidity in the plane of curvature, ϖ and ϖ_0 the strained and unstrained curvatures, and τ the torsion of the wire in the strained condition, the torsion being reckoned as zero in the unstrained condition. The axial force, and the couple, required to hold the spring to any given length reckoned along the axis of the spiral, and to any given angle between planes through its ends and the axes, are of course (§ 272) equal to the rates of variation of the potential energy, per unit of variation of these co-ordinates respectively. It must be carefully remarked, however, that, if the terminal rigidly attached to one end of the spring be held fast so as to fix the tangent at this end, and the motion of the other terminal be so regulated as to keep the figure of the intermediate spring always truly spiral, this motion will be somewhat complicated; as the radius of the cylinder, the inclination of the axis of the spiral to the fixed direction of the tangent at the fixed end, and the position of the point in the axis in which it is cut by the plane perpendicular to it through the fixed end of the spring, all vary as the spring changes in figure. The *effective components* of any infinitely small motion of the moveable terminal are its component translation along, and rotation round, the instantaneous position of the axis of the spiral (two degrees of freedom), along with which it will generally have an infinitely small translation in some direction

and rotation round some line, each perpendicular to this axis, to be determined from the two degrees of arbitrary motion, by the condition that the curve remains a true spiral.

Spiral springs.

607. In the practical use of spiral springs, this condition is not rigorously fulfilled: but, instead, either of two plans is generally followed:—(1) Force, without any couple, is applied pulling out or pressing together two definite points of the two terminals, each as nearly as may be in the axis of the unstrained spiral; or (2) One terminal being held fixed, the other is allowed to slide, without any turning, in a fixed direction, being as nearly as may be the direction of the axis of the spiral when unstrained. The preceding investigation is applicable to the infinitely small displacement in either case: the couple being put equal to zero for case (1), and the instantaneous rotatory motion round the axis of the spiral equal to zero for case (2).

For infinitely small displacements let $\phi = \phi_0 + \delta\phi$, and $x = x_0 + \delta x$, in (10), so that now

$$L = \frac{dE}{d\delta\phi}, \quad R = \frac{dE}{d\delta x}.$$

Then, retaining only terms of the lowest degree relative to δx and $\delta\phi$ in each formula, and writing x and ϕ instead of x_0 and ϕ_0, we have

$$\left.\begin{array}{r} E = \frac{1}{2l^3}\left\{\left(B\frac{x^2}{l^2-x^2}+A\right)\phi^2\delta x^2 + 2(A-B)x\phi\delta x\delta\phi + [B(l^2-x^2)+Ax^2]\delta\phi^2\right\} \\ R = \frac{1}{l^3}\left\{\left(B\frac{x^2}{l^2-x^2}+A\right)\phi^2\delta x + (A-B)x\phi\delta\phi\right\} \\ L = \frac{1}{l^3}\{(A-B)x\phi\delta x + [B(l^2-x^2)+Ax^2]\delta\phi\} \end{array}\right\} (11).$$

Example 1.—For a spiral of 45° inclination we have

$$x^2 = \tfrac{1}{2}l^2 \text{ and } \phi^2 = \tfrac{1}{2}\frac{l^2}{r^2}:$$

and the formulæ become

$$\left.\begin{array}{l} R = \frac{1}{2}\frac{1}{lr^2}[(A+B)\,\delta x + (A-B)\,r\delta\phi] \\ L = \frac{1}{2}\frac{1}{lr}[(A-B)\,\delta x + (A+B)\,r\delta\phi] \end{array}\right\} \ldots\ldots\ldots (12).$$

Spiral springs.

A careful study of this case, illustrated if necessary by a model easily made out of ordinary iron or steel wire, will be found very instructive.

Spiral spring of infinitely small inclination:

Example 2.—Let $\frac{x}{l}$ be very small. Neglecting, therefore, its square, we have $\phi = \frac{l}{r}$, and $L = \frac{B}{l}\delta\phi = B\delta\frac{1}{r}$; and $R = \frac{A}{lr^2}\delta x$. The first of these is simply the equation of direct flexure (§ 595). The interpretation of the second is as follows:—

608. In a spiral spring of infinitely small inclination to the plane perpendicular to its axis, the displacement produced in the moveable terminal by a force applied to it in the axis of the spiral is a simple rectilineal translation in the direction of the axis, and is equal to the length of the circular arc through which an equal force carries one end of a rigid arm or crank equal in length to the radius of the cylinder, attached perpendicularly to one end of the wire of the spring supposed straightened and held with the other end absolutely fixed, and the end which bears the crank free to turn in a collar. This statement is due to J. Thomson*, who showed that in pulling out a spiral spring of infinitely small inclination the action exercised and the elastic quality used are the same as in a torsion-balance with the same wire straightened (§ 433). This theory is, as he proved experimentally, sufficiently approximate for most practical applications; spiral springs, as commonly made and used, being of very small inclination. There is no difficulty in finding the requisite correction, for the actual inclination in any case, from the preceding formulæ. The fundamental principle that spiral springs act chiefly by torsion seems to have been first discovered by Binet in 1814†.

virtually a torsion-balance.

Elastic curve transmitting force and couple.

609. In continuation of §§ 590, 593, 597, we now return to the case of a uniform wire straight and untwisted (that is, cylindrical or prismatic) when free from stress. Let us suppose one end to be held fixed in a given direction, and no force from without to influence the wire except that transmitted to it by a rigid frame attached to its other end and acted on by a

* *Camb. and Dub. Math. Jour.* 1848.

† St Venant, *Comptes Rendus.* Sept. 1864.

force, R, in a given line, AB, and a couple, G, in a plane perpendicular to this line. The form and twist it will have when in equilibrium are determined by the condition that the torsion and flexure at any point, P, of its length are those due to the couple G compounded with the couple obtained by bringing R to P. It follows that the rigid body of § 597 will move exactly as there specified if it be set in motion with the proper angular velocity, and, $\mathfrak{O}$ being held fixed, a force equal and parallel to R be applied at a point $\mathfrak{Q}$, fixed relatively to the body at unit distance from $\mathfrak{O}$, in the line $\mathfrak{O}\mathfrak{T}$.

Kirchhoff's kinetic comparison.

This beautiful theorem was discovered by Kirchhoff; to whom also the first thoroughly general investigation of the equations of equilibrium and motion of an elastic wire is due*.

To prove the theorem, it is only necessary to remark that the rate of change of the moment of R round any line through P, kept parallel to itself as P moves along the curve, in the elastic problem, is equal simply to the moment round the parallel line through $\mathfrak{O}$, of R at $\mathfrak{Q}$ in the kinetic analogue. It may be added that G of the elastic problem corresponds to the constant moment of momentum round the line through $\mathfrak{O}$ parallel to the constant direction of R in the kinetic analogue.

610. The comparison thus established between the static problem of the bending and twisting of a wire, and the kinetic problem of the rotation of a rigid body, affords highly interesting illustrations, and, as it were, graphic representations, of the circumstances of either by aid of the other; the usefulness of which in promoting a thorough mental appropriation of both must be felt by every student who values rather the physical subject than the mechanical process of working through mathematical expressions, to which so many minds able for better things in science have unhappily been devoted of late years.

When particularly occupied with the kinetic problem in chap. IX., we shall have occasion to examine the rotations corresponding to the spirals of §§ 601—603, and to point out also the general character of the elastic curves corresponding to some of the less simple cases of rotatory motion.

* *Crelle's Journal*, 1859, Ueber das Gleichgewicht und die Bewegung eines unendlich dünnen elastischen Stabes.

Common pendulum and plane elastic curve.

611. For the present we confine ourselves to one example, which, so far as the comparison between the static and kinetic problems is concerned, is the simplest of all—the *Elastic Curve* of James Bernoulli, and the common pendulum. A uniform straight wire, either equally flexible in all planes through its length, or having its directions of maximum and minimum flexural rigidity in two planes through its whole length, is acted on by a force and couple in one of these planes, applied either directly to one end, or by means of an arm rigidly attached to it, the other end being held fast. The force and couple may, of course (§ 558), be reduced to a single force, the extreme case of a couple being mathematically included as an infinitely small force at an infinitely great distance. To avoid any restriction of the problem, we must suppose this force applied to an arm rigidly attached to the wire, although in any case in which the line of the force cuts the wire, the force may be applied directly at the point of intersection, without altering the circumstances of the wire between this point and the fixed end. The wire will, in these circumstances, be bent into a curve lying throughout in the plane through its fixed end and the line of the force, and (§ 599) its curvatures at different points will, as was first shown by James Bernoulli, be simply as their distances from this line. The curve fulfilling this condition has clearly just two independent parameters, of which one is conveniently regarded as the mean proportional, a, between the radius of curvature at any point and its distance from the line of force, and the other, the maximum distance, b, of the wire from the line of force. By choosing any value for each of these parameters it is easy to trace the corresponding curve with a very high approximation to accuracy, by commencing with a small circular arc touching at one extremity a straight line at the given maximum distance from the line of force, and continuing by small circular arcs, with the proper increasing radii, according to the diminishing distances of their middle points from the line of force. The annexed diagrams are, however, not so drawn; but are simply traced from the forms actually assumed by a flat steel spring, of small enough breadth not to be much disturbed by tortuosity in the cases in which different

Graphic construction of elastic curve transmitting force in one plane.

parts of it cross one another. The mode of application of the force is sufficiently explained by the indications in the diagram.

Equation of the plane elastic curve.

Let the line of force be the axis of x, and let ρ be the radius of curvature at any point (x, y) of the curve. The dynamical condition stated above becomes

$$\rho y = \frac{B}{T} = a^2 \text{.............................} (1),$$

where B denotes the flexural rigidity, T the tension of the cord, and a a linear parameter of the curve depending on these elements. Hence, by the ordinary formula for ρ^{-1},

$$y = \frac{a^2 \dfrac{d^2y}{dx^2}}{\left(1+\dfrac{dy^2}{dx^2}\right)^{\frac{3}{2}}} \text{.............................} (2).$$

Multiplying by $2dy$ and integrating, we have

$$y^2 = C - \frac{2a^2}{\left(1+\dfrac{dy^2}{dx^2}\right)^{\frac{1}{2}}} \text{.........................} (3);$$

and finally,

$$x = \int \frac{(y^2 - C)\,dy}{(4a^4 - C^2 + 2Cy^2 - y^4)^{\frac{1}{2}}} \text{.....................} (4),$$

which is the equation of the curve expressed in terms of an elliptic integral.

If, in the first integral, (3), we put $\dfrac{dy}{dx} = 0$, we find

$$y = \pm (C \pm 2a^2)^{\frac{1}{2}} \text{.............................} (5),$$

the upper sign within the bracket giving points of maximum, and the lower, points, if any real, of minimum distance from the axis. Hence there are points of equal maximum distance from the line of force on its two sides, but no real minima when $C < 2a^2$; which therefore comprehends the cases of diagrams 1...5. But there are real minima as well as maxima when $C > 2a^2$, which is therefore the case of diagram 7. In this case it may be remarked that the analytical equations comprehend two equal and similar de-

10—2

Equation of the plane elastic curve.

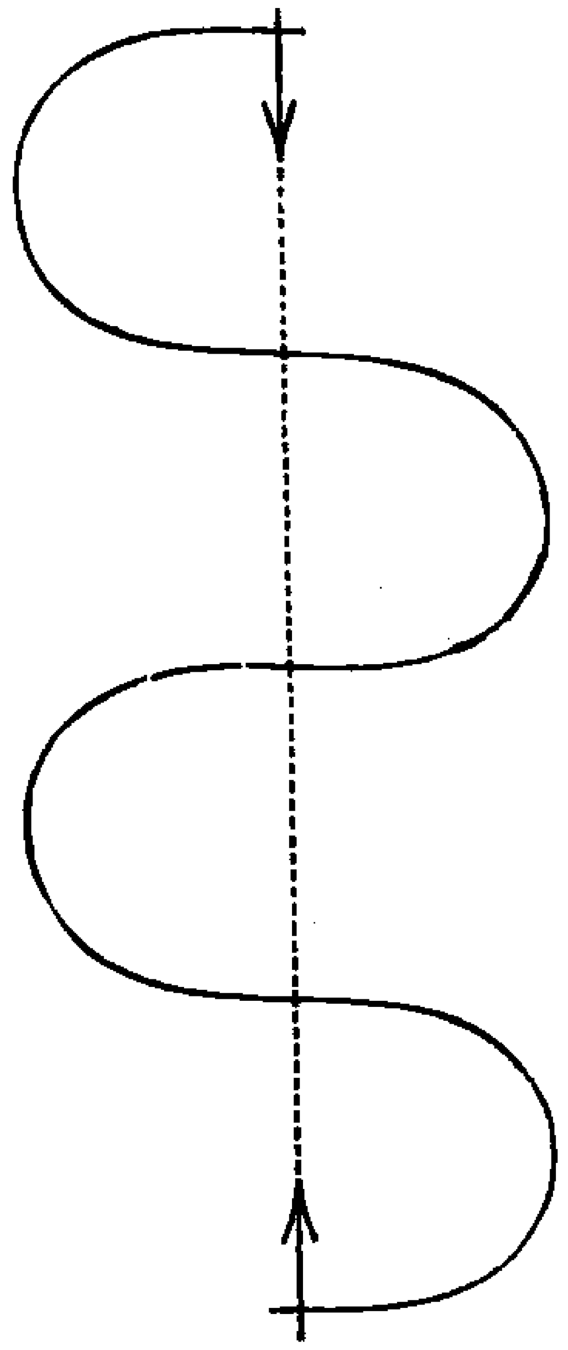


Fig. 2.

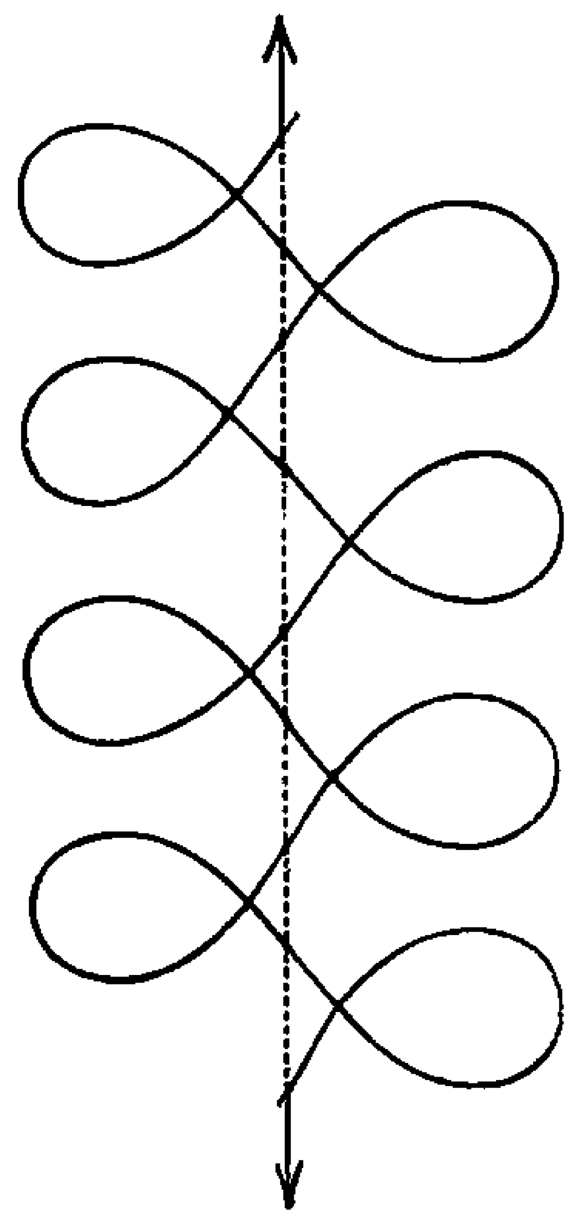


Fig. 5.

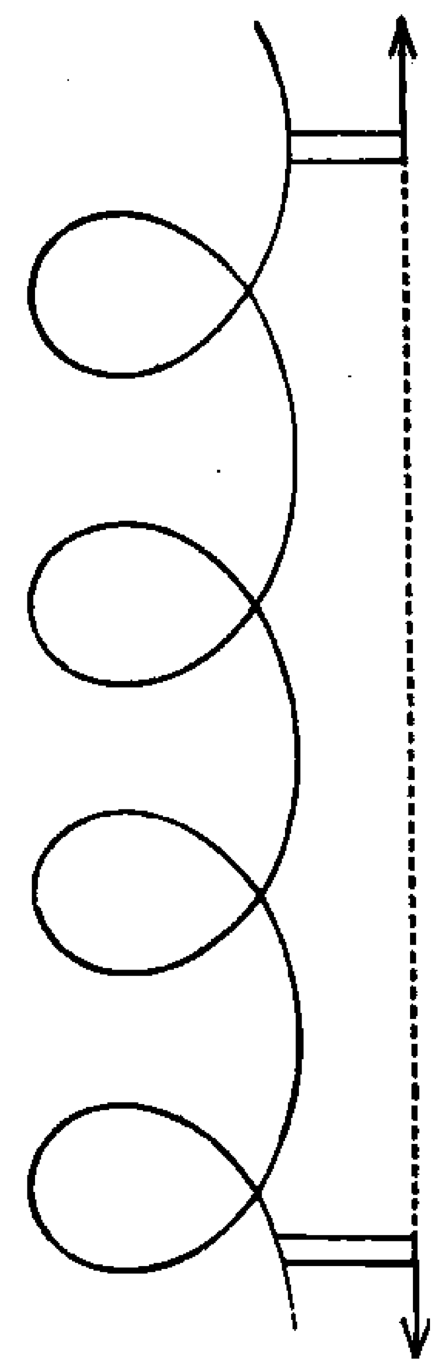


Fig. 7.

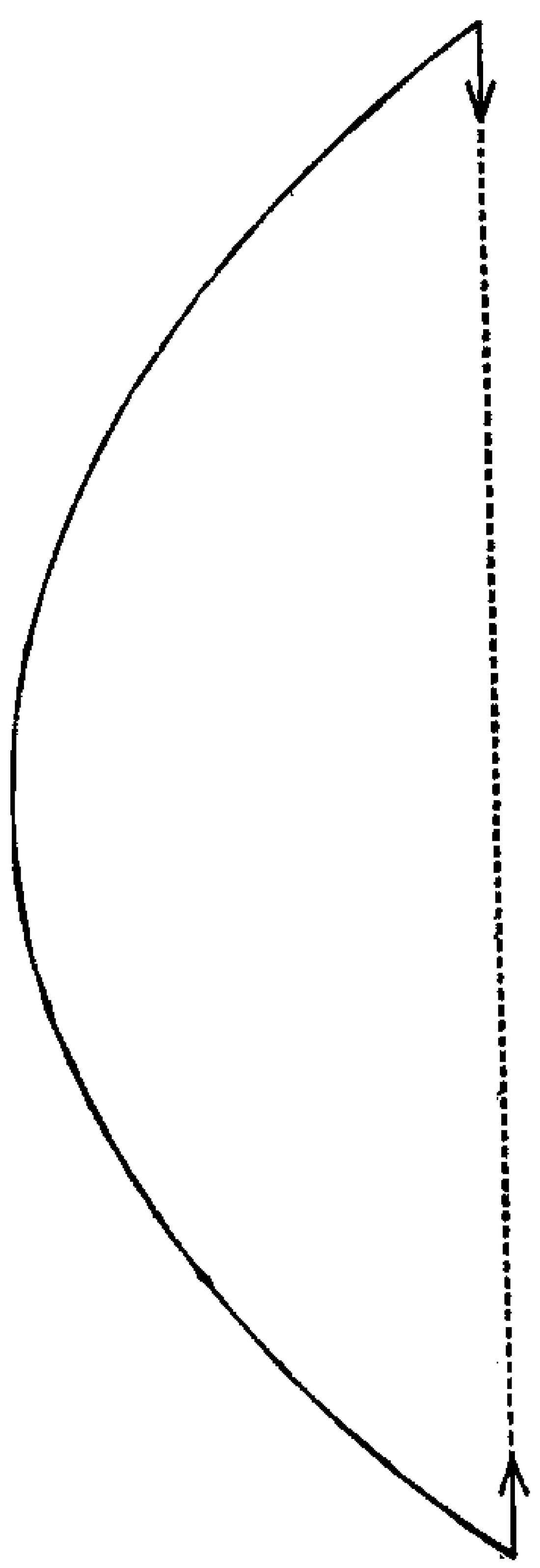


Fig. 1.

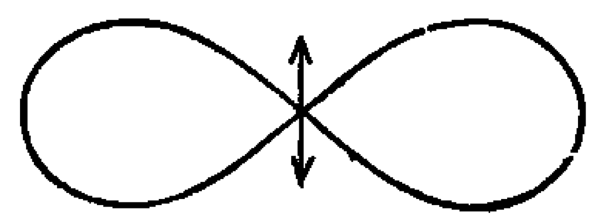


Fig. 4.

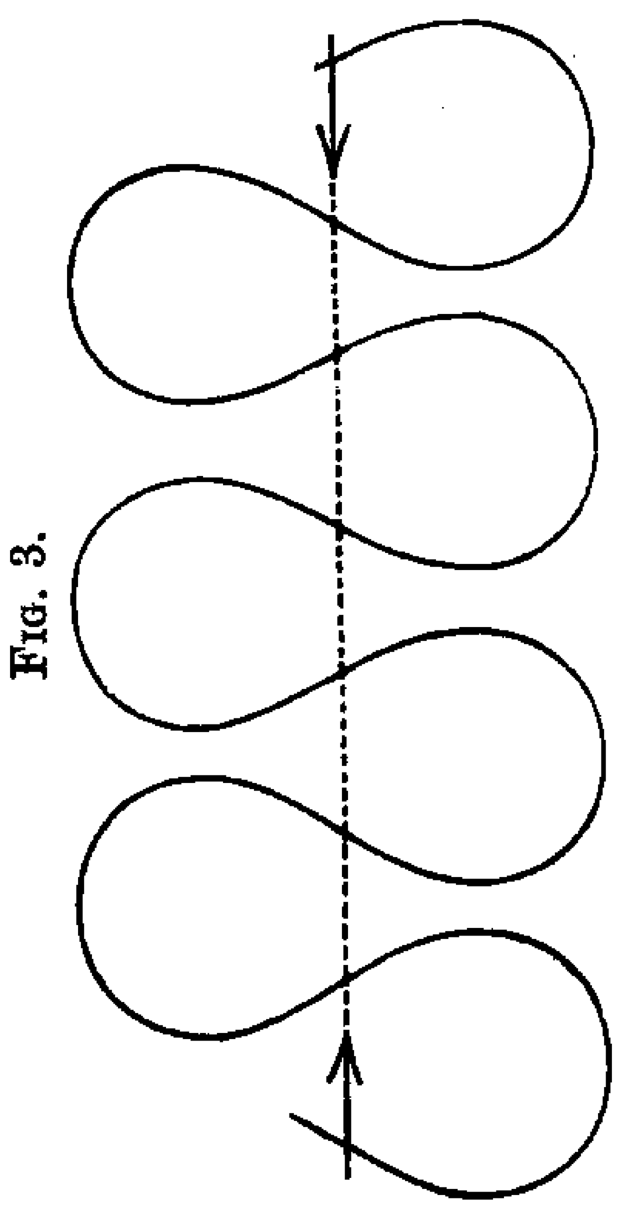


Fig. 3.

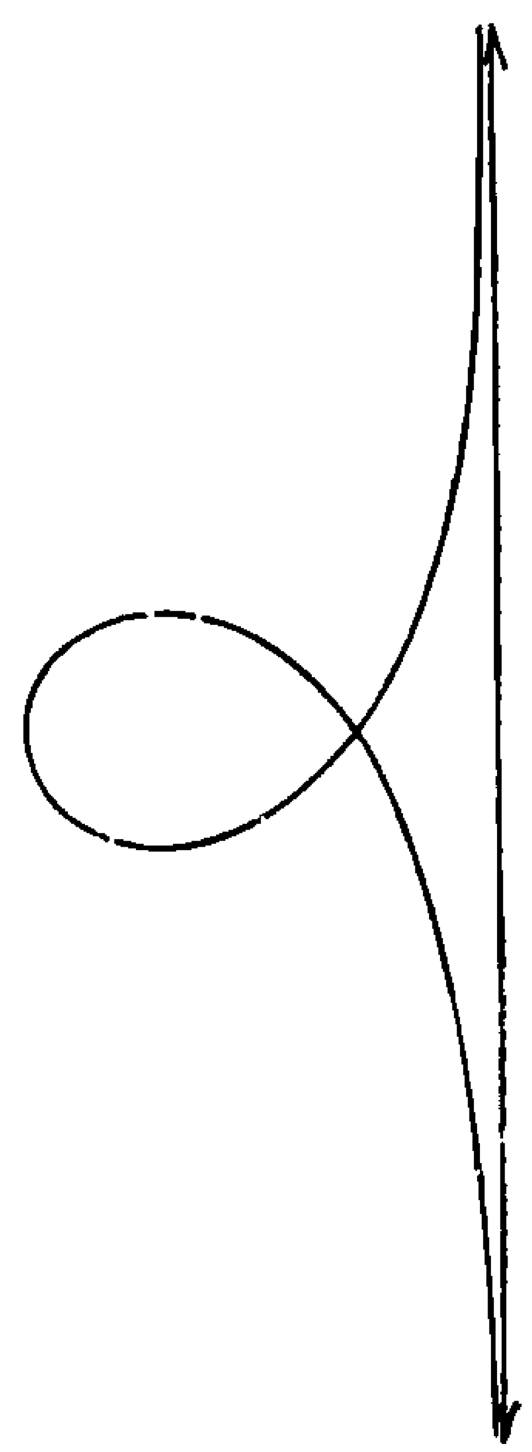


Fig. 6.

Equation of the plane elastic curve.

tached curves symmetrically situated on the two sides of the line of force; of which one only is shown in the diagram.

The intermediate case, $C=2a^2$, is that of diagram 6. For it the final integral degrades into a logarithmic form, as follows:

$$x=\int\frac{y\,dy}{(4a^2-y^2)^{\frac{1}{2}}}-\int\frac{2a^2dy}{y\,(4a^2-y^2)^{\frac{1}{2}}};$$

or, with the integrations effected, and the constant assigned to make the axis of y be that of symmetry,

$$x=-(4a^2-y^2)^{\frac{1}{2}}+a\log\frac{2a+(4a^2-y^2)^{\frac{1}{2}}}{y}\;\ldots\ldots\ldots\ldots(6).$$

This equation, when the radical is taken with the sign indicated, represents the branch proceeding from the vertex, first to the negative side of the axis of y, crossing it at the double point, and going to infinity towards the positive axis of x as an asymptote. The other branch is represented by the same equation with the sign of the radical reversed in each place.

It may be remarked that in (3) the sign of $\left(1+\frac{dy^2}{dx^2}\right)^{\frac{1}{2}}$ can only change, for a point moving continuously along the curve, when $\frac{dy}{dx}$ becomes infinite. The interpretation is facilitated by putting

$$\frac{dy}{dx}=\tan\theta,\ \text{or}\ \left(1+\frac{dy^2}{dx^2}\right)^{\frac{1}{2}}=-\cos\theta,$$

which reduces (3) to

$$y^2=2a^2\cos\theta+C\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(7).$$

Here, when $C>2a^2$ (the case in which, as we have seen above, there are minimum as well as maximum values of y on one side of the line of force), there is no limit to the value of θ. It increases, of course, continuously for a point moving continuously along the curve; the augmentation being 2π for one complete period (diagram 7).

When $C<2a^2$, θ has equal positive and negative values at the points in which the curve cuts the line of force. These values being given by the equation

$$\cos\theta=-\frac{C}{2a^2}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(8),$$

Equation of the plane elastic curve.

are obtuse when C is positive (diagram 3), and acute when C is negative (diagram 1). The extreme negative value of C is of course $-2a^2$.

If we take $$C=-2a^2+b^2,$$

$\pm b$ will be the maximum positive or negative value of y, as we see by (7); and if we suppose b to be small in comparison with a, we have the case of a uniform spring bent, as a bow, but slightly, by a string stretched between its ends.

Bow slightly bent.

612. An important particular case is that of figure 1, which corresponds to a bent bow having the same flexural rigidity throughout. If the amount of bending be small, the equation is easily integrated to any requisite degree of approximation. We will merely sketch the process of investigation.

Let e be the maximum distance from the axis, corresponding to $x=0$. Then $y=e$ gives $\frac{dy}{dx}=0$, and (3) becomes

$$e^2-y^2=2a^2\left(1-\frac{1}{\sqrt{1+\frac{dy^2}{dx^2}}}\right);$$

whence $$\frac{dy}{dx}=\frac{\sqrt{e^2-y^2}\sqrt{4a^2-e^2+y^2}}{2a^2-e^2+y^2}\;\ldots\ldots\ldots\ldots\ldots\ldots(9).$$

For a first approximation, omit e^2-y^2 in comparison with a^2 where they occur in the same factors, and we have

$$\frac{dy}{dx}=\frac{\sqrt{e^2-y^2}}{a},$$

or, since $y=e$ when $x=0$,

$$y=e\cos\frac{x}{a}\;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(10),$$

the harmonic curve, or curve of sines, which is the simplest form assumed by a vibrating cord or pianoforte wire.

For a closer approximation we may substitute for y, in those factors where it was omitted, the value given by (10); and so on. Thus we have

$$\frac{dy}{dx}=\frac{\sqrt{e^2-y^2}}{a}\left(1+\frac{3e^2}{8a^2}\sin^2\frac{x}{a}\right),\text{ nearly,}$$

Bow slightly bent.

or $$\frac{dy}{\sqrt{e^2-y^2}}=\frac{dx}{a}\left(1+\frac{3e^2}{16a^2}-\frac{3e^2}{16a^2}\cos\frac{2x}{a}\right),$$

from which, by integration,

$$\cos^{-1}\frac{y}{e}=\frac{x}{a}\left(1+\frac{3e^2}{16a^2}\right)-\frac{3e^2}{32a^2}\sin\frac{2x}{a}$$

and $$y=e\cos\left\{\frac{x}{a}\left(1+\frac{3e^2}{16a^2}\right)\right\}+\frac{3e^3}{32a^2}\sin\frac{x}{a}\sin\frac{2x}{a}.$$

Plane elastic curve and common pendulum.

613. As we choose particularly the common pendulum for the corresponding kinetic problem, the force acting on the rigid body in the comparison must be that of gravity in the vertical through its centre of gravity. It is convenient, accordingly, not to take *unity* as the velocity of the point travelling along the bent wire, but the velocity gravity would generate in a body falling through a height equal to half the constant, a, of § 611: and this constant, a, will then be the length of the isochronous simple pendulum. Thus if an elastic curve be held with its line of force vertical, and if a point, P, be moved along it with a constant velocity equal to $\sqrt{ga}$, (a denoting the mean proportional between the radius of curvature at any point and its distance from the line of force,) the tangent at P will keep always parallel to a simple pendulum, of length a, placed at any instant parallel to it, and projected with the same angular velocity. Diagrams 1...5 correspond to *vibrations* of the pendulum. Diagram 6 corresponds to the case in which the pendulum would just reach its position of unstable equilibrium in an infinite time. Diagram 7 corresponds to cases in which the pendulum flies round continuously in one direction, with periodically increasing and diminishing velocity. The extreme case, of the circular elastic curve, corresponds to a pendulum flying round with infinite angular velocity, which of course experiences only infinitely small variation in the course of the revolution. A conclusion worthy of remark is, that the rectification of the elastic curve is the same analytical problem as finding the time occupied by a pendulum in describing any given angle.

Wire of any shape disturbed by forces and couples applied through its length.

614. Hitherto we have confined our investigation of the form and twist of a wire under stress to a portion of the whole wire not itself acted on by force from without, but merely engaged in transmitting force between two equilibrating systems applied to the wire beyond this portion; and we have, thus, not included the very important practical cases of a curve deformed by its own weight or centrifugal force, or fulfilling such conditions of equilibrium as we shall have to use afterwards in finding its equations of motion according to D'Alembert's principle. We therefore proceed now to a perfectly general investigation of the equilibrium of a curve, uniform or not uniform throughout its length; either straight, or bent and twisted in any way, when free from stress; and not restricted by any condition as to the positions of the three principal flexure-torsion axes (§ 596); under the influence of any distribution whatever of force and couple through its whole length.

Let α, β, γ be the components of the mutual force, and ξ, η, ζ those of the mutual couple, acting between the matter on the two sides of the normal section through (x, y, z). Those for the normal section through $(x+\delta x,\ y+\delta y,\ z+\delta z)$ will be

$$\alpha + \frac{d\alpha}{ds}\delta s,\quad \beta + \frac{d\beta}{ds}\delta s,\quad \gamma + \frac{d\gamma}{ds}\delta s,$$

$$\xi + \frac{d\xi}{ds}\delta s,\quad \eta + \frac{d\eta}{ds}\delta s,\quad \zeta + \frac{d\zeta}{ds}\delta s.$$

Hence, if $X\delta s$, $Y\delta s$, $Z\delta s$, and $L\delta s$, $M\delta s$, $N\delta s$ be the components of the applied force, and applied couple, on the portion δs of the wire between those two normal sections, we have (§ 551) for the equilibrium of this part of the wire

$$-X = \frac{d\alpha}{ds},\quad -Y = \frac{d\beta}{ds},\quad -Z = \frac{d\gamma}{ds} \quad\ldots\ldots\ldots\ldots\ldots(1),$$

and (neglecting, of course, infinitely small terms of the second order, as $\delta y \delta s$)

$$-L\delta s = \frac{d\xi}{ds}\delta s + \gamma\delta y - \beta\delta z,\ \text{etc.};$$

or

$$-L = \frac{d\xi}{ds} + \gamma\frac{dy}{ds} - \beta\frac{dz}{ds},\quad -M = \frac{d\eta}{ds} + \alpha\frac{dz}{ds} - \gamma\frac{dx}{ds},\quad -N = \frac{d\zeta}{ds} + \beta\frac{dx}{ds} - \alpha\frac{dy}{ds} \ldots(2).$$

We may eliminate α, β, γ from these six equations by means of the following convenient assumption—

$$\alpha\frac{dx}{ds} + \beta\frac{dy}{ds} + \gamma\frac{dz}{ds} = T \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3),$$

Longitudinal tension.

T meaning the component of the force acting across the normal section, along the tangent to the middle line. From this, and the second and third of (2), we have

$$\alpha = T\frac{dx}{ds} - \left(M + \frac{d\eta}{ds}\right)\frac{dz}{ds} + \left(N + \frac{d\zeta}{ds}\right)\frac{dy}{ds}.$$

This, and the symmetrical expressions for β and γ, used in (1), give

$$\left.\begin{aligned} X &= -\frac{d}{ds}\left\{T\frac{dx}{ds} - \left(M+\frac{d\eta}{ds}\right)\frac{dz}{ds} + \left(N+\frac{d\zeta}{ds}\right)\frac{dy}{ds}\right\} \\ Y &= -\frac{d}{ds}\left\{T\frac{dy}{ds} - \left(N+\frac{d\zeta}{ds}\right)\frac{dx}{ds} + \left(L+\frac{d\xi}{ds}\right)\frac{dz}{ds}\right\} \\ Z &= -\frac{d}{ds}\left\{T\frac{dz}{ds} - \left(L+\frac{d\xi}{ds}\right)\frac{dy}{ds} + \left(M+\frac{d\eta}{ds}\right)\frac{dx}{ds}\right\} \end{aligned}\right\} \ldots\ldots(4).$$

We have besides, from (2),

$$0 = \frac{dx}{ds}\left(L + \frac{d\xi}{ds}\right) + \frac{dy}{ds}\left(M + \frac{d\eta}{ds}\right) + \frac{dz}{ds}\left(N + \frac{d\zeta}{ds}\right) \ldots\ldots\ldots(5).$$

To complete the mathematical expression of the circumstances, it only remains to introduce the equations of torsion-flexure. For this purpose, let any two lines of reference for the substance of the wire, PK, PL, be chosen at right angles to one another in the normal section through P. Let κ_0, λ_0 be the components of the curvature (§ 589) in the planes perpendicular to these lines, and through the tangent, PT, when the wire is unstrained; and κ, λ what they become under the actual stress. Let τ_0 denote the rate of twist (§ 119) of either line of reference round the tangent from point to point along the wire in the unstrained condition, and τ in the strained, so that $\tau - \tau_0$ is the rate of twist produced at P by the actual stress. Thus [§ 595 (3)] we have

$$\left.\begin{aligned} \xi l + \eta m + \zeta n &= A(\kappa-\kappa_0) + c(\lambda-\lambda_0) + b(\tau-\tau_0) \\ \xi l' + \eta m' + \zeta n' &= c(\kappa-\kappa_0) + B(\lambda-\lambda_0) + a(\tau-\tau_0) \\ \xi\frac{dx}{ds} + \eta\frac{dy}{ds} + \zeta\frac{dz}{ds} &= b(\kappa-\kappa_0) + a(\lambda-\lambda_0) + C(\tau-\tau_0) \end{aligned}\right\} \ldots\ldots(6),$$

Equations of torsion-flexure.

Equations of torsion-flexure.

where (l, m, n), (l', m', n'), $\left(\frac{dx}{ds}, \frac{dy}{ds}, \frac{dz}{ds}\right)$ denote the directions of PK, PL, PT; so that

$$\left.\begin{array}{c} l\frac{dx}{ds}+m\frac{dy}{ds}+n\frac{dz}{ds}=0, \quad l'\frac{dx}{ds}+m'\frac{dy}{ds}+n'\frac{dz}{ds}=0 \\ ll'+mm'+nn'=0 \\ l^2+m^2+n^2=1, \qquad l'^2+m'^2+n'^2=1 \end{array}\right\}\ldots\ldots(7).$$

Now if lines $O_{\prime}K$, $O_{\prime}L$, $O_{\prime}T$, each of unit length, be drawn, as in § 593, always parallel to PK, PL, PT, and if P be carried at unit velocity along the curve, the component velocity of $_{\prime}L$ parallel to $O_{\prime}T$, or that of $_{\prime}T$ parallel to $O_{\prime}K$ with its sign changed, is (§ 593) equal to κ; and similar statements apply to λ and τ. Hence,

Torsion, and two components of curvature, of wire (or component angular velocities of rotating solid).

$$\left.\begin{array}{l} \kappa=-\left\{l'\frac{d}{ds}\left(\frac{dx}{ds}\right)+m'\frac{d}{ds}\left(\frac{dy}{ds}\right)+n'\frac{d}{ds}\left(\frac{dz}{ds}\right)\right\} \\ \lambda=+\left\{l\frac{d}{ds}\left(\frac{dx}{ds}\right)+m\frac{d}{ds}\left(\frac{dy}{ds}\right)+n\frac{d}{ds}\left(\frac{dz}{ds}\right)\right\} \\ \tau=+\left(l'\frac{dl}{ds}+m'\frac{dm}{ds}+n'\frac{dn}{ds}\right) \end{array}\right\}\ldots(8).$$

Equations (7) reduce (l, m, n), (l', m', n') to one variable element, being the co-ordinate by which the position of the substance of the wire, round the tangent at any point of the central curve, is specified: and (8) express κ, λ, τ in terms of this co-ordinate, and the three Cartesian co-ordinates x, y, z of P. The specification of the unstrained condition of the wire gives κ_0, λ_0, τ_0 as functions of s. Thus (6) gives ξ, η, ζ each in terms of s, and the four co-ordinates, and their differential coefficients relatively to s. Substituting these in (4) and (5) we have four differential equations which, with

$$\frac{dx^2}{ds^2}+\frac{dy^2}{ds^2}+\frac{dz^2}{ds^2}=1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots(9),$$

constitute the five equations by which the five unknown functions (the four co-ordinates, and the tension, T) are to be determined in terms of s, or by means of which, with s and T eliminated, the two equations of the curve may be found, and the co-ordinate for the position of the normal section round the tangent determined in terms of x, y, z.

Terminal conditions.

The terminal conditions for any specified circumstances are easily expressed in the proper mathematical terms, by aid of equations (2). Thus, for instance, if a given force and a given couple be directly applied to a free end, or if the problem be limited to a portion of the wire terminated in one direction at a point Q, and if, in virtue of actions on the wire beyond, we have a given force $(\alpha_0, \beta_0, \gamma_0)$ and a given couple (ξ_0, η_0, ζ_0) acting on the normal section through Q of the portion under consideration, and if s_0 is the length of the wire from the zero of reckoning for s up to the point Q, and L_0, M_0, N_0 the values of L, M, N at this point, the equations expressing the terminal conditions will be

$$\left.\begin{array}{l} \xi=\xi_0, \ -\frac{d\xi}{ds}=L_0+\left(\gamma_0\frac{dy}{ds}-\beta_0\frac{dz}{ds}\right) \\ \eta=\eta_0, \ -\frac{d\eta}{ds}=M_0+\left(\alpha_0\frac{dz}{ds}-\gamma_0\frac{dx}{ds}\right) \\ \zeta=\zeta_0, \ -\frac{d\zeta}{ds}=N_0+\left(\beta_0\frac{dx}{ds}-\alpha_0\frac{dy}{ds}\right) \end{array}\right| \text{ when } s=s_0 \ \Bigg\}\ldots\ldots(10).$$

From these we see, by taking $L_0=0$, $M_0=0$, $N_0=0$, $\alpha_0=0$, $\beta_0=0$, $\gamma_0=0$, $\xi_0=0$, $\eta_0=0$, $\zeta_0=0$, that

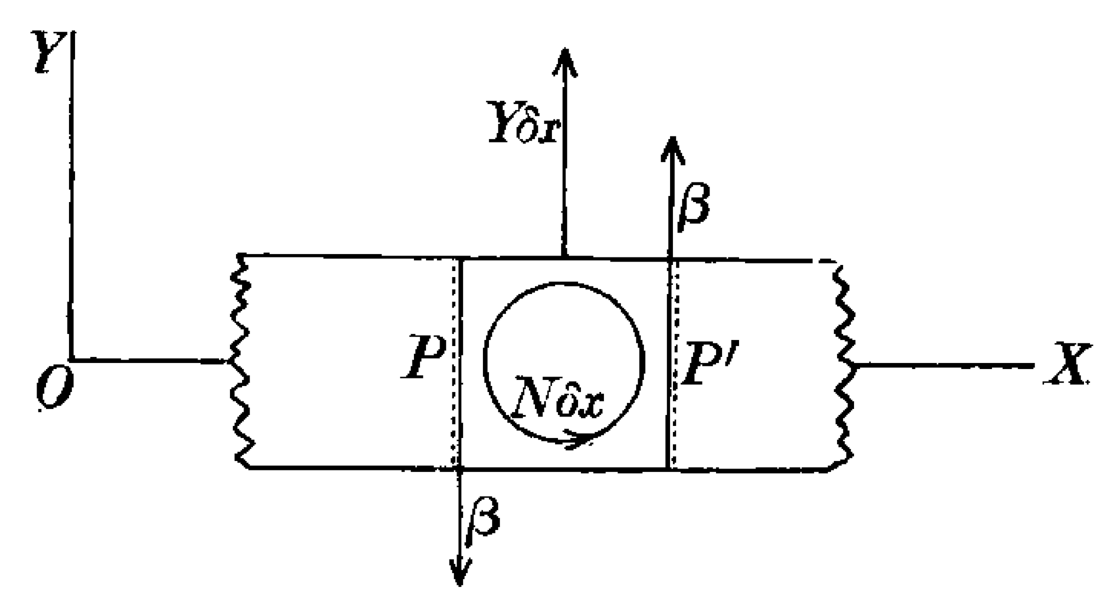
615. For the simple and important case of a naturally straight wire, acted on by a distribution of force, but not of couple, through its length, the condition fulfilled at a perfectly free end, acted on by neither force nor couple, is that the curvature is zero at the end, and its rate of variation from zero, per unit of length from the end, is, at the end, zero. In other words, the curvatures at points infinitely near the end are as the squares of their distances from the end in general (or, as some higher power of these distances, in singular cases). The same statements hold for the *change* of curvature produced by the stress, if the unstrained wire is not straight, but the other circumstances the same as those just specified.

Straight beam infinitely little bent.

616. As a very simple example of the equilibrium of a wire subject to forces through its length, let us suppose the natural form to be straight, and the applied forces to be in lines, and the couples to have their axes all perpendicular to its length, and to be not great enough to produce more than an infinitely small deviation from the straight line. Further, Straight beam infinitely little bent.

in order that these forces and couples may produce no twist, let the three flexure-torsion axes be perpendicular to and along the wire. But we shall not limit the problem further by supposing the section of the wire to be uniform, as we should thus exclude some of the most important practical applications, as to beams of balances, levers in machinery, beams in architecture and engineering. It is more instructive to investigate the equations of equilibrium directly for this case than to deduce them from the equations worked out above for the much more comprehensive general problem. The particular principle for the present case is simply that the rate of variation of the rate of variation, per unit of length along the wire, of the bending couple in any plane through the length, is equal, at any point, to the applied force per unit of length, with the simple rate of variation of the applied couple subtracted. This, together with the direct equations (§ 599) between the component bending couples, gives the required equations of equilibrium.

The diagram representing a section of the wire in the plane *xy*, let $OP = x$, $PP' = \delta x$. Let Y and N be the components



in the plane of the diagram, of the applied force and couple, each reckoned per unit of length of the wire; so that $Y\delta x$ and $N\delta x$ will be the amounts of force and couple in this plane, actually applied to the portions of the wire between P and P'.

Let, as before (§ 614), β and γ denote the components parallel to OY and OZ of the mutual force*, and ζ and η the components

* These forces, being each *in* the plane of section of the solid separating the portions of matter between which they act, are of the kind called *shearing forces.* See below, § 662.

Straight beam infinitely little bent.

in the plane XOY, XOZ, of the mutual couple, between the portions of matter on the two sides of the normal section through P; and β', γ' and ζ', η' the same for P'. The matter between these two sections is balanced under these actions from the matter contiguous to it beyond them, and the force and couple applied to it from without. These last have, in the plane XOY, components respectively equal to $Y\delta x$ and $N\delta x$: and hence for the equilibrium of the portion PP',

$$-\beta + Y\delta x + \beta' = 0, \text{ by forces parallel to } OY,$$

and $$-\zeta + N\delta x + \zeta' + \beta\delta x = 0, \text{ by couples in plane } XOY,$$

the term $\beta\delta x$ in this second equation being the moment of the couple formed by the infinitely nearly equal forces β, β' in the dissimilar parallel directions through P and P'. Now

$$\beta' - \beta = \frac{d\beta}{dx}\delta x, \text{ and } \zeta' - \zeta = \frac{d\zeta}{dx}\delta x.$$

Hence the preceding equations give

$$\left.\begin{aligned}\frac{d\beta}{dx} &= -Y\\ \frac{d\zeta}{dx} &= -N - \beta\end{aligned}\right\}\text{.........................} (1);$$

and these, by the elimination of β,

$$\frac{d^2\zeta}{dx^2} = -\frac{dN}{dx} + Y \text{.........................}(2).$$

Similarly, by forces and couples in the plane XOZ,

$$\frac{d^2\eta}{dx^2} = -\frac{dM}{dx} + Z \text{.........................}(3),$$

couples in this plane being reckoned positive when they tend to turn from the direction of OX to that of OZ; which is opposite to the convention (551) generally adopted as being proper when the three axes are dealt with symmetrically.

Since the wire deviates infinitely little from the straight line OX, the component curvatures are

$$\frac{d^2y}{dx^2} \text{ in the plane } XOY,$$

and $$\frac{d^2z}{dx^2} \quad \text{,,} \quad \text{,,} \quad XOZ.$$

Straight beam infinitely little bent.

Hence the equations of flexure are

$$\left.\begin{aligned}\zeta &= B\frac{d^2y}{dx^2} + a\frac{d^2z}{dx^2}\\ \eta &= a\frac{d^2y}{dx^2} + C\frac{d^2z}{dx^2}\end{aligned}\right\}\dots\dots\dots\dots(4),$$

where B and C are the flexural rigidities (§ 596) in the planes xy and xz, and a the coefficient expressing the couple in either produced by unit curvature in the other; three quantities which are to be regarded, in general, as given functions of x. Substituting these expressions for ζ and η, in (2) and (3), we have the required equations of equilibrium.

Case of independent flexure in two planes.

617. If the directions of maximum and minimum flexural rigidity lie throughout the wire in two planes, the equations of equilibrium become simplified by these planes being chosen as planes of reference, XOY, XOZ. The flexure in either plane then depends simply on the forces in it, and thus the problem divides itself into the two quite independent problems of integrating the equations of flexure in the two principal planes, and so finding the projections of the curve on two fixed planes agreeing with their position when the rod is straight.

In this case, and with XOY, XOZ so chosen, we have $a=0$. Hence the equations of flexure (4) become simply

$$\zeta = B\frac{d^2y}{dx^2},\quad \eta = C\frac{d^2z}{dx^2};$$

and the differential equations of the curve, found by using these in (2) and (3),

$$\frac{d^2}{dx^2}\left(B\frac{d^2y}{dx^2}\right) = \mathfrak{Y},\ \frac{d^2}{dx^2}\left(C\frac{d^2z}{dx^2}\right) = \mathfrak{Z}\dots\dots\dots(5),$$

where $$\mathfrak{Y} = -\frac{dN}{dx} + Y,\ \mathfrak{Z} = -\frac{dM}{dx} + Z\dots\dots\dots(6).$$

Here $\mathfrak{Y}$ and $\mathfrak{Z}$ are to be generally regarded as known functions of x, given explicitly by (6), being the amounts of component simple forces perpendicular to the wire, reckoned per unit of its length, that would produce the same figure as the distribution of force and couple we have supposed actually applied throughout

the length. Later, when occupied with the theory of magnetism, we shall meet with a curious instance of the relation expressed by (6). In the meantime it may be remarked that although the figure of the wire does not sensibly differ when the simple distribution of force is substituted for any given distribution of force and couple, the shearing forces in normal sections become thoroughly altered by this change of circumstances, as is shown by (1). When the wire is uniform, B and C are constant, and the equations of equilibrium become

Case of independent flexure in two planes.

$$\frac{d^4y}{dx^4} = \frac{\mathfrak{Y}}{B},\ \frac{d^4z}{dx^4} = \frac{\mathfrak{Z}}{C}\dots\dots\dots\dots(7).$$

Plank bent by its own weight.

The simplest example is obtained by taking $\mathfrak{Y}$ and $\mathfrak{Z}$ each constant, a very interesting and useful case, being that of a uniform beam influenced only by its own weight, except where held or pressed by its supports. Confining our attention to flexure in the one principal plane, XOY, and supposing this to be vertical, so that $\mathfrak{Y} = gw$, if w be the mass per unit of length; we have, for the complete integral, of course

$$y = \frac{gw}{B}\left(\tfrac{1}{24}x^4 + Kx^3 + K'x^2 + K''x + K'''\right)\dots\dots\dots(8),$$

where K, K', etc., denote constants of integration. These, four in number, are determined by the terminal conditions; which, for instance, may be that the value of y and of $\frac{dy}{dx}$ is given for each end. Or, as for instance in the case of a plank simply resting with its ends on two edges or trestles, and free to turn round either, the condition may be that the curvature vanishes at each end: so that if OX be taken as the line through the points of support, we have

Plank supported by its ends.

$$\left.\begin{aligned}y&=0\\ \frac{d^2y}{dx^2}&=0\end{aligned}\right\}\text{ when } x=0 \text{ and when } x=l,$$

l being the length of the plank. The solution then is

$$y = \frac{gw}{B}\cdot\tfrac{1}{24}(x^4 - 2lx^3 + l^3x)\dots\dots\dots\dots(9).$$

Hence, by putting $x=\tfrac{1}{2}l$, we find $y = \frac{gw}{B}\cdot\frac{5l^4}{16\times 24}$ for the distance

Plank supported by its ends;

by which the middle point is deflected from the straight line joining the points of support.

Or, as in the case of a plank balanced on a trestle at its middle (taken as zero of x), or hung by a rope tied round it there, we may have

by its middle.

$$\left.\begin{array}{l} y=0 \\ \dfrac{dy}{dx}=0 \end{array}\right\} \text{ when } x=0,$$

and
$$\left.\begin{array}{l} \dfrac{d^2y}{dx^2}=0 \\ \dfrac{d^3y}{dx^3}=0 \end{array}\right\} \text{ when } x=\tfrac{1}{2}l \text{ [see above, § 614 (10)].}$$

The solution in this case is, for the positive half of the plank,

$$y=\frac{gw}{B}\cdot\tfrac{1}{24}(x^4-2lx^3+\tfrac{3}{2}l^2x^2)\dots\dots\dots\dots\dots\dots(10).$$

By putting $x=\frac{1}{2}l$, we find $y=\dfrac{gw}{B}\cdot\dfrac{3l^4}{16.24}$. Hence

Droops compared.

618. When a uniform bar, beam, or plank is balanced on a single trestle at its middle, the droop of its ends is only $\frac{3}{8}$ of the droop which its middle has when the bar is supported on trestles at its ends. From this it follows that the former is $\frac{3}{8}$ and the latter $\frac{5}{8}$ of the droop or elevation produced by a force equal to half the weight of the bar, applied vertically downwards or upwards to one end of it, if the middle is held fast in a horizontal position. For let us first suppose the whole to rest on a trestle under its middle, and let two trestles be placed under its ends and gradually raised till the pressure is entirely taken off from the middle. During this operation the middle remains fixed and horizontal, while a force increasing to half the weight, applied vertically upwards on each end, raises it through a height equal to the sum of the droops in the two cases above referred to. This result is of course proved directly by comparing the absolute values of the droop in those two cases as found above, with the deflection from the tangent at the end of the cord in the elastic curve, figure 2, of § 611, which is cut by the cord at right angles. It may be stated otherwise

Plank supported by its ends or middle.

thus: the droop of the middle of a uniform beam resting on trestles at its ends is increased in the ratio of 5 to 13 by laying a mass equal in weight to itself on its middle: and, if the beam is hung by its middle, the droop of the ends is increased in the ratio of 3 to 11 by hanging on each of them a mass equal to half the weight of the beam.

Plank supported by its ends or middle:

619. The important practical problem of finding the distribution of the weight of a solid on points supporting it, when more than two of these are in one vertical plane, or when there are more than three altogether, which (§ 568) is indeterminate* if the solid is perfectly rigid, may be completely solved for a uniform elastic beam, naturally straight, resting on three or more points in rigorously fixed positions all nearly in one horizontal line, by means of the preceding results.

by three or more points.

If there are i points of support, the $i-1$ parts of the rod between them in order and the two end parts will form $i+1$ curves expressed by distinct algebraic equations [§ 617 (8)], each involving four arbitrary constants. For determining these constants we have $4i+4$ equations in all, expressing the following conditions:—

I. The ordinates of the inner ends of the projecting parts of the rod, and of the two ends of each intermediate part, are respectively equal to the given ordinates of the corresponding points of support [$2i$ equations].

II. The curves on the two sides of each support have coincident tangents and equal curvatures at the point of transition from one to the other [$2i$ equations].

III. The curvature and its rate of variation per unit of length along the rod, vanish at each end [4 equations].

Thus the equation of each part of the curve is completely determined: and then, by § 616, we find the shearing force in any normal section. The difference between these in the

* It need scarcely be remarked that indeterminateness does not exist in nature. How it may occur in the problems of abstract dynamics, and is obviated by taking something more of the properties of matter into account, is instructively illustrated by the circumstances referred to in the text.

neighbouring portions of the rod on the two sides of a point of support, is of course equal to the pressure on this point.

Plank supported by its ends and middle.

620. The solution for the case of this problem in which two of the points of support are at the ends, and the third midway between them either exactly in the line joining them, or at any given very small distance above or below it, is found at once, without analytical work, from the particular results stated in § 618. Thus if we suppose the beam, after being first supported wholly by trestles at its ends, to be gradually pressed up by a trestle under its middle, it will bear a force simply proportional to the space through which it is raised from the zero point, until all the weight is taken off the ends, and borne by the middle. The whole distance through which the middle rises during this process is, as we found, $\frac{gw}{B} \cdot \frac{8l^4}{16.24}$; and this whole elevation is $\frac{8}{5}$ of the droop of the middle in the first position. If therefore, for instance, the middle trestle be fixed exactly in the line joining those under the ends, it will bear $\frac{5}{8}$ of the whole weight, and leave $\frac{3}{16}$ to be borne by each end. And if the middle trestle be lowered from the line joining the end ones by $\frac{7}{15}$ of the space through which it would have to be lowered to relieve itself of all pressure, it will bear just $\frac{1}{3}$ of the whole weight, and leave the other two thirds to be equally borne by the two ends.

Rotation of a wire round its elastic central line.

621. A wire of equal flexibility in all directions, and straight when freed from stress, offers, when bent and twisted in any manner whatever, not the slightest resistance to being turned round its elastic central curve, as its conditions of equilibrium are in no way affected by turning the whole wire thus equally throughout its length. The useful application of this principle, to the maintenance of equal angular motion in two bodies rotating round different axes, is rendered somewhat difficult in practice by the necessity of a perfect attachment and adjustment of each end of the wire, so as to have the tangent to its elastic central curve exactly in line with the axis of rotation. But if this condition is rigorously fulfilled, and the wire is of exactly equal flexibility in every direction, and

Elastic universal flexure joint; § 109.

exactly straight when free from stress, it will give, against any constant resistance, an accurately uniform motion from one to another of two bodies rotating round axes which may be inclined to one another at any angle, and need not be in one plane. If they are in one plane, if there is no resistance to the rotatory motion, and if the action of gravity on the wire is insensible, it will take some of the varieties of form (§ 612) of the plane elastic curve of James Bernoulli. But however much it is altered from this; whether by the axes not being in one plane; or by the torsion accompanying the transmission of a couple from one shaft to the other, and necessarily, when the axes are in one plane, twisting the wire out of it; or by gravity; the elastic central curve will remain at rest, the wire in every normal section rotating round it with uniform angular velocity, equal to that of each of the two bodies which it connects. Under Properties of Matter, we shall see, as indeed may be judged at once from the performances of the vibrating spring of a chronometer for twenty years, that imperfection in the elasticity of a metal wire does not exist to any such degree as to prevent the practical application of this principle, even in mechanism required to be durable.

Equable elastic rotating joint.

It is right to remark, however, that if the rotation be too rapid, the equilibrium of the wire rotating round its unchanged elastic central curve may become unstable, as is immediately discovered by experiments (leading to very curious phenomena), when, as is often done in illustrating the kinetics of ordinary rotation, a rigid body is hung by a steel wire, the upper end of which is kept turning rapidly.

Practical inequalities.

622. If the wire is not of rigorously equal flexibility in all directions, there will be a periodic inequality in the communicated angular motion, having for period a half turn of either body: or if the wire, when unstressed, is not exactly straight, there will be a periodic inequality, having the whole turn for its period. In other words, if ϕ and ϕ' be angles simultaneously turned through by the two bodies, with a constant working couple transmitted from one to the other through the wire, $\phi - \phi'$ will not be zero, as in the proper elastic universal

Practical inequalities. flexure joint, but will be a function of $\sin 2\phi$ and $\cos 2\phi$ if the first defect alone exists; or it will be a function of $\sin\phi$ and $\cos\phi$ if there is the second defect whether alone or along with the first. It is probable that, if the bend in the wire when unstressed is not greater than can be easily provided against in actual construction, the inequality of action caused by it may be sufficiently remedied without much difficulty in practice, by setting it at one or at each end, somewhat inclined to the axis of the rotating body to which it is attached. But these considerations lead us to a subject of much greater interest in itself than any it can have from the possibility of usefulness in practical applications. The simple cases we shall choose illustrate three kinds of action which may exist, each either alone or with one or both the others, in the equilibrium of a wire not equally flexible in all directions, and straight when unstressed.

Elastic rotating joint.

Rotation round its elastic central circle, of a straight wire made into a hoop.

623. A uniform wire, straight when unstressed, is bent till its two ends meet, which are then attached to one another, with the elastic central curve through each touching one straight line: so that whatever be the form of the normal section, and the quality, crystalline or non-crystalline, of the substance, the whole wire must become, when in equilibrium, an exact circle (gravity being not allowed to produce any disturbance). It is required to find what must be done to turn the whole wire uniformly through any angle round its elastic central circle.

If the wire is of exactly equal flexibility in all directions*, it will, as we have seen (§ 621), offer no resistance at all to this action, except of course by its own inertia; and if it is once set to rotate thus uniformly with any angular velocity, great or small, it would continue so for ever were the elasticity perfect, and were there no resistance from the air or other matter touching the axis.

To avoid restricting the problem by any limitation, we must suppose the wire to be such that, if twisted and bent in any way, the potential energy of the elastic action developed, per

* In this case, clearly it might have been twisted before its ends were put together, without altering the circular form taken when left with its ends joined.

unit of length, is a quadratic function of the twist, and two components of the curvature (§§ 590, 595), with six arbitrarily given coefficients. But as the wire has no twist*, three terms of this function disappear in the case before us, and there remain only three terms,—those involving the squares and the product of the components of curvature in planes perpendicular to two rectangular lines of reference in the normal section through any point. The position of these lines of reference may be conveniently chosen so as to make the product of the components of curvature disappear: and the planes perpendicular to them will then be the planes of maximum and minimum flexural rigidity when the wire is kept free from twist†. There is no difficulty in applying the general equations of § 614 to express these circumstances and answer the proposed question. Leaving this as an analytical exercise to the student, we take a shorter way to the conclusion by a direct application of the principle of energy.

Rotation round its elastic central circle, of a straight wire made into a hoop.

Let the potential energy per unit of length be $\frac{1}{2}(B\kappa^2 + C\lambda^2)$, when κ and λ are the component curvatures in the planes of maximum and minimum flexural rigidity: so that, as in § 617, B and C are the measures of the flexural rigidities in these planes. Now if the wire be held in any way at rest with these planes through each point of it inclined at the angles ϕ and $\frac{1}{2}\pi - \phi$ to the plane of its elastic central circle, the radius of this circle being r, we should have $\kappa = \frac{1}{r}\cos\phi$, $\lambda = \frac{1}{r}\sin\phi$. Hence, since $2\pi r$ is the whole length,

$$E = \pi\left(\frac{B}{r}\cos^2\phi + \frac{C}{r}\sin^2\phi\right) \qquad (1).$$

* Which we have supposed, in order that it may take a circular form; although in the important case of equal flexibility in all directions this condition would obviously be fulfilled, even with twist.

† When, as in ordinary cases, the wire is either of isotropic material (see § 677 below), or has a normal axis (§ 596) in the direction of its elastic central line, flexure will produce no tendency to twist: in other words, the products of twist into the components of curvature will disappear from the quadratic expressing the potential energy: or the elastic central line is an axis of pure torsion. But, as shown in the text, the case under consideration gains no simplicity from this restriction.

Rotation round its elastic central circle, of a straight wire made into a hoop.

Let us now suppose every infinitely small part of the wire to be acted on by a couple in the normal plane, and let L be the amount of this couple per unit of length, which must be uniform all round the ring in order that the circular form may be retained, and let this couple be varied so that, rotation being once commenced, ϕ may increase at any uniform angular velocity. The equation of work done per unit of time (§§ 240, 287) is

$$2\pi r L\dot\phi = \frac{dE}{dt} = \frac{dE}{d\phi}\dot\phi.$$

And therefore, by (1),

$$-L = \frac{B-C}{r^2}\sin\phi\cos\phi = \frac{B-C}{2r^2}\sin 2\phi,$$

which shows that the couple required in the normal plane through every point of the ring, to hold it with the planes of greatest flexural rigidity touching a cone inclined at any angle, ϕ, to the plane of the circle, is proportional to $\sin 2\phi$; is in the direction to prevent ϕ from increasing; and when $\phi = \frac{1}{4}\pi$, amounts to $\frac{B-C}{2r^2}$ per unit length of the circumference. From this we see that there are two positions of stable equilibrium, —being those in which the plane of least flexural rigidity lies in the plane of the ring; and two positions of unstable equilibrium,—being those in which the plane of greatest flexural rigidity is in the plane of the ring.

Rotation round its elastic central circle, of a hoop of wire equally flexible in all directions, but circular when unstrained.

624. A wire of uniform flexibility in all directions, so shaped as to be a circular arc of radius a when free from stress, is bent till its ends meet, and these are joined as in § 623, so that the whole becomes a circular ring of radius r. It is required to find the couple which will hold this ring turned round the central curve through any angle ϕ in every normal section, from the position of stable equilibrium (which is of course that in which the naturally concave side of the wire is on the concave side of the ring, the natural curvature being either increased or diminished, but not reversed, when the wire is bent into the ring). Applying the principle of energy exactly as in the preceding section, we find that in this case the couple

is proportional to $\sin\phi$, and that when $\phi = \frac{1}{2}\pi$, its amount per unit of length of the circumference is $\frac{B}{ar}$, if B denote the flexural rigidity.

For in this case we have the potential energy

$$E = \pi r B\left\{\left(\frac{1}{a} - \frac{1}{r}\cos\phi\right)^2 + \left(\frac{1}{r}\sin\phi\right)^2\right\} = \pi r B\left(\frac{1}{a^2} - \frac{2}{ar}\cos\phi + \frac{1}{r^2}\right) \quad (2),$$

and

$$L = \frac{1}{2\pi r}\frac{dE}{d\phi} = \frac{B}{ar}\sin\phi \quad \ldots\ldots\ldots\ldots\ldots\ldots (3).$$

If every part of the ring is turned half round, so as to bring the naturally concave side of the wire to the convex side of the ring, we have of course a position of unstable equilibrium.

Wire unequally flexible in different directions, and circular when unstrained, bent to another circle by balancing couples applied to its ends.

625. A wire of unequal flexibility in different directions is formed so that, when free from stress, it constitutes a circular arc of radius a, with the plane of greatest flexural rigidity at each point touching a cone inclined to its plane at an angle α. Its ends are then brought together and joined, as in §§ 623, 624, so that the whole becomes a closed circular ring, of any given radius r. It is required to find the changed inclination, ϕ, to the plane of the ring, which the plane of greatest flexural rigidity assumes, and the couple, G, in the plane of the ring, which acts between the portions of matter on each side of any normal section.

The two equations between the components of the couple and the components of the curvature in the planes of greatest and least flexural rigidity determine the two unknown quantities of the problem.

These equations are

$$\left.\begin{aligned} B\left(\frac{1}{r}\cos\phi - \frac{1}{a}\cos\alpha\right) &= G\cos\phi \\ C\left(\frac{1}{r}\sin\phi - \frac{1}{a}\sin\alpha\right) &= G\sin\phi \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots (4),$$

since $\frac{1}{a}\cos\alpha$ and $\frac{1}{a}\sin\alpha$ are the components of natural curvature in the principal planes, and therefore $\frac{1}{r}\cos\phi - \frac{1}{a}\cos\alpha$, and

Wire unequally flexible in different directions, and circular when unstrained, bent to another circle by balancing couples applied to its ends.

$\frac{1}{r}\sin\phi - \frac{1}{a}\sin\alpha$, are the changes from the natural to the actual curvatures in these planes maintained by the corresponding components $G\cos\phi$ and $G\sin\phi$ of the couple G.

The problem, so far as the position into which the wire turns round its elastic central curve, may be solved by an application of the principle of energy, comprehending those of §§ 623, 624 as particular cases.

Let L be the amount, per unit of length of the ring, of the couple which must be applied from without, in each normal section, to hold it with the plane of maximum flexural rigidity at each point inclined at any given angle, ϕ, to the plane of the ring. We have, as before (§§ 623, 624), for the potential energy of the elastic action in the ring when held so,

$$E = \pi r\left\{B\left(\frac{\cos\phi}{r} - \frac{\cos\alpha}{a}\right)^2 + C\left(\frac{\sin\phi}{r} - \frac{\sin\alpha}{a}\right)^2\right\} \ldots\ldots(5).$$

Hence

$$L = \frac{1}{2\pi r}\frac{dE}{d\phi} = \left\{-B\left(\frac{\cos\phi}{r} - \frac{\cos\alpha}{a}\right)\frac{\sin\phi}{r} + C\left(\frac{\sin\phi}{r} - \frac{\sin\alpha}{a}\right)\frac{\cos\phi}{r}\right\}. \ (6).$$

This equated to zero is the same as (4) with G eliminated, and determines the relation between ϕ and r, in order that the ring when altered to radius r instead of a may be in equilibrium in itself (that is, without any application of couple in the normal section). The present method has the advantage of facilitating the distinction between the solutions, as regards stability or instability of the equilibrium, since (§ 291) for stable equilibrium E is a minimum, and for unstable equilibrium a maximum.

As a particular case, let $C = \infty$, which simplifies the problem very much. The terms involving C as a factor in (5) and (6) become nugatory in this case, and require of course that

$$\frac{\sin\phi}{r} - \frac{\sin\alpha}{a} = 0.$$

But the former method is clearer and better for the present case; as this result is at once given by the second of equations (4); and then the value of G, if required, is found from the first. We conclude what is stated in the following section:—

Conical bendings of developable surface.

626. Let a uniform hoop, possessing flexibility only in one tangent plane to its elastic central line at each point, be given, so shaped that when under no stress (for instance, when cut through in any normal section and uninfluenced by force from other bodies) it rests in the form of a circle of radius a, with its planes of inflexibility all round touching a cone inclined to the plane of this circle. This is very nearly the case with a common hoop of thin sheet-iron fitted upon a conical vat, or on either end of a barrel of ordinary shape. Let such a hoop be shortened (or lengthened), made into a circle of radius a by riveting its ends together (§ 623) in the usual way, and left with no force acting on it from without. It will rest with its plane of inflexibility inclined at the angle $\phi = \sin^{-1}(r\sin\alpha/a)$ to the plane of its circular form, and the elastic couple acting in this plane between the portions of matter on the two sides of any normal section will be

$$G = \frac{B}{\cos\phi}\left(\frac{\cos\phi}{r} - \frac{\cos\alpha}{a}\right).$$

These results we see at once, by remarking that the component curvature in the plane of inflexibility at each point must be invariably of the same value, $\sin\alpha/a$, as in the given unstressed condition of the hoop: and that the component couple, $G\cos\phi$, in the plane perpendicular to that of inflexibility at each point, must be such as to change the component curvature in this plane from $\cos\alpha/a$ to $\cos\phi/r$.

The greatest circle to which such a hoop can be changed is of course that whose radius is $a/\sin\alpha$: and for this $\phi = \frac{1}{2}\pi$, or the surface of inflexibility at each point (the surface of the sheet-metal in the practical case) becomes the plane of the circle: and therefore $G = \infty$, showing that if a hoop approaching infinitely nearly to this condition be made, in the manner explained, the internal couple acting across each normal section will be infinitely great, which is obviously true.

627. Another very important and interesting case readily dealt with by a method similar to that which we have applied to the elastic wire, is the equilibrium of a plane elastic plate

Flexure of a plane elastic plate. bent to a shape differing infinitely little from the plane, by any forces subject to certain conditions stated below (§ 632). Some definitions and preliminary considerations may be conveniently taken first.

Definitions. (1) A *surface of a solid* is a surface passing through always the same particles of the solid, however it is strained.

(2) The middle surface of a plate is the surface passing through all those of its particles which, when it is free from stress, lie in a plane midway between its two plane sides.

(3) A normal section of a plate, or a surface normal to a plate, is a surface which, when the plate is free from stress, cuts its sides and all planes parallel to them at right angles, being therefore, when unstrained, necessarily either a single plane or a cylindrical (or prismatic) surface.

(4) The *deflection* of any point or small part of the plate, is the distance of its middle surface there from the tangent plane to the middle surface at any conveniently chosen point of reference in it.

(5) The *inclination* of the plate, at any point, is the inclination of the tangent plane of the middle surface there to the tangent plane at the point of reference.

(6) The *curvature of a plate* at any point, or in any part, is the curvature of its middle surface there.

(7) In a surface infinitely nearly plane the curvature is said to be *uniform,* if the curvatures in every two parallel normal sections are equal.

(8) Any diameter of a plate, or distance in a plate infinitely nearly plane, is called finite, unless it is an infinitely great multiple of the least radius of curvature multiplied by the greatest inclination.

Geometrical preliminaries. Choosing XOY as the tangent plane at the point of reference, let (x, y, z) be any point of its middle surface, i its inclination there, and $\frac{1}{r}$ its curvature in a normal section through that

point, inclined at an angle ϕ to ZOX. We have

Geometrical preliminaries.

$$\tan i = \sqrt{\left(\frac{dz^2}{dx^2} + \frac{dz^2}{dy^2}\right)} \quad\text{.........................(1)},$$

and, if i be infinitely small,

$$\frac{1}{r} = \frac{d^2z}{dx^2}\cos^2\phi + 2\frac{d^2z}{dxdy}\sin\phi\cos\phi + \frac{d^2z}{dy^2}\sin^2\phi \quad\text{..... (2)}.$$

To prove these, let ξ, η, ζ be the co-ordinates of any point of the surface infinitely near (x, y, z). Then, by the elements of the differential calculus,

$$\zeta = \frac{dz}{dx}\xi + \frac{dz}{dy}\eta + \tfrac{1}{2}\left(\frac{d^2z}{dx^2}\xi^2 + 2\frac{d^2z}{dxdy}\xi\eta + \frac{d^2z}{dy^2}\eta^2\right).$$

Let $\xi = \rho\cos\phi, \quad \eta = \rho\sin\phi,$

so that we have

$$\left.\begin{aligned}\zeta &= A\rho + \tfrac{1}{2}B\rho^2, \text{ where } A = \frac{dz}{dx}\cos\phi + \frac{dz}{dy}\sin\phi \\ \text{and } B &= \frac{d^2z}{dx^2}\cos^2\phi + 2\frac{d^2z}{dxdy}\sin\phi\cos\phi + \frac{d^2z}{dy^2}\sin^2\phi\end{aligned}\right\} \text{.......(3)}.$$

Then by the formula for the curvature of a plane curve (§ 9),

$$\frac{1}{r} = \frac{B}{(1+A^2)^{\frac{3}{2}}}, \text{ or, as } A \text{ is infinitely small, } \frac{1}{r} = B,$$

and thus (2) is proved.

It follows that the surface represented by

$$z = \tfrac{1}{2}(Ax^2 + 2cxy + By^2) \text{.....................(4)},$$

is a surface of uniform curvature if A, B, c be constant throughout the admitted range of values of (x, y); these being limited by the condition that $Ax + cy$, and $cx + By$ must be everywhere infinitely small.

628. When a plane surface is bent to any other shape than a developable surface (§ 139), it must experience some degree of stretching or contraction. But an essential condition for the theory of elastic plates on which we are about to enter, is that the amount of the stretching or contraction thus *necessary* in the middle surface is at most incomparably smaller than the stretching and contraction of the two sides (§ 141) due to curvature. It will be shown in § 629 that this condition, if we exclude the case of bending into a surface differing infinitely little from a developable surface, is equivalent to the following:—

Limitation of flexure not to imply a stretching of middle surface comparable with that of either side.

The deflection [§ 627 Def. (4)] *is, at all places finitely* [§ 627 Def. (8)] *distant from the point of reference, incomparably smaller than the thickness.*

And if we extend the signification of "deflection" from that defined in (4) of § 627, to distance from some true developable surface, the excluded case is of course brought under the statement.

Although the truth of this is obvious, it is satisfactory to prove it by investigating the actual degrees of stretching and contraction referred to.

Stretching of a plane by synclastic or anticlastic flexure.

629. Let us suppose a given plane surface to be bent to some curved form without any stretching or contracting of lines radiating from some particular point of it, O; and let it be required to find the stretching or contraction in the circumference of a circle described from O as centre, with any radius a, on the unstrained plane. If the stretching in each part of the circumference, and not merely on the whole, is to be found, something more as to the mode of the bending must be specified; which, for simplicity, in the first place, we shall suppose to be, that any point P of the given surface moves in a plane perpendicular to the tangent plane through O, during the straining.

Let a, θ be polar co-ordinates of P in its primitive position, and r, θ those of the projection on the tangent plane through O, of its position in the bent surface, and let z be the distance of this position from the tangent plane through O. An element, $ad\theta$, of the unstrained circle, becomes

$$(r^2d\theta^2+dr^2+dz^2)^{\frac{1}{2}}$$

on the bent surface; and, therefore, for the stretching* of this element we have

$$\epsilon=\left(\frac{r^2}{a^2}+\frac{dr^2}{a^2d\theta^2}+\frac{dz^2}{a^2d\theta^2}\right)^{\frac{1}{2}}-1 \quad\text{..................(1).}$$

* Ratio of the elongation to the unstretched length.

Stretching of a plane by synclastic or anticlastic flexure.

Hence if e denote the ratio of the elongation of the whole circumference to its unstretched length, or the mean stretching of the circumference,

$$e=\frac{1}{2\pi}\int_0^{2\pi}d\theta\left\{\left(\frac{r^2}{a^2}+\frac{dr^2}{a^2d\theta^2}+\frac{dz^2}{a^2d\theta^2}\right)^{\frac{1}{2}}-1\right\} \quad\text{............(2),}$$

where we must suppose z and r known functions of θ. Confining ourselves now to distances from O within which the curvature of the surface is sensibly uniform, we have

$$z=\frac{a^2}{2\rho},\ \text{ and }\ r=\rho\sin\frac{a}{\rho}=a\left(1-\tfrac{1}{6}\frac{a^2}{\rho^2}+\text{etc.}\right) \quad\text{.........(3),}$$

if ρ be the radius of curvature of the normal section through O and P: and, if we take as the zero line for θ that in which the tangent plane is cut by one of the principal normal planes (§ 130),

$$\frac{1}{\rho}=\frac{1}{\rho_1}\cos^2\theta+\frac{1}{\rho_2}\sin^2\theta=\tfrac{1}{2}\left(\frac{1}{\rho_1}+\frac{1}{\rho_2}\right)+\tfrac{1}{2}\left(\frac{1}{\rho_1}-\frac{1}{\rho_2}\right)\cos 2\theta \quad\text{...(4),}$$

where ρ_1, ρ_2 are the principal radii of curvature. Hence the term $dr^2/a^2d\theta^2$ under the radical sign disappears if we include no terms involving higher powers than the first, of the small fraction a^2/ρ^2; and, to this degree of approximation

$$\epsilon=\left\{1-\tfrac{1}{3}\frac{a^2}{\rho^2}+a^2\left(\frac{1}{\rho_2}-\frac{1}{\rho_1}\right)^2\sin^2\theta\cos^2\theta\right\}^{\frac{1}{2}}-1=-\tfrac{1}{6}\frac{a^2}{\rho^2}+\frac{a^2}{2}\left(\frac{1}{\rho_2}-\frac{1}{\rho_1}\right)^2\sin^2\theta\cos^2\theta,$$

or, by (4), and reductions, finally

$$\epsilon=-\tfrac{1}{6}a^2\left\{\left(\frac{1}{\rho_1\rho_2}+\tfrac{1}{2}\left(\frac{1}{\rho_1^2}-\frac{1}{\rho_2^2}\right)\cos 2\theta+\tfrac{1}{2}\left(\frac{1}{\rho_1}-\frac{1}{\rho_2}\right)^2\cos 4\theta\right\} \quad\text{...(5).}$$

Using this in (2) we find

$$e=-\tfrac{1}{6}\frac{a^2}{\rho_1\rho_2} \quad\text{..............................(6).}$$

The whole amount of stretching thus expressed will, it follows from (5), be distributed uniformly through the circumference, if, instead of compelling each point P to remain in the plane through O, perpendicular to XOY, we allow it to yield in the direction of the circumference through a space equal to

$$\frac{a^3}{24}\left\{\left(\frac{1}{\rho_1^2}-\frac{1}{\rho_2^2}\right)\sin 2\theta+\tfrac{1}{2}\left(\frac{1}{\rho_1}-\frac{1}{\rho_2}\right)^2\sin 4\theta\right\} \quad\text{..........(7).}$$

From (6) we conclude that

Stretching of a plane by synclastic or anticlastic flexure.

630. If a plane area be bent to a uniform degree of curvature throughout, without any stretching in any radius through a certain point of it, and with uniform stretching or contraction over the circumference of every circle described from the same point as centre, the amount of this contraction (reckoned negative where the actual effect is stretching) is equal to the ratio of one-sixth of the square of the radius of the circle, to the rectangle under the maximum and minimum radii of curvature of normal sections of the surface; or which is the same thing, the ratio of two-thirds of the rectangle under the maximum and minimum deflections of the circumference from the tangent plane of the surface at the centre, to the square of the radius; or, which is the same, the ratio one-third of the maximum deflection to the maximum radius of curvature.

If the surface thus bent be the middle surface of a plate of uniform thickness, and if each line of particles perpendicular to this surface in the unstrained plate remain perpendicular to it when bent, the stretching on the convex side, and the contraction on the concave side, in any normal section, is obviously equal to the ratio of half the thickness, to the radius of curvature. The comparison of this, with the last form of the preceding statement, proves that the second of the two conditions stated in § 628 secures the fulfilment of the first.

Stretching of a curved surface by flexure not fulfilling Gauss's condition.

631. If a surface already bent as specified, be again bent to a different shape still fulfilling the prescribed conditions, or if a surface given curved be altered to any other shape by bending according to the same conditions, the contraction produced in the circumferences of the concentric circles by this bending, will of course be equal to the increment in the value of the ratio stated in the preceding section. Hence if a curved surface be bent to any other figure, without stretching in any part of it, the rectangle under the two principal radii of curvature at every point remains unchanged. This is Gauss's celebrated theorem regarding the bending of curved surfaces, of which we gave a more elementary demonstration in our introductory Chapter (see §§ 138, 150).

Gauss's theorem regarding flexure.

Limitations as to the forces and flexures to be admitted in elementary theory of elastic plate.

632. Without further preface we now commence the theory of the flexure of a plane elastic plate with the promised (§ 627) statement of restricting conditions.

(1) Of the forces applied from without to any part of the plate, bounded by a normal surface [§ 627 (3)], the components parallel to any line in the plane of the plate are either evanescent or are reducible to *couples*. In other words the algebraic sum of such components, for any part of the plate bounded by a normal surface, is zero.

(2) The principal radii of curvature of the middle surface are everywhere infinitely great multiples of the thickness of the plate.

(3) The deflection is nowhere, within finite distance from the point of reference, more than an infinitely small fraction of the thickness. This condition has a definite meaning for an infinitely large plate, which may be explained thus:—it would be necessary to go to a distance equal to a large multiple of the product of the least radius of curvature into the greatest inclination, to reach a place where the deflection is more than a very small fraction of the thickness of the plate. The consideration of this condition, is of great importance in connection with the theory of the propagation of waves through an infinite plane elastic plate, but scarcely belongs to our present subject.

(4) Neither the thickness of the plate nor the moduluses of elasticity of its substance need be uniform throughout, but if they vary at all they must vary continuously from place to place; and must not any of them be incomparably greater in one place than in another within any finite area of the plate.

Results of general theory stated in advance.

633. The general theory of elastic solids investigated later shows that when these conditions are fulfilled the distribution of strain through the plate possesses the following properties, the statement of which at present, although not necessary for the particular problem on which we are entering, will promote a thorough understanding and appreciation of the principles involved.

Results of general theory stated in advance.

(1) The stretching of any part of the middle surface is infinitely small in comparison with that of either side, in every part of the plate where the curvature is finite.

(2) The particles in any straight line perpendicular to the plate when plane, remain in a straight line perpendicular to the curved surfaces into which its sides, and parallel planes of the substance between them, become distorted when it is bent. And hence the curves in which these surfaces are cut by any plane through that line, have one point in it for centre of curvature of them all.

(3) The whole thickness of the plate remains unchanged, at every point; but the half thickness on one side (which when the curvature is synclastic is the convex side) of the middle surface becomes diminished and on the other side increased, by equal amounts comparable with the elongations and shortenings of lengths equal to the half thickness, measured on the two side surfaces of the plate.

634. The conclusions from the general theory on which we shall found the equations of equilibrium and motion of an elastic plate are as follows:—

Laws for flexure of elastic plate assumed in advance.

Let a naturally plane plate be bent to any surface of uniform curvature [§ 627 (7)] throughout, the applied forces and the extents of displacement fulfilling the conditions and restrictions of § 632: Then—

(1) The force across any section of the plate is, at each point of it, in a line parallel to the tangent plane to the middle surface in the neighbourhood.

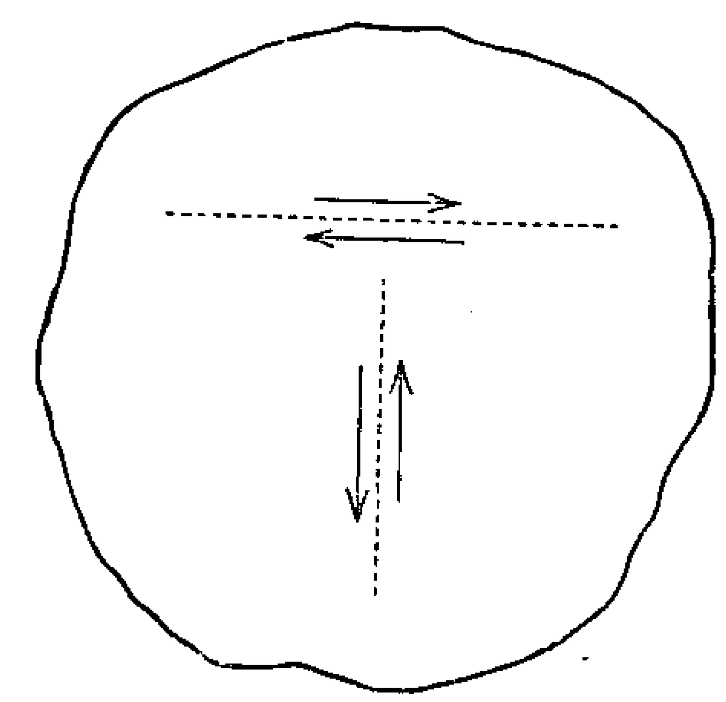
(2) The forces across any set of parallel normal sections are equally inclined to the directions of the normal sections at all points (that is to say, are in directions which would be parallel if the plate were bent, and which deviate actually from parallelism only by the infinitely small deviations produced in the normal sections of the flexure).

(3) The amounts of force across one normal section, or any

set of parallel normal sections, on equal infinitely small areas, are simply proportional to the distances of these areas from the middle surface of the plate.

Laws for flexure of elastic plate assumed in advance.

(4) The component forces in the tangent planes of the normal sections are equal and in dissimilar directions in sections which are perpendicular to one another. For proof, see § 661. The meaning of "dissimilar directions" in this expression is explained by the diagram; where the arrow-heads indicate the directions in which the portions of matter on the two sides of each normal section would yield if the substance were actually divided, half way through the plate from one side, by each of the normal sections indicated by dotted lines.



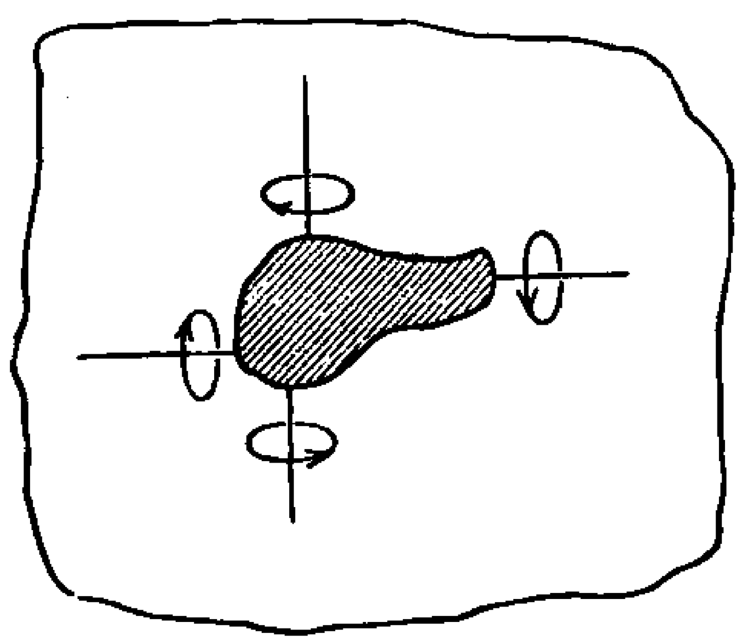
(5) By the law of superposition, we see that if the applied forces be all doubled, or altered in any other ratio, the curvature in every normal section, and all the internal forces specified in (1), (2), (3), (4), are changed in the same ratio; and the potential energy of the internal forces becomes changed according to the square of the same ratio.

635. From § 634 (3) it follows immediately that the forces experienced by any portion of the plate bounded by a normal section through the circumference of a closed polygon or curve of the middle surface, from the action of the contiguous matter of the plate all round it, may be reduced to a set of couples by taking them in groups over infinitely small rectangles into which the bounding normal section may be imagined as divided by normal lines. From § 634 (2) it follows that the distribution of couple thus obtained is uniform along each straight portion, if any there is, of the boundary, and equal per equal lengths in all parallel parts of the boundary.

Stress-couple acting across a normal section.

Twisting components proved equal round any two perpendicular axes.

636. From § 634 (4) it follows that the component couples round axes perpendicular to the boundary are equal in parts of the boundary at right angles to one another, and are in

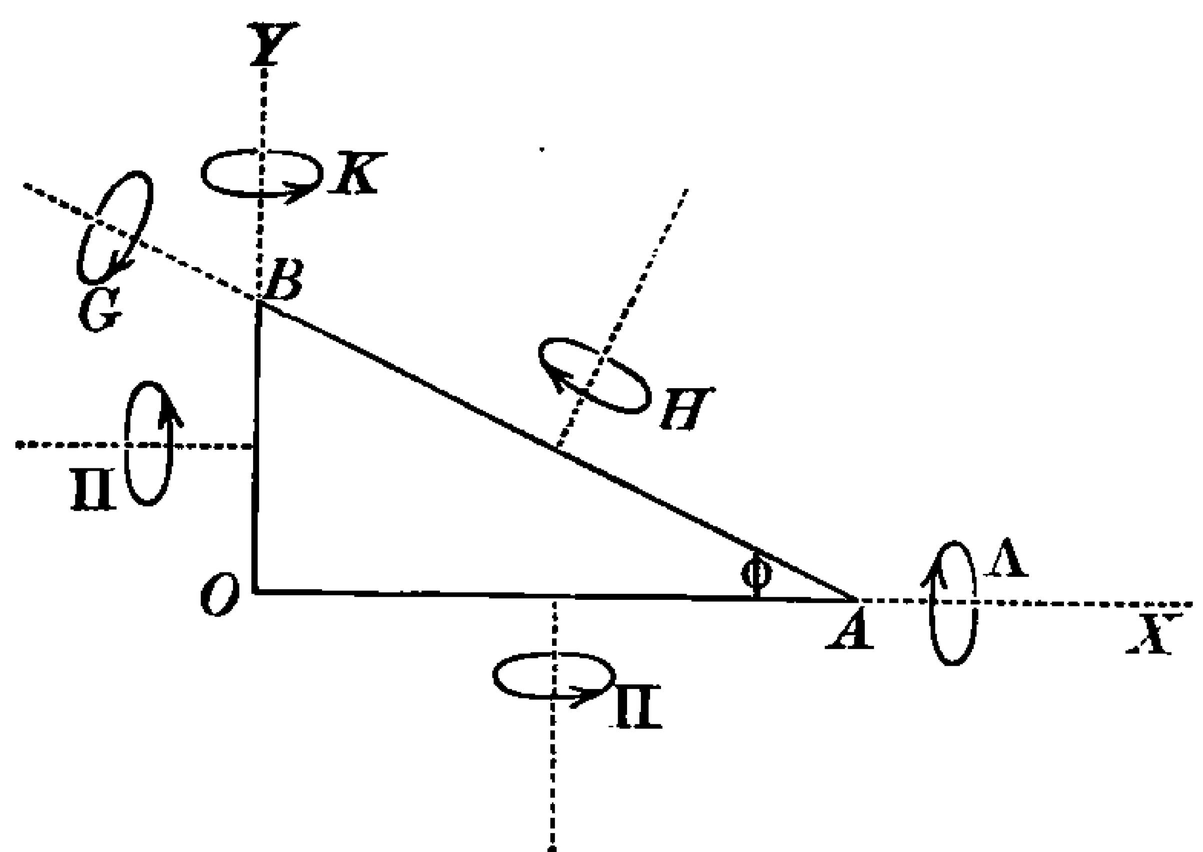


directions related to one another in the manner indicated by the circular arrows in the diagram; that is to say, in such directions that if the axis is, according to the rule of § 234, drawn *outwards* from the portion of the plate under consideration, for one point of the boundary it must be drawn *inwards* for every point where the boundary is perpendicular to its direction at that point.

Principal axes of bending stress.

637. We may now prove that there are two normal sections, at right angles to one another, in which the component couples round axes perpendicular to them vanish, and that in these sections the component couples round axes coincident with the sections are of maximum and minimum values.

Let OAB be a right-angled triangle of the plate. Let Λ and Π be the two component couples acting on the side OA; K and Π those on the side OB; and G and H those on the side AB; the amount of each couple being reckoned per unit of length of the side on which it acts, and the axes and directions of the several couples being as indicated by the circular arrows when each is reckoned as positive. Then, if $AB=a$, and $BAO=\phi$, the whole amounts of the couples on the three sides are respectively



Principal axes of bending stress investigated.

$$\Lambda a \cos\phi,\quad \Pi a \cos\phi,$$
$$\mathrm{K} a \sin\phi,\quad \Pi a \sin\phi,$$
$$Ga,\quad Ha.$$

Resolving the two latter round OX and OY, we have

$$Ga\cos\phi - Ha\sin\phi \text{ round } OX,$$
and
$$Ga\sin\phi + Ha\cos\phi \quad ,, \quad OY.$$

But if the portion in question, of the plate, were to become rigid, its equilibrium would not be disturbed (§ 564); and therefore we must have

$$\left.\begin{array}{l} Ga\cos\phi - Ha\sin\phi = \Lambda a\cos\phi + \Pi a\sin\phi \text{ by couples round } OX \\ \text{and} \\ Ga\sin\phi + Ha\cos\phi = \mathrm{K}a\sin\phi + \Pi a\cos\phi \quad ,, \quad ,, \quad OY \end{array}\right\} (1).$$

From these we find immediately

$$\left.\begin{array}{l} G = \Lambda\cos^2\phi + 2\Pi\sin\phi\cos\phi + \mathrm{K}\sin^2\phi, \\ H = (\mathrm{K}-\Lambda)\sin\phi\cos\phi + \Pi(\cos^2\phi - \sin^2\phi) \end{array}\right\} \text{.........(2).}$$

Hence the values of ϕ, which make H vanish, give to G its maximum and minimum values, and being determined by the equation

$$\tan 2\phi = -\frac{\Pi}{\frac{1}{2}(\mathrm{K}-\Lambda)} \text{.........................(3),}$$

differ from one another by $\frac{1}{2}\pi$.

A modification of these formulæ, which we shall find valuable, is obtained by putting

$$\Sigma = \tfrac{1}{2}(\mathrm{K}+\Lambda),\quad \Theta = \tfrac{1}{2}(\mathrm{K}-\Lambda) \text{..................(4).}$$

This reduces (2) to

$$\left.\begin{array}{l} G = \Sigma + \Pi\sin 2\phi - \Theta\cos 2\phi \\ H = \quad\ \Pi\cos 2\phi + \Theta\sin 2\phi \end{array}\right\} \text{...............(5),}$$

which again become

$$\left.\begin{array}{l} G = \Sigma + \Omega\cos 2(\phi - a) \\ H = -\Omega\sin 2(\phi - a) \end{array}\right\} \text{.....................(6),}$$

where a [being a value of ϕ given by (3)], and Ω are taken so

$$\left.\begin{array}{ll} \text{that} & \Pi = \Omega\sin 2a,\quad \Theta = -\,\Omega\cos 2a, \\ \text{so that, of course,} & \Omega = (\Pi^2 + \Theta^2)^{\frac{1}{2}} \end{array}\right\} \text{..............(7).}$$

This analysis demonstrates the following convenient synthesis of the whole system of internal force in question:—

Synclastic and anticlastic stresses defined.

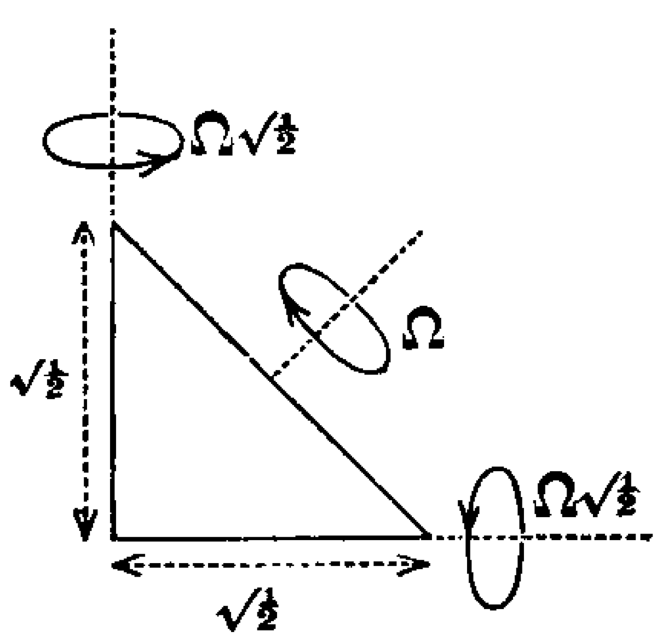
638. The action experienced by each part of the plate, in virtue of the internal forces between it and the surrounding contiguous matter of the plate, being called a *stress* [in accordance with the general use of this term defined below (§ 658)], may be regarded as made up of two distinct elements—(1) a synclastic stress, and (2) an anticlastic stress; as we shall call them.

(1) Synclastic stress consists of equal direct bending action round every straight line in the plane of the plate. Its amount may be conveniently regarded as measured by the amount, Σ, of the mutual couple between the portions of matter on the two sides of any straight normal section of unit length. Its effect would be to produce equal curvature in all normal sections (that is to say, a spherical figure) if the plate were equally flexible in all directions.

Anticlastic stress referred to its principal axes;

(2) Anticlastic stress consists of two simple bending stresses of equal amounts in opposite directions round two sets of parallel straight lines perpendicular to one another in the plane of the plate. Its effect would be uniform anticlastic curvature, with equal convexities and concavities, if the plate were equally flexible in all directions. Its amount is reckoned as the amount, Ω, of the mutual couple between the portions of matter on the two sides of a straight normal section of unit length, parallel to either of these two sets of lines. It gives rise to couples of the same amount, Ω, between the portions of matter on each side of a normal section of unit length parallel to either of the sets of lines bisecting the right angles between those; but the couples now referred to are *in* the plane of the normal section instead of perpendicular to it. This is proved and illustrated by the annexed diagram, representing [a particular case of the diagram and equations (1) of § 637] the equilibrium of an isosceles right-angled triangle under the influence of couples, each equal to $\Omega\sqrt{\tfrac{1}{2}}$, applied to it round axes coinciding with

referred to axes inclined to them at 45°.



its legs, and a third couple, Ω round an axis perpendicular to its hypotenuse.

Octantal resolution and composition of anticlastic stress.

If two pairs of rectangular axes, each bisecting the right angles formed by the other, be chosen as axes of reference, an anticlastic stress having any third pair of rectangular lines for its axes may, as the preceding formulæ [§ 637 (5)] show, be resolved into two having their axes coincident with the two pairs of axes of reference respectively, by the ordinary cosine formula with each angle doubled. Hence it follows that any two anticlastic stresses may be compounded into one by the same geometrical construction as the parallelogram of forces, made upon lines inclined to one another at an angle equal to twice that between the corresponding axes of the two given stresses; and the position of the axes of the resultant stress will be indicated by the angles of this diagram each halved.

Construction by parallelogram.

Geometrical analogues.

639. Precisely the same set of statements are of course applicable to the curvature of a surface. Thus the proposition proved in § 637 (3) for bending stresses has, for its analogue in curvature, Euler's theorem proved formerly in § 130; and analogues to the series of definitions and propositions founded on it and derived from it may be at once understood without more words or proof.

Two cylindrical curvatures round perpendicular axes, and an anticlastic curvature round axis bisecting their right angles;

Let $$z = \tfrac{1}{2}(\kappa x^2 + 2\varpi xy + \lambda y^2) \quad\ldots\ldots(1)$$

be the equation of a curved surface infinitely near a point O at which it is touched by the plane YOX. Its curvature may be regarded as compounded of a cylindrical curvature, λ, with axis parallel to OX, a cylindrical curvature, κ, with axis parallel to OY, and an anticlastic curvature, ϖ, with axis bisecting the angles XOY, YOX'. Thus, if ϖ and λ each vanished, the surface would be cylindrical, with $1/\kappa$ for radius of curvature and generating lines parallel to OY. Or, if κ and λ each vanished, there would be anticlastic curvature, with sections of equal maximum curvature in the two directions, bisecting the angles XOY and YOX', and radius of curvature in those sections equal to $1/\varpi$.

If now we put

$$\sigma = \tfrac{1}{2}(\kappa + \lambda), \quad \vartheta = \tfrac{1}{2}(\kappa - \lambda) \quad\ldots\ldots(2),$$

or a spherical curvature and two anticlastic curvatures;

the equation of the surface becomes

$$z=\tfrac{1}{2}\{\sigma(x^2+y^2)+\vartheta(x^2-y^2)+2\varpi xy\}\quad\ldots\ldots(3);$$

or, if

$$\left.\begin{array}{l} x=r\cos\phi,\ y=r\sin\phi, \\ z=\tfrac{1}{2}\{\sigma+\vartheta\cos 2\phi+\varpi\sin 2\phi\}r^2 \end{array}\right\}\quad\ldots\ldots(4);$$

or a spherical and one anticlastic curvature.

or, lastly,

$$\left.\begin{array}{l} z=\tfrac{1}{2}\{\sigma+\omega\cos 2(\phi-\alpha)\}r^2, \\ \vartheta=\omega\cos 2\alpha,\ \varpi=\omega\sin 2\alpha \end{array}\right\}\quad\ldots\ldots(5).$$

In these formulæ σ measures the spherical curvature; and ϑ and ϖ two components of anticlastic curvature, referred to the pair of axes $X'X$, $Y'Y$, and the other pair bisecting their angles. The resultant of ϑ and ϖ is an anticlastic curvature ω, with axes inclined, in the angle XOY at angle α to OX, and in YOX' at angle α to OY.

Work done in bending.

640. The notation of §§ 637, 639 being retained, the work done on any area A of the plate experiencing a change of curvature $(\delta\kappa, \delta\lambda, \delta\varpi)$ under the action of a stress $(\mathrm{K}, \Lambda, \Pi)$, is

$$(\mathrm{K}\delta\kappa+\Lambda\delta\lambda+2\Pi\delta\varpi)A\quad\ldots\ldots(1);$$

or

$$(2\Sigma\delta\sigma+2\Theta\delta\vartheta+2\Pi\delta\varpi)A\quad\ldots\ldots(2),$$

if, as before,

$$\Sigma=\tfrac{1}{2}(\mathrm{K}+\Lambda),\ \Theta=\tfrac{1}{2}(\mathrm{K}-\Lambda),\ \sigma=\tfrac{1}{2}(\kappa+\lambda),\ \vartheta=\tfrac{1}{2}(\kappa-\lambda)\ldots(3).$$

Let $PQP'Q'$ be a rectangular portion of the plate with its centre at O, and its sides $Q'P$, $P'Q$ parallel to OX, and $Q'P'$, PQ parallel to OY. If

$$z=\tfrac{1}{2}(\kappa x^2+2\varpi xy+\lambda y^2)$$

be the equation of the curved surface, we have

$$\frac{dz}{dx}=\kappa x+\varpi y,\quad \frac{dz}{dy}=\varpi x+\lambda y;$$

and therefore the tangent plane at (x, y) deviates in direction from XOY by an infinitely small rotation

$$\left.\begin{array}{ll} \kappa x+\varpi y & \text{round } OY \\ \text{and}\quad \varpi x+\lambda y & \quad\text{,,}\quad OX \end{array}\right\}\quad\ldots\ldots(4).$$

Hence the rotation from XOY to the mean tangent plane for all points of the side PQ or $Q'P'$ is

$$\mp\tfrac{1}{2}Q'P\,.\,\kappa \text{ round } OY,$$

and

$$\mp\tfrac{1}{2}Q'P\,.\,\varpi\quad\text{,,}\quad OX.$$

Work done in bending.

Hence if the tangent plane, XOY, at O remains fixed, while the curvature changes from $(\kappa, \varpi, \lambda)$ to $(\kappa+\delta\kappa, \varpi+\delta\varpi, \lambda+\delta\lambda)$, the work done by the couples $PQ\,.\,\mathrm{K}$ round OY, and $PQ\,.\,\Pi$ round OX, distributed over the side PQ, will be

$$\tfrac{1}{2}Q'P\,.\,PQ\,.\,(\mathrm{K}\delta\kappa+\Pi\delta\varpi),$$

and an equal amount will be done by the equal and opposite couples distributed over the side $Q'P'$ undergoing an equal and opposite rotation. Similarly, we find for the whole work done on the sides $P'Q$ and $Q'P$,

$$PQ\,.\,Q'P\,.\,(\Pi\delta\varpi+\Lambda\delta\lambda).$$

Hence the whole work done on all the four sides of the rectangle is

$$PQ\,.\,Q'P\,.\,(\mathrm{K}\delta\kappa+2\Pi\delta\varpi+\Lambda\delta\lambda):$$

whence the proposition to be proved, since any given area of the plate may be conceived as divided into infinitely small rectangles.

It is an instructive exercise to verify the result by beginning with the consideration of a portion of plate bounded by any given curve, and using the expressions (1) of § 637, by which we find, for the couples on any infinitely short portion, ds, of its boundary, specified in position by (x, y),

$$\left.\begin{array}{ll} \left(-\Lambda\dfrac{dx}{ds}+\Pi\dfrac{dy}{ds}\right)ds & \text{round } OX \\ \text{and}\quad \left(\mathrm{K}\dfrac{dy}{ds}-\Pi\dfrac{dx}{ds}\right)ds & \quad\text{,,}\quad OY \end{array}\right\}\quad\ldots\ldots(5).$$

But, as we have just seen in (4), the rotation experienced by the tangent plane to the plate at (x, y), when the curvature changes from $(\kappa, \varpi, \lambda)$ to $(\kappa+\delta\kappa, \varpi+\delta\varpi, \lambda+\delta\lambda)$, is

$$\left.\begin{array}{ll} x\delta\varpi+y\delta\lambda & \text{round } OX \\ \text{and}\quad x\delta\kappa+y\delta\varpi & \quad\text{,,}\quad OY \end{array}\right\}\quad\ldots\ldots(6),$$

the tangent plane to the plate at O being supposed to remain unchanged in position; and therefore the work done on the portion ds of the edge is

$$\left\{\left(\mathrm{K}\frac{dy}{ds}-\Pi\frac{dx}{ds}\right)(x\delta\kappa+y\delta\varpi)+\left(\Pi\frac{dy}{ds}-\Lambda\frac{dx}{ds}\right)(x\delta\varpi+y\delta\lambda)\right\}ds.$$

The required work, being the integral of this over the whole of the bounding curve, is therefore

$$(\mathrm{K}\delta\kappa+2\Pi\delta\varpi+\Lambda\delta\lambda)A;$$

Work done in bending.

since $$\int x \frac{dy}{ds}\, ds = -\int y \frac{dx}{ds}\, ds = A,$$

and $$\int x \frac{dx}{ds}\, ds = 0, \quad \int y \frac{dy}{ds}\, ds = 0,$$

each integral being round the whole closed curve.

Partial differential equations for work done in bending an elastic plate.

641. Considering now the elastic forces called into action by the flexure $(\kappa, \varpi, \lambda)$ reckoned from the unstressed condition of the plate (plane, or infinitely nearly plane), and denoting by w the whole amount of their potential energy, per unit area of the plate, we have, as in the case of the wire treated in § 594,

$$\mathrm{K}\delta\kappa = \delta_\kappa w,\ \ \Lambda\delta\lambda = \delta_\lambda w,\ \ 2\Pi\delta\varpi = \delta_\varpi w \text{.........}(7);$$

or, according to the other notation,

$$2\Sigma\delta\sigma = \delta_\sigma w,\ \ 2\Theta\delta\vartheta = \delta_\vartheta w,\ \ 2\Pi\delta\varpi = \delta_\varpi w \text{.......}(8);$$

where, as above explained, K and Λ denote the simple bending stresses (measured by the amount of bending couple, per unit of length) round lines parallel to OY and OX respectively: Π the anticlastic stress with axes at 45° to OX and OY: and Σ and Θ the synclastic stress and the anticlastic stress with OX and OY for axes, together equivalent to K and Λ. Also, as in § 595, we see that whatever be the character, eolotropic or isotropic, § 677, of the substance of the plate, it must be a homogeneous quadratic function of the three components of curvature, whether $(\kappa, \lambda, \varpi)$ or $(\sigma, \vartheta, \varpi)$. From this and (7), or (8), it follows that the coefficients in the linear functions of the three components of curvature which express the components of the stress required to maintain it, must fulfil the ordinary conservative relations of equality in three pairs, reducing the whole number from nine to six.

Potential energy of an elastic plate held bent.

Thus A, B, C, a, b, c denoting six constants depending on the quality of the solid substance and the thickness of the plate, we have $w = \frac{1}{2}(A\kappa^2 + B\lambda^2 + C\varpi^2 + 2a\lambda\varpi + 2b\varpi\kappa + 2c\kappa\lambda)$.......(9); and hence, by (7),

$$\left.\begin{aligned} \mathrm{K} &= A\kappa + c\lambda + b\varpi \\ \Lambda &= c\kappa + B\lambda + a\varpi \\ 2\Pi &= b\kappa + a\lambda + C\varpi \end{aligned}\right\} \text{........................}(10).$$

Potential energy of an elastic plate held bent.

Transforming these by § 640 (3) we have, in terms of $\sigma, \vartheta, \varpi$,

$$\begin{aligned} w = \tfrac{1}{2}\{&(A+B+2c)\sigma^2 + (A+B-2c)\vartheta^2 + C\varpi^2 \\ &+ 2(b-a)\vartheta\varpi + 2(b+a)\sigma\varpi + 2(A-B)\sigma\vartheta\}\text{......}(11), \end{aligned}$$

and $$\left.\begin{aligned} 2\Sigma &= (A+B+2c)\sigma + (A-B)\vartheta + (b+a)\varpi \\ 2\Theta &= (A-B)\sigma + (A+B-2c)\vartheta + (b-a)\varpi \\ 2\Pi &= (b+a)\sigma + (b-a)\vartheta + C\varpi \end{aligned}\right\} \text{.........}(12).$$

These second forms are chiefly useful as showing immediately the relations which must be fulfilled among the coefficients for the important case considered in the following section.

Case of equal flexibility in all directions.

642. If the plate be equally flexible in all directions, a synclastic stress must produce spherical curvature: an anticlastic stress having any pair of rectangular lines in the plate for its axes must produce anticlastic curvature having these lines for sections of equal greatest curvature on the opposite sides of the tangent plane: and in either action the amount of the curvature is simply proportional to the amount of the stress. Hence if $\mathfrak{h}$ and $\mathfrak{k}$ denote two coefficients depending on the bulk-modulus and rigidity of the substance if isotropic (see §§ 677, 680, below), and on the thickness of the plate, we have

Synclastic and anticlastic rigidities of a plate.

$$\Sigma = \mathfrak{h}\sigma,\ \ \Theta = \mathfrak{k}\vartheta,\ \ \pi = \mathfrak{k}\varpi \text{.............}(13).$$

And therefore [§ 640 (2)]

$$w = \mathfrak{h}\sigma^2 + \mathfrak{k}(\vartheta^2 + \varpi^2) \text{..................}(14).$$

Hence the coefficients in the general expressions of § 641 fulfil, in the case of equal flexibility in all directions, the following conditions:—

$$a = 0,\ \ b = 0,\ \ A = B,\ \ 2(A - c) = C \text{.............}(15);$$

and the newly-introduced coefficients $\mathfrak{h}$ and $\mathfrak{k}$ are related to them thus:— $$A + c = \mathfrak{h},\ \ \tfrac{1}{2}C = A - c = \mathfrak{k} \text{..................}(16).$$

Plate bent by any forces.

643. Let us now consider the equilibrium of an infinite plate, disturbed from its natural plane by forces applied to it in any way, subject only to the conditions of § 632. The substance may be of any possible quality as regards elasticity in different directions: and the plate itself need not be homogeneous either as to this quality, or as to its thickness, in different parts; provided only that round every point it is in both respects sensibly homogeneous [§ 632 Def. (4)] to distances great in comparison with the thickness at that point.

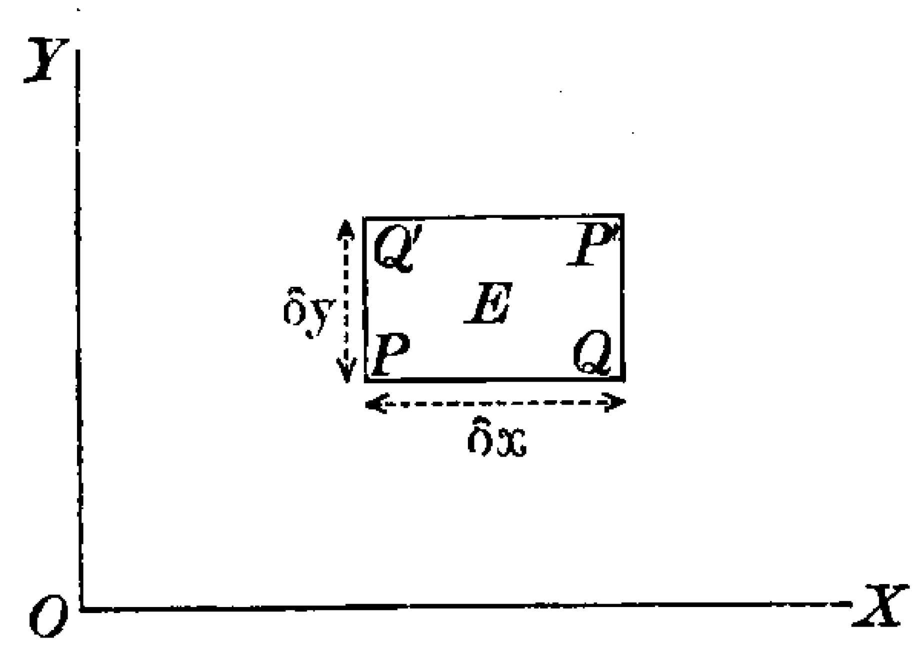
Plate bent by any forces.

644. Let OX, OY be rectangular axes of reference in the plane of the undisturbed plate; and let z be the infinitely small displacement from this plane, of the point (x, y) of the plate, when disturbed by any forces, specified in their effective components as follows:—Take a portion, E, of the plate bounded by a normal surface cutting the middle surface in a line enclosing an infinitely small area σ in the neighbourhood of the point (x, y), and let $Z\sigma$ denote the sum of the component forces perpendicular to XOY on all the matter of E in the neighbourhood of the point (x, y): and $L\sigma$, $M\sigma$ the component couples round OX and OY obtained by transferring, according to Poinsot, the forces from all points of the portion E, supposed for the moment rigid, to one point of it which it is convenient to take at the centre of inertia of the area, σ, of the part of the middle surface belonging to it. This force and these couples, along with the internal forces of elasticity exerted on the matter of E, across its boundary, by the matter surrounding it, must (§ 564) fulfil the conditions of equilibrium for E treated as a rigid body. And E, being not really rigid, must have the curvature due, according to § 641, to the bending stress constituted by the last-mentioned forces. These conditions expressed mathematically supply five equations from which, four elements specifying the internal forces being eliminated, we have a single partial differential equation for z in terms of x and y, which is the required equation of equilibrium.

Conditions of equilibrium.

Equations of equilibrium of plate bent by any forces, investigated.

Let σ be a rectangle $PQP'Q'$, with sides δx parallel to OX and δy parallel to OY. Let $\alpha\delta y$, $\alpha'\delta y$ be the infinitely nearly equal shearing forces perpendicular to the plate in the normal surfaces through PQ' and QP' respectively: and let β, β' be the corresponding notation for PQ, $P'Q'$.



We shall have, of course,

$$\alpha' - \alpha = \frac{d\alpha}{dx}\delta x, \text{ and } \beta' - \beta = \frac{d\beta}{dy}\delta y.$$

The results of these actions on the portion, E, of the plate, considered as rigid, are forces $\alpha'\delta y$, $\beta'\delta x$ through the middle points of QP', $Q'P'$, in the direction of z positive, and forces $\alpha\delta y$, $\beta\delta x$ through the middle points of PQ', PQ, in the direction of z negative. Hence, towards the equilibrium of E as a rigid body, they contribute

Equations of equilibrium of plate bent by any forces, investigated.

$$(\alpha'-\alpha)\delta y+(\beta'-\beta)\delta x, \text{ or } \left(\frac{d\alpha}{dx}+\frac{d\beta}{dy}\right)\delta x\delta y, \text{ component force parallel to } OZ,$$

$$\alpha\delta y \, . \, \delta x \text{ couple round } OY,$$

and $$\beta\delta x \, . \, \delta y \quad ,, \quad ,, \quad OX;$$

(in these last two expressions the difference between α and α' and between β and β' are of course neglected). Again, if K, Λ, Π specify, according to the system of § 637, the bending stress at (x, y), we shall have couples infinitely nearly equal and opposite, on the pairs of opposite sides, of which, estimated in components round OX and OY, the differences, representing the residual turning tendencies on E as a rigid body, are as follows:—

$$\text{round } OX, \begin{cases} \text{from sides } PQ,\ Q'P',\ \dfrac{d\Lambda}{dy}\delta y \, . \, \delta x, \\ \quad ,, \quad ,, \quad PQ',\ QP',\ \dfrac{d\Pi}{dx}\delta x \, . \, \delta y, \end{cases}$$

$$\text{round } OY, \begin{cases} \text{from sides } PQ,\ Q'P',\ \dfrac{d\Pi}{dy}\delta y \, . \, \delta x, \\ \quad ,, \quad ,, \quad PQ',\ QP',\ \dfrac{dK}{dx}\delta x \, . \, \delta y; \end{cases}$$

or in all, $$\text{round } OX, \left(\frac{d\Lambda}{dy}+\frac{d\Pi}{dx}\right)\delta x\delta y,$$

and $$,, \quad OY, \left(\frac{d\Pi}{dy}+\frac{dK}{dx}\right)\delta x\delta y.$$

The equations of equilibrium, therefore, between these and the applied forces on E treated as a rigid body give, if we remove the common factor, $\delta x\delta y$,

Equations of stress in plate bent by any forces.

$$\left.\begin{aligned} Z + \frac{d\alpha}{dx} + \frac{d\beta}{dy} &= 0 \\ L + \beta + \frac{d\Lambda}{dy} + \frac{d\Pi}{dx} &= 0 \\ M + \alpha + \frac{d\Pi}{dy} + \frac{dK}{dx} &= 0 \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots (1).$$

Equations of stress in plate bent by any forces.

The first of these, with α and β replaced in it by their values from the second and third, becomes

$$\frac{d^2K}{dx^2} + 2\frac{d^2\Pi}{dxdy} + \frac{d^2\Lambda}{dy^2} = Z - \frac{dM}{dx} - \frac{dL}{dy} \quad\ldots\ldots\ldots\ldots\ldots (2).$$

Now κ, λ, ϖ denoting component curvatures of the plate, according to the system of § 639, we have of course

$$\kappa = \frac{d^2z}{dx^2}, \quad \lambda = \frac{d^2z}{dy^2}, \quad \varpi = \frac{d^2z}{dxdy} \quad\ldots\ldots\ldots\ldots\ldots\ldots (3),$$

and hence (10) of § 641 give

Equations connecting stress and curvature.

$$\left.\begin{aligned} K &= A\frac{d^2z}{dx^2} + c\frac{d^2z}{dy^2} + b\frac{d^2z}{dxdy} \\ \Lambda &= c\frac{d^2z}{dx^2} + B\frac{d^2z}{dy^2} + a\frac{d^2z}{dxdy} \\ 2\Pi &= b\frac{d^2z}{dx^2} + a\frac{d^2z}{dy^2} + C\frac{d^2z}{dxdy} \end{aligned}\right\} \quad\ldots\ldots\ldots\ldots\ldots\ldots (4).$$

Using these in (2) we find the required differential equation of the disturbed surface. On the general supposition (§ 643) we must regard A, B, C, a, b, c as given functions of x and y. In the important practical case of a homogeneous plate they are constants; and the required equation becomes the linear partial differential equation of the fourth order with constant coefficients, as follows:—

$$A\frac{d^4z}{dx^4} + 2b\frac{d^4z}{dx^3dy} + (C+2c)\frac{d^4z}{dx^2dy^2} + 2a\frac{d^4z}{dxdy^3} + B\frac{d^4z}{dy^4} = Z - \frac{dM}{dx} - \frac{dL}{dy} \quad (5).$$

For the case of equal flexibility in all directions, according to § 642 (13), this becomes

Partial differential equation of the bent surface.

$$\left.\begin{aligned} &A\left(\frac{dz^4}{dx^2} + 2\frac{d^4z}{dx^2dy^2} + \frac{d^4z}{dy^4}\right) = Z - \frac{dM}{dx} - \frac{dL}{dy} \\ \text{or}\quad &A\left(\frac{d^2}{dx^2} + \frac{d^2}{dy^2}\right)^2 z = Z - \frac{dM}{dx} - \frac{dL}{dy} \end{aligned}\right\} \quad\ldots\ldots\ldots (6).$$

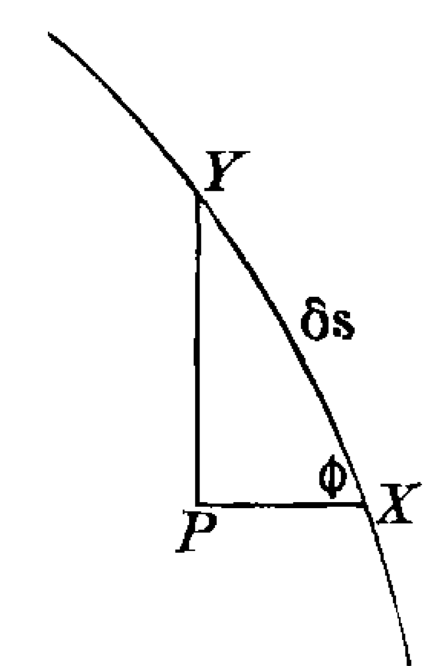
645. To investigate the boundary conditions for a plate of limited dimensions, we may first consider it as forming part of an infinite plate bounded by a normal surface drawn through a closed curve traced on its middle surface. The preceding investigation leads immediately to expressions for the force and couple on any portion of the normal bounding surface. If then

Boundary conditions;

the portion in question be actually cut out from the surrounding sheet, and if a distribution of force and couple identical with that so found be applied to its edge, its elastic condition will remain absolutely unchanged throughout up to the very normal edge. To fulfil this condition requires three equations, expressing (1) that the shearing force applied to the edge (that is, the applied tangential force in the normal surface constituting the edge), which is necessarily in the direction of the normal line to the plate, must be equal to the required amount, and (2) and (3) that the couple applied to any small part of the edge must have components of the proper amounts round any two lines in the plane of the plate. These three equations were given by Poisson as necessary for the full expression of the boundary condition; but Kirchhoff has demonstrated that they express too much, and has shown that two equations suffice. This we shall prove by showing that when a finite plate is given in any condition of stress, or free from stress, we may apply, round axes everywhere perpendicular to its normal surface-edge, any arbitrary distribution of couple without producing any change except at infinitely small distances from the edge, provided a certain distribution of force also, calculated from the distribution of couple, be applied to the edge, perpendicularly to the plate.

Boundary conditions; Poisson's three: two sufficient, proved by Kirchhoff.

Let $XY, = \delta s$, be an infinitely small element at a point (x, y) of a curve traced on the middle surface of an infinite plate; and, PX and PY being parallel to the axes of x and y, let $YXP = \phi$. Then, if $\zeta\delta s$ denote the shearing force in the normal surface to the plate through δs, and, as before (§ 644), $\alpha \,.\, PY$ and $\beta \,.\, PX$ be those in normal surfaces through PY and PX, we must have, for the equilibrium of the triangle YPX supposed rigid (§ 564),



$\zeta\delta s = \alpha \,.\, PY + \beta \,.\, PX$, whence $\zeta = \alpha \sin\phi + \beta\cos\phi$.

Using here for α and β their values by (1) of § 644, we have

Kirchhoff's boundary equations investigated.

$$\zeta = -\left(M + \frac{d\Pi}{dy} + \frac{dK}{dx}\right)\sin\phi - \left(L + \frac{d\Lambda}{dy} + \frac{d\Pi}{dx}\right)\cos\phi \ldots\ldots (1).$$

Kirchhoff's boundary equations investigated.

Next, if $G\delta s$ and $H\delta s$ denote the components round XY, and round an axis perpendicular to it in the plane of the plate, of the couple acting across the normal surface through δs, we have [(2) of § 637],

$$G = \Lambda \cos^2\phi + 2\Pi \sin\phi \cos\phi + \mathrm{K} \sin^2\phi \ldots\ldots\ldots\ldots\ldots (2),$$

$$H = (\mathrm{K} - \Lambda) \sin\phi \cos\phi + \Pi (\cos^2\phi - \sin^2\phi) \ldots\ldots\ldots\ldots (3).$$

If (ζ, G, H) denoted the action experienced by the edge in virtue of applied forces, all the plate outside a closed curve, of which δs is an element, being removed, these three equations would express the same as the three boundary equations given by Poisson. Lastly, let $\mathfrak{Z}\delta s$, $G\delta s$, $\mathfrak{H}\delta s$ denote the force perpendicular to the plate, and the components of couple, actually applied at any point (x, y) of a free edge on the length δs of the middle curve. As we shall immediately see (§ 648), if

$$\mathfrak{Z} - \zeta + \frac{d}{ds}(\mathfrak{H} - H) = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (4),$$

the plate will be in the same condition of stress throughout, except infinitely near the edge, as with (ζ, G, H) for the action on the edge. Hence, eliminating ζ and H between these four equations, there remain to us (2) unchanged and another, or in all these two—

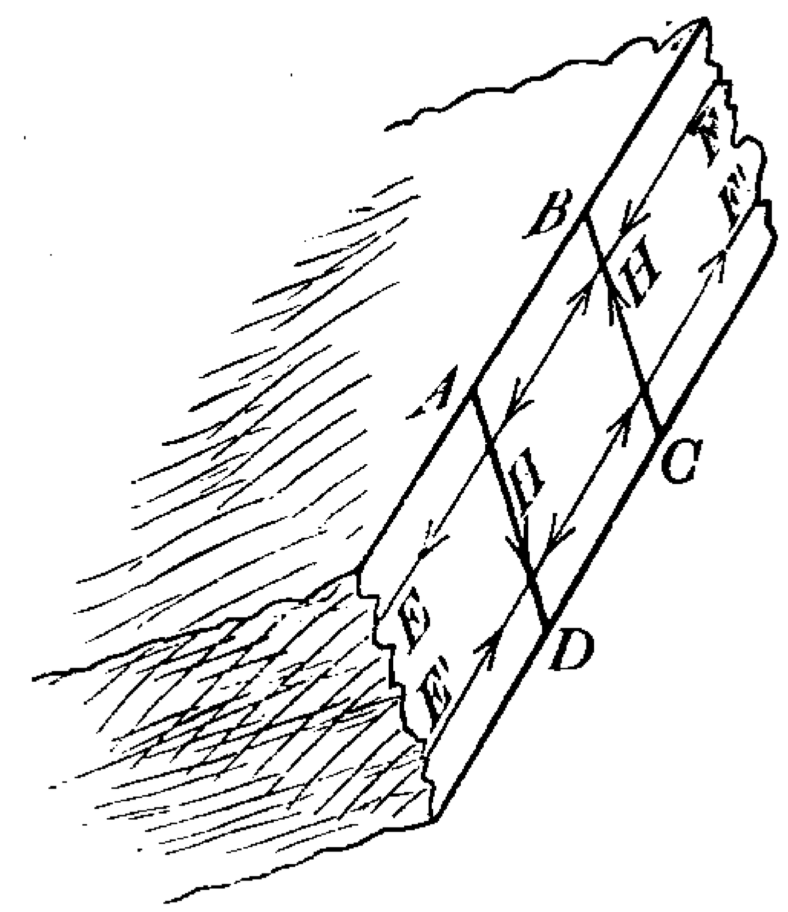
$$\left.\begin{aligned} &G = \Lambda \cos^2\phi + 2\Pi \sin\phi \cos\phi + \mathrm{K} \sin^2\phi, \text{ and} \\ &\mathfrak{Z} + \frac{d\mathfrak{H}}{ds} = -\left(M + \frac{d\Pi}{dy} + \frac{d\mathrm{K}}{dx}\right)\sin\phi - \left(L + \frac{d\Lambda}{dy} + \frac{d\Pi}{dx}\right)\cos\phi + \frac{d}{ds}[(\mathrm{K}-\Lambda)\sin\phi\cos\phi + \Pi(\cos^2\phi - \sin^2\phi)] \end{aligned}\right\} (5),$$

which are Kirchhoff's boundary equations.

Distribution of shearing force determined, which produces same flexure as a given distribution of couple round axes perpendicular to boundary.

646. The proposition stated at the end of last section is equivalent to this:—That a certain distribution of normal shearing force on the bounding edge of a finite plate may be determined which shall produce the same effect as any given distribution of couple, round axes everywhere perpendicular to the normal surface supposed to constitute the edge. To prove this let equal forces act in opposite directions in lines EF, $E'F'$ on each side of the middle line and parallel to it, constituting the supposed distribution of couple. It must be understood that the forces are actually distributed along their lines of action, and not, as in the abstract dynamics of ideal rigid bodies, applied indifferently at any points of these lines; but the

amount of the force per unit of length, though equal in the neighbouring parts of the two lines, must differ from point to point along the edge, to constitute any other than a uniform distribution of couple. Lastly, we may suppose the forces in the opposite directions to be not confined to two lines, as shown in the diagram, but to be diffused over the two halves of the edge on the two sides of its middle line; and further, the amount of them in equal infinitely small breadths at different distances from the middle line must be proportional to these distances, as stated in § 634 (3), if the given distribution of couple is to be thoroughly such as H of § 645.



Distribution of shearing force determined, which produces same flexure as a given distribution of couple round axes perpendicular to boundary.

Let now the whole edge be divided into infinitely small rectangles, such as $ABCD$ in the diagram, by lines drawn perpendicularly across it. In one of these rectangles apply a balancing system of couples consisting of a diffused couple equal and opposite to the part of the given distribution of couple belonging to the area of the rectangle, and a couple of single forces in the lines AD, CB, of equal and opposite moment. This balancing system obviously cannot cause any sensible disturbance (stress or strain) in the plate, except within a distance comparable with the sides of the rectangle; and, therefore, when the same thing is done in all the rectangles into which the edge is divided, the plate is only disturbed to an infinitely small distance from the edge inwards all round. But the given distribution of couple is thus removed (being directly balanced by a system of diffused force equal and opposite everywhere to that constituting it), and there remains only the set of forces applied in the cross lines. Of these there are two in each cross line, derived from the operations performed in the two rectangles of which it is a common side, and their difference alone remains effective. Thus we see that if the given distribution of couple be uniform along the edge, it may be removed without disturbing the condition of the plate except infinitely near the edge: in other words,

Uniform distribution of twisting couple produces no flexure.

647. *A uniform distribution of couple along the whole edge of a finite plate, everywhere round axes in the plane of the plate, and perpendicular to the edge, produces distortion, spreading to only infinitely small distances inwards from the edge all round, and no stress or distortion of the plate as a whole.* The truth of this remarkable proposition is also obvious when we consider that the tendency of such a distribution of couple can only be to drag the two sides of the edge infinitesimally in opposite directions round the area of the plate. Later (§ 728) we shall investigate strictly the strain, in the neighbourhood of the edge, produced by it, and we shall find (§ 729) that it diminishes with extreme rapidity inwards from the edge, becoming practically insensible at distances exceeding twice the thickness of the plate.

The distribution of shearing force that produces same flexure as from distribution of twisting couple.

648. *A distribution of couple on the edge of a plate, round axes everywhere in the plane of the plate, and perpendicular to the edge, of any given amount per unit of length of the edge, may be removed, and, instead, a distribution of force perpendicular to the plate, equal in amount per unit length of the edge, to the rate of variation per unit length of the amount of the couple, without altering the flexure of the plate as a whole, or producing any disturbance in its stress or strain except infinitely near the edge.*

In the diagram of § 646 let $AB = \delta s$. Then if H be the amount of the given couple per unit length along the edge, between AD, BC, the amount of it on the rectangle $ABCD$ is $H\delta s$, and therefore H must be the amount of the forces introduced along AD, CB, in order that they may constitute a couple of the requisite moment. Similarly, if $H'\delta s$ denote the amount of the couple in the contiguous rectangle on the other side of BC, the force in BC derived from it will be H' in the direction opposite to H. There remains effective in BC a single force equal to the difference, $H'-H$.

If from A to B be the direction in which we suppose s a length measured along the edge from any zero point, to increase, we have

$$H' - H = \frac{dH}{ds}\,\delta s.$$

Thus we are left with single forces, equal to $\frac{dH}{ds}\,\delta s$, applied in lines perpendicularly across the edge, at consecutive distances δs from one another; and for this we may substitute, without causing disturbance except infinitely near the edge, a continuous distribution of transverse force, amounting to dH/ds per unit length; which is the proposition to be proved. The direction of this force, when dH/ds is positive, is that of z negative: whence immediately the form of it expressed in (4) of § 645.

Case of circular strain.

649. As a first example of the application of these equations, we shall consider the very simple case of a uniform plate of finite or infinite extent, symmetrically influenced in concentric circles by a load distributed symmetrically, and by proper boundary appliances if required.

Let the origin of co-ordinates be chosen at the centre of symmetry, and let r, θ be polar co-ordinates of any point P, so that

$$x = r\cos\theta, \quad y = r\sin\theta.$$

The second member of (6), § 644, will be a function of r, which for brevity we may now denote simply by Z (being the amount of load per unit area when the applied forces on each small part are reducible to a single normal force through some point of it). Since z is now a function of r, and, as we have seen before [§ 491 (e)],

$$\nabla^2 u = \frac{1}{r}\frac{d}{dr}\left(r\frac{du}{dr}\right)$$

when u is any function of r, equation (6) of § 644 becomes

$$\frac{A}{r}\frac{d}{dr}\left\{r\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{dz}{dr}\right)\right]\right\} = Z \text{..............(1).}$$

Hence

$$z = \frac{1}{A}\int\frac{dr}{r}\int\quad r\int\frac{dr}{r}\int rZdr + \tfrac{1}{4}C(\log r - 1)r^2 + \tfrac{1}{4}C'r^2 + C''\log r + C''' \text{...(2),}$$

which is the complete integral, with the four arbitrary constants explicitly shown. The following expressions, founded on intermediate integrals, deserve attention now, as promoting a thorough comprehension of the solution; and some of them will be required later for expressing the boundary conditions. The notation of (7) will be explained in § 650:—

Plate circularly strained.

$$\left.\begin{array}{c}\left(\text{inclination, divided by radius; or curvature in} \atop \text{normal section perpendicular to radius}\right)\\ \dfrac{1}{r}\dfrac{dz}{dr}=\dfrac{1}{Ar^2}\displaystyle\int rdr\int\frac{dr}{r}\int rZdr+\tfrac{1}{2}C(\log r-\tfrac{1}{2})+\tfrac{1}{2}C'+\frac{C''}{r^2}\end{array}\right\}\ldots(3),$$

$$\left.\begin{array}{c}\text{(curvature in radial section)}\\ \dfrac{d^2z}{dr^2}=-\dfrac{1}{Ar^2}\displaystyle\int rdr\int\frac{dr}{r}\int rZdr+\frac{1}{A}\int\frac{dr}{r}\int rZdr+\tfrac{1}{2}C(\log r+\tfrac{1}{2})+\tfrac{1}{2}C'-\frac{C''}{r^2}\end{array}\right\}\ldots(4),$$

$$\left.\begin{array}{c}\text{(sum of curvatures in rectangular sections)}\\ \nabla^2 z=\dfrac{1}{A}\displaystyle\int\frac{dr}{r}\int rZdr+C\log r+C'\end{array}\right\}\ldots\ldots(5),$$

$$\left.\begin{array}{l}A\dfrac{d^2z}{dr^2}+c\dfrac{dz}{rdr}=G\\ =-\dfrac{A-c}{Ar^2}\displaystyle\int rdr\int\frac{dr}{r}\int rZdr+\int\frac{dz}{r}\int rZdr+\tfrac{1}{2}C\{(A+c)\log r+\tfrac{1}{2}(A-c)\}\\ \qquad\qquad +\tfrac{1}{2}C'(A+c)-C''(A-c)\dfrac{1}{r^2}\\ H=0\end{array}\right\}\ldots(6),$$

$$L=c\frac{d^2z}{dr^2}+A\frac{dz}{rdr}\ \ldots\ldots\ldots\ldots\ldots\ldots(7),$$

$$\left.\begin{array}{l}(A-c)\dfrac{d}{dr}\left(\dfrac{1}{r}\dfrac{dz}{dr}\right)+\dfrac{dG}{dr}=A\dfrac{d}{dr}\nabla^2 z=-\zeta\\ =\dfrac{1}{r}\displaystyle\int rZdr+C\frac{A}{r}\end{array}\right\}\ldots\ldots\ldots(8).$$

Of these (6) and (8) express, according to the notation of § 645, the couple and the shearing force acting on the normal surface cutting the middle surface of the plate in the circle of radius r. They are derivable analytically from our solution (2) by means of (2), (3), and (1) of § 645, with (4) of § 644, and (15) of § 642. The work is of course much shortened by taking $y=0$, and $x=r$, and using (3) and (4) of the present section. The student may go through this process, with or without the abbreviation, as an analytical exercise; but it is more instructive, as well as more direct, to investigate *ab initio* the equilibrium of a plate symmetrically strained in concentric circles, and so, in the course of an independent demonstration of (6) § 644, for this case, or (1) § 649, to find expressions for the flexural and shearing stresses.

Independent investigation for circular strain.

650. It is clear that, in every part of the plate, the normal sections (§ 637) of maximum and minimum, or minimum and maximum, bending couples are those through and perpendicular to the radius drawn from O the centre of symmetry. At distance r from O, let L and G be the bending couples in the section through the radius, and in the section perpendicular to it; so that, if λ and κ be the curvatures in these sections, we have, by (10) of § 641 and (15) of § 642,

$$\left.\begin{array}{l}L=A\lambda+c\kappa\\ G=c\lambda+A\kappa\end{array}\right\}\ldots\ldots\ldots\ldots\ldots\ldots\ldots(9).$$

Let also ζ be the shearing force (§ 616, footnote) in the circular normal section of radius r. The symmetry requires that there be no shearing force in radial normal sections.

Considering now an element, E, bounded by two radii making an infinitely small angle $\delta\theta$ with one another, and two concentric circles of radii $r-\frac{1}{2}\delta r$ and $r+\frac{1}{2}\delta r$; we see that the equal couples $L\delta r$ on its radial normal sections, round axes falling short of direct opposition by the infinitely small angle $\delta\theta$, have a resultant equal to $L\delta r\delta\theta$ round an axis perpendicular to the middle radius, in the negative direction when L is positive; and the infinitely nearly equal couples on its outer and inner circular edges have a resultant round the same axis, equal to $\dfrac{d}{dr}(Gr\delta\theta)\,\delta r$, being the difference of the values taken by $Gr\delta\theta$ when $r-\frac{1}{2}\delta r$ and $r+\frac{1}{2}\delta r$ are put for r. There is also the couple of the shearing forces on the outer and inner edges, each infinitely nearly equal to $\zeta r\delta\theta$; of which the moment is $\zeta r\delta\theta\delta r$. Hence, for the equilibrium of E under the action of these couples,

$$-L\delta r\delta\theta+\frac{d}{dr}(Gr)\,\delta r\delta\theta+\zeta r\delta\theta\delta r=0,$$

or

$$-L+\frac{d}{dr}(Gr)+\zeta r=0\ldots\ldots\ldots\ldots\ldots(10),$$

if, as we may now conveniently do, we suppose no couples to be applied from without to any part of the plate except its bounding edges. Again, considering normal forces on E, we

Independent investigation for circular strain.

have $\frac{d}{dr}(\zeta r\delta\theta)\,\delta r$ for the sum of those acting on it from the contiguous matter of the plate, and $Zr\delta\theta\delta r$ from external matter if, as above, Z denote the amount of applied normal force per unit area of the plate. Hence, for the equilibrium of these forces,

$$\frac{d}{dr}(\zeta r) + Zr = 0 \qquad\qquad (11).$$

Substituting for ζ in (11) by (10); for L and G in the result by (9); and, in the result of this, for λ and κ their expressions by the differential calculus, which are dz/rdr and d^2z/dr^2, since the plate is a surface of revolution differing infinitely little from a plane perpendicular to the axis, we arrive finally at (1) the differential equation of the problem. Of the other formulæ of § 649, (6), (7), (8) follow immediately from (9) and (10) now proved: except $H=0$, which follows from the fact that the radial and circular normal sections are the sections of maximum and minimum, or minimum and maximum, curvature.

Interpretation of terms in integral.

651. We are now able to perceive the meaning of each of the four arbitrary constants.

(1) C''' is of course merely a displacement of the plate without strain.

(2) $C''\log r$ is a displacement which produces anticlastic curvature throughout, with $\pm C''/r^2$ for the curvatures in the two principal sections: corresponding to which the bending couples, L, G, are equal to $\pm(A-c)\,C''/r^2$. An infinite plane plate, with a circular aperture, and a uniform distribution of bending couple applied to the edge all round, in each part round the tangent as axis, would experience this effect; as we see from the fact that the stress in the plate, due to C'', diminishes according to the inverse square of the distance from the centre of symmetry. It is remarkable that although the absolute value of the deflection, $C''\log r$, is infinite for infinite values of r, the restrictive condition (3) of § 632 is not violated provided C'' is infinitely small in comparison with the thickness: and it may be readily proved that the law (1) of § 633 is, in point of fact, fulfilled by

this deflection, even if the whole displacement has rigorously this value, $C''\log r$, and is precisely in the direction perpendicular to the undisturbed plane. For this case $\zeta=0$, or there is no shear.

Interpretation of terms in integral.

(3) $\frac{1}{4}C'r^2$ is a displacement corresponding to spherical curvature: and therefore involving simply a uniform synclastic stress [§ 638 (1)], of which the amount is of course [§ 641 (10) or (11)] equal to $A+c$ divided by the radius of curvature, or $(A+c)\times\frac{1}{2}C'$, agreeing with the equal values given for L and G by (6) and (7) of § 649. In this case also $\zeta=0$, or there is no shearing force. A finite plate of any shape acted on by a uniform bending couple all round its edge, becomes bent thus spherically.

(4) $\frac{1}{4}C(\log r-1)r^2$ is a deflection involving a shearing force equal to $-AC/r$, and a bending couple,

$$\tfrac{1}{2}C\{(A+c)\log r + \tfrac{1}{2}(A-c)\},$$

in the circle of distance r from the centre of symmetry.

Symmetrical flexure of flat ring.

652. It is now a problem of the merest algebra to find the flexure of a flat ring, or portion of plane plate bounded by two concentric circles, when acted on by any given bending couples and transverse forces applied uniformly round its outer and inner edges. For equilibrium, the forces on the outer and inner edges must be in contrary directions, and of equal amounts. Thus we have three arbitrary data: the amounts of the couple applied to the two edges, each reckoned per unit of length, and the whole amount, F, of the force on either edge. By (4), § 651, or (8) of § 649, we see that

$$-C = \frac{F}{2\pi A} \qquad\qquad (12);$$

and there remain unknown the two constants, C' and C'', to be determined from the two equations given by putting the expression for G [(6) of § 649] equal to the equal values for the values of r at the outer and inner edges respectively.

Example.—A circular table (of isotropic material), with a concentric circular aperture, is supported by its outer edge, *Symmetrical flexure of flat ring.*

which rests simply on a horizontal circle; and is deflected by a load uniformly distributed over its inner edge (or *vice versâ*, inner for outer). To find the deflection due to this load (which of course is simply added to the deflection due to the weight, determined below). Here G must vanish at each edge.

The radii of the outer and inner edges being a and a', the equations are

$$\tfrac{1}{2}C\{(A+c)\log a+\tfrac{1}{2}(A-c)\}+\tfrac{1}{2}C'(A+c)-C''(A-c)\frac{1}{a^2}=0,$$

and the same with a' for a. Hence

Flexure of flat ring equilibrated by forces symmetrically distributed over its edges;

$$C''(A-c)\left(\frac{1}{a'^2}-\frac{1}{a^2}\right)=-\tfrac{1}{2}C(A+c)\log\frac{a}{a'},$$

and

$$\tfrac{1}{2}C'(A+c)(a^2-a'^2)=-\tfrac{1}{2}C[(A+c)(a^2\log a-a'^2\log a')+\tfrac{1}{2}(A-c)(a^2-a'^2)]:$$

and thus, using for C its value (12), we find [(2) § 649]

$$z=\frac{F}{2\pi A}\left[\tfrac{1}{4}\left(-\log r+1+\frac{a^2\log a-a'^2\log a'}{a^2-a'^2}+\tfrac{1}{2}\frac{A-c}{A+c}\right)r^2+\tfrac{1}{2}\frac{A+c}{A-c}\frac{a^2a'^2\log\frac{a}{a'}}{a^2-a'^2}\log r+C'''\right].$$

Putting the factor of r^2 into a more convenient form, and assigning C''' so that the deflection may be reckoned from the level of the inner edge, we have finally

$$z=\frac{F}{2\pi A}\left\{\tfrac{1}{4}\left(-\log\frac{r}{a'}+\frac{a^2}{a^2-a'^2}\log\frac{a}{a'}+\tfrac{1}{2}\frac{3A+c}{A+c}\right)r^2\right.$$

$$\left.+\tfrac{1}{2}\frac{A+c}{A-c}\frac{a^2a'^2\log\frac{a}{a'}}{a^2-a'^2}\log\frac{r}{a'}-\tfrac{1}{4}\frac{a^2a'^2}{a^2-a'^2}\log\frac{a}{a'}-\tfrac{1}{8}\frac{3A+c}{A+c}a'^2\right\}\,..(13).$$

Towards showing the distribution of stress through the breadth of the ring, we have from this, by § 649 (6),

$$G=\frac{F}{2\pi a}\cdot\tfrac{1}{2}(A+c)\left(\frac{a^2}{a^2-a'^2}\log\frac{a}{a'}-\log\frac{r}{a'}-\frac{a^2a'^2}{a^2-a'^2}\log\frac{a}{a'}\frac{1}{r^2}\right)..(14),$$

which, as it ought to do, vanishes when $r=a'$, and when $r=a$. Further, by § 649 (8),

$$\zeta=\frac{F}{2\pi r}\,\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(15),$$

which shows that, as is obviously true, the whole amount of the transverse force in any concentric circle of the ring is equal to F.

653. The problem of § 652, extended to admit a load distributed in any symmetrical manner over the surface of the ring instead of merely confined to one edge, is solved algebraically in precisely the same manner, when the terms dependent on Z, and exhibited in the several expressions of § 649, are found by integration. One important remark we have to make however: that much needless labour is avoided by treating Z as a discontinuous function in these integrations in cases in which one continuous algebraic or transcendental function does not express the distribution of load over the whole portion of plate considered. Unless this plan were followed, the expression for z, dz/dr, G, and ζ, would have to be worked out separately for each annular portion of plate through which Z is continuous, and their values equated on each side of each separating circle. Hence if there were i annular portions to be thus treated separately there would be $4i$ arbitrary constants, to be determined by the $4(i-1)$ equations so obtained, and the 4 equations expressing that at the outer and inner bounding circular edges G has the prescribed values (whether zero or not) of the applied bending couples, and that z and ζ have each a prescribed value at one or other of these circles. But by the more artful method (due to Fourier and Poisson), the complication of detail required in virtue of the discontinuity of Z is confined to the successive integrations; and the arbitrary constants, of which there are now but four, are determined by the conditions for the two extreme bounding edges.

and with load symmetrically spread over its area.

Example.—A circular table (of isotropic material), with a concentric circular aperture, is borne by its outer or inner edge which rests simply on a horizontal circular support, and is loaded by matter uniformly distributed over an annular area of its surface, extending from its inner edge outwards to a concentric circle of given radius, c. It is required to find the flexure.

First, supposing the aperture filled up, and the plate uniform from outer edge to centre, let the whole circle of radius c be uniformly loaded at the rate w, a constant, per unit of its area.

We have

	$Z=$	$\int rZdr=$	$\int\frac{dr}{r}\int rZdr=$	$\int rdr\int\frac{dr}{r}\int rZdr=$	$\int\frac{dr}{r}\int rdr\int\frac{dr}{r}\int rZdr=$
When $r=0$	w	0	0	0	0
„ $<c$	w	$\frac{1}{2}wr^2$	$\frac{1}{4}wr^2$	$\frac{1}{16}wr^4$	$\frac{1}{64}wr^4$
„ $>c$	0	$\frac{1}{2}wc^2$	$\frac{1}{4}wc^2\left(2\log\frac{r}{c}+1\right)$	$\frac{1}{16}wc^2\left(4r^2\log\frac{r}{c}+c^2\right)$	$\frac{1}{16}wc^2\left(2r^2\log\frac{r}{c}-r^2+c^2\log\frac{r}{c}+\frac{5}{4}c^2\right)$
	I.	II.	III.	IV.	V.

Of these results, V. used in (2) gives the general solution; and IV., III., and II. in (6) and (8) give the corresponding expressions for G and ζ. If, first, we suppose the value of G thus found to have any given value for each of two values, r', r'', of r, and ζ to have a given value for one of these values of r, we have three simple algebraic equations to find C, C', C''; and we solve a more general problem than that proposed; to which we descend by making the prescribed values of G and ζ zero. The power of mathematical expression and analysis in dealing with discontinuous functions, is strikingly exemplified in the applicability of the result not only to the contemplated case, in which c is intermediate between r' and r''; but also to cases in which c is less than either (when we fall back on the previous case, of § 652), or c greater than either (when we have a solution more directly obtainable by taking $Z=w$ for all values of r).

Circular table of isotropic material, supported symmetrically on its edge, and strained only by its own weight.

If the plate is in reality continuous to its centre, and uniformly loaded over the whole area of the circle of radius c, we must have $C=0$ and $C''=0$ to avoid infinite values of ζ and G at the centre: and the equation $G=0$ for the outer boundary of the disc gives C' at once, completing the determination. If, lastly, we suppose c to be not less than the radius of the disc, we have the solution for a uniform circular disc uniformly supported round its edge, and strained only by its own weight.

Reduction of general problem to case of no load over area.

654. If now we consider the general problem,—to determine the flexure of a plate of any form, with an arbitrary distribution of load over it, and with arbitrary boundary appliances, subject of course to the condition that all the applied forces, when the data are entirely of force, must constitute an equilibrating system; we may immediately reduce this problem to the simpler one in which there is no load distributed over the area, but arbitrary boundary appliances only. We shall merely sketch the mathematical investigation.

First it is easily proved, as for a corresponding expression for three independent variables in § 491 (c), that

$$\left(\frac{d^2}{dx^2}+\frac{d^2}{dy^2}\right)\iint\rho'\log D\,dx'dy'=2\pi\rho\,\ldots\ldots\ldots\ldots\ldots(1),$$

where ρ' is any function of two independent variables, x', y'; ρ the same function of x, y; D denotes $\sqrt{\{(x-x')^2+(y-y')^2\}}$; and $\iint$ denotes integration over an area comprehending all values of x', y', for which ρ' does not vanish. Hence

$$\left(\frac{d^2}{dx^2}+\frac{d^2}{dy^2}\right)^2u=Z\,\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2),$$

if

$$u=\frac{1}{4\pi^2}\iint dx'dy'\log D\iint dx''dy''Z''\log D'\ldots\ldots\ldots(3),$$

where $D'=\sqrt{\{(x''-x')^2+(y''-y')^2\}}$; and if Z'' and Z denote the values for (x'', y'') and (x, y) of any arbitrary function of two independent variables. Let this function denote the amount of load per unit of area, which we may suppose to vanish for all values of the co-ordinates not included in the plate; and to avoid trouble regarding limits, let all the integrals be supposed to extend from $-\infty$ to $+\infty$. We thus have, in $z=u$, a solution of our equation (2): and therefore $z-u$ must satisfy the same equation with the second member replaced by zero: or, if $\mathfrak{z}$ denote a general solution of

$$\left(\frac{d^2}{dx^2}+\frac{d^2}{dy^2}\right)^2\mathfrak{z}=0\,\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4),$$

then

$$z=u+\mathfrak{z}\,\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5)$$

is the general solution of (2). The boundary conditions for $\mathfrak{z}$ are of course had by substituting $u+\mathfrak{z}$ for z in the directly prescribed boundary equations, whatever they may be.

Flat circular ring the only case hitherto solved.

655. Mathematicians have not hitherto succeeded in solving this problem with complete generality, for any other form of plate than the circular ring (or circular disc with concentric circular aperture). Having given (§§ 640, 653) a detailed

Flat circular ring the only case hitherto solved.

solution of the problem for this case, subject to the restriction of symmetry, we shall merely indicate the extension of the analysis to include any possible non-symmetrical distribution of strain. The same analysis, under much simpler conditions, will occur to us again and again, and will be on some points more minutely detailed, when we shall be occupied with important practical problems regarding electric influence, fluid motion, and electric and thermal conduction, through cylindrical spaces.

Taking the centre of the circular bounding edges as origin for polar co-ordinates, let

$$x = r\cos\theta,\ \ y = r\sin\theta.$$

We easily find by transformation

$$\frac{d^2\mathfrak{z}}{dx^2}+\frac{d^2\mathfrak{z}}{dy^2}=\frac{1}{r}\frac{d}{dr}\left(r\frac{d\mathfrak{z}}{dr}\right)+\frac{1}{r^2}\frac{d^2\mathfrak{z}}{d\theta^2}\ \ldots\ldots\ldots\ldots\ldots\ (6).$$

If we put $\log r = \vartheta$, or $r = \epsilon^{\vartheta}$ ....(7),

this becomes

$$\frac{d^2\mathfrak{z}}{dx^2}+\frac{d^2\mathfrak{z}}{dy^2}=\epsilon^{-2\vartheta}\left(\frac{d^2\mathfrak{z}}{d\vartheta^2}+\frac{d^2\mathfrak{z}}{d\theta^2}\right)\ \ldots\ldots\ldots\ldots\ldots\ (8).$$

Hence if, as before, ∇^2 denote $\dfrac{d^2}{dx^2}+\dfrac{d^2}{dy^2}$,

$$\nabla^4\mathfrak{z}=\epsilon^{-2\vartheta}\left(\frac{d^2}{d\vartheta^2}+\frac{d^2}{d\theta^2}\right)\epsilon^{-2\vartheta}\left(\frac{d^2}{d\vartheta^2}+\frac{d^2}{d\theta^2}\right)\mathfrak{z}\ \ldots\ldots\ldots\ldots(9).$$

This equated to zero gives

$$\frac{d^2\mathfrak{z}}{d\vartheta^2}+\frac{d^2\mathfrak{z}}{d\theta^2}=\epsilon^{2\vartheta}v\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(10),$$

if v denote any solution of

$$\frac{d^2v}{d\vartheta^2}+\frac{d^2v}{d\theta^2}=0\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(11).$$

We shall see, when occupied with the electric and other problems referred to above, that a general solution of this equation, appropriate for our present problem as for all involving the expression of arbitrary functions of θ for particular values of ϑ, is

$$v=\sum_0^\infty\{(A_i\cos i\theta+B_i\sin i\theta)\epsilon^{i\vartheta}+(\mathfrak{A}_i\cos i\theta+\mathfrak{B}_i\sin i\theta)\epsilon^{-i\vartheta}\}\ldots(12),$$

where A_i, B_i, $\mathfrak{A}_i$, $\mathfrak{B}_i$ are constants. That this is a solution, is of course verified in a moment by differentiation. From it we

readily find (and the result of course is verified also by differentiation),

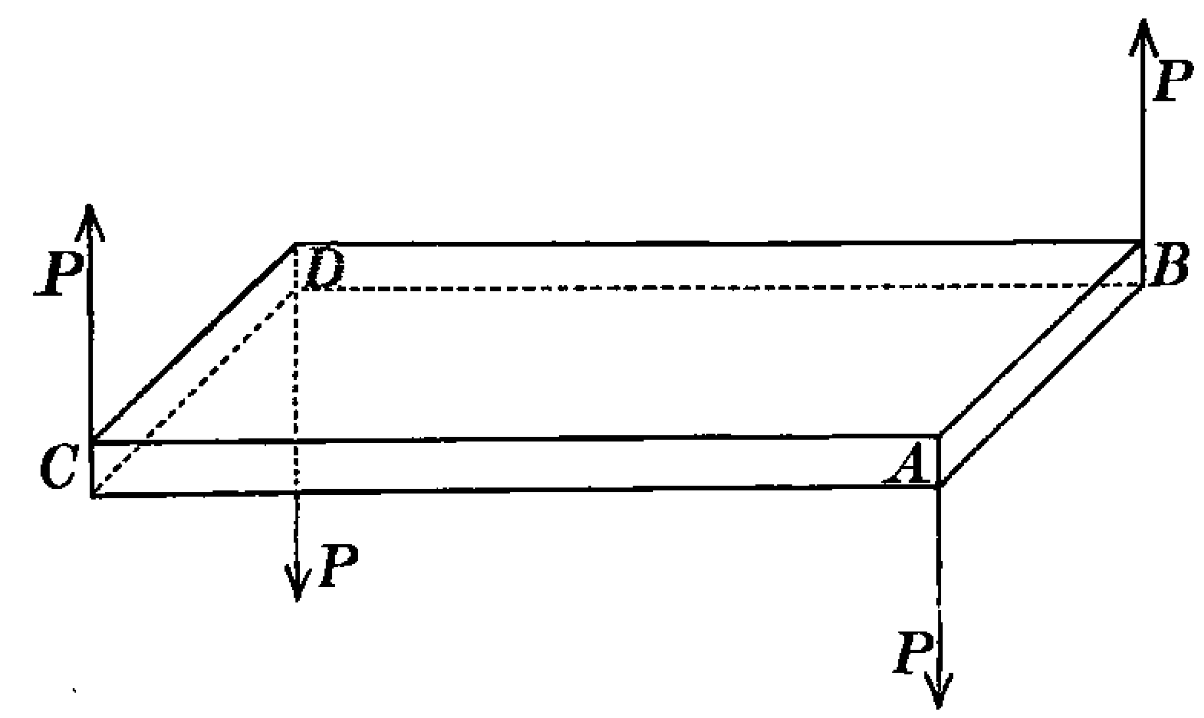
Flat circular ring the only case hitherto solved.

$$\mathfrak{z}=\sum_{i=0}^{i=\infty}\left\{\frac{1}{(i+2)^2-i^2}(A_i\cos i\theta+B_i\sin i\theta)\,\epsilon^{(i+2)\vartheta}\right\}$$
$$+\sum_{i=2}^{i=\infty}\left\{\frac{-1}{i^2-(i-2)^2}(\mathfrak{A}_i\cos i\theta+\mathfrak{B}_i\sin i\theta)\,\epsilon^{-(i-2)\vartheta}\right\}-\tfrac{1}{2}(\mathfrak{A}_1\cos\theta+\mathfrak{B}_1\sin\theta)\,\vartheta\epsilon^{\vartheta}+v'$$
$$\ldots\ldots\ldots\ (13),$$

v' being any solution of (11), which may be conveniently taken as given by (12) with accented letters A_i', etc., to denote four new constants. If now the arbitrary periodic functions of θ, with 2π for period, given as the values whether of displacement, or shearing force, or couple, for the outer and inner circular edges, be expressed by Fourier's theorem [§ 77 (14)] in simple harmonic series; the two equations [§ 645 (5)] for each edge, applied separately to the coefficients of $\cos i\theta$ and $\sin i\theta$ in the expressions thus obtained, give eight equations for determining the eight constants A_i, $\mathfrak{A}_i$, B_i, $\mathfrak{B}_i$, A_i', $\mathfrak{A}_i'$, B_i', $\mathfrak{B}_i'$.

Rectangular plate, held and loaded by diagonal pairs of corners.

656. Although the problem of fulfilling arbitrary boundary conditions has not yet been solved for rectangular plates, there is one remarkable case of it which deserves particular notice; not only as interesting in itself, and important in practical application, but as curiously illustrating one of the most difficult points [§§ 646, 648] of the general theory. A rectangular plate acted on perpendicularly by a balancing system of four equal parallel forces applied at its four



corners, becomes strained to a condition of uniform anticlastic curvature throughout, with the sections of no-flexure parallel to its sides, and therefore with sections of equal opposite maximum curvature in the normal planes inclined to the sides at 45°. This follows immediately from § 648, if we suppose the corners rounded off ever so little, and the forces diffused over them.

Rectangular plate, held and loaded by diagonal pairs of corners.

Or, in each of an infinite number of normal lines in the edge AB, let a pair of opposite forces each equal to $\frac{1}{2}P$ be applied; which cannot disturb the plate. These, with halves of the single forces P in the dissimilar directions at the corners A and B, constitute a diffused couple over the whole edge AB, amounting in moment per unit of length to $\frac{1}{2}P$, round axes perpendicular to the plane of the edge. Similarly, the other halves of the forces P at the corners A, B, with halves of those at C and D and introduced balancing forces, constitute diffused couples over the edges CA and DB; and the remaining halves of the corner forces at C and D, with introduced balancing forces, constitute a diffused couple over CD; each having $\frac{1}{2}P$ for the amount of moment per unit length of the edge over which it is diffused. Their directions are mutually related in the manner specified in § 638 (2), and thus taken all together, they constitute an anticlastic stress of value $\Omega = \frac{1}{2}P$. Hence (§ 642) the result is uniform anticlastic strain amounting to $\frac{1}{2}P/k$, and having its axes inclined at 45° to the edges; that is to say (§ 639), a flexure with maximum curvatures on the two sides of the tangent plane each equal to $\frac{1}{2}P/k$, and in normal sections in the positions stated.

Transition to finite flexures indicated.

657 Few problems of physical mathematics are more curious than that presented by the transition from this solution, founded on the supposition that the greatest deflection is but a small fraction of the thickness of the plate, to the solution for larger flexures, in which corner portions will bend approximately as developable surfaces (cylindrical, in fact), and a central quadrilateral part will remain infinitely nearly plane; and thence to the extreme case of an infinitely thin perfectly flexible rectangle of inextensible fabric. This extreme case may be easily observed and experimented on by taking a carefully cut rectangle of paper (§ 145), supporting it by fine threads attached to two opposite corners, and kept parallel, while two equal weights are hung by threads from the other corners.

Transmission of force through an elastic solid.

658. The definitions and investigations regarding strain of §§ 154—190 constitute a kinematical introduction to the theory of elastic solids. We must now, in commencing the elementary dynamics of the subject, consider the forces called into play through the interior of a solid when brought into a condition of strain. We adopt, from Rankine*, the term *stress* to designate such forces, as distinguished from strain defined (§ 154) to express the merely geometrical idea of a change of volume or figure.

Homogeneous stress.

659. When through any space in a body under the action of force, the mutual force between the portions of matter on the two sides of any plane area is equal and parallel to the mutual force across any equal, similar, and parallel plane area, the stress is said to be homogeneous through that space. In other words, the stress experienced by the matter is homogeneous through any space if all equal similar and similarly turned portions of matter within this space are similarly and equally influenced by force.

Force transmitted across any surface in elastic solid.

660. To be able to find the distribution of force over the surface of any portion of matter homogeneously stressed, we must know the direction, and the amount per unit area, of the force across a plane area cutting through it in any direction. Now if we know this for any three planes, in three different directions, we can find it for a plane in any direction, as we see in a moment by considering what is necessary for the equilibrium of a tetrahedron of the substance. The resultant force on one of its faces must be equal and opposite to the resultant of the forces on the three others, which is known if these faces are parallel to the three planes for each of which the force is given.

Specification of a stress;

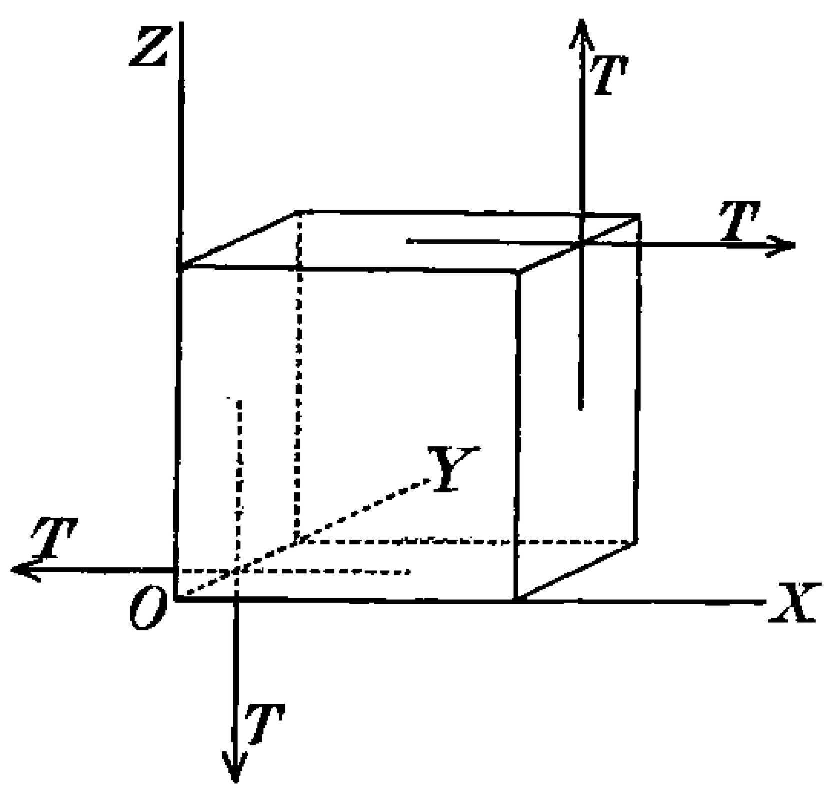
661. Hence the stress, in a body homogeneously stressed, is completely specified when the direction, and the amount per unit area, of the force on each of three distinct planes is given. It is, in the analytical treatment of the subject, generally convenient to take these planes of reference at right angles to one another. But we should immediately fall into error did we not remark that the specification here indicated consists not of nine but in reality only of six independent elements. For if the equilibrating forces on the six faces of a cube be each resolved into three components parallel to its three edges OX, OY, OZ, we have in all 18 forces; of which each pair acting perpendicularly

by six independent elements.

* *Cambridge and Dublin Mathematical Journal*, 1850.

on a pair of opposite faces, being equal and directly opposed, balance one another. The twelve tangential components that remain constitute three pairs of couples having their axes in the direction of the three edges, each of which must separately be in equilibrium. The diagram shows the pair of equilibrating couples having OY for axis; from the consideration of which we infer that the forces on the faces (zy), parallel to OZ, are equal to the forces on the faces (yx), parallel to OX. Similarly, we see that the forces on the faces (yx), parallel to OY, are equal to those of the faces (xz), parallel to OZ; and that the forces on (xz), parallel to OX, are equal to those on (zy), parallel to OY.

Relations between pairs of tangential tractions necessary for equilibrium.



Specification of a stress; by six independent elements: three simple longitudinal stresses, and three simple shearing stresses.

662. Thus, any three rectangular planes of reference being chosen, we may take six elements thus, to specify a stress: P, Q, R the normal components of the forces on these planes; and S, T, U the tangential components, respectively perpendicular to OX, of the forces on the two planes meeting in OX, perpendicular to OY, of the forces on the planes meeting in OY, and perpendicular to OY, of the forces on the planes meeting in OY; each of the six forces being reckoned per unit of area. A normal component will be reckoned as positive when it is a traction tending to separate the portions of matter on the two sides of its plane. P, Q, R are sometimes called longitudinal stresses, sometimes simple normal tractions, and S, T, U shearing stresses.

Simple longitudinal, and shearing, stresses.

Force across any surface in terms of rectangular specification of stress.

From these data, to find in the manner explained in § 660, the force on any plane, specified by l, m, n, the direction-cosines of its normal; let such a plane cut OX, OY, OZ in the three points X, Y, Z. Then, if the area XYZ be denoted for a moment by A, the areas YOZ, ZOX, XOY, being its projections on the three rectangular planes, will be respectively equal to Al, Am, An. Hence, for the equilibrium of the tetrahedron of matter bounded by those four triangles, we have, if F, G, H denote the com-

ponents of the force experienced by the first of them, XYZ, per unit of its area,

Force across any surface in terms of rectangular specification of stress.

$$F \cdot A = P \cdot lA + U \cdot mA + T \cdot nA,$$

and the two symmetrical equations for the components parallel to OY and OZ. Hence, dividing by A, we conclude

$$\left.\begin{aligned} F &= Pl + Um + Tn \\ G &= Ul + Qm + Sn \\ H &= Tl + Sm + Rn \end{aligned}\right\} \dots\dots\dots\dots\dots\dots (1).$$

These expressions stand in the well-known relation to the ellipsoid

$$Px^2 + Qy^2 + Rz^2 + 2(Syz + Tzx + Uxy) = 1 \dots\dots\dots (2),$$

according to which, if we take

$$x = lr, \quad y = mr, \quad z = nr,$$

and if λ, μ, ν denote the direction-cosines and p the length of the perpendicular from the centre to the tangent plane at (x, y, z) of the ellipsoid, we have

$$F = \frac{\lambda}{pr}, \quad G = \frac{\mu}{pr}, \quad H = \frac{\nu}{pr}.$$

We conclude that

Stress-quadric.

663. For any fully specified state of stress in a solid, a quadric surface may always be determined, which shall represent the stress graphically in the following manner:—

To find the direction, and the amount per unit area, of the force acting across any plane in the solid, draw a radius perpendicular to this plane from the centre of the quadric to its surface. The required force will be equal to the reciprocal of the product of the length of this radius into the perpendicular from the centre to the tangent plane at the extremity of the radius, and will be perpendicular to this tangent plane.

Principal planes and axes of a stress.

664. From this it follows that for any stress whatever there are three determinate planes at right angles to one another such that the force acting in the solid across each of them is precisely perpendicular to it. These planes are called the principal or normal planes of the stress; the forces upon them, per unit area, —its principal or normal tractions; and the lines perpendicular

Principal planes and axes of a stress.

to them,—its principal or normal axes, or simply its axes. The three principal semi-diameters of the quadric surface are equal to the reciprocals of the square roots of the principal tractions. If, however, in any case each of the three principal tractions is negative, it will be convenient to reckon them rather as *pressures;* the reciprocals of the square roots of which will be the semi-axes of a real stress-ellipsoid representing the distribution of force in the manner explained above, with pressure substituted throughout for traction.

Varieties of stress-quadric.

665. When the three principal tractions are all of one sign, the stress-quadric is an ellipsoid; the cases of an ellipsoid of revolution and a sphere being included, as those in which two, or all three, are equal. When one of the three is negative and the two others positive, the surface is a hyperboloid of one sheet. When one of the normal tractions is positive and the two others negative, the surface is a hyperboloid of two sheets.

666. When one of the three principal tractions vanishes, while the other two are finite, the stress-quadric becomes a cylinder, circular, elliptic, or hyperbolic, according as the other two are equal, unequal, of one sign, or of contrary signs. When two of the three vanish, the quadric becomes two planes; and the stress in this case is (§ 662) called a simple longitudinal stress. The theory of principal planes, and principal or normal tractions, just stated (§ 664), is then equivalent to saying that any stress whatever may be regarded as made up of three simple longitudinal stresses in three rectangular directions. The geometrical interpretations are obvious in all these cases.

Composition of stresses.

667. The composition of stresses is of course to be effected by adding the component tractions thus:—If $(P_1, Q_1, R_1, S_1, T_1, U_1)$, $(P_2, Q_2, R_2, S_2, T_2, U_2)$, etc., denote, according to § 662, any given set of stresses acting simultaneously in a substance, their joint effect is the same as that of a single resultant stress of which the specification in corresponding terms is $(\Sigma P, \Sigma Q, \Sigma R, \Sigma S, \Sigma T, \Sigma U)$.

Laws of strain and stress compared.

668. Each of the statements that have now been made (§§ 659, 667) regarding stresses, is applicable to *infinitely small* strains, if for traction perpendicular to any plane, reckoned per

unit of its area, we substitute *elongation*, in the lines of the traction, reckoned per unit of length; and for *half the tangential traction* parallel to any direction, *shear* in the same direction reckoned in the manner explained in § 175. The student will find it a useful exercise to study in detail this transference of each one of those statements, and to justify it by modifying in the proper manner the results of §§ 171, 172, 173, 174, 175, 185, to adapt them to infinitely small strains. It must be remarked that the strain-quadric thus formed according to the rule of § 663, which may have any of the varieties of character mentioned in §§ 665, 666, is not the same as the strain-ellipsoid of § 160, which is always essentially an ellipsoid, and which, for an infinitely small strain, differs infinitely little from a sphere.

Laws of strain and stress compared.

The comparison of § 172, with the result of § 661 regarding tangential tractions, is particularly interesting and important.

669. The following schedule of the meaning of the elements constituting the corresponding rectangular specifications of a strain and stress explained in preceding sections, will be found convenient:—

Rectangular elements of strains and stresses.

Components of the strain.	Components of the stress.	Planes; of which relative motion, or across which force is reckoned.	Direction of relative motion or of force.
e	P	yz	x
f	Q	zx	y
g	R	xy	z
a	S	$\begin{cases} yx \\ zx \end{cases}$	y z
b	T	$\begin{cases} zy \\ xy \end{cases}$	z x
c	U	$\begin{cases} xz \\ yz \end{cases}$	x y

Work done by a stress within a varying solid.

670. If a unit cube of matter, given under any stress (P, Q, R, S, T, U), be subjected further to such infinitesimal change of this stress as shall produce an infinitely small simple longitudinal strain e alone, the work done on it will be Pe; since, of

Work done by a stress within a varying solid.

the component forces P, U, T parallel to OX, U and T do no work in virtue of this strain. Similarly Qf, Rg are the works done if, the same stress acting, infinitesimal strains f or g are produced, either of them alone. Again, if the cube experiences a simple shear, a, whether we regard it (§ 172) as a differential sliding of the planes yx, parallel to y, or of the planes zx, parallel to z, we see that the work done is Sa: and similarly, Tb if the strain is simply a shear b, parallel to OZ, of planes zy, or parallel to OX, of planes xy: and Uc if the strain is a shear c, parallel to OX, of planes xz, or parallel to OY, of planes yz. Hence the whole work done by the stress (P, Q, R, S, T, U) on a unit cube taking the additional infinitesimal strain (e, f, g, a, b, c), while the stress varies only infinitesimally, is

Compare § 673, (20).

$$Pe + Qf + Rg + Sa + Tb + Uc \dots\dots\dots\dots\dots (3).$$

It is to be remarked that, inasmuch as the action called a stress is a system of forces which balance one another if the portion of matter experiencing it is rigid, it cannot (§ 551) do any work when the matter moves in any way without change of shape: and therefore no amount of translation or rotation of the cube taking place along with the strain can render the amount of work done different from that just found.

If the side of the cube be of any length p, instead of unity, each force will be p^2 times, and each relative displacement p times; and therefore the work done p^3 times the respective amounts reckoned above. Hence a body of any shape, and of cubic content C, subjected throughout to a uniform stress (P, Q, R, S, T, U) while taking uniformly throughout an additional strain (e, f, g, a, b, c), experiences an amount of work equal to

$$(Pe + Qf + Rg + Sa + Tb + Uc)\,C \dots\dots\dots\dots (4).$$

Work done on the surface of a varying solid.

It is to be remarked that this is necessarily equal to the work done on the bounding surface of the body by forces applied to it from without. For the work done on any portion of matter within the body is simply that done on its surface by the matter touching it all round, as no force acts at a distance from without on the interior substance. Hence if we imagine the whole body divided into any number of parts, each of any shape, the sum

of the works done on all these parts is, by the disappearance of equal positive and negative terms expressing the portions of the work done on each part by the contiguous parts on all its sides, and spent by these other parts in this action, reduced to the integral amount of work done by force from without, applied all round the outer surface.

Work done on the surface of a varying solid.

The analytical verification of this is instructive with regard to the syntax of the mathematical language in which the theory of the transmission of force is expressed. Let x, y, z be the coordinates of any point within the body; W the whole amount of work done in the circumstances specified above; and $\iiint$ integration extended throughout the space occupied by the body: so that

$$W = \iiint (Pe + Qf + Rg + Sa + Tb + Uc)\, dxdydz \dots\dots\dots (5).$$

If now we denote by α, β, γ the component displacements of any point of the matter infinitely near the point (x, y, z), experienced when the additional strain (e, f, g, a, b, c) takes place, whether non-rotationally (§ 182) and with some point of the body fixed, or with any motion of translation whatever and any infinitely small rotation, by adapting § 181 (5) to infinitely small strains according to our present notation (§ 669), and using in it § 190 (e), we have

Strain-components in terms of displacement.

$$\left.\begin{array}{lll} e = \dfrac{d\alpha}{dx}, & f = \dfrac{d\beta}{dy}, & g = \dfrac{d\gamma}{dz}, \\ a = \dfrac{d\beta}{dz} + \dfrac{d\gamma}{dy}, & b = \dfrac{d\gamma}{dx} + \dfrac{d\alpha}{dz}, & c = \dfrac{d\alpha}{dy} + \dfrac{d\beta}{dx} \end{array}\right\} \dots\dots\dots (6).$$

With these, (5) becomes

Work done through interior;

$$W = \iiint \left(P\frac{d\alpha}{dx} + U\frac{d\beta}{dx} + T\frac{d\gamma}{dx} + U\frac{d\alpha}{dy} + Q\frac{d\beta}{dy} + S\frac{d\gamma}{dy} + T\frac{d\alpha}{dz} + S\frac{d\beta}{dz} + R\frac{d\gamma}{dz} \right) dxdydz \dots (7).$$

Hence by integration

$$W = \iint [(P\alpha + U\beta + T\gamma)dydz + (U\alpha + Q\beta + S\gamma)dzdx + (T\alpha + S\beta + R\gamma)dxdy] \dots\dots\dots (8),$$

the limits of the integrations being so taken that, if $d\sigma$ denote an element of the bounding surface, $\iint$ integration all over it, and l, m, n the direction-cosines of the normal at any point of it, the expression means the same as

$$W = \iint \{(P\alpha + U\beta + T\gamma)l + (U\alpha + Q\beta + S\gamma)m + (T\alpha + S\beta + R\gamma)n\} d\sigma \dots (9);$$

which, with the terms grouped otherwise, becomes

$$W = \iint\{(Pl+Um+Tn)\alpha+(Ul+Qm+Sn)\beta+(Tl+Sm+Rn)\gamma\}d\sigma \ldots (10).$$

agrees with work done on surface.

The second member of this, in virtue of (1), expresses directly the work done by the forces applied from without to the bounding surface.

Differential equation of work done by a stress.

671. If, now, we suppose the body to yield to a stress (P, Q, R, S, T, U), and to oppose this stress only with its innate resistance to change of shape, the differential equation of work done will [by (4) with de, df, etc., substituted for e, f, etc.] be

$$dw = Pde + Qdf + Rdg + Sda + Tdb + Udc \ldots\ldots\ldots (11),$$

if w denote the whole amount of work done per unit of volume in any part of the body while the substance in this part experiences a strain (e, f, g, a, b, c) from some initial state regarded as a state of no strain. This equation, as we shall see later, under Properties of Matter, expresses the work done in a natural fluid, by distorting stress (or difference of pressure in different directions) working against its innate viscosity; and w is then, according to Joule's discovery, the dynamic value of the heat generated in the process. The equation may also be applied to express the work done in straining an imperfectly elastic solid, or an elastic solid of which the temperature varies during the process. In all such applications the stress will depend partly on the speed of the straining motion, or on the varying temperature, and not at all, or not solely, on the state of strain at any moment, and the system will not be dynamically conservative.

Physical application.

Perfectly elastic body defined, in abstract dynamics.

672. *Definition.*—A perfectly elastic body is a body which, when brought to any one state of strain, requires at all times the same stress to hold it in this state; however long it be kept strained, or however rapidly its state be altered from any other strain, or from no strain, to the strain in question. Here, according to our plan (§§ 443, 448) for Abstract Dynamics, we ignore variation of temperature in the body. If, however, we add a condition of absolutely no variation of temperature, or of recurrence to one specified temperature after changes of strain, we have a definition of that property of perfect elasticity

towards which highly elastic bodies in nature approximate; and which is rigorously fulfilled by all fluids, and may be so by some real solids, as homogeneous crystals. But inasmuch as the elastic reaction of every kind of body against strain varies with varying temperature, and (a thermodynamic consequence of this, as we shall see later) any increase or diminution of strain in an elastic body is necessarily accompanied by a change of temperature; even a perfectly elastic body could not, in passing through different strains, act as a rigorously conservative system, but, on the contrary, must give rise to dissipation of energy in consequence of the conduction or radiation of heat induced by these changes of temperature.

Its conditional fulfilment in nature.

But by making the changes of strain quickly enough to prevent any sensible equalization of temperature by conduction or radiation (as, for instance, Stokes has shown, is done in sound of musical notes travelling through air); or by making them slowly enough to allow the temperature to be maintained sensibly constant* by proper appliances; any highly elastic, or perfectly elastic body in nature may be got to act very nearly as a conservative system.

Potential energy of an elastic solid held strained.

673. In nature, therefore, the integral amount, w, of work defined as above, is for a perfectly elastic body, independent (§ 274) of the series of configurations, or states of strain, through which it may have been brought from the first to the second of the specified conditions, provided it has not been allowed to change sensibly in temperature during the process.

The analytical statement is that the expression (11) for dw must be the differential of a function of e, f, g, a, b, c, regarded as independent variables; or, which means the same, w is a function of these elements, and

$$\left.\begin{array}{lll} P=\dfrac{dw}{de}, & Q=\dfrac{dw}{df}, & R=\dfrac{dw}{dg}, \\ S=\dfrac{dw}{da}, & T=\dfrac{dw}{db}, & U=\dfrac{dw}{dc}. \end{array}\right\}\ldots\ldots\ldots\ldots(12).$$

* "On the Thermoelastic and Thermomagnetic Properties of Matter" (W. Thomson). *Quarterly Journal of Mathematics.* April, 1855; Mathematical and Physical Papers, Art. XLVIII. Part VII.

Potential energy of an elastic solid held strained.

In Appendix C, we shall return to the comprehensive analytical treatment of this theory, not confining it to infinitely small strains for which alone the notation $(e, f, \ldots)$, as defined in § 669, is convenient. In the meantime, we shall only say that when the whole amount of strain is infinitely small, and the stress-components are therefore all altered in the same ratio as the strain-components if these are altered all in any one ratio; w must be a homogeneous quadratic function of the six variables e, f, g, a, b, c, which, if we denote by (e, e), $(f, f)\ldots(e, f)\ldots$ constants depending on the quality of the substance and on the directions chosen for the axes of co-ordinates, we may write as follows:—

$$w = \tfrac{1}{2}\left\{\begin{aligned}&(e,e)e^2+(f,f)f^2+(g,g)g^2+(a,a)a^2+(b,b)b^2+(c,c)c^2\\&+2(e,f)ef+2(e,g)eg+2(e,a)ea+2(e,b)eb+2(e,c)ec\\&+2(f,g)fg+2(f,a)fa+2(f,b)fb+2(f,c)fc\\&+2(g,a)ga+2(g,b)gb+2(g,c)gc\\&+2(a,b)ab+2(a,c)ac\\&+2(b,c)bc\end{aligned}\right\} \quad (13).$$

The 21 coefficients (e, e), $(f, f)\ldots(b, c)$, in this expression constitute the 21 "coefficients of elasticity," which Green first showed to be proper and essential for a complete theory of the dynamics of an elastic solid subjected to infinitely small strains. The only condition that can be theoretically imposed upon these coefficients is that they must not permit w to become negative for any values, positive or negative, of the strain-components $e, f, \ldots$. Under Properties of Matter, we shall see that an untenable theory (Boscovich's), falsely worked out by mathematicians, has led to relations among the coefficients of elasticity which experiment has proved to be false.

Eliminating w from (12) by (13) we have

Stress-components expressed in terms of strain.

$$\left.\begin{aligned}P &= (e,e)e+(e,f)f+(e,g)g+(e,a)a+(e,b)b+(e,c)c\\ Q &= (e,f)e+(f,f)f+(f,g)g+(f,a)a+(f,b)b+(f,c)c\\ &\quad \text{etc.} \qquad\qquad \text{etc.}\\ &\quad \text{etc.} \qquad\qquad \text{etc.}\end{aligned}\right\} \quad (14).$$

These equations express the six components of stress (P, Q, R, S, T, U) as linear functions of the six components of strain (e, f, g, a, b, c) with 15 equalities [namely $(e, f) = (f, e)$, etc.] among their 36 coefficients, which leave only 21 of them independent. The mere principle of superposition (which we have used above in establishing the quadratic form for w) might have been directly applied to demonstrate linear formulæ for the stress-components. Thus it is that some authors have been led to lay down, as the foundation of the most general possible theory of elasticity, six equations involving 36 coefficients supposed to be independent. But it is only by the principle of energy that, as first discovered by Green, the fifteen pairs of these coefficients are proved to be equal.

Stress-components expressed in terms of strain.

Strain-components expressed in terms of stress.

The algebraic transformation of equations (14) to express the strain-components singly, by linear functions of the stress-components, may be directly effected of course by forming the proper determinants from the 36 coefficients, and taking the 36 proper quotients. From a known determinantal theorem, used also above [§ 313 (d)], it follows that there are 15 equalities between pairs of these 36 quotients, because of the 15 equalities in pairs of the coefficients of e, f, etc., in (14). Thus, if we denote by

$$[P, P], [Q, Q], \ldots [P, Q], \ldots [Q, P] \ldots$$

the set of 36 determinantal quotients found by that process (being, therefore, known algebraic functions of the original coefficients (e, e), (f, f), ... etc.), we have

$$\left.\begin{aligned}e &= [P,P]P+[P,Q]Q+[P,R]R+[P,S]S+[P,T]T+[P,U]U\\ f &= [Q,P]P+[Q,Q]Q+[Q,R]R+[Q,S]S+[Q,T]T+[Q,U]U\\ &\quad \text{etc.} \qquad\qquad\qquad \text{etc.}\end{aligned}\right\}\ldots(16);$$

and these new coefficients satisfy 15 equations

$$[P, Q] = [Q, P], \; [P, R] = [R, P] \ldots\ldots\ldots\ldots\ldots\ldots (17).$$

By what we proved in § 313 (d) when engaged with precisely the same algebraic transformation, we see that $[P, P]$, $[Q, Q]$, ..., $[P, Q]$, ... are simply the coefficients of $P^2, Q^2, \ldots, 2PQ, \ldots$ in the expression for $2w$ obtained by eliminating $e, f, \ldots$ from (13), so that

$$w = \tfrac{1}{2}\{[P,P]P^2+[Q,Q]Q^2+\ldots+2[P,Q]PQ+2[P,R]PR+\ldots\}\ldots(18);$$

and

$$\left.\begin{aligned}e &= \left[\frac{dw}{dP}\right], & f &= \left[\frac{dw}{dQ}\right], & g &= \left[\frac{dw}{dR}\right],\\ a &= \left[\frac{dw}{dS}\right], & b &= \left[\frac{dw}{dT}\right], & c &= \left[\frac{dw}{dU}\right],\end{aligned}\right\}\ldots\ldots\ldots\ldots(19);$$