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TREATISE

ON

NATURAL PHILOSOPHY

BY

LORD KELVIN, LL.D., D.C.L., F.R.S.

AND

PETER GUTHRIE TAIT, M.A.

PART II.

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PREFACE.

THE original design of the Authors in commencing this work about twenty years ago has not been carried out beyond the production of the first of a series of volumes, in which it was intended that the various branches of mathematical and experimental physics should be successively treated. The intention of proceeding with the other volumes is now definitely abandoned; but much new matter has been added to the first volume, and it has been divided into two parts, in the second edition now completed in this second part. The original first volume contained many references to the intended future volumes; and these references have been allowed to remain in the present completion of the new edition of the first volume, because the plan of treatment followed depended on the expectation of carrying out the original design.

Throughout the latter part of the book extensive use has, according to Prof. Stokes' revival of this valuable notation, been made of the "solidus" to replace the horizontal stroke in fractions; for example $\frac{a}{b}$ is printed a/b. This notation is (as is illustrated by the spacing between these lines) advantageous for the introduction of isolated analytical expressions in the midst of the text, and its use in printing complex fractional and exponential expressions permits the printer to dispense with much of the troublesome process known as "justification," and effects a considerable saving in space and expense.

An index to the *whole* of the first volume has been prepared by Mr BURNSIDE, and is placed at the end.

A schedule is also given below of all the amendments and additions (excepting purely verbal changes and corrections) made in the present edition of the first volume.

Inspection of the schedules on pages xxii. to xxv. will shew that much new matter has been imported into the present edition, both in Part I. and Part II. These additions are indicated by the word "new."

The most important part of the labour of editing Part II. has been borne by Mr G. H. DARWIN, and it will be seen from the schedule below that he has made valuable contributions to the work.

NOTE TO NEW IMPRESSION, 1912

A few slight additions and corrections have been made by Sir GEORGE DARWIN and Prof. H. LAMB, but, substantially, the work remains as last passed by the authors. The additions can be identified by the initials attached in brackets.

1912

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DIVISION II.

ABSTRACT DYNAMICS.

CHAPTER V.

INTRODUCTORY.

438. UNTIL we know thoroughly the nature of matter and the forces which produce its motions, it will be utterly impossible to submit to mathematical reasoning the *exact* conditions of any physical question. It has been long understood, however, that approximate solutions of problems in the ordinary branches of Natural Philosophy may be obtained by a species of *abstraction*, or rather *limitation of the data*, such as enables us easily to solve the modified form of the question, while we are well assured that the circumstances (so modified) affect the result only in a superficial manner. Approximate treatment of physical questions.

439. Take, for instance, the very simple case of a crowbar employed to move a heavy mass. The accurate mathematical investigation of the action would involve the simultaneous treatment of the motions of every part of bar, fulcrum, and mass raised; but our ignorance of the nature of matter and molecular forces, precludes any such complete treatment of the problem.

It is a result of observation that the particles of the bar, fulcrum, and mass, separately, retain throughout the process nearly the same relative positions. Hence the idea of solving, Approximate treatment of physical questions. instead of the complete but infinitely transcendent problem, another, in reality quite different, but which, while amply simple, obviously leads to practically the same results so far as concerns the equilibrium and motions of the bodies as a whole.

440. The new form is given at once by the experimental result of the trial. Imagine the masses involved to be *perfectly rigid*, that is, incapable of changing form or dimensions. Then the infinite series of forces, really acting, may be left out of consideration; so that the mathematical investigation deals with a finite (and generally small) number of forces instead of a practically infinite number. Our warrant for such a substitution is to be established thus.

441. The effects of the intermolecular forces could be exhibited only in alterations of the form or volume of the masses involved. But as these (practically) remain almost unchanged, the forces which produce, or tend to produce, them may be left out of consideration. Thus we are enabled to investigate the action of machinery supposed to consist of separate portions whose form and dimensions are unalterable.

Further approximations. **442.** If we go a little further into the question, we find that the lever *bends*, some parts of it are extended and others compressed. This would lead us into a very serious and difficult inquiry if we had to take account of the whole circumstances. But (by experience) we find that a sufficiently accurate solution of this more formidable case of the problem may be obtained by supposing (what can *never* be realized in practice) the mass to be homogeneous, and the forces consequent on a dilatation, compression, or distortion, to be proportional in magnitude, and opposed in direction, to these deformations respectively. By this further assumption, close approximations may be made to the vibrations of rods, plates, etc., as well as to the statical effect of springs, etc.

443. We may pursue the process further. Compression, in general, produces heat, and extension, cold. The elastic forces of the material are thus rendered sensibly different from what they would be with the same changes of bulk and shape, but

with no change of temperature. By introducing such considerations, we reach, without great difficulty, what may be called a *third* approximation to the solution of the physical problem considered. Further approximations.

444. We might next introduce the conduction of the heat, so produced, from point to point of the solid, with its accompanying modifications of elasticity, and so on; and we might then consider the production of thermo-electric currents, which (as we shall see) are always developed by unequal heating in a mass if it be not perfectly homogeneous. Enough, however, has been said to show, *first*, our utter ignorance as to the true and complete solution of any physical question by the only perfect method, that of the consideration of the circumstances which affect the motion of every portion, separately, of each body concerned; and, *second*, the practically sufficient manner in which practical questions may be attacked by limiting their generality, *the limitations introduced being themselves deduced from experience*, and being therefore Nature's own solution (to a less or greater degree of accuracy) of the infinite additional number of equations by which we should otherwise have been encumbered.

445. To take another case: in the consideration of the propagation of waves at the surface of a fluid, it is impossible, not only on account of mathematical difficulties, but on account of our ignorance of *what* matter is, and what forces its particles exert on each other, to form the equations which would give us the separate motion of each. Our first approximation to a solution, and one sufficient for most practical purposes, is derived from the consideration of the motion of a homogeneous, incompressible, and perfectly plastic mass; a hypothetical substance which may have no existence in nature.

446. Looking a little more closely, we find that the actual motion differs considerably from that given by the analytical solution of the restricted problem, and we introduce further considerations, such as the *compressibility* of fluids, their *internal friction*, the heat generated by the latter, and its effects in dilating the mass, etc. etc. By such successive corrections we

Further approximations.

attain, at length, to a mathematical result which (at all events in the present state of experimental science) agrees, within the limits of experimental error, with observation.

447. It would be easy to give many more instances substantiating what has just been advanced, but it seems scarcely necessary to do so. We may therefore at once say that there is no question in physical science which can be *completely and accurately* investigated by mathematical reasoning, but that there are different degrees of approximation, involving assumptions more and more nearly coincident with observation, which may be arrived at in the solution of any particular question.

Object of the present division of the work.

448. *The object of the present division of this volume is to deal with the first and second of these approximations.* In it we shall suppose all solids either RIGID, *i.e.*, unchangeable in form and volume, or ELASTIC; but in the latter case, we shall assume the law, connecting a compression or a distortion with the force which causes it, to have a particular form deduced from experiment. And we shall in the latter case neglect the thermal or electric effects which compression or distortion generally cause. We shall also suppose fluids, whether liquids or gases, to be either INCOMPRESSIBLE or compressible according to certain known laws; and we shall omit considerations of fluid friction, although we admit the consideration of friction between solids. Fluids will therefore be supposed *perfect*, *i.e.*, such that any particle may be moved amongst the others by the slightest force.

449. When we come to Properties of Matter and the various forms of Energy, we shall give in detail, as far as they are yet known, the modifications which further approximations have introduced into the previous results.

Laws of friction.

450. The laws of friction between solids were very ably investigated by Coulomb; and, as we shall require them in the succeeding chapters, we give a brief summary of them here; reserving the more careful scrutiny of experimental results to our chapter on Properties of Matter.

451. To produce and to maintain sliding of one solid body on another requires a tangential force which depends—(1) upon

Laws of friction.

the nature of the bodies; (2) upon their polish, or the species and quantity of lubricant which may have been applied; (3) upon the normal pressure between them, to which it is in general directly proportional. It does not (except in some extreme cases where scratching or excessive abrasion takes place) depend sensibly upon the area of the surfaces in contact. When two bodies are pressed together without being caused to slide one on another, the force which prevents sliding is called Statical Friction. It is capable of opposing a tangential resistance to motion which may be of any amount less than or at most equal to μR; where R is the whole normal pressure between the bodies; and μ (which depends mainly upon the nature of the surfaces in contact) is what is commonly called the *coefficient of Statical Friction.* This coefficient varies greatly with the circumstances, being in some cases as low as 0·03, in others as high as 0·80. Later, we shall give a table of its values. When the applied forces are insufficient to produce motion, the whole amount of statical friction is not called into play; its amount then just reaches what is sufficient to equilibrate the other forces, and its direction is the opposite of that in which their resultant tends to produce motion.

452. When the statical friction has been overcome, and sliding is produced, experiment shows that a force of friction continues to act, opposing the motion; that this force of *Kinetic Friction* is in most cases considerably less than the extreme force of static friction which had to be overcome before the sliding commenced; that it too is sensibly proportional to the normal pressure; and that it is approximately the same whatever be the velocity of the sliding.

Rejection of merely curious speculations.

453. In the following Chapters on Abstract Dynamics we confine ourselves mainly to the general principles, and the fundamental formulas and equations of the mathematics of this extensive subject; and, neither seeking nor avoiding mathematical exercitations, we enter on special problems solely with a view to possible usefulness for physical science, whether in the way of the *material* of experimental investigation, or for illustrating physical principles, or for aiding in speculations of Natural Philosophy.

CHAPTER VI.

STATICS OF A PARTICLE.—ATTRACTION.

Objects of the chapter.

454. We naturally divide Statics into two parts—the equilibrium of a particle, and that of a rigid or elastic body or system of particles whether solid or fluid. In a very few sections we shall dispose of the first of these parts, and the rest of this chapter will be devoted to a digression on the important subject of Attraction.

Conditions of equilibrium of a particle.

455. By § 255, forces acting at the same point, or on the same material particle, are to be compounded by the same laws as velocities. Hence, evidently, the sum of their components in any direction must vanish if there is equilibrium; and there is equilibrium if the sums of the components in each of three lines not in one plane are each zero. And thence the necessary and sufficient mathematical equations of equilibrium.

Thus, for the equilibrium of a material particle, it is *necessary*, and *sufficient*, that the (algebraic) sums of the components of the applied forces, resolved in any three rectangular directions, should vanish.

Equilibrium of a particle.

If P be one of the forces, l, m, n its direction-cosines, we have

$$\Sigma lP = 0, \quad \Sigma mP = 0, \quad \Sigma nP = 0.$$

If there be not equilibrium, suppose R, with direction-cosines λ, μ, ν, to be the resultant force. If reversed in direction, it will, with the other forces, produce equilibrium. Hence

$$\Sigma lP - \lambda R = 0, \quad \Sigma mP - \mu R = 0, \quad \Sigma nP - \nu R = 0.$$

Equilibrium of a particle

And $$R^2 = (\Sigma lP)^2 + (\Sigma mP)^2 + (\Sigma nP)^2,$$

while $$\frac{\lambda}{\Sigma lP} = \frac{\mu}{\Sigma mP} = \frac{\nu}{\Sigma nP}.$$

456. We may take one or two particular cases as examples of the general results above. Thus,

(1) If the particle rest on a frictionless curve, the component force along the curve must vanish.

If x, y, z be the co-ordinates of the point of the curve at which the particle rests, we have evidently

$$\Sigma P\left(l\frac{dx}{ds} + m\frac{dy}{ds} + n\frac{dz}{ds}\right) = 0.$$

When P, l, m, n are given in terms of x, y, z, this, with the *two* equations to the curve, determines the position of equilibrum.

(2) If the curve be frictional, the resultant force along it must be balanced by the friction.

If F be the friction, the condition is

$$\Sigma P\left(l\frac{dx}{ds} + m\frac{dy}{ds} + n\frac{dz}{ds}\right) - F = 0.$$

This gives the amount of friction which will be called into play; and equilibrium will subsist until, as a limit, the friction is μ times the normal pressure on the curve. But the normal pressure is

$$\Sigma P\left\{\left(m\frac{dz}{ds} - n\frac{dy}{ds}\right)^2 + \left(n\frac{dx}{ds} - l\frac{dz}{ds}\right)^2 + \left(l\frac{dy}{ds} - m\frac{dx}{ds}\right)^2\right\}^{\frac{1}{2}}.$$

Hence, the limiting positions, between which equilibrium is possible, are given by the two equations to the curve, combined with

$$\Sigma P\left(l\frac{dx}{ds} + m\frac{dy}{ds} + n\frac{dz}{ds}\right) \pm \mu\Sigma P\left\{\left(m\frac{dz}{ds} - n\frac{dy}{ds}\right)^2 + \left(n\frac{dx}{ds} - l\frac{dz}{ds}\right)^2 + \left(l\frac{dy}{ds} - m\frac{dx}{ds}\right)^2\right\}^{\frac{1}{2}} = 0.$$

(3) If the particle rest on a smooth surface, the resultant of the applied forces must evidently be perpendicular to the surface.

If $\phi(x, y, z) = 0$ be the equation of the surface, we must therefore have

$$\frac{\frac{d\phi}{dx}}{\Sigma lP} = \frac{\frac{d\phi}{dy}}{\Sigma mP} = \frac{\frac{d\phi}{dz}}{\Sigma nP},$$

and these three equations determine the position of equilibrium.

Equilibrium of a particle.

(4) If it rest on a rough surface, friction will be called into play, resisting motion along the surface; and there will be equilibrium at any point within a certain boundary, determined by the condition that at *it* the friction is μ times the normal pressure on the surface, while within it the friction bears a less ratio to the normal pressure. When the only applied force is gravity, we have a very simple result, which is often practically useful. Let θ be the angle between the normal to the surface and the vertical at any point; the normal pressure on the surface is evidently $W\cos\theta$, where W is the weight of the particle; and the resolved part of the weight parallel to the surface, which must of course be balanced by the friction, is $W\sin\theta$. In the limiting position, when sliding is just about to commence, the greatest possible amount of statical friction is called into play, and we have

$$W\sin\theta=\mu W\cos\theta,$$

or

$$\tan\theta=\mu.$$

Angle of repose.

The value of θ thus found is called the *Angle of Repose*.

Let $\phi(x, y, z)=0$ be the surface: P, with direction-cosines l, m, n, the resultant of the applied forces. The normal pressure is

$$P\frac{l\dfrac{d\phi}{dx}+m\dfrac{d\phi}{dy}+n\dfrac{d\phi}{dz}}{\sqrt{\left(\dfrac{d\phi}{dx}\right)^2+\left(\dfrac{d\phi}{dy}\right)^2+\left(\dfrac{d\phi}{dz}\right)^2}}.$$

The resolved part of P parallel to the surface is

$$P\sqrt{\frac{\left(m\dfrac{d\phi}{dz}-n\dfrac{d\phi}{dy}\right)^2+\left(n\dfrac{d\phi}{dx}-l\dfrac{d\phi}{dz}\right)^2+\left(l\dfrac{d\phi}{dy}-m\dfrac{d\phi}{dx}\right)^2}{\left(\dfrac{d\phi}{dx}\right)^2+\left(\dfrac{d\phi}{dy}\right)^2+\left(\dfrac{d\phi}{dz}\right)^2}}.$$

Hence, for the boundary of the portion of the surface within which equilibrium is possible, we have the additional equation

$$\left(m\frac{d\phi}{dz}-n\frac{d\phi}{dy}\right)^2+\left(n\frac{d\phi}{dx}-l\frac{d\phi}{dz}\right)^2+\left(l\frac{d\phi}{dy}-m\frac{d\phi}{dx}\right)^2=\mu^2\left(l\frac{d\phi}{dx}+m\frac{d\phi}{dy}+n\frac{d\phi}{dz}\right)^2.$$

Attraction.

457. A most important case of the composition of forces acting at one point is furnished by the consideration of the attraction of a body of any form upon a material particle any-

where situated. Experiment has shown that the attraction exerted by any portion of matter upon another is not modified by the proximity, or even by the interposition, of other matter; and thus the attraction of a body on a particle is the resultant of the attractions exerted by its several parts. To treatises on applied mathematics we must refer for the examination of the consequences, often very curious, of various laws of attraction; but, dealing with Natural Philosophy, we confine ourselves mainly, (and except where we give the mathematics of Laplace's beautiful and instructive and physically important, though unreal, theory of capillary attraction,) to the law of the inverse square of the distance which Newton discovered for gravitation. This, indeed, furnishes us with an ample supply of most interesting as well as useful results.

Attraction.

Universal law of attraction.

458. The law, which (as a property of matter) is to be carefully considered in the next proposed Division of this Treatise, may be thus enunciated.

Every particle of matter in the universe attracts every other particle, with a force whose direction is that of the line joining the two, and whose magnitude is directly as the product of their masses, and inversely as the square of their distance from each other.

Experiment shows (as will be seen further on) that the same law holds for electric and magnetic attractions under properly defined conditions.

Special unit of quantity of matter.

459. For the special applications of Statical principles to which we proceed, it will be convenient to use a special unit of mass, or quantity of matter, and corresponding units for the measurement of electricity and magnetism.

Thus if, in accordance with the physical law enunciated in § 458, we take as the expression for the forces exerted on each other by masses M and m, at distance D,

$$\frac{Mm}{D^2};$$

it is obvious that our *unit* force is the mutual attraction of two units of mass placed at unit of distance from each other.

Linear, surface, and volume, densities.

460. It is convenient for many applications to speak of the *density* of a distribution of matter, electricity, etc., along a line, over a surface, or through a volume.

Here line-density = quantity of matter per unit of length.
surface-density = " " " area.
volume-density = " " " volume.

Electric and magnetic reckonings of *quantity*.

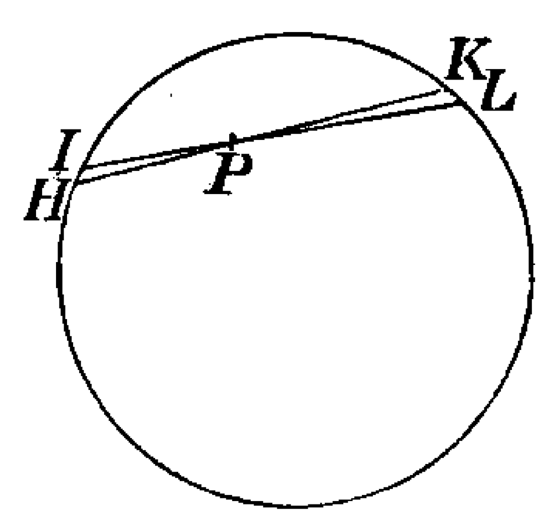
461. In applying the succeeding investigations to electricity or magnetism, it is only necessary to premise that M and m stand for *quantities* of free electricity or magnetism, whatever these may be, and that here the idea of *mass* as depending on *inertia* is not necessarily involved. The formula $\frac{Mm}{D^2}$ will still represent the mutual action, if we take as unit of imaginary electric or magnetic matter, such a quantity as exerts unit force on an equal quantity at unit distance. Here, however, one or both of M, m may be negative; and, as in these applications like kinds *repel* each other, the mutual action will be attraction or repulsion, according as its sign is negative or positive. With these provisos, the following theory is applicable to any of the above-mentioned classes of forces. We commence with a few simple cases which can be completely treated by means of elementary geometry.

Positive and negative masses admitted in abstract theory of attraction.

Uniform spherical shell. Attraction on *internal* point.

462. *If the different points of a spherical surface attract equally with forces varying inversely as the squares of the distances, a particle placed within the surface is not attracted in any direction.*

Let $HIKL$ be the spherical surface, and P the particle within it. Let two lines HK, IL, intercepting very small arcs HI, KL, be drawn through P; then, on account of the similar triangles HPI, KPL, those arcs will be proportional to the distances HP, LP; and any small elements of the spherical surface at HI and KL, each bounded all round by straight lines passing through P [and very nearly coinciding with HK], will be in the duplicate ratio of those lines.



Uniform spherical shell. Attraction on *internal* point.

Hence the forces exercised by the matter of these elements on the particle P are equal; for they are as the quantities of matter directly, and the squares of the distances, inversely; and these two ratios compounded give that of equality. The attractions therefore, being equal and opposite, balance one another: and a similar proof shows that the attractions due to all parts of the whole spherical surface are balanced by contrary attractions. Hence the particle P is not urged in any direction by these attractions.

Digression on the division of surfaces into elements.

463. The division of a spherical surface into infinitely small elements will frequently occur in the investigations which follow: and Newton's method, described in the preceding demonstration, in which the division is effected in such a manner that all the parts may be taken together in *pairs of opposite elements with reference to an internal point;* besides other methods deduced from it, suitable to the special problems to be examined; will be repeatedly employed. The present digression, in which some definitions and elementary geometrical propositions regarding this subject are laid down, will simplify the subsequent demonstrations, both by enabling us, through the use of convenient terms, to avoid circumlocution, and by affording us convenient means of reference for elementary principles, regarding which repeated explanations might otherwise be necessary.

Explanations and definitions regarding cones.

464. If a straight line which constantly passes through a fixed point be moved in any manner, it is said to describe, or generate, a *conical surface* of which the fixed point is the vertex.

If the generating line be carried from a given position continuously through any series of positions, no two of which coincide, till it is brought back to the first, the entire line on the two sides of the fixed point will generate a complete conical surface, consisting of two sheets, which are called *vertical or opposite cones.* Thus the elements HI and KL, described in Newton's demonstration given above, may be considered as being cut from the spherical surface by two *opposite cones* having P for their common vertex.

The solid angle of a cone, or of a complete conical surface.

465. If any number of spheres be described from the vertex of a cone as centre, the segments cut from the concentric spherical surfaces will be similar, and their areas will be as the squares of the radii. The quotient obtained by dividing the area of one of these segments by the square of the radius of the spherical surface from which it is cut, is taken as the measure of the *solid angle of the cone.* The segments of the same spherical surfaces made by the opposite cone, are respectively equal and similar to the former (but "perverted"). Hence the solid angles of two vertical or opposite cones are equal: either may be taken as the solid angle of the complete conical surface, of which the opposite cones are the two sheets.

Sum of all the solid angles round a point $=4\pi$.

466. Since the area of a spherical surface is equal to the square of its radius multiplied by 4π, it follows that the sum of the solid angles of all the distinct cones which can be described with a given point as vertex, is equal to 4π.

Sum of the solid angles of all the complete conical surfaces $=2\pi$.

467. The solid angles of vertical or opposite cones being equal, we may infer from what precedes that the sum of the solid angles of all the complete conical surfaces which can be described without mutual intersection, with a given point as vertex, is equal to 2π.

Solid angle subtended at a point by a terminated surface.

468. The solid angle subtended at a point by a superficial area of any kind, is the solid angle of the cone generated by a straight line passing through the point, and carried entirely round the boundary of the area.

Orthogonal and oblique sections of a small cone.

469. A very small cone, that is, a cone such that any two positions of the generating line contain but a very small angle, is said to be cut at right angles, or orthogonally, by a spherical surface described from its vertex as centre, or by any surface, whether plane or curved, which touches the spherical surface at the part where the cone is cut by it.

A very small cone is said to be cut obliquely, when the section is inclined at any finite angle to an orthogonal section; and this angle of inclination is called the *obliquity of the section.*

The area of an orthogonal section of a very small cone is equal

to the area of an oblique section in the same position, multiplied by the cosine of the obliquity.

Orthogonal and oblique sections of a small cone.

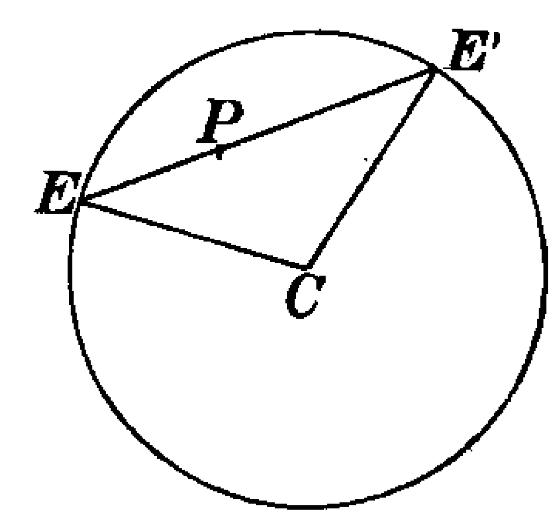
Hence the area of an oblique section of a small cone is equal to the quotient obtained by dividing the product of the square of its distance from the vertex, into the solid angle, by the cosine of the obliquity.

Area of segment cut from spherical surface by small cone.

470. Let E denote the area of a very small element of a spherical surface at the point E (that is to say, an element every part of which is very near the point E), let ω denote the solid angle subtended by E at any point P, and let PE, produced if necessary, meet the surface again in E': then, a denoting the radius of the spherical surface, we have

$$E = \frac{2a \, . \, \omega \, . \, PE^2}{EE'} \, .$$

For, the obliquity of the element E, considered as a section of the cone of which P is the vertex and the element E a section; being the angle between the given spherical surface and another described from P as centre, with PE as radius; is equal to the angle between the radii, EP and EC, of the two spheres. Hence, by considering the isosceles triangle ECE', we find that the cosine of the obliquity is equal to $\frac{\frac{1}{2}EE'}{EC}$ or to $\frac{EE'}{2a}$, and we arrive at the preceding expression for E.



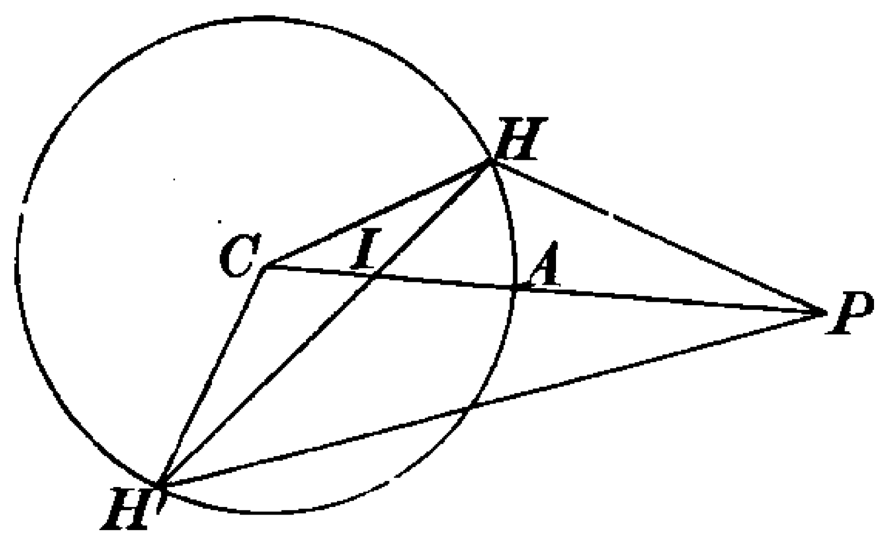
Uniform spherical shell. Attraction on *external* point.

471. *The attraction of a uniform spherical surface on an external point is the same as if the whole mass were collected at the centre*.*

* This theorem, which is more comprehensive than that of Newton in his first proposition regarding attraction on an external point (Prop. LXXI.), is fully established as a corollary to a subsequent proposition (Prop. LXXIII. cor. 2). If we had considered the proportion of the forces exerted upon two external points at different distances, instead of, as in the text, investigating the absolute force on one point, and if besides we had taken together all the pairs of elements which would constitute two narrow annular portions of the surface, in planes perpendicular to PC, the theorem and its demonstration would have coincided precisely with Prop. LXXI. of the *Principia*.

Uniform spherical shell. Attraction on *external* point.

Let P be the external point, C the centre of the sphere, and CAP a straight line cutting the spherical surface in A. Take I in CP, so that CP, CA, CI may be continual proportionals, and let the whole spherical surface be divided into *pairs of opposite elements with reference to the point I.*



Let H and H' denote the magnitudes of a pair of such elements, situated respectively at the extremities of a chord HH'; and let ω denote the magnitude of the solid angle subtended by either of these elements at the point I.

We have (§ 469),

$$H = \frac{\omega . IH^2}{\cos CHI}, \quad \text{and} \quad H' = \frac{\omega . IH'^2}{\cos CH'I}.$$

Hence, if ρ denote the density of the surface, the attractions of the two elements H and H' on P are respectively

$$\rho \frac{\omega}{\cos CHI} \cdot \frac{IH^2}{PH^2}, \quad \text{and} \quad \rho \frac{\omega}{\cos CH'I} \cdot \frac{IH'^2}{PH'^2}.$$

Now the two triangles PCH, HCI have a common angle at C, and, since $PC : CH :: CH : CI$, the sides about this angle are proportional. Hence the triangles are similar; so that the angles CPH and CHI are equal, and

$$\frac{IH}{HP} = \frac{CH}{CP} = \frac{a}{CP}.$$

In the same way it may be proved, by considering the triangles PCH', $H'CI$, that the angles CPH' and $CH'I$ are equal, and that

$$\frac{IH'}{H'P} = \frac{CH'}{CP} = \frac{a}{CP}.$$

Hence the expressions for the attractions of the elements H and H' on P become

$$\rho \frac{\omega}{\cos CHI} \cdot \frac{a^2}{CP^2}, \quad \text{and} \quad \rho \frac{\omega}{\cos CH'I} \cdot \frac{a^2}{CP^2},$$

which are equal, since the triangle HCH' is isosceles; and, for

Uniform spherical shell. Attraction on *external* point.

the same reason, the angles CPH, CPH', which have been proved to be respectively equal to the angles CHI, $CH'I$, are equal. We infer that the resultant of the forces due to the two elements is in the direction PC, and is equal to

$$2\omega . \rho . \frac{a^2}{CP^2}.$$

To find the total force on P, we must take the sum of all the forces along PC due to the pairs of opposite elements; and, since the multiplier of ω is the same for each pair, we must add all the values of ω, and we therefore obtain (§ 467), for the required resultant,

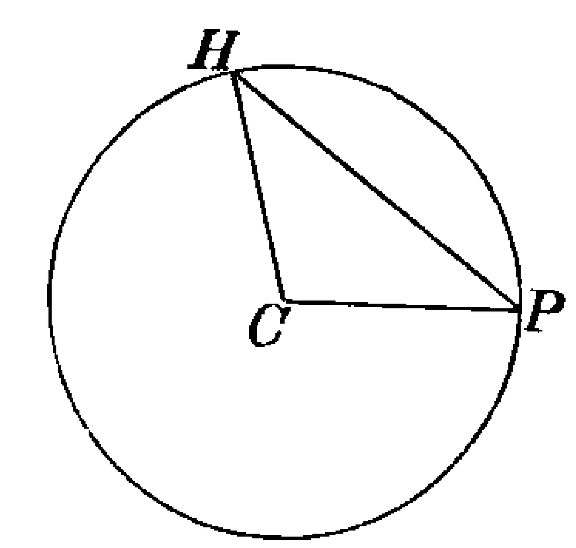
$$\frac{4\pi\rho a^2}{CP^2}.$$

The numerator of this expression; being the product of the density, into the area of the spherical surface; is equal to the whole mass; and therefore the force on P is the same as if the whole mass were collected at C.

Cor. The force on an external point, infinitely near the surface, is equal to $4\pi\rho$, and is in the direction of a normal at the point. The force on an internal point, however near the surface, is, by a preceding proposition, *nil.*

Attraction on an element of the surface.

472. Let σ be the area of an infinitely small element of the surface at any point P, and at any other point H of the surface let a small element subtending a solid angle ω, at P, be taken. The area of this element will be equal to



$$\frac{\omega . PH^2}{\cos CHP},$$

and therefore the attraction along HP, which it exerts on the element σ at P, will be equal to

$$\frac{\rho\omega . \rho\sigma}{\cos CHP}, \text{ or } \frac{\omega}{\cos CHP} \rho^2\sigma.$$

Now the total attraction on the element at P is in the direction CP; the component in this direction of the attraction due to the element H, is

$$\omega . \rho^2\sigma;$$

Attraction on an element of the surface.

and, since all the cones corresponding to the different elements of the spherical surface lie on the same side of the tangent plane at P, we deduce, for the resultant attraction on the element σ,

$$2\pi\rho^2\sigma.$$

From the corollary to the preceding proposition, it follows that this attraction is half the force which would be exerted on an external point, possessing the same quantity of matter as the element σ, and placed infinitely near the surface.

473. In some of the most important elementary problems of the theory of electricity, spherical surfaces with densities varying inversely as the cubes of distances from eccentric points occur: and it is of fundamental importance to find the attraction of such a shell on an internal or external point. This may be done synthetically as follows; the investigation being, as we shall see below, virtually the same as that of § 462, or § 471.

Attraction of a spherical surface of which the density varies inversely as the cube of the distance from a given point.

474. Let us first consider the case in which the given point S and the attracted point P are separated by the spherical surface. The two figures represent the varieties of this case in which, the point S being without the sphere, P is within; and, S being within, the attracted point is external. The same demonstration is applicable literally with reference to the two figures; but, to avoid the consideration of negative quantities, some of the expressions may be conveniently modified to suit the second figure. In such instances the two expressions are given in a double line, the upper being that which is most convenient for the first figure, and the lower for the second.

Let the radius of the sphere be denoted by a, and let f be the distance of S from C, the centre of the sphere (not represented in the figures).

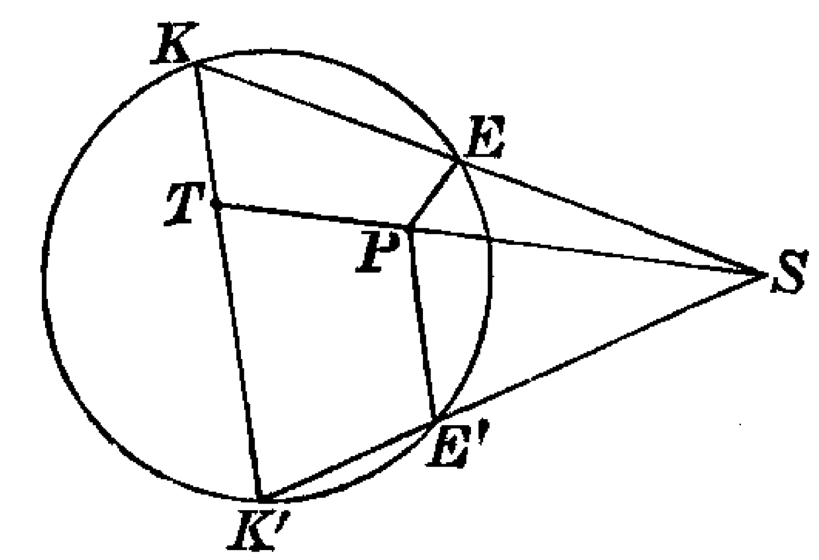
Join SP and take T in this line (or its continuation) so that

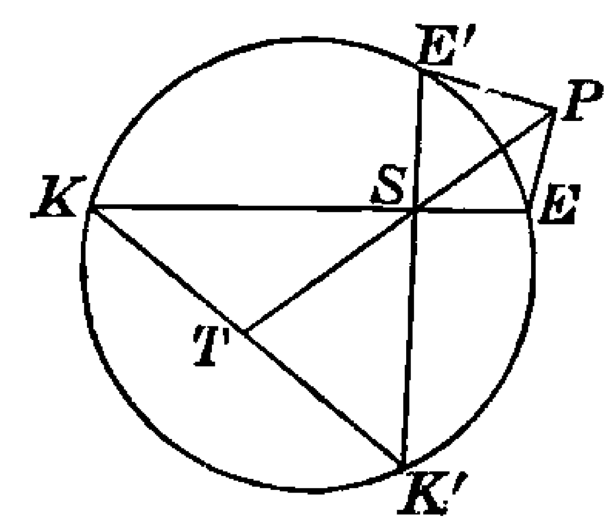
$$\text{(fig. 1)}\quad SP\,.\,ST = f^2 - a^2.$$
$$\text{(fig. 2)}\quad SP\,.\,TS = a^2 - f^2.$$

Through T draw any line cutting the spherical surface at K, K'. Join SK, SK', and let the lines so drawn cut the spherical surface again in E, E'.

Attraction of a spherical surface of which the density varies inversely as the cube of the distance from a given point.

Let the whole spherical surface be divided into pairs of opposite elements with reference to the point T. Let K and K' be a pair of such elements situated at the extremities of the chord KK', and subtending the solid angle ω at the point T; and let elements E and E' be taken subtending at S the same solid angles respectively as the elements K and K'. By this means we may divide the whole spherical surface into pairs of conjugate elements, E, E', since it is easily seen that when we have taken every pair of elements, K, K', the whole surface





will have been exhausted, without repetition, by the deduced elements, E, E'. Hence the attraction on P will be the final resultant of the attractions of all the pairs of elements, E, E'.

Now if ρ be the surface density at E, and if F denote the attraction of the element E on P, we have

$$F = \frac{\rho\,.\,E}{EP^2}.$$

According to the given law of density we shall have

$$\rho = \frac{\lambda}{SE^3},$$

where λ is a constant. Again, since SEK is equally inclined to the spherical surface at the two points of intersection, we have

$$E = \frac{SE^2}{SK^2}.K = \frac{SE^2}{SK^2}.\frac{2a\omega\,.\,TK^2}{KK'};$$

and hence

$$F = \frac{\dfrac{\lambda}{SE^3}.\dfrac{SE^2}{SK^2}.\dfrac{2a\omega\,.\,TK^2}{KK'}}{EP^2} = \lambda\,.\,\frac{2a}{KK'}.\frac{TK^2}{SE\,.\,SK^2\,.\,EP^2}.\omega.$$

Attraction of a spherical surface of which the density varies inversely as the cube of the distance from a given point.

Now, by considering the great circle in which the sphere is cut by a plane through the line SK, we find that

$$\text{(fig. 1) } SK \cdot SE = f^2 - a^2,$$

$$\text{(fig. 2) } KS \cdot SE = a^2 - f^2,$$

and hence $SK \cdot SE = SP \cdot ST$, from which we infer that the triangles KST, PSE are similar; so that $TK : SK :: PE : SP$.

Hence $$\frac{TK^2}{SK^2 \cdot PE^2} = \frac{1}{SP^2},$$

and the expression for F becomes

$$F = \lambda \cdot \frac{2a}{KK'} \cdot \frac{1}{SE \cdot SP^2} \cdot \omega.$$

Modifying this by preceding expressions we have

$$\text{(fig. 1) } F = \lambda \cdot \frac{2a}{KK'} \cdot \frac{\omega}{(f^2 - a^2)\, SP^2} \cdot SK,$$

$$\text{(fig. 2) } F = \lambda \cdot \frac{2a}{KK'} \cdot \frac{\omega}{(a^2 - f^2)\, SP^2} \cdot KS.$$

Similarly, if F' denote the attraction of E' on P, we have

$$\text{(fig. 1) } F' = \lambda \frac{2a}{KK'} \cdot \frac{\omega}{(f^2 - a^2)\, SP^2} \cdot SK',$$

$$\text{(fig. 2) } F' = \lambda \frac{2a}{KK'} \cdot \frac{\omega}{(a^2 - f^2)\, SP^2} \cdot K'S.$$

Now in the triangles which have been shown to be similar, the angles TKS, EPS are equal; and the same may be proved of the angles $TK'S$, $E'PS$. Hence the two sides SK, SK' of the triangle KSK' are inclined to the third at the same angles as those between the line PS and directions PE, PE' of the two forces on the point P; and the sides SK, SK' are to one another as the forces, F, F', in the directions PE, PE'. It follows, by "the triangle of forces," that the resultant of F and F' is along PS, and that it bears to the component forces the same ratios as the side KK' of the triangle bears to the other two sides. Hence the resultant force due to the two elements E and E' on the point P, is towards S, and is equal to

$$\lambda \cdot \frac{2a}{KK'} \cdot \frac{\omega}{(f^2 \sim a^2) \cdot SP^2} \cdot KK', \text{ or } \frac{\lambda \cdot 2a \cdot \omega}{(f^2 \sim a^2)\, SP^2}.$$

The total resultant force will consequently be towards S; and we find, by summation (§ 467) for its magnitude,

Attraction of a spherical surface of which the density varies inversely as the cube of the distance from a given point.

$$\frac{\lambda \cdot 4\pi a}{(f^2 \sim a^2)\, SP^2}.$$

Hence we infer that the resultant force at any point P, separated from S by the spherical surface, is the same as if a quantity of matter equal to $\dfrac{\lambda \cdot 4\pi a}{f^2 \sim a^2}$ were concentrated at the point S.

475. To find the attraction when S and P are either both without or both within the spherical surface.

Take in CS, or in CS produced through S, a point S_1, such that $$CS \cdot CS_1 = a^2.$$

Then, by a well-known geometrical theorem, if E be any point on the spherical surface, we have

$$\frac{SE}{S_1E} = \frac{f}{a}.$$

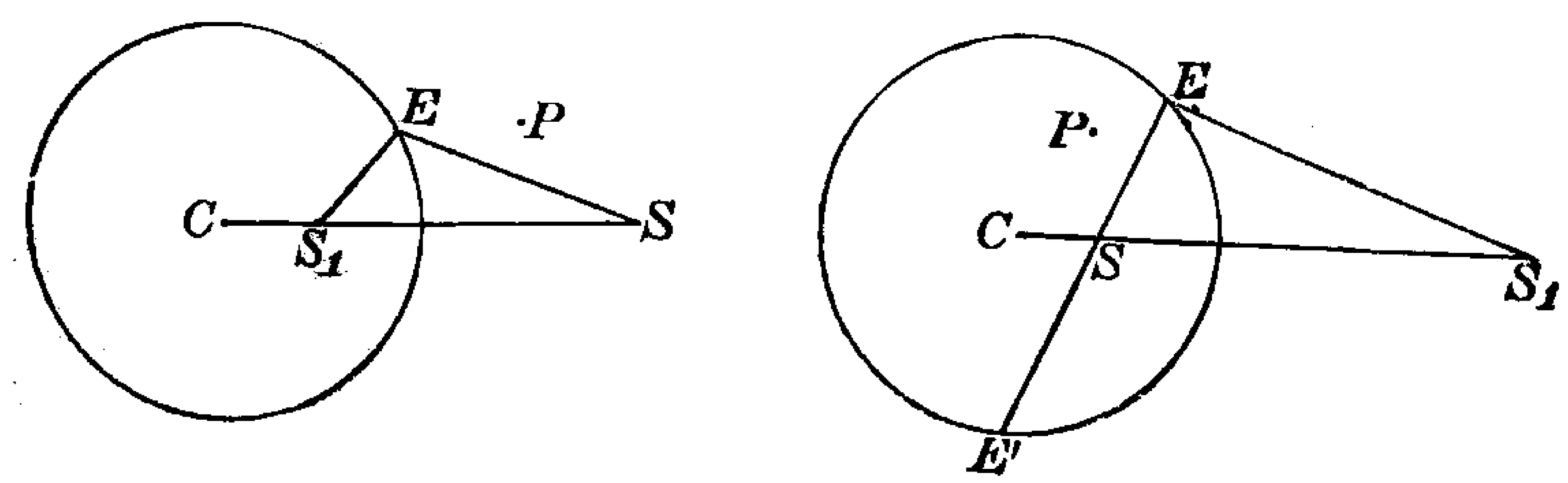
Hence we have

$$\frac{\lambda}{SE^3} = \frac{\lambda a^3}{f^3 \cdot S_1E^3}.$$

Hence, ρ being the surface-density at E, we have

$$\rho = \frac{\dfrac{\lambda a^3}{f^3}}{S_1E^3} = \frac{\lambda_1}{S_1E^3},$$

if $$\lambda_1 = \frac{\lambda a^3}{f^3}.$$



Hence, by the investigation in the preceding section, the attraction on P is towards S_1, and is the same as if a quantity

Attraction of a spherical surface of which the density varies inversely as the cube of the distance from a given point.

of matter equal to $\frac{\lambda_1 . 4\pi a}{f_1^2 \sim a^2}$ were concentrated at that point; f_1 being taken to denote CS_1. If for f_1 and λ_1 we substitute their values, $\frac{a^2}{f}$ and $\frac{\lambda a^3}{f^3}$, we have the modified expression

$$\frac{\lambda \frac{a}{f} . 4\pi a}{a^2 \sim f^2}$$

for the quantity of matter which we must conceive to be collected at S_1.

Uninsulated sphere under the influence of an electric point.

476. If a spherical surface be electrified in such a way that the electrical density varies inversely as the cube of the distance from an internal point S, or from the corresponding external point S_1, it will attract any external point, as if its whole electricity were concentrated at S, and any internal point, as if a quantity of electricity greater than its own in the ratio of a to f were concentrated at S_1.

Let the density at E be denoted, as before, by $\frac{\lambda}{SE^3}$. Then, if we consider two opposite elements at E and E', which subtend a solid angle ω at the point S, the areas of these elements being $\frac{\omega . 2a\, SE^2}{EE'}$ and $\frac{\omega . 2a . SE'^2}{EE'}$, the quantity of electricity which they possess will be

$$\frac{\lambda . 2a . \omega}{EE'}\left(\frac{1}{SE} + \frac{1}{SE'}\right) \text{ or } \frac{\lambda . 2a . \omega}{SE . SE'}.$$

Now $SE . SE'$ is constant (Euc. III. 35) and its value is $a^2 - f^2$. Hence, by summation, we find for the total quantity of electricity on the spherical surface

$$\frac{\lambda . 4\pi a}{a^2 - f^2}.$$

Hence, if this be denoted by m, the expressions in the preceding paragraphs, for the quantities of electricity which we must suppose to be concentrated at the point S or S_1, according as P is without or within the spherical surface, become respectively

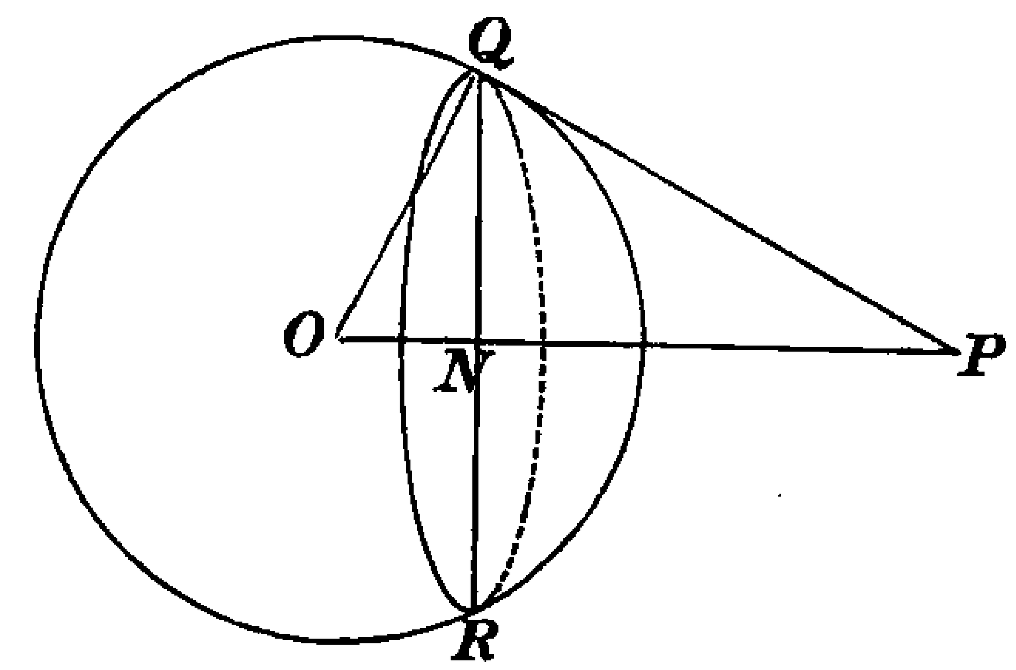
$$m, \text{ and } \frac{a}{f} m.$$

477. The *direct* analytical solution of such problems consists in the expression, by § 455, of the three components of the whole attraction as the sums of its separate parts due to the several particles of the attracting body; the transformation, by the usual methods, of these sums into definite integrals; and the evaluation of the latter. This is, in general, inferior in elegance and simplicity to the less direct mode of solution depending upon the determination of the potential energy of the attracted particle with reference to the forces exerted upon it by the attracting body, a method which we shall presently develop with peculiar care, as being of incalculable value in the theories of Electricity and Magnetism as well as in that of Gravitation. But before we proceed to it, we give some instances of the direct method, beginning with the case of a spherical shell.

Direct analytical calculation of attractions.

Uniform spherical shell.

(*a*) Let P be the attracted point, O the centre of the shell. Let any plane perpendicular to OP cut it in N, and the sphere in the small circle QR. Let $QOP = \theta$, $OQ = a$, $OP = D$. Then as the whole attraction is evidently along PO, we may at once resolve the parts of it in that direction. The circular band corresponding to θ, $\theta + d\theta$ has for area $2\pi a^2 \sin\theta d\theta$. Hence if M be the mass of the shell, the component attraction of the band on P, along PO, is



$$\frac{M}{2} \sin\theta d\theta . \frac{PN}{PQ^3}; \text{ and } PQ^2 = a^2 + D^2 - 2aD\cos\theta.$$

Hence if $PQ = x$, $\quad xdx = aD \sin\theta d\theta.$

Also $\quad PN = D - a\cos\theta = \frac{x^2 - a^2 + D^2}{2D};$

hence the attraction of the band is

$$\frac{M}{4D^2}\frac{x^2 - a^2 + D^2}{ax^2} dx.$$

Uniform spherical shell.

This divides itself, on integration, into two cases,

(1) P external, *i.e.*, $D > a$. Here the limits of x are $D - a$ and $D + a$, and the attraction is $\frac{M}{4D^2}\left[\frac{x}{a} - \frac{D^2 - a^2}{ax}\right]_{D-a}^{D+a} = \frac{M}{D^2}$, as before.

(2) P internal, *i.e.*, $D < a$. Here the limits are $a - D$ and $a + D$, and the attraction is $\frac{M}{4D^2}\left[\frac{x}{a} + \frac{a^2 - D^2}{ax}\right]_{a-D}^{a+D} = 0$.

Uniform circular disc, on particle in its axis.

(*b*) A useful case is that of the attraction of a circular plate of uniform surface density on a point in a line through its centre, and perpendicular to its plane.

If a be the radius of the plate, h the distance of the point from it, and M its mass, the attraction (which is evidently in a direction perpendicular to the plate) is easily seen to be

$$\frac{M}{a^2}\int_0^a \frac{2hrdr}{(h^2+r^2)^{\frac{3}{2}}} = \frac{2M}{a^2}\left\{1 - \frac{h}{\sqrt{h^2+a^2}}\right\}.$$

If ρ denote the surface density of the plate, this becomes

$$2\pi\rho\left(1 - \frac{h}{\sqrt{h^2+a^2}}\right);$$

which, for an infinite plate, becomes

$$2\pi\rho.$$

From the preceding formula many useful results may easily be deduced: thus,

Cylinder on particle in axis.

(*c*) A uniform *cylinder* of length l, and diameter a, attracts a point in its axis at a distance x from the nearest end with a force

$$2\pi\rho\int_x^{x+l}\left(1 - \frac{h}{\sqrt{h^2+a^2}}\right)dh = 2\pi\rho\left\{l - \sqrt{(x+l)^2+a^2} + \sqrt{x^2+a^2}\right\}.$$

When the cylinder is of infinite length (in one direction) the attraction is therefore

$$2\pi\rho\left(\sqrt{x^2+a^2} - x\right);$$

and, when the attracted particle is in contact with the centre of the end of the infinite cylinder, this is

$$2\pi\rho a.$$

Right cone on particle at vertex.

(*d*) A right cone, of semivertical angle α, and length l, attracts a particle at its vertex. Here we have at once for the attraction, the expression

$$2\pi\rho l(1 - \cos\alpha),$$

which is simply proportional to the length of the axis.

It is of course easy, when required, to find the necessarily less simple expression for the attraction on any point of the axis.

Positive and negative discs.

(*e*) For magnetic and electro-magnetic applications a very useful case is that of two equal discs, each perpendicular to the line joining their centres, on any point in that line—their masses (§ 461) being of opposite sign—that is, one repelling and the other attracting.

Let a be the radius, ρ the mass of a superficial unit, of either, c their distance, x the distance of the attracted point from the nearest disc. The whole action is evidently

$$2\pi\rho\left\{\frac{x+c}{\sqrt{(x+c)^2+a^2}} - \frac{x}{\sqrt{x^2+a^2}}\right\}.$$

In the particular case when c is diminished without limit, this becomes

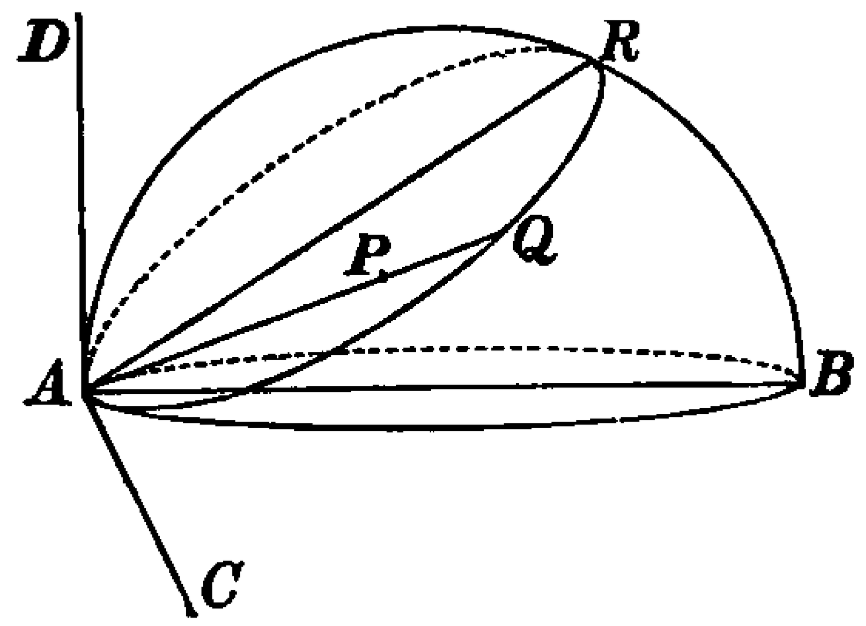
$$2\pi\rho c\frac{a^2}{(x^2+a^2)^{\frac{3}{2}}}.$$

Variation of force in crossing an attracting surface.

478. Let P and P' be two points infinitely near one another on two sides of a surface over which matter is distributed; and let ρ be the density of this distribution on the surface in the neighbourhood of these points. Then whatever be the resultant attraction, R, at P, due to all the attracting matter, whether lodging on this surface, or elsewhere, the resultant force, R', on P' is the resultant of a force equal and parallel to R, and a force equal to $4\pi\rho$, in the direction from P' perpendicularly towards the surface. For, suppose PP' to be perpendicular to the surface, which will not limit the generality of the proposition, and consider a circular disc, of the surface, having its centre in PP', and radius infinitely small in comparison with the radii of curvature of the surface but infinitely great in comparison with PP'. This disc will [§ 477, (*b*)] attract P and P' with forces, each equal to $2\pi\rho$ and opposite to one another in the line PP'. Whence the proposition. It is one of much importance in the theory of electricity.

Uniform hemisphere attracting particle at edge.

(*a*) As a further example of the direct analytical process, let us find the components of the attraction exerted by a uniform *hemisphere* on a particle at its edge. Let A be the particle, AB a diameter of the base, AC the tangent to the base at A; and AD perpendicular to AC, and AB. Let RQA be a section by a plane passing through AC; AQ any radius-vector of this section; P a point in AQ. Let $AP = r$, $CAQ = \theta$, $RAB = \phi$. The volume of an element at P is



$$r d\theta \,.\, r\sin\theta d\phi \,.\, dr = r^2 \sin\theta d\phi d\theta dr.$$

The resultant attraction on unit of matter at A has zero component along AC. Along AB the component is

$$\rho \iiint \sin\theta d\phi d\theta dr \cos\phi \sin\theta,$$

between proper limits. The limits of r are 0 and $2a\sin\theta\cos\phi$, those of ϕ are 0 and $\frac{\pi}{2}$, and those of θ are 0 and π. Hence,

Attraction along $AB = \frac{2}{3}\pi\rho a$.

Along AD the component is

$$\rho\int_0^{+\pi}\int_0^{\frac{\pi}{2}}\int_0^{2a\sin\theta\cos\phi} \sin\theta d\theta d\phi dr \sin\phi\sin\theta = \tfrac{4}{3}\rho a.$$

Alteration of latitude; by hemispherical hill or cavity.

(*b*) Hence at the southern base of a hemispherical hill of radius a and density ρ, the true latitude (as measured by the aid of the plumb-line, or by reflection of starlight in a trough of mercury) is diminished by the attraction of the mountain by the angle

$$\frac{\frac{2}{3}\pi\rho a}{G - \frac{4}{3}\rho a}$$

where G is the attraction of the earth, estimated in the same units. Hence, if R be the radius and σ the mean density of the earth, the angle is

$$\frac{\frac{2}{3}\pi\rho a}{\frac{4}{3}\pi\sigma R - \frac{4}{3}\rho a}, \text{ or } \tfrac{1}{2}\frac{\rho a}{\sigma R} \text{ approximately.}$$

Alteration of latitude; by hemispherical hill or cavity,

Hence the latitudes of stations at the base of the hill, north and south of it, differ by $\frac{a}{R}\left(2 + \frac{\rho}{\sigma}\right)$; instead of by $\frac{2a}{R}$, as they would do if the hill were removed.

In the same way the latitude of a place at the southern edge of a hemispherical *cavity* is increased on account of the cavity by $\frac{1}{2}\frac{\rho a}{\sigma R}$ where ρ is the density of the superficial strata.

(*c*) For mutual attraction between two segments of a homogeneous solid sphere, investigated indirectly on a hydrostatic principle, see § 753 below.

by crevasse.

479. As a curious additional example of the class of questions considered in § 478 (*a*) (*b*), a deep crevasse, extending east and west, increases the latitude of places at its southern edge by (approximately) the angle $\frac{3}{4}\frac{\rho a}{\sigma R}$ where ρ is the density of the crust of the earth, and a is the width of the crevasse. Thus the north edge of the crevasse will have a *lower* latitude than the south edge if $\frac{3}{2}\frac{\rho}{\sigma} > 1$, which might be the case, as there are rocks of density $\frac{2}{3} \times 5{\cdot}5$ or $3{\cdot}67$ times that of water. At a considerable depth in the crevasse, this change of latitudes is nearly *doubled,* and then the southern side has the greater latitude if the density of the crust be not less than $1{\cdot}83$ times that of water. The reader may exercise himself by drawing lines of equal latitude in the neighbourhood of the crevasse in this case: and by drawing meridians for the corresponding case of a crevasse running north and south.

Attraction of a sphere composed of concentric shells of uniform density.

480. It is interesting, and will be useful later, to consider as a particular case, the attraction of a sphere whose mass is composed of concentric layers, each of uniform density.

Let R be the radius, r that of any layer, $\rho = F(r)$ its density. Then, if σ be the mean density,

$$\tfrac{4}{3}\pi\sigma R^3 = 4\pi\int_0^R \rho r^2 dr,$$

from which σ may be found.

The surface attraction is $\frac{4}{3}\pi\sigma R, = G$, suppose.

At a distance r from the centre the attraction is $\frac{4\pi}{r^2}\int_0^r \rho r^2 dr$.

Attraction of a sphere composed of concentric shells of uniform density.

If it is to be the same for all points inside the sphere

$$\int_0^r \rho r^2 dr = \frac{G}{4\pi} r^2.$$

Hence $\rho = F(r) = \frac{1}{2\pi} \cdot \frac{G}{r}$ is the requisite law of density.

If the density of the upper crust be τ, the attraction at a depth h, small compared with the radius, is

$$\tfrac{4}{3}\pi\sigma_1 (R-h) = G_1,$$

where σ_1 is the mean density of nucleus when a shell of thickness h is removed from the sphere. Also, evidently,

$$\tfrac{4}{3}\pi\sigma_1 (R-h)^3 + 4\pi\tau (R-h)^2 h = \tfrac{4}{3}\pi\sigma R^3,$$

or

$$G_1 (R-h)^2 + 4\pi\tau (R-h)^2 h = GR^2,$$

whence

$$G_1 = G\left(1 + \frac{2h}{R}\right) - 4\pi\tau h.$$

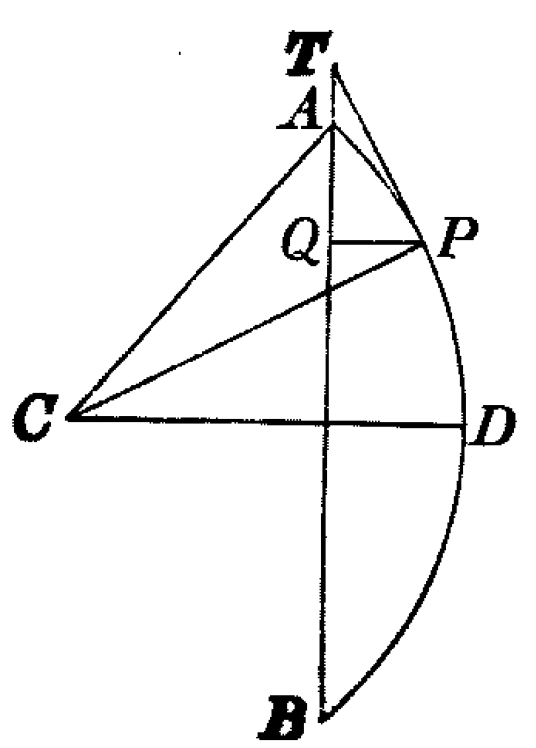
The attraction is therefore unaltered at a depth h if

$$\frac{G}{R} = \tfrac{4}{3}\pi\sigma = 2\pi\tau.$$

481. Some other simple cases may be added here, as their results will be of use to us subsequently.

Attraction of a uniform circular arc,

(*a*) The attraction of a circular arc, AB, of uniform density, on a particle at the centre, C, of the circle, lies evidently in the line CD bisecting the arc. Also the resolved part parallel to CD of the attraction of an element at P is



$$\frac{\text{mass of element at } P}{CD^2} \cos . \widehat{PCD}.$$

Now suppose the density of the chord AB to be the same as that of the arc. Then for (mass of element at $P \times \cos \widehat{PCD}$) we may put mass of projection of element on AB at Q; since, if PT be the tangent at P, $\widehat{PTQ} = \widehat{PCD}$.

Hence attraction along $CD = \dfrac{\text{Sum of projected elements}}{CD^2}$

$$= \frac{\rho AB}{CD^2},$$

if ρ be the density of the given arc,

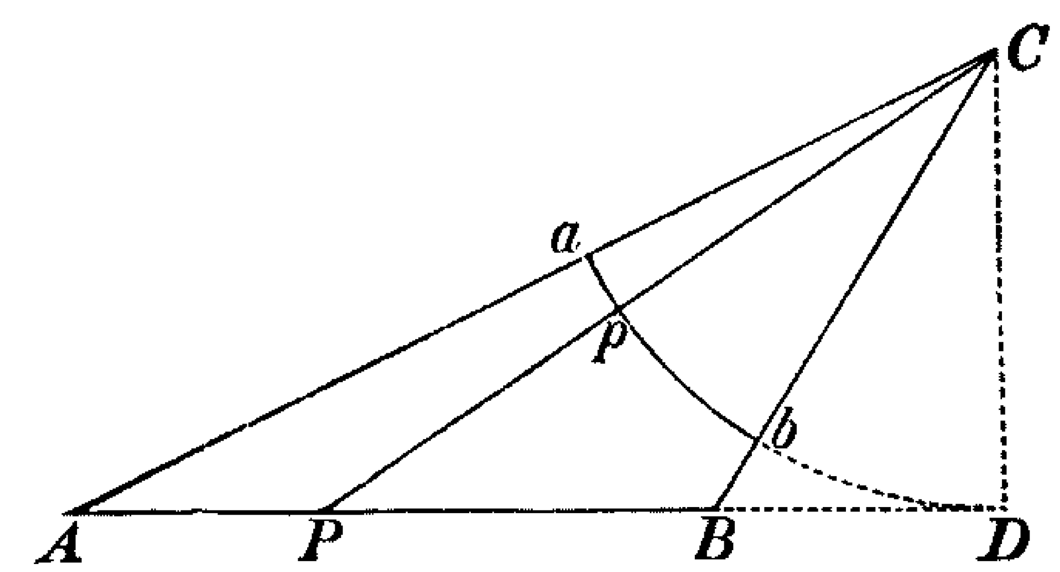
Attraction of a uniform circular arc,

$$= \frac{2\rho \sin \widehat{ACD}}{CD}.$$

It is therefore the same as the attraction of a mass equal to the chord, with the arc's density, concentrated at the point D.

straight line.

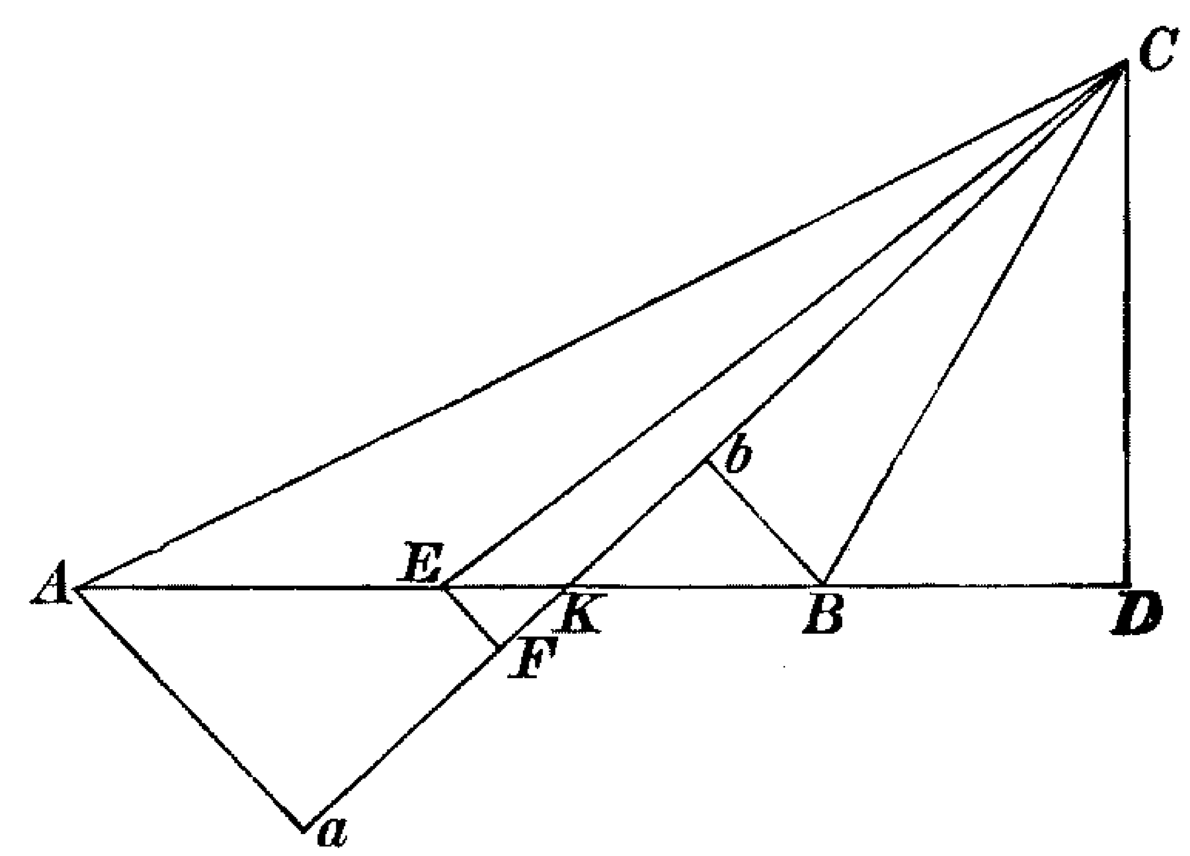
(*b*) Again a limited straight line of uniform density attracts any external point in the same direction and with the same force as the corresponding arc of a circle of the same density, which has the point for centre, and touches the straight line.



For if CpP be drawn cutting the circle in p and the line in P; Element at p : element at P :: Cp : $CP\frac{CP}{CD}$; that is, as $Cp^2 : CP^2$. Hence the attractions of these elements on C are equal and in the same line. Thus the arc ab attracts C as the line AB does; and, by the last proposition, the attraction of AB bisects the angle ACB, and is equal to

$$\frac{2\rho}{CD} \sin \tfrac{1}{2}\widehat{ACB}.$$

(*c*) This may be put into other useful forms — thus, let CKF bisect the angle ACB, and let Aa, Bb, EF, be drawn perpendicular to CF from the ends and middle point of AB. We have



$$\sin \widehat{KCB} = \frac{KB}{CB} \sin \widehat{CKD} = \frac{AB}{AC + CB}\,\frac{CD}{CK}.$$

Attraction of a uniform straight line.

Hence the attraction, which is along CK, is

$$\frac{2\rho AB}{(AC+CB)\,CK} = \frac{\rho AB}{8\,(AC+CB)\,(\overline{AC+CB}^2 - AB^2)} \,.\, CF. \qquad (1)$$

For, evidently,

$$bK : Ka :: BK : KA :: BC : CA :: bC : Ca,$$

i.e., ab is divided, externally in C, and internally in K, in the same ratio. Hence, by geometry,

$$KC \,.\, CF = aC \,.\, Cb = \tfrac{1}{4}\{\overline{AC+CB}^2 - AB^2\},$$

which gives the transformation in (1).

(*d*) CF is obviously the tangent at C to a hyperbola, passing through that point, and having A and B as foci. Hence, if in *any* plane through AB any hyperbola be described, with foci A and B, it will be a line of force as regards the attraction of the line AB; that is, as will be more fully explained later, a curve which at every point indicates the direction of attraction.

(*e*) Similarly, if a prolate spheroid be described with foci A and B, and passing through C, CF will evidently be the normal at C; thus the force on a particle at C will be perpendicular to the spheroid; and the particle would evidently rest in equilibrium on the surface, even if it were smooth. This is an instance of (what we shall presently develop at some length) a surface of equilibrium, a level surface, or an equipotential surface.

(*f*) We may further prove, by a simple application of the preceding theorem, that the lines of force due to the attraction of two infinitely long rods in the line AB produced, one of which is attractive and the other repulsive, are the series of ellipses described from the extremities, A and B, as foci, while the surfaces of equilibrium are generated by the revolution of the confocal hyperbolas.

Potential.

482. As of immense importance, in the theory not only of gravitation but of electricity, of magnetism, of fluid motion, of the conduction of heat, etc., we give here an investigation of the most important properties of the *Potential.*

483. This function was introduced for gravitation by Laplace, but the name was first given to it by Green, who may almost be said to have in 1828 created the theory, as we now have it.

Potential.

Green's work was neglected till 1846, and before that time most of its important theorems had been re-discovered by Gauss, Chasles, Sturm, and Thomson.

In § 273, the *potential energy* of a conservative system in any configuration was defined. When the forces concerned are forces acting, either really or apparently, at a distance, as attraction of gravitation, or attractions or repulsions of electric or magnetic origin, it is in general most convenient to choose, for the zero configuration, infinite distance between the bodies concerned. We have thus the following definition :—

484. The mutual potential energy of two bodies in any relative position is the amount of work obtainable from their mutual repulsion, by allowing them to separate to an infinite distance asunder. When the bodies attract mutually, as for instance when no other force than gravitation is operative, their mutual potential energy, according to the convention for zero now adopted, is negative, or (§ 547 below) their *exhaustion of potential energy* is positive.

485. The *Potential* at any point, due to any attracting or repelling body, or distribution of matter, is the mutual potential energy between it and a unit of matter placed at that point. But in the case of gravitation, to avoid defining the potential as a negative quantity, it is convenient to change the sign. Thus the gravitation potential, at any point, due to any mass, is the quantity of work required to remove a unit of matter from that point to an infinite distance.

486. Hence if V be the potential at any point P, and V_1 that at a proximate point Q, it evidently follows from the above definition that $V - V_1$ is the work required to remove an independent unit of matter from P to Q; and it is useful to note that this is altogether independent of the form of the path chosen between these two points, as it gives us a preliminary idea of the power we acquire by the introduction of this mode of representation.

Suppose Q to be so near to P that the attractive forces exerted on unit of matter at these points, and therefore at any

Potential. point in the line PQ, may be assumed to be equal and parallel. Then if F represent the resolved part of this force along PQ, $F.PQ$ is the work required to transfer unit of matter from P to Q. Hence

$$V - V_1 = F.PQ,$$

or

$$F = \frac{V - V_1}{PQ},$$

Force in terms of the potential. that is, the attraction on unit of matter at P in any direction PQ, is the rate at which the potential at P increases per unit of length of PQ.

Equipotential surface. **487.** A surface, at every point of which the potential has the same value, and which is therefore called an *Equipotential Surface*, is such that the attraction is everywhere in the direction of its normal. For in no direction along the surface does the potential change in value, and therefore there is no force in any such direction. Hence if the attracted particle be placed on such a surface (supposed smooth and rigid), it will rest in any position, and the surface is therefore sometimes called a *Surface of Equilibrium*. We shall see later, that the force on a particle of a liquid at the free surface is always in the direction of the normal, hence the term *Level Surface*, which is often used for the other terms above.

Relative intensities of force at different points of an equipotential surface. **488.** If a series of equipotential surfaces be constructed for values of the potential increasing by equal small amounts, it is evident from § 486 that the attraction at any point is inversely proportional to the normal distance between two successive surfaces close to that point; since the numerator of the expression for F is, in this case, constant.

Line of force. **489.** A line drawn from any origin, so that at every point of its length its tangent is the direction of the attraction at that point, is called a *Line of Force;* and it obviously cuts at right angles every equipotential surface which it meets.

These three last sections are true *whatever* be the law of attraction; in the next we are restricted to the law of the inverse square of the distance.

Variation of intensity along a line of force. **490.** If, through every point of the boundary of an infinitely small portion of an equipotential surface, the corresponding lines of force be drawn, we shall evidently have a tubular surface of infinitely small section. The force in any direction, at any point within such a tube, so long as it does not cut through attracting matter, is inversely as the section of the tube made by a plane passing through the point and perpendicular to the given direction. Or, more simply, the whole force is at every point tangential to the direction of the tube, and inversely as its transverse section: from which the more general statement above is easily seen to follow.

This is an immediate consequence of a most important theorem, which will be proved later, § 492. *The surface integral of the attraction exerted by any distribution of matter in the direction of the normal at every point of any closed surface is $4\pi M$; where M is the amount of matter within the surface, while the attraction is considered positive or negative according as it is inwards or outwards at any point of the surface.*

For in the present case the force perpendicular to the tubular part of the surface vanishes, and we need consider the ends only. When none of the attracting mass is within the portion of the tube considered, we have at once

$$F\varpi - F'\varpi' = 0,$$

F being the force at any point of the section whose area is ϖ. This is equivalent to the celebrated equation of Laplace—App. B (a); and below, § 491 (c).

When the attracting body is symmetrical about a point, the lines of force are obviously straight lines drawn from this point. Hence the tube is in this case a cone, and, by § 469, ϖ is proportional to the square of the distance from the vertex. Hence F is inversely as the square of the distance for points external to the attracting mass.

When the mass is symmetrically disposed about an axis in infinitely long cylindrical shells, the lines of force are evidently perpendicular to the axis. Hence the tube becomes a *wedge*, whose section is proportional to the distance from the axis, and the attraction is therefore inversely as the distance from the axis.

Variation of intensity along a line of force.

When the mass is arranged in infinite parallel planes, each of uniform density, the lines of force are obviously perpendicular to these planes; the tube becomes a *cylinder;* and, since its section is constant, the force is the same at all distances.

If an infinitely small length l of the portion of the tube considered pass through matter of density ρ, and if ϖ be the area of the section of the tube in this part, we have

$$F\varpi - F'\varpi' = 4\pi l\omega\rho.$$

This is equivalent to Poisson's extension of Laplace's equation [§ 491 (*c*)].

Potential due to an attracting point.

491. In estimating work done against a force which varies inversely as the square of the distance from a fixed point, the mean force is to be reckoned as the geometrical mean between the forces at the beginning and end of the path: and, whatever may be the path followed, the effective space is to be reckoned as the difference of distances from the attracting point. Thus the work done in any course is equal to the product of the difference of distances of the extremities from the attracting point, into the geometrical mean of the forces at these distances; or, if O be the attracting point, and m its force on a unit mass at unit distance, the work done in moving a particle, of unit mass, from any position P to any other position P', is

$$(OP' - OP)\sqrt{\frac{m^2}{OP^2\,OP'^2}}, \text{ or } \frac{m}{OP} - \frac{m}{OP'}.$$

To prove this it is only necessary to remark, that for any infinitely small step of the motion, the effective space is clearly the difference of distances from the centre, and the working force may be taken as the force at either end, or of any intermediate value, the geometrical mean for instance: and the preceding expression applied to each infinitely small step shows that the same rule holds for the sum making up the whole work done through any finite range, and by any path.

Hence, by § 485, it is obvious that the potential at P, of a mass m situated at O, is $\frac{m}{OP}$; and thus that the potential of any

mass at a point P is to be found by adding the quotients of every portion of the mass, each divided by its distance from P.

Analytical investigation of the value of the potential.

a. For the analytical proof of these propositions, consider, first, a pair of particles, O and P, whose masses are m and unity, and co-ordinates abc, xyz. If D be their distance

$$D^2 = (x-a)^2 + (y-b)^2 + (z-c)^2.$$

The components of the mutual attraction are

$$X = m\frac{x-a}{D^3}, \quad Y = m\frac{y-b}{D^3}, \quad Z = m\frac{z-c}{D^3};$$

and therefore the work required to remove P to infinity is

$$m\int\frac{(x-a)\,dx + (y-b)\,dy + (z-c)\,dz}{D^3}$$

$$= m\int\frac{dD}{D^2}$$

which, since the superior limit is $D = \infty$, is equal to

$$\frac{m}{D}.$$

The mutual potential energy is therefore, in this case, the product of the masses divided by their mutual distance; and therefore the potential at x, y, z, due to m, is $\frac{m}{D}$.

Again, if there be more than one fixed particle m, the same investigation shows us that the potential at xyz is

$$\Sigma\frac{m}{D}.$$

And if the particles form a continuous mass, whose density at a, b, c is ρ, we have of course for the potential the expression

$$\iiint\rho\frac{da\,db\,dc}{D},$$

the limits depending on the boundaries of the mass.

Force at any point.

If we call V the potential at any point P (x, y, z), it is evident (from the way in which we have obtained its value) that the components of the attraction on unit of matter at P are

$$X = -\frac{dV}{dx}, \quad Y = -\frac{dV}{dy}, \quad Z = -\frac{dV}{dz}.$$

Force at any point.

Hence the force, resolved along any curve of which s is the arc, is
$$X\frac{dx}{ds}+Y\frac{dy}{ds}+Z\frac{dz}{ds}=-\left(\frac{dV}{dx}\frac{dx}{ds}+\frac{dV}{dy}\frac{dy}{ds}+\frac{dV}{dz}\frac{dz}{ds}\right)$$
$$=-\frac{dV}{ds}.$$

All this is evidently independent of the question whether P lies within the attracting mass or not.

Force within a homogeneous sphere.

b. If the attracting mass be a sphere of density ρ, and centre a, b, c, and if P be within its surface, we have, since the exterior shell has no effect,
$$X=-\frac{dV}{dx}=\frac{4}{3}\pi\rho D^3 \cdot \frac{x-a}{D^3}$$
$$=\frac{4}{3}\pi\rho\,(x-a).$$

Rate of increase of the force in any direction.

Hence
$$\frac{dX}{dx}=-\frac{d^2V}{dx^2}=\frac{4}{3}\pi\rho.$$

c. Now if
$$\nabla^2=\frac{d^2}{dx^2}+\frac{d^2}{dy^2}+\frac{d^2}{dz^2},$$
we have $\nabla^2\frac{1}{D}=0$, as was proved before, App. B g (14) as a particular case of g. The proof for this case alone is as follows:
$$\frac{d}{dx}\frac{1}{D}=-\frac{x-a}{D^3};\qquad \frac{d^2}{dx^2}\frac{1}{D}=-\frac{1}{D^3}+\frac{3\,(x-a)^2}{D^5}:$$
and from this, and the similar expressions for the second differentials in y and z, the theorem follows by summation.

Hence as
$$V=\iiint\rho\,\frac{da\,db\,dc}{D}$$
and ρ does not involve x, y, z, we see that *as long as D does not vanish within the limits of integration*, *i.e.*, as long as P is not a point of the attracting mass
$$\nabla^2 V=0;$$
or, in terms of the components of the force,

Laplace's equation.

$$\frac{dX}{dx}+\frac{dY}{dy}+\frac{dZ}{dz}=0.$$

Laplace's equation.

If P be within the attracting mass, suppose a small sphere to be described so as to contain P. Divide the potential into two parts, V_1 that of the sphere, V_2 that of the rest of the body.

The expression above shows that
$$\nabla^2 V_2=0.$$
Also the expressions for $\frac{d^2V}{dx^2}$, etc., in the case of a sphere (b) give
$$\nabla^2 V_1=-4\pi\rho,$$
where ρ is the density of the sphere.

Poisson's extension of Laplace's equation.

Hence as
$$V=V_1+V_2$$
$$\nabla^2 V=-4\pi\rho,$$
which is the general equation of the potential, and includes the case of P being wholly external to the attracting mass, since there $\rho=0$. In terms of the components of the force, this equation becomes
$$\frac{dX}{dx}+\frac{dY}{dy}+\frac{dZ}{dz}=4\pi\rho.$$

d. We have already, in these most important equations, the means of verifying various former results, and also of adding new ones.

Potential of matter arranged in concentric spherical shells of uniform density.

Thus, to find the attraction of a hollow sphere composed of concentric shells, each of uniform density, on an external point (by which we mean a point *not* part of the mass). In this case symmetry shows that V must depend upon the distance from the centre of the sphere alone. Let the centre of the sphere be origin, and let
$$r^2=x^2+y^2+z^2.$$
Then V is a function of r alone, and consequently
$$\frac{dV}{dx}=\frac{dV}{dr}\frac{dr}{dx}=\frac{x}{r}\frac{dV}{dr},$$
$$\frac{d^2V}{dx^2}=\frac{1}{r}\frac{dV}{dr}-\frac{x^2}{r^3}\frac{dV}{dr}+\frac{x^2}{r^2}\frac{d^2V}{dr^2},$$
and
$$\nabla^2 V=\frac{2}{r}\frac{dV}{dr}+\frac{d^2V}{dr^2}.$$

Hence, when P is outside the sphere, or in the hollow space within it,
$$\frac{2}{r}\frac{dV}{dr}+\frac{d^2V}{dr^2}=0.$$

Potential of matter arranged in concentric spherical shells of uniform density.

A first integral of this is $r^2 \dfrac{dV}{dr} = C$.

For a point outside the shell C has a finite value, which is easily seen to be $-M$, where M is the mass of the shell.

For a point in the internal cavity $C=0$, because evidently at the centre there is no attraction—*i.e.*, there $r=0$, $\dfrac{dV}{dr}=0$ together. Hence there is no attraction on *any* point in the cavity.

We need not be surprised at the apparent discontinuity of this solution. It is owing to the *discontinuity of the given distribution of matter*. Thus it appears, by § 491 *c*, that the true general equation of the potential is not what we have taken above, but

$$\frac{d^2V}{dr^2} + \frac{2}{r}\frac{dV}{dr} = -4\pi\rho,$$

where ρ, the density of the matter at distance r from the centre, is zero when $r < a$ the radius of the cavity: has a finite value σ, which for simplicity we may consider constant, when $r > a$ and $< a'$ the radius of the outer bounding surface: and is zero, again, for all values of r exceeding a'. Hence, integrating from $r=0$, to $r=r$, any value, we have (since $r^2 \dfrac{dV}{dr} = 0$ when $r=0$),

$$r^2 \frac{dV}{dr} = -4\pi \int_0^r \rho r^2 dr = -M_1,$$

if M_1 denote the whole amount of matter within the spherical surface of radius r; which is the discontinuous function of r specified as follows:—

From $r=0$ to $r=a$, $\quad r=a$ to $r=a'$, $\quad r=a'$ to $r=\infty$,

$$M_1 = 0, \qquad M_1 = \frac{4\pi\sigma}{3}(r^3 - a^3), \qquad M_1 = \frac{4\pi\sigma}{3}(a'^3 - a^3).$$

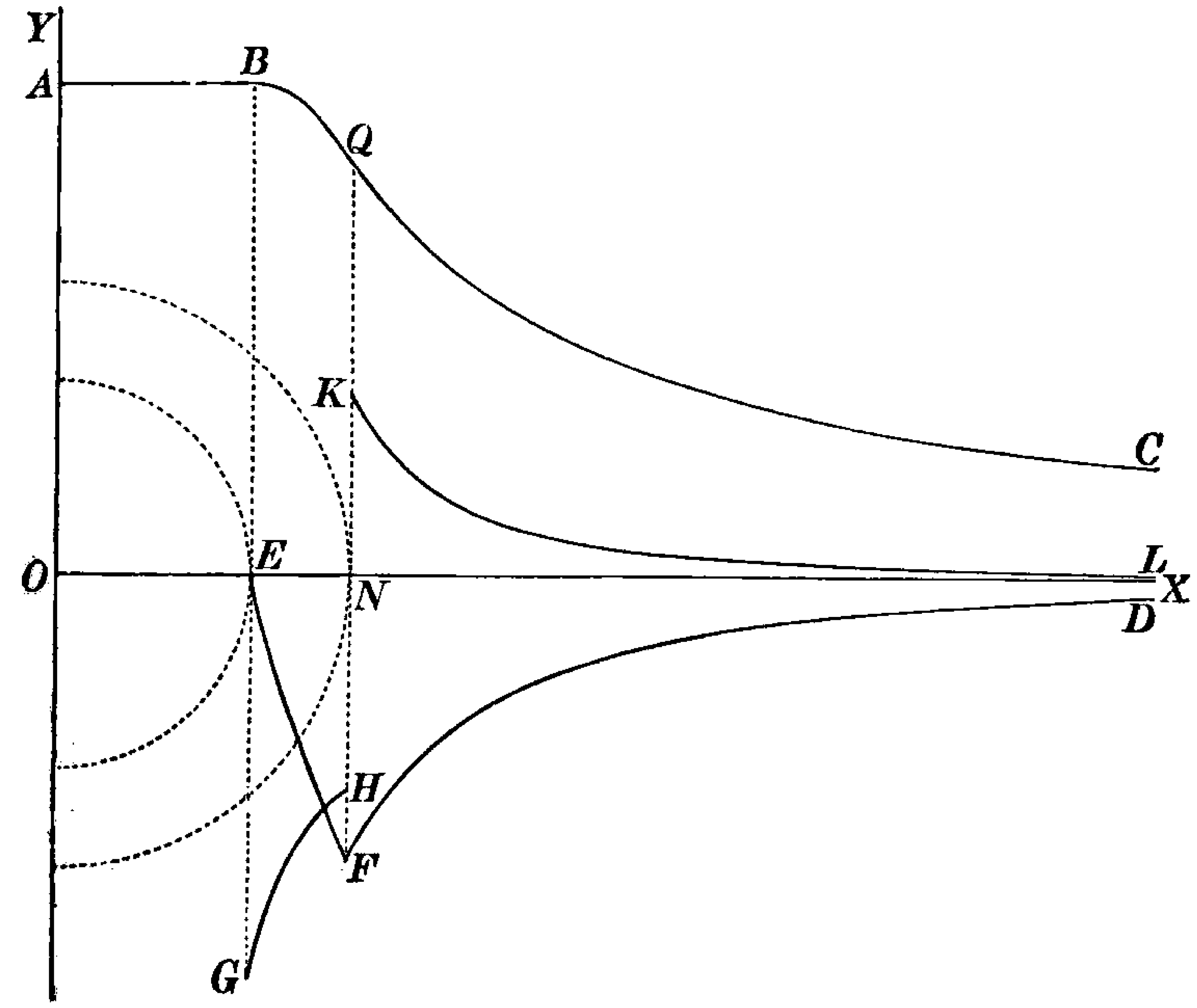
The corresponding values of V are, in order,

$$V = 2\pi\sigma(a'^2 - a^2), \quad V = \frac{4\pi\sigma}{3}\left(\frac{3a'^2 - r^2}{2} - \frac{a^3}{r}\right), \quad V = \frac{4\pi\sigma}{3r}(a'^3 - a^3).$$

We have entered thus into detail in this case, because such apparent anomalies are very common in the analytical solution of physical questions. To make this still more clear, we subjoin a graphic representation of the values of V, $\dfrac{dV}{dr}$, and $\dfrac{d^2V}{dr^2}$ for this case. $ABQC$, the curve for V, is partly a straight line, and has a point of inflection at Q: but there is no discontinuity

and no abrupt change of direction. $OEFD$, that for $\dfrac{dV}{dr}$, is continuous, but its direction twice changes abruptly. That for $\dfrac{d^2V}{dr^2}$ consists of three detached portions, OE, GH, KL.

Potential of matter arranged in concentric spherical shells of uniform density.



e. For a mass disposed in infinitely long concentric cylindrical shells, each of uniform density, if the axis of the cylinders be z, we must evidently have V a function of x^2+y^2 only.

Coaxal right cylinders of uniform density and infinite length.

Hence $\dfrac{dV}{dz}=0$, or the attraction is wholly perpendicular to the axis.

Also, $\dfrac{d^2V}{dz^2}=0$; and therefore by (d)

$$\nabla^2 V = \frac{d^2V}{dr^2} + \frac{1}{r}\frac{dV}{dr} = -4\pi\rho.$$

Hence
$$r\frac{dV}{dr} = C - 4\pi\int \rho r\, dr,$$

from which conclusions similar to the above may be drawn.

f. If, finally, the mass be arranged in infinite parallel planes, each of uniform density, and perpendicular to the axis

Matter arranged in infinite parallel planes of uniform density.

of x; the resultant force must be parallel to this direction: that is to say, $Y=0$, $Z=0$, and therefore

$$\frac{dX}{dx}=4\pi\rho,$$

which, if ρ is known in terms of x, is completely integrable.

Outside the mass, $\rho=0$, and therefore

$$X=C,$$

or the attraction is the same at all distances, a result easily verified by the direct methods.

If within the mass the density is constant, we have

$$X=C'+4\pi\rho x;$$

and if the origin be in the middle of the lamina, we have, obviously, $C'=0$. Hence if t denote the thickness, the values of X at the two sides and in the spaces beyond are respectively $-2\pi\rho t$ and $+2\pi\rho t$. The difference of these is $4\pi\rho t$ (§ 478).

Equipotential surface.

g. Since in any case $\dfrac{dV}{ds}$ is the component of the attraction in the direction of the tangent to the arc s, the attraction will be perpendicular to that arc if

$$\frac{dV}{ds}=0,$$

or $$V=C.$$

This is the equation of an *equipotential* surface.

If n be the normal to such a surface, measured outwards, the whole force at any point is evidently

$$\frac{dV}{dn},$$

and its direction is that in which V increases.

Integral of normal attraction over a closed surface.

492. Let S be any closed surface, and let O be a point, either external or internal, where a mass, m, of matter is collected. Let N be the component of the attraction of m in the direction of the normal drawn inwards from any point P, of S. Then, if $d\sigma$ denotes an element of S, and $\iint$ integration over the whole of it,

$$\iint N d\sigma=4\pi m, \text{ or } =0 \text{ ....(1)},$$

according as O is internal or external.

Integral of normal attraction over a closed surface. Equivalent to Poisson's extension of Laplace's equation, § 491 *c*.

Case 1, *O internal*. Let $OP_1P_2P_3$... be a straight line drawn in any direction from O, cutting S in P_1, P_2, P_3, etc., and therefore passing out at P_1, in at P_2, out again at P_3, in again at P_4, and so on. Let a conical surface be described by lines through O, all infinitely near OP_1P_2..., and let ω be its solid angle (§ 465). The portions of $\iint Nd\sigma$ corresponding to the elements cut from S by this case will be clearly each equal in absolute magnitude to ωm, but will be alternately positive and negative. Hence as there is an odd number of them their sum is $+\omega m$. And the sum of these, for all solid angles round O is (§ 466) equal to $4\pi m$; that is to say, $\iint Nd\sigma=4\pi m$.

Equivalent to Laplace's equation, § 491 *c*.

Case 2, *O external*. Let $OP_1P_2P_3$... be a line drawn from O passing across S, inwards at P_1, outwards at P_2, and so on. Drawing, as before, a conical surface of infinitely small solid angle, ω, we have still ωm for the absolute value of each of the portions of $\iint Nd\sigma$ corresponding to the elements which it cuts from S; but their signs are alternately negative and positive: and therefore as their number is even, their sum is zero. Hence $\iint Nd\sigma=0$.

From these results it follows immediately that if there be any distribution of matter, partly within and partly without a closed surface S, and N and $d\sigma$ be still used with the same signification, we have

$$\iint Nd\sigma=4\pi M \text{..........................(2)}$$

if M denote the whole amount of matter within S.

This, with M eliminated from it by Poisson's theorem, § 491 *c*, is the particular case of the analytical theorem of Chap. I. App. A (*a*), found by taking $\alpha=1$, and $U'=1$, by which it becomes

$$0=\iint d\sigma\partial U-\iiint\nabla^2 U dx\,dy\,dz \text{...............(3)}.$$

For let U be the potential at (x, y, z), due to the distribution of matter in question. Then, according to the meaning of ∂, we have $\partial U=-N$. Also, let ρ be the density of the matter at (x, y, z). Then [§ 491 (*c*)] we have

$$\nabla^2 U=-4\pi\rho.$$

Hence (3) gives

$$\iint Nd\sigma=4\pi\iiint\rho\,dx\,dy\,dz=4\pi M.$$

Integral of normal attraction over a closed surface.

493. If in crossing any surface K we find an abrupt change in the value of the component force perpendicular to K, it follows from (2) that there must be a condensation of matter on K, and that the surface-density of this distribution is $N/4\pi$, if N be the difference of the values of the normal component on the two sides of K; as we see by taking for our closed surface S an infinitely small rectangular parallelepiped with two of its faces parallel to K and on opposite sides of it. This result was found in § 478, in a thoroughly synthetical manner. The same result is found by the proper analytical interpretation of Poisson's equation

$$\frac{dX}{dx}+\frac{dY}{dy}+\frac{dZ}{dz}=4\pi\rho.$$

It is to be remarked that in travelling across K abrupt change in the value of the component force along any line parallel to K is forbidden by the Conservation of Energy.

494. The theorem of Laplace and Poisson, § 492, for the present application most conveniently taken (§ 491 c) in its differential form

$$\rho=-\frac{1}{4\pi}\left(\frac{d^2V}{dx^2}+\frac{d^2V}{dy^2}+\frac{d^2V}{dz^2}\right)\ldots\ldots\ldots\ldots\ldots (1),$$

Inverse problem.

is explicitly the solution of the inverse problem,—*given the potential at every point of space*, or, which is virtually the same, *given the direction and magnitude of the resultant force at every point of space,—it is required to find the distribution of matter by which it is produced.*

494 a. Example. Let the potential be given equal to zero for all space external to a given closed surface S, and let

$$V=\phi\,(x,\ y,\ z)\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

for all space within this surface; $\phi\,(x, y, z)$ being any arbitrary function subject to no other condition than that its value is zero at S, and that it has no abrupt changes of value within S. Abrupt changes in the values of differential coefficients,

$$\frac{d\phi}{dx},\ \frac{d\phi}{dy},\ \frac{d\phi}{dz},$$

are not excluded, but are subject to interpretations, as in § 493, if they occur.

Inverse problem.

494 b. The required distribution of matter must include a surface distribution on S, because there is abrupt change in the value of the normal component force from

$$\sqrt{\left(\frac{d\phi^2}{dx^2}+\frac{d\phi^2}{dy^2}+\frac{d\phi^2}{dz^2}\right)}$$

at the inside of S to zero at the outside. Thus, by § 493, and by § 494 (1), we have for our complete solution (compare §§ 501, 505, 506, 507 below),

$$\left.\begin{array}{ll} & \rho=0, \text{ for space external to } S \\ & \sigma=\dfrac{1}{4\pi}\left(\dfrac{d\phi^2}{dx^2}+\dfrac{d\phi^2}{dy^2}+\dfrac{d\phi^2}{dz^2}\right)^{\frac{1}{2}} \text{ on } S, \\ \text{and} & \rho=-\dfrac{1}{4\pi}\left(\dfrac{d^2\phi}{dx^2}+\dfrac{d^2\phi}{dy^2}+\dfrac{d^2\phi}{dz^2}\right) \\ \text{for space enclosed by } S. & \end{array}\right\}\ldots\ldots\ldots(2).$$

494 c. From § 492 (2), remembering that $N=0$ outside of S, we infer that the total mass on and within S is zero, and therefore the quantity of matter condensed on S is equal and of opposite sign to the quantity enclosed by it.

494 d. Sub-Example. Let the potential be given equal to zero for all space external to the ellipsoidal surface

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1,$$

and equal to

$$\tfrac{1}{2}\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{z^2}{c^2}\right)\ldots\ldots\ldots\ldots\ldots\ldots(3),$$

for the space enclosed by it: in other words let the potential be zero wherever the value of (3) is negative, and equal to the value of (3) wherever it is positive.

494 e. The solution (2) becomes

$$\left.\begin{array}{ll} \rho=0, & \text{wherever } \dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}>1; \\ \sigma=-\dfrac{1}{4\pi p}, & \text{at the surface } \dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}=1; \\ \text{and } \rho=\dfrac{1}{4\pi}\left(\dfrac{1}{a^2}+\dfrac{1}{b^2}+\dfrac{1}{c^2}\right) & \text{wherever } \dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}<1. \end{array}\right\}\ldots(4);$$

Inverse problem.

p denoting the perpendicular from the centre to the tangent plane of the ellipsoidal surface.

494 *f*. Let q be an infinitely small quantity. The equation

$$\frac{x^2}{a^2-q}+\frac{y^2}{b^2-q}+\frac{z^2}{c^2-q}=1 \quad \text{..................(5)}$$

represents an ellipsoidal surface confocal with the given one, and infinitely near it. The distance between the two surfaces infinitely near any point (x, y, z) of either is easily proved to be equal to $\frac{1}{2} q/p$. Calling this t, we have, from (4),

$$\sigma=-\frac{1}{4\pi}\cdot\frac{2t}{q} \quad \text{.......................(6).}$$

We conclude from (6) and (4) and the theorem (§ 494 *c*) of masses that

Attractions of solid homogeneous ellipsoid and circumscribed focaloid of equal mass found equal.

494 *g*. The attraction of a homogeneous solid ellipsoid is the same through all external space as the attraction of a homogeneous focaloid* of equal mass coinciding with its surface.

Homoeoids and Focaloids defined.

* To avoid complexity of diction we now propose to introduce two new words, "focaloid" and "homoeoid," according to the following definitions:—

(1) A *homoeoid* is an infinitely thin shell bounded by two similar surfaces similarly oriented.

The one point which is situated similarly relatively to the two similar surfaces of a homoeoid is called the homoeoidal centre. Supposing the homoeoid to be a finite closed surface, the homoeoidal centre may be any internal or external point. In the extreme case of two equal surfaces, the homoeoidal centre is at an infinite distance. The homoeoid in this extreme case (which is interesting as representing the surface-distribution of ideal magnetic matter constituting the free polarity of a body magnetized uniformly in parallel lines) may be called a homoeoidal couple. In every case the thickness of the homoeoid is directly proportional to the perpendicular from the centre to the tangent plane at any point. When (the surface being still supposed to be finite and closed) the centre is external, the thickness is essentially negative in some places, and positive in others.

The bulk of a homoeoid is the excess of the bulk of the part where the thickness is positive above that where the thickness is negative. The bulk of a homoeoidal couple is essentially zero. Its moment and its axis are important qualities, obvious in their geometric definition, and useful in magnetism as representing the magnetic moment and the magnetic axis of a piece of matter uniformly magnetized in parallel lines.

(2) An *elliptic homoeoid* is an infinitely thin shell bounded by two concentric similar ellipsoidal surfaces.

(3) A *focaloid* is an infinitely thin shell bounded by two confocal ellipsoidal surfaces.

(4) The terms "thick homoeoid" and "thick focaloid" may be used in the comparatively rare cases (see for example §§ 494*i*, 519, 522) when forms satisfying the definitions (1) and (3) except that they are not infinitely thin, are considered.

494 *h*. Take now a homogeneous solid ellipsoid and divide it into an infinite number of focaloids, numbered 1, 2, 3, ... from the surface inwards. Take the mass of No. 1 and distribute it uniformly through the space enclosed by its inner boundary. This makes no difference in the attraction through space external to the original ellipsoid. Take the infinitesimally increased mass of No. 2 and distribute it uniformly through the space enclosed by *its* inner boundary. And so on with Nos. 3, 4, &c., till instead of the given homogeneous ellipsoid we have another of the same mass and correspondingly greater density enclosed by any smaller confocal ellipsoidal surface.

Proof of Maclaurin's Theorem.

494 *i*. We conclude that

Any two confocal homogeneous solid ellipsoids of equal masses produce equal attraction through all space external to both.

Maclaurin's Theorem.

This is Maclaurin's splendid theorem. It is tantamount to the following, which presents it in a form specially interesting in some respects:

Any two thick or thin confocal focaloids of equal masses, each homogeneous, produce equal attraction through all space external to both.

Equivalent in shells of Maclaurin's Theorem.

494 *j*. Maclaurin's theorem reduces the problem of finding the attraction of an ellipsoid* on any point in external space, (which when attempted by direct integration presents difficulties not hitherto directly surmounted,) to the problem of

Digression on the attraction of an ellipsoid.

* To avoid circumlocutions we call simply "an ellipsoid" a homogeneous solid ellipsoid.

Digression on the attraction of an ellipsoid.

finding the attraction of an ellipsoid on a point at its surface which, as the limiting case of the attraction of an ellipsoid on an internal point, is easily solved by direct integration, thus:

To find the potential of an ellipsoid at any interior point.

494 k. Divide the whole solid into pairs of vertically opposite infinitesimal cones or pyramids, having the attracted point P for common vertex.

Let $E'PE$ be any straight line through P, cut by the surface at E' and E, and let $d\sigma$ be the solid angle of the pair of cones lying along it. The potentials at P of the two are easily shown to be $\frac{1}{2} PE^2 d\sigma$ and $\frac{1}{2} PE'^2 d\sigma$, and therefore the whole contribution of potential at P by the pair is $\frac{1}{2}(PE'^2 + PE^2)\, d\sigma$.

Hence, if V denote the potential at P of the whole ellipsoid, the density being taken as unity, we have

$$V = \iint \tfrac{1}{2}(PE^2 + PE'^2)\, d\sigma \text{..................(7)},$$

where $\iint$ denotes integration over a hemisphere of spherical surface of unit radius.

Now if x, y, z be the co-ordinates of P relative to the principal axes of the ellipsoid; and l, m, n the direction cosines of PE, we have, by the equation of the ellipsoid,

$$\frac{(x + lPE)^2}{a^2} + \frac{(y + mPE)^2}{b^2} + \frac{(z + mPE)^2}{c^2} = 1;$$

whence

$$\left(\frac{l^2}{a^2} + \frac{m^2}{b^2} + \frac{n^2}{c^2}\right) PE^2 + 2\left(\frac{lx}{a^2} + \frac{my}{b^2} + \frac{nz}{c^2}\right) PE - \left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2}\right) = 0.$$

When (x, y, z) is within the ellipsoid this equation, viewed as a quadratic in PE, has its roots of opposite signs; the positive one is PE, the negative is $-PE'$.

Now if r_1, r_2 be the two roots of $gr^2 + 2fr - e = 0$, we have

$$\tfrac{1}{2}(r_1^2 + r_2^2) = (2f^2 + ge)/g^2.$$

Hence

$$\left.\begin{aligned} \tfrac{1}{2}(PE^2 + PE'^2) &= \frac{\frac{l^2}{a^2}\left(\frac{2x^2}{a^2} + e\right) + \frac{m^2}{b^2}\left(\frac{2y^2}{b^2} + e\right) + \frac{n^2}{c^2}\left(\frac{2z^2}{c^2} + e\right) + Q}{\left(\frac{l^2}{a^2} + \frac{m^2}{b^2} + \frac{n^2}{c^2}\right)^2}, \\ \text{where}\quad e &= 1 - \frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2}, \\ \text{and}\quad Q &= 4\left(\frac{mnyz}{b^2c^2} + \frac{nlzx}{c^2a^2} + \frac{lmxy}{a^2b^2}\right) \end{aligned}\right\} \text{..(8)}.$$

Now in the $\iint$ integration of (7), as we see readily by taking for example one of the hemispheres into which the whole sphere round P is cut by the plane through P perpendicular to z, it is clear that

Digression on the attraction of an ellipsoid.

$$\iint \frac{Q\, d\sigma}{\frac{l^2}{a^2} + \frac{m^2}{b^2} + \frac{n^2}{c^2}} = 0 \text{..................(9)};$$

and therefore (7) and (8) give

$$V = \iint d\sigma \frac{\frac{l^2}{a^2}\left(\frac{2x^2}{a^2} + e\right) + \frac{m^2}{b^2}\left(\frac{2y^2}{b^2} + e\right) + \frac{n^2}{c^2}\left(\frac{2z^2}{c^2} + e\right)}{\left(\frac{l^2}{a^2} + \frac{m^2}{b^2} + \frac{n^2}{c^2}\right)^2} \text{...(10)};$$

or

$$V = e\Phi + \frac{x^2}{a}\frac{d\Phi}{da} + \frac{y^2}{b}\frac{d\Phi}{db} + \frac{z^2}{c}\frac{d\Phi}{dc} \text{..........(11)},$$

where

$$\Phi = \iint \frac{d\sigma}{\frac{l^2}{a^2} + \frac{m^2}{b^2} + \frac{n^2}{c^2}} \text{.......................(12)}.$$

494 l. A symmetrical evaluation of Φ not being obvious, we may be content to take

$$l = \cos\theta, \quad m = \sin\theta\cos\phi, \quad n = \sin\theta\sin\phi,$$

and

$$d\sigma = \sin\theta\, d\theta\, d\phi.$$

Using these, replacing l, and putting

$$\frac{1}{b^2} - \left(\frac{1}{b^2} - \frac{1}{a^2}\right) l^2 = H, \quad\text{and}\quad \frac{1}{c^2} - \left(\frac{1}{c^2} - \frac{1}{a^2}\right) l^2 = K,$$

we find

$$\Phi = \int_0^1 dl \int_0^{2\pi} \frac{d\phi}{H\cos^2\phi + K\sin^2\phi}.$$

$$\int_0^{2\pi} \frac{d\phi}{H\cos^2\phi + K\sin^2\phi} = 4\int_0^\infty \frac{dt}{H + Kt^2} = \frac{2\pi}{\sqrt{(HK)}}.$$

Hence

$$\Phi = 2\pi \int_0^1 \frac{dl}{\left[\frac{1}{b^2} - \left(\frac{1}{b^2} - \frac{1}{a^2}\right) l^2\right]^{\frac{1}{2}} \left[\frac{1}{c^2} - \left(\frac{1}{c^2} - \frac{1}{a^2}\right) l^2\right]} \text{...(13)}.$$

By (12) we know that Φ is a symmetrical function of a, b, c.

Digression on the attraction of an ellipsoid.

To bring (12) to this form, take

$$l=\frac{a}{\sqrt{(a^2+u)}}\text{......................}(14),$$

which reduces (13) to

$$\Phi=\pi abc\int_0^\infty\frac{du}{(a^2+u)^{\frac{1}{2}}(b^2+u)^{\frac{1}{2}}(c^2+u)^{\frac{1}{2}}}\text{.......}(15).$$

The expression (11) for V, with (15) for Φ, is worth preserving for its own sake and for some applications; but the following, derived from it by performing the indicated differentiations, is simpler and is generally preferable:

$$V=\pi abc\int_0^\infty\left(1-\frac{x^2}{a^2+u}-\frac{y^2}{b^2+u}-\frac{z^2}{c^2+u}\right)\frac{du}{(a^2+u)^{\frac{1}{2}}(b^2+u)^{\frac{1}{2}}(c^2+u)^{\frac{1}{2}}}\text{..}(16);$$

or, if M denote the mass of the ellipsoid,

$$V=\frac{3M}{4}\int_0^\infty\left(1-\frac{x^2}{a^2+u}-\frac{y^2}{b^2+u}-\frac{z^2}{c^2+u}\right)\frac{du}{(a^2+u)^{\frac{1}{2}}(b^2+u)^{\frac{1}{2}}(c^2+u)^{\frac{1}{2}}}\text{...}(17).$$

This, or (16), expresses the potential at any point (x, y, z) within the ellipsoid (a, b, c) or on its surface.

494 *m*. The potential at any external point is deduced from (17) through Maclaurin's theorem [§§ 494 *i*] simply by substituting for a, b, c the semi-axes of the ellipsoid confocal with (a, b, c), and passing through x, y, z: these semi-axes are $\sqrt{(a^2+q)}$, $\sqrt{(b^2+q)}$, $\sqrt{(c^2+q)}$, where q denotes the positive root of the equation

$$\frac{x^2}{a^2+q}+\frac{y^2}{b^2+q}+\frac{z^2}{c^2+q}=1\text{.............}(18);$$

which is a cubic in q. Thus, for an external point, we find

$$V=\frac{3M}{4}\int_0^\infty\left(1-\frac{x^2}{a^2+q+u}-\frac{y^2}{b^2+q+u}-\frac{z^2}{c^2+q+u}\right)\frac{du}{(a^2+q+u)^{\frac{1}{2}}(b^2+q+u)^{\frac{1}{2}}(c^2+q+u)^{\frac{1}{2}}}$$
$$\text{.........}(19);$$

which may be written shorter as follows:

$$V=\frac{3M}{4}\int_q^\infty\left(1-\frac{x^2}{a^2+u}-\frac{y^2}{b^2+u}-\frac{z^2}{c^2+u}\right)\frac{du}{(a^2+u)^{\frac{1}{2}}(b^2+u)^{\frac{1}{2}}(c^2+u)^{\frac{1}{2}}}\text{...}(20).$$

494 *n*. These formulas, (17) and (20), are, we believe, due to Lejeune Dirichlet*, who proves them (Crelle's *Journal*, 1846, Vol. XXXII.) by showing that they satisfy the equation

Digression on the attraction of an ellipsoid.

$$\frac{d^2V}{dx^2}+\frac{d^2V}{dy^2}+\frac{d^2V}{dz^2}=-4\pi,$$

when

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}<1,$$

and

$$\frac{d^2V}{dx^2}+\frac{d^2V}{dy^2}+\frac{d^2V}{dz^2}=0,$$

when

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}>1;$$

and that

$$\frac{dV}{dx},\ \frac{dV}{dy},\ \frac{dV}{dz}$$

have equal values at points infinitely near the surface

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1,$$

outside and inside it. His first step towards this proof (the completion of which we leave as an exercise to our readers) is the evaluation of dV/dx, dV/dy, dV/dz. In this it is necessary to remark that, for the external point, terms depending on the variation of q as it appears in (20) vanish because of (18): and taking the results which we then get instantly by plain differentiation, and remembering that $X=-dV/dx$, &c., we have, for the principal components of the resultant force,

$$\left.\begin{aligned}X&=\frac{3Mx}{2}\int_q^\infty\frac{du}{(a^2+u)^{\frac{3}{2}}(b^2+u)^{\frac{1}{2}}(c^2+u)^{\frac{1}{2}}}\\Y&=\frac{3My}{2}\int_q^\infty\frac{du}{(a^2+u)^{\frac{1}{2}}(b^2+u)^{\frac{3}{2}}(c^2+u)^{\frac{1}{2}}}\\Z&=\frac{3Mz}{2}\int_q^\infty\frac{du}{(a^2+u)^{\frac{1}{2}}(b^2+u)^{\frac{1}{2}}(c^2+u)^{\frac{3}{2}}}\end{aligned}\right\}\text{.....}(21),$$

where $q=0$ when (x, y, z) is internal, and q is the positive root of the cubic (18), when (x, y, z) is external.

Using (21) in (20) and (17), we see that

$$V=\frac{3M}{4}\int_q^\infty\frac{du}{(a^2+u)^{\frac{1}{2}}(b^2+u)^{\frac{1}{2}}(c^2+u)^{\frac{1}{2}}}-\tfrac{1}{2}(Xx+Yy+Zz)\text{...}(22).$$

* [An equivalent formula appears to have been given by Plana in 1840. (Todhunter, *Hist. of Th. of Attractions*, Vol. II., p. 433.) H. L.]

Digression on the attraction of an ellipsoid.

494 *o*. For the case of an internal point or a point on the surface, by putting $q=0$, we fall back on the original expressions (16) for V, and the proper differential coefficients of it for X, Y, Z.

These results may be written as follows:

$$\left.\begin{aligned} &X=\frac{4\pi}{3}\mathfrak{A}x,\quad Y=\frac{4\pi}{3}\mathfrak{B}y,\quad Z=\frac{4\pi}{3}\mathfrak{C}z,\\ &V=\Phi-\frac{2\pi}{3}(\mathfrak{A}x^2+\mathfrak{B}y^2+\mathfrak{C}z^2)\end{aligned}\right\}\ldots(23),$$

where Φ, $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ are constants, of which Φ is given by (12), or (13), or (15), and the others by (21) with $q=0$; all expressed in terms of elliptic integrals.

It follows that the internal equipotential surfaces are concentric similar ellipsoids with axes proportional to $\mathfrak{A}^{-\frac{1}{2}}$, $\mathfrak{B}^{-\frac{1}{2}}$, $\mathfrak{C}^{-\frac{1}{2}}$; and that the internal surfaces of equal resultant force are concentric similar ellipsoids with axes proportional to $\mathfrak{A}^{-1}$, $\mathfrak{B}^{-1}$, $\mathfrak{C}^{-1}$.

The external equipotentials are transcendental plinthoids * of an interesting character. So are the equipotentials partly internal (where they are ellipsoidal) and external (where they are not ellipsoidal).

It is interesting, and useful in helping to draw the external equipotentials, to remark the following relations between the internal equipotentials, the external equipotentials, and the surface of the attracting ellipsoid.

(1) The external equipotential $V=C$ is the envelope of the series of ellipsoidal surfaces obtained by giving an infinite number of constant values to q in the equation

$$\int_q^\infty\left(1-\frac{x^2}{a^2+u}-\frac{y^2}{b^2+u}-\frac{z^2}{c^2+u}\right)\frac{du}{(a^2+u)^{\frac{1}{2}}(b^2+u)^{\frac{1}{2}}(c^2+u)^{\frac{1}{2}}}=\frac{4C}{3M}\ldots(\alpha).$$

(2) This envelope is cut by the ellipsoidal surface

$$\frac{x^2}{a^2+q}+\frac{y^2}{b^2+q}+\frac{z^2}{c^2+q}=1\ \ldots\ldots\ldots\ldots\ldots(\beta),$$

* From πλινθοειδής, brick-like. Plinthoid, as we now use the term, denotes as it were a sea-worn brick; any figure with three rectangular axes, and surfaces everywhere convex, such as an ellipsoid, or a perfectly symmetrical bale of cotton with slightly rounded sides and rounded edges and corners. One extreme of plinthoidal figure is a rectangular parallelepiped; another extreme, just not excluded by our definition, is a figure composed of two equal and similar right rectangular pyramids fixed together base to base, that is a "regular octohedron."

Digression on the attraction of an ellipsoid.

for any particular value of q in the line along which it is touched by the particular one of the series of consecutive ellipsoidal surfaces (β) corresponding to this value of q.

(3) If the ellipsoidal surface (β) be filled with homogeneous matter, the complete equipotential for any particular value of C is composed of an interior ellipsoidal surface passing tangentially to the external plinthoidal (but not ellipsoidal) surface across the transitional line defined in (2).

It is easy to make graphic illustrations for the case of ellipsoids of revolution, by aid of § 527 below.

Attraction of an infinitely long elliptic cylinder.

494 *p*. In the case of an elliptic cylinder, which is important in many physical investigations, replace M by $4\pi abc/3$, and put $c=\infty$.

Thus we find

$$\left.\begin{aligned} X&=2\pi abx\int_q^\infty\frac{du}{(a^2+u)^{\frac{3}{2}}(b^2+u)^{\frac{1}{2}}}=\frac{4\pi ab[\surd(a^2+q)-\surd(b^2+q)]x}{(a^2-b^2)\surd(a^2+q)}\\ &=\frac{4\pi abx}{\surd(a^2+q)[\surd(a^2+q)+\surd(b^2+q)]}\\ Y&=2\pi aby\int_q^\infty\frac{du}{(a^2+u)^{\frac{1}{2}}(b^2+u)^{\frac{3}{2}}}=\frac{4\pi ab[\surd(a^2+q)-\surd(b^2+q)]y}{(a^2-b^2)\surd(b^2+q)}\\ &=\frac{4\pi aby}{\surd(b^2+q)[\surd(a^2+q)+\surd(b^2+q)]}\\ &\text{where}\qquad q=0,\ \text{when}\ \frac{x^2}{a^2}+\frac{y^2}{b^2}<1;\\ &\text{and } q \text{ is the positive root of the quadratic}\\ &\frac{x^2}{a^2+q}+\frac{y^2}{b^2+q}=1,\ \text{when}\ \frac{x^2}{a^2}+\frac{y^2}{b^2}>1.\end{aligned}\right\}\ldots(24).$$

For the case of $q=0$, that is to say, the case of an internal point, (24) becomes

$$X=\frac{4\pi ab}{a+b}\,\frac{x}{a},\quad\text{and}\quad Y=\frac{4\pi ab}{a+b}\,\frac{y}{b}\ldots\ldots\ldots(25).$$

Internal isodynamic surfaces are similar to the bounding surface.

494 *q*. For the magnitude of the resultant force we deduce

$$R=\surd(X^2+Y^2)=\frac{4\pi ab}{a+b}\sqrt{\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)}\ \ldots\ldots(26);$$

Attraction of an infinitely long elliptic cylinder.

and it is remarkable that this is constant for all points on the surface of the elliptic cylinder $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, and on each similar internal surface, and that its values on different ones of these surfaces are as their linear magnitudes.

Potential in free space cannot have a maximum or minimum value;

495 *a*. At any point of zero force, the potential is a *maximum* or a *minimum*, or a "*minimax*." Now from § 492 (2) it follows that the potential cannot be a maximum or a minimum at a point in free space. For if it were so, a closed surface could be described about the point, and indefinitely near it, so that at every point of it the value of the potential would be less than, or greater than, that at the point; so that N would be negative or positive all over the surface, and therefore $\iint N d\sigma$ would be finite, which is impossible, as the surface encloses none of the attracting mass.

is a minimax at a point of zero force in free space.

495 *b*. Consider, now, a point of zero force in free space:—the potential, if it varies at all in the neighbourhood, must be a minimax at the point, because, as has just been proved, it cannot be a maximum or a minimum. Hence a material particle placed at a point of zero force under the action of any attracting bodies, and free from all constraint, is in unstable equilibrium, a result due to Earnshaw*.

Earnshaw's theorem of unstable equilibrium.

495 *c*. If the potential be constant over a closed surface which contains none of the attracting mass, it has the same constant value throughout the interior. For if not, it must have a maximum or a minimum value somewhere within the surface, which (§ 495, *a*) is impossible.

Mean potential over a spherical surface equal to that at its centre.

496. The mean potential over any spherical surface, due to matter entirely without it, is equal to the potential at its centre; a theorem apparently first given by Gauss. See also *Cambridge Mathematical Journal*, Feb. 1845 (Vol. IV. p. 225). It is one of the most elementary propositions of spherical harmonic analysis, applied to potentials, found by applying App. B. (16) to the formulæ of § 539, below. But the following proof taken from the paper now referred to is noticeable as independent of the harmonic expansion.

* *Cambridge Phil. Trans.*, March, 1839.

Mean potential over a spherical surface equal to that at its centre.

Let, in Chap. I. App. A. (*a*), S be a spherical surface, of radius a; and let U be the potential at (x, y, z), due to matter altogether external to it; let U' be the potential of a unit of matter uniformly distributed through a smaller concentric spherical surface; so that, outside S and to some distance within it, $U'=\frac{1}{r}$; and lastly, let $a=1$. The middle member of App. A (*a*) (1) becomes

$$\frac{1}{a}\iint \partial U d\sigma - \iiint U'\nabla^2 U dx dy dz,$$

which is equal to zero, since $\nabla^2 U=0$ for the whole internal space, and (§ 492) $\iint \partial U d\sigma = 0$. Equating therefore the third member to zero we have

$$\iint d\sigma U \partial U' = \iiint U\nabla^2 U' dx dy dz.$$

Now at the surface, S, $\partial U' = -\frac{1}{a^2}$; and for all points external to the sphere of matter to which U' is due, $\nabla^2 U'=0$, and for all internal points $\nabla^2 U' = -4\pi\rho'$, if ρ' be the density of the matter. Hence the preceding equation becomes

$$\frac{1}{a^2}\iint U d\sigma = 4\pi \iiint \rho' U dx dy dz.$$

Let now the density ρ' increase without limit, and the spherical space within which the triple integral extends, therefore become infinitely small. If we denote by U_0 the value of U at its centre, which is also the centre of S, we shall have

$$\iiint \rho' U dx dy dz = U_0 \iiint \rho' dx dy dz = U_0.$$

Hence the equation becomes

$$\frac{\iint U d\sigma}{4\pi a^2} = U_0,$$

which was to be proved.

The following more elementary proof is preferable:—imagine any quantity of matter to be uniformly distributed over the spherical surface. The mutual potential (§ 547 below) of this and the external mass is the same as if the matter were condensed from the spherical surface to its centre.

Theorem of Gauss:

497. If the potential of any masses has a constant value, V, through any finite portion, K, of space, unoccupied by matter, it is equal to V through every part of space which can be reached

Theorem of Gauss, proved.

in any way without passing through any of those masses: a very remarkable proposition, due to Gauss, proved thus:—If the potential differ from V in space contiguous to K, we may, from any point C within K, as centre, in the neighbourhood of a place where the potential differs from V, describe a spherical surface not large enough to contain any part of any of the attracting masses, nor to include any of the space external to K except such as has potential all greater than V, or all less than V. But this is impossible, since we have just seen (§ 496) that the mean potential over the spherical surface must be V. Hence the supposition that the potential differs from V in any place contiguous to K and not including masses, is false.

498. Similarly we see that in any case of symmetry round an axis, if the potential is constant through a certain finite distance, however short, along the axis, it is constant throughout the whole space that can be reached from this portion of the axis, without crossing any of the masses. (See § 546, below.)

Green's problem.

499. Let S be any finite portion of a surface, or a complete closed surface, or an infinite surface; and let E be any point on S. (*a*) It is possible to distribute matter over S so as to produce, over the whole of S, potential equal to $F(E)$, any arbitrary function of the position of E. (*b*) There is only one whole quantity of matter, and one distribution of it, which can do this.

In Chap. I. App. A. (*b*) (*e*), etc., let $a=1$. By (*e*) we see that there is one, and that there is only one, solution of the equation

$$\nabla^2 U = 0$$

for all points not belonging to S, subject to the condition that U shall have a value arbitrarily given over the whole of S. Continuing to denote by U the solution of this problem, and considering first the case of S an open shell, that is to say, a finite portion of curved surface (including a plane, of course, as a particular case), let, in Chap. I. App. A. (*a*), U' be the potential at (x, y, z) due to a distribution of matter, having $\varpi\,(Q)$ for density at any point, Q. Let the triple integration extend throughout infinite space, exclusive of the infinitely thin shell S. Although

Green's problem:

in the investigation referred to [App. A. (*a*)] the triple integral extended only through the finite space contained within a closed surface, the same process shows that we have now, instead of the second and third members of (1) of that investigation, the following equated expressions:—

$$\iint d\sigma U'\{[\partial U]-(\partial U)\} - \iiint dxdydz\,U'\nabla^2 U$$
$$= \iint d\sigma U\{[\partial U']-(\partial U')\} - \iiint dxdydz\,U\nabla^2 U'$$

where $[\partial U]$ denotes the rate of variation of U on either side of S, infinitely near E, reckoned per unit of length *from* S; and (∂U) denotes the rate of variation of U infinitely near E, on the other side of S, reckoned per unit of length *towards* S; and $[\partial U']$, $(\partial U')$ denote the same for U'. Now we shall suppose the matter of which U' is the potential not to be condensed in finite quantities on any finite areas of S, which will make

$$[\partial U'] = (\partial U'):$$

and the conditions defining U and U' give, throughout the space of the triple integral,

$$\nabla^2 U = 0, \text{ and } \nabla^2 U' = -4\pi\varpi;$$

ϖ denoting the value of $\varpi\,(Q)$ when Q is the point (x, y, z). Hence the preceding equation becomes

$$\iint d\sigma U'\{[\partial U]-(\partial U)\} = 4\pi\iiint dxdydz\varpi U\text{...........}(1).$$

Let now the matter of which U' is the potential be equal in amount to unity and be confined to an infinitely small space round a point Q. We shall have

$$\iiint dxdydz\varpi U = U(Q)\iiint \varpi dxdydz = U(Q),$$

if we denote the value of U at (Q) by $U(Q)$:

also $$U' = \frac{1}{EQ}.$$

Hence (1) becomes

$$\iint \frac{[\partial U]-(\partial U)}{EQ}\,d\sigma = 4\pi U(Q)\text{..............}(2).$$

Hence a distribution of matter over S, having

reduced to the proper general solution of Laplace's equation.

$$\frac{1}{4\pi}\{[\partial U]-(\partial U)\}\text{ ....}(3)$$

for density at the point E, gives U as its potential at (x, y, z). We conclude, therefore, that it is possible to find one, but only one, distribution of matter over S which shall produce an arbi-

Green's problem;

trarily given potential, $F(E)$, over the whole of S; and in (2) we have the solution of this problem, when the problem of finding U to fulfil the conditions stated above, has been solved.

If S is any finite closed surface, any group of surfaces, open or closed, or an infinite surface, the same conclusions clearly hold. The triple integration used in the investigation must then be separately carried out through all the portions of space separated from one another by S, or by portions of S.

If the solution, ρ, of the problem has been obtained for the case in which the arbitrary function is the potential at any point of S, due to a unit of matter at any point P not belonging to S, that is to say, for the case of $F(E)=\frac{1}{EP}$, the solution of the general problem was shown by Green to be deducible from it thus:—

solved synthetically in terms of particular solution of Laplace's equation.

$$U=\iint \rho F(E)\, d\sigma \qquad (4).$$

The proof is obvious: For let, for a moment, ${}_{,}\rho$ denote the superficial density required to produce U, then ${}_{,}\rho'$ denoting the value of ${}_{,}\rho$ for any other element, E', of S, we have

$$F(E)=\iint \frac{{}_{,}\rho' d\sigma'}{E'E}.$$

Hence the preceding double integral becomes

$$\iint d\sigma \rho \iint d\sigma' \frac{{}_{,}\rho'}{E'E}, \text{ or } \iint d\sigma' {}_{,}\rho' \iint d\sigma \frac{\rho}{E'E}.$$

But, by the definition of ρ,

$$\iint d\sigma \frac{\rho}{E'E}=\frac{1}{E'P} \qquad (5);$$

and therefore

$$\iint \rho F(E)\, d\sigma = \iint d\sigma' \frac{{}_{,}\rho'}{E'P} \qquad (6).$$

The second member of this is equal to U, according to the definition of ${}_{,}\rho$.

The expression (46) of App. B., from which the spherical harmonic expansion of an arbitrary function was derived, is a case of the general result (4) now proved.

Isolation of effect by closed portion of surface.

500. It is important to remark that, if S consist, in part, of a closed surface, Q, the determination of U within it will be independent of those portions of S, if any, which lie without it; and, *vice versa*, the determination of U through external

space will be independent of those portions of S, if any, which lie within Q. Or if S consist, in part, of a surface Q, extending infinitely in all directions, the determination of U through all space on either side of Q, is independent of those portions of S, if any, which lie on the other side. This follows from the preceding investigation, modified by confining the triple integration to one of the two portions of space separated completely from one another by Q.

Isolation of effect by closed portion of surface.

501. Another remark of extreme importance is this:—If $F(E)$ be the potential at E of any distribution, M, of matter, and if S be such as to separate perfectly any portion or portions of space, H, from all of this matter; that is to say, such that it is impossible to pass into H from any part of M without crossing S; then, throughout H, the value of U will be the potential of M.

Green's problem; applied to a given distribution of electricity, M, influencing a conducting surface, S.

For if V denote this potential, we have, throughout H, $\nabla^2 V=0$; and at every point of the boundary of H, $V=F(E)$. Hence, considering the theorem of Chap. I. App. A. (*c*), for the space H alone, and its boundary alone, instead of S, we see that, through this space, V satisfies the conditions prescribed for U, and therefore, through this space, $U=V$.

Solved Examples. (1) Let M be a homogeneous solid ellipsoid; and let S be the bounding surface, or any of the external ellipsoidal surfaces confocal with it. The required surface-density is proved in § 494 *g* to be *inversely* proportional to the perpendicular from the centre to the tangent-plane; or, which is the same, directly proportional to the distance between S and another confocal ellipsoid surface infinitely near it. In other words, the attraction of a focaloid (§ 494 *g*, foot-note) of homogeneous matter is, for all points external to it, the same as that of a homogeneous solid of equal mass bounded by any confocal ellipsoid interior to it.

Virtually Maclaurin's theorem, § 494 *i*.

(2) Let M be an elliptic homoeoid (§ 494 *g*, foot-note) of homogeneous matter; and let S be any external confocal ellipsoidal surface. The required surface-density is proved in § 519 below to be *directly* proportional to the perpendicular from the centre to the tangent-plane; and, which is

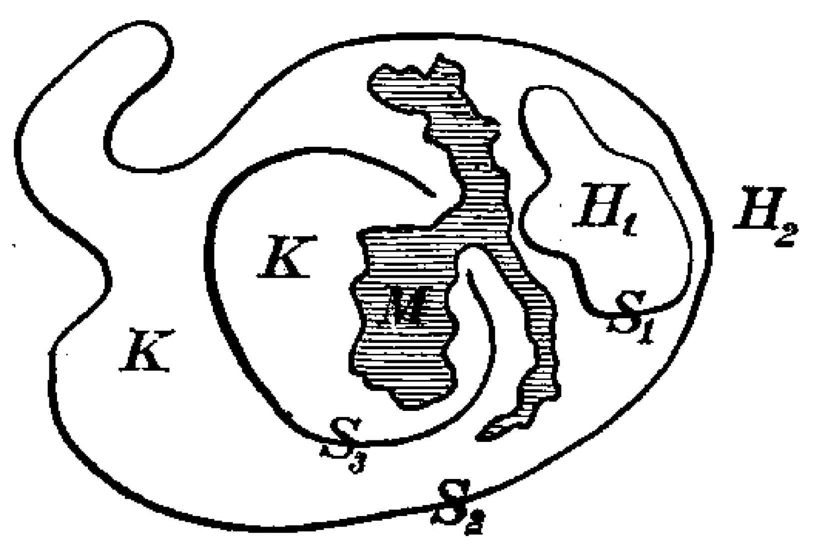
Elliptic homoeoid, an example belonging to the reducible case, § 505, of Green's problem.

Green's problem.

the same, directly proportional to the distance between S and a similar concentric ellipsoidal surface infinitely near it. In other words, the attractions of confocal infinitely thin elliptic homoeoids of homogeneous matter are the same for all external points, if their masses are equal.

Complex application of § 501.

502. To illustrate more complicated applications of § 501, let S consist of three detached surfaces, S_1, S_2, S_3, as in the diagram, of which S_1, S_2 are closed, and S_3 is an open shell, and if $F(E)$ be the potential due to M, at any point, E, of any of these portions of S; then throughout H_1, and H_2, the spaces within S_1 and without S_2, the value of U is simply the potential of M. The value of U through K, the remainder of space, depends, of course, on the character of the composite surface S, and is a case of the general problem of which the solution was proved to be possible and single in Chap. I. App. A.



General problem of electric influence possible and determinate.

503. From § 500 follows the grand proposition:—*It is possible to find one, but no other than one, distribution of matter over a surface S which shall produce over S, and throughout all space H separated by S from every part of M, the same potential as any given mass M.*

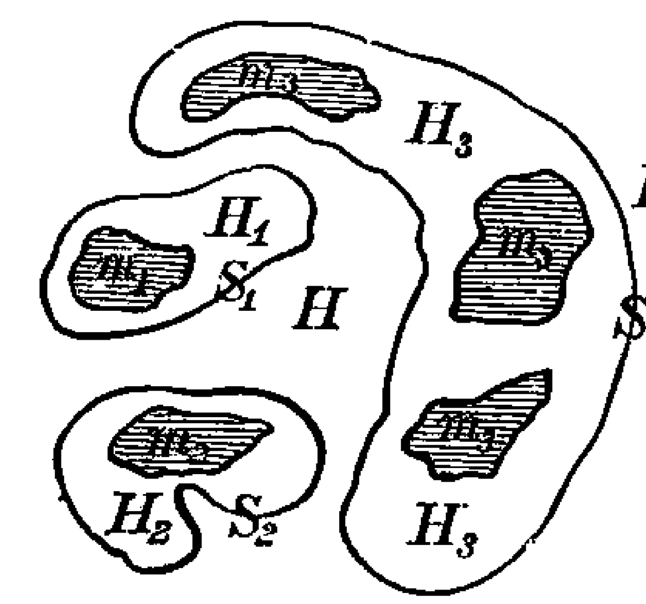
Thus, in the preceding diagram, it is possible to find one, and but one, distribution of matter over S_1, S_2, S_3 which shall produce over S_3 and through H_1 and H_2 the same potential as M.

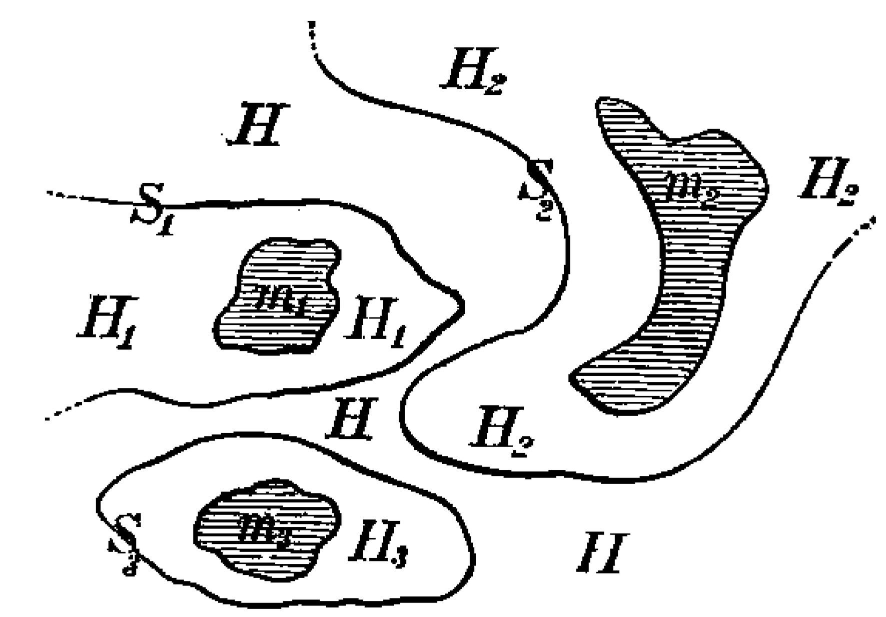
The statement of this proposition most commonly made is: *It is possible to distribute matter over any surface, S, completely enclosing a mass M, so as to produce the same potential as M through all space outside S;* which, though seemingly more limited, is, when interpreted with proper mathematical comprehensiveness, equivalent to the foregoing.

Simultaneous electric influences in spaces separated by infinitely thin conducting surfaces.

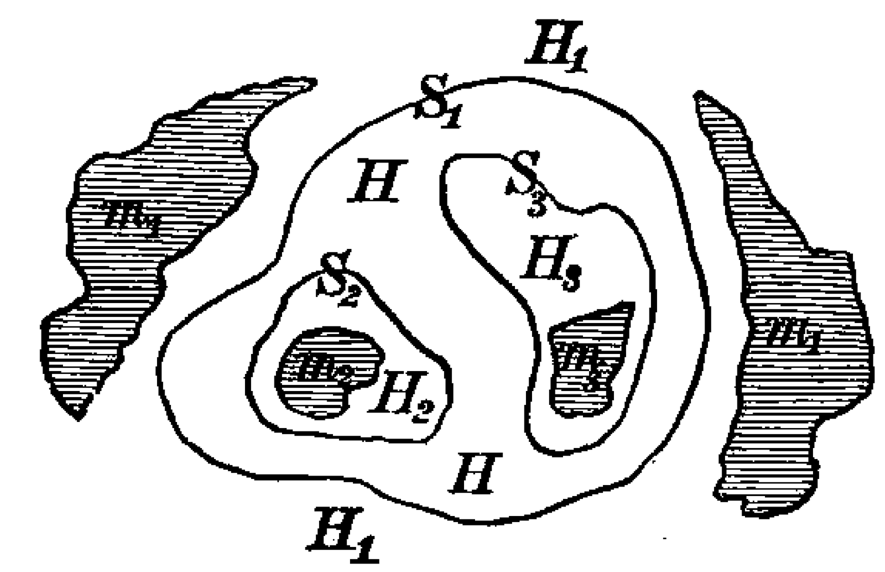
504. If S consist of several closed or infinite surfaces, S_1, S_2, S_3, respectively separating certain isolated spaces H_1, H_2, H_3, from

H, the remainder of all space, and if $F(E)$ be the potential of masses m_1, m_2, m_3, lying in the spaces H_1, H_2, H_3; the portions of U due to S_1, S_2, S_3, respectively will throughout H be equal respectively to the potentials of m_1, m_2, m_3, separately. For as we have just seen, it is possible to find one, but only





one, distribution of matter over S_1 which shall produce the potential of m_1, throughout all the space H_1, H_2, H_3, etc., and one, but only one, distribution over S_2 which shall produce the potential of m_2 throughout H, H_1, H_3, etc.; and so on. But these distributions on S_1, S_2, etc., jointly constitute a distribution producing the potential $F(E)$ over every part of S, and therefore the sum of the potentials due to them all, at any point, fulfils the conditions presented for U. This is therefore (§ 503) *the* solution of the problem.



Reducible case of Green's problem;

505. Considering still the case in which $F(E)$ is prescribed to be the potential of a given mass, M: let S be an equipotential surface enclosing M, or a group of isolated surfaces enclosing all the parts of M, and each equipotential for the whole of M. The potential due to the supposed distribution over S will be the same as that of M, through all external space, and will be constant (§ 497) through each enclosed portion of space. Its resultant attraction will therefore be the same as that of M on all external points, and zero on all internal points. Hence we see at once that the density of the matter distributed over it, Reducible case of Green's problem:

to produce $F(E)$, is equal to $\frac{R}{4\pi}$ where R denotes the resultant force of M, at the point E.

We have $[\partial U] = -R$ and $(\partial U) = 0$. Using this in § 500 (2), we find the preceding formula for the required surface-density.

applied to the invention of solved problems of electric influence.

506. Considering still the case of §§ 501, 505, let S be the equipotential not of M alone, as in § 505, but of M and another mass m completely separated by it from M; so that $V+v=C$ at S, if V and v denote the potentials of M and m respectively.

The potential of the supposed distribution of matter on S, which, (§ 501), is equal to V through all space separated from M by S, is equal to $C-v$ at S, and therefore equal to $C-v$ throughout the space separated from m by S.

Thus, passing from potentials to attractions, we see that the resultant attraction of S alone, on all points on one side of it is the same as that of M; and on the other side is equal and opposite to that of m. The most direct and simple complete statement of this result is as follows:—

If masses m, m', in portions of space, H, H', completely separated from one another by one continuous surface S, whether closed or infinite, are known to produce tangential forces equal and in the same direction at each point of S, one and the same distribution of matter over S will produce the force of m throughout H', and that of m' throughout H. The density of this distribution is equal to $\frac{R}{4\pi}$, if R denote the resultant force due to one of the masses, and the other with its *sign* changed. And it is to be remarked that the direction of this resultant force is, at every point, E, of S, perpendicular to S, since the potential due to one mass, and the other with its sign changed, is constant over the whole of S.

Examples.

507. Green, in first publishing his discovery of the result stated in § 505, remarked that it shows a way to find an infinite variety of closed surfaces for any one of which we can solve the problem of determining the distribution of matter over it which shall produce a given uniform potential at each point of its surface, and consequently the same also throughout

its interior. Thus, an example which Green himself gives, let M be a uniform bar of matter, AA'. The equipotential surfaces round it are, as we have seen above (§ 481 c), prolate ellipsoids of revolution, each having A and A' for its foci; and the resultant force at any point P was found to be

Reducible case of Green's problem:—examples.

$$\frac{mp}{l(l^2-a^2)},$$

the whole mass of the bar being denoted by m, and its length by $2a$; $A'P+AP$ by $2l$; and the perpendicular from the centre to the tangent plane at P of the ellipsoid, by p. We conclude that a distribution of matter over the surface of the ellipsoid, having

$$\frac{1}{4\pi}\,\frac{mp}{l(l^2-a^2)}$$

for density at P, produces on all external space the same resultant force as the bar, and zero force or a constant potential through the internal space. This is a particular case of the Example (2) § 501 above, founded on the general result regarding ellipsoidal homoeoids proved below, in §§ 519, 520, 521.

508. As a second example, let M consist of two equal particles, at points I, I'. If we take the mass of each as unity, the potential at P is $\frac{1}{IP}+\frac{1}{I'P}$; and therefore

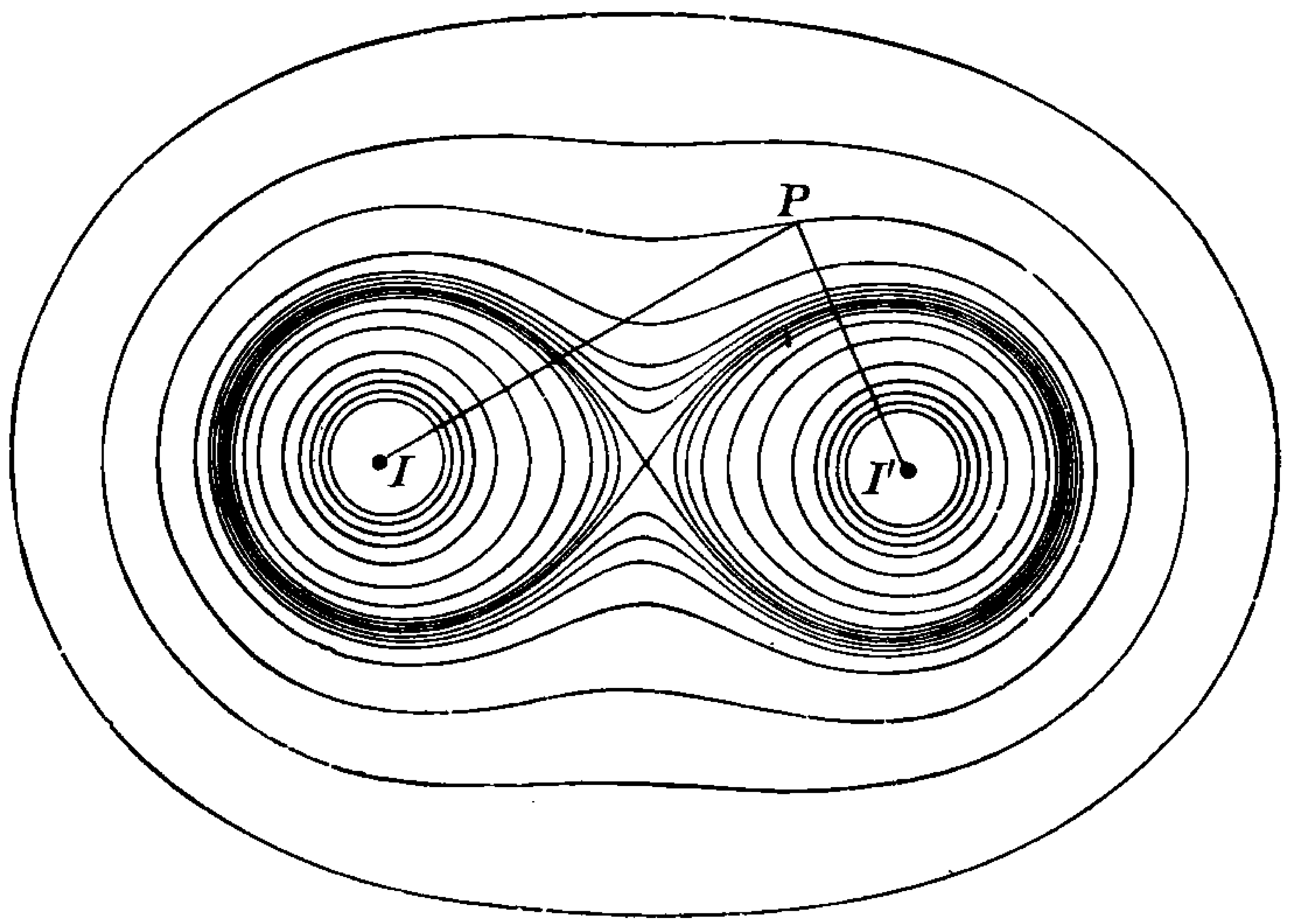
$$\frac{1}{IP}+\frac{1}{I'P}=C$$

is the equation of an equipotential surface; it being understood that negative values of IP and $I'P$ are inadmissible, and that any constant value, from ∞ to 0, may be given to C. The curves in the annexed diagram have been drawn, from this equation, for the cases of C equal respectively to 10, 9, 8, 7, 6, 5, 4·5, 4·3, 4·2, 4·1, 4, 3·9, 3·8, 3·7, 3·5, 3, 2·5, 2; the value of II' being unity.

The corresponding equipotential surfaces are the surfaces traced by these curves, if the whole diagram is made to rotate round II' as axis. Thus we see that for any values of C less than 4 the equipotential surface is one closed surface. Choosing

Reducible case of Green's problem:—examples.

any one of these surfaces, let R denote the resultant of forces equal to $\frac{1}{IP^2}$ and $\frac{1}{I'P^2}$ in the lines PI and PI'. Then if



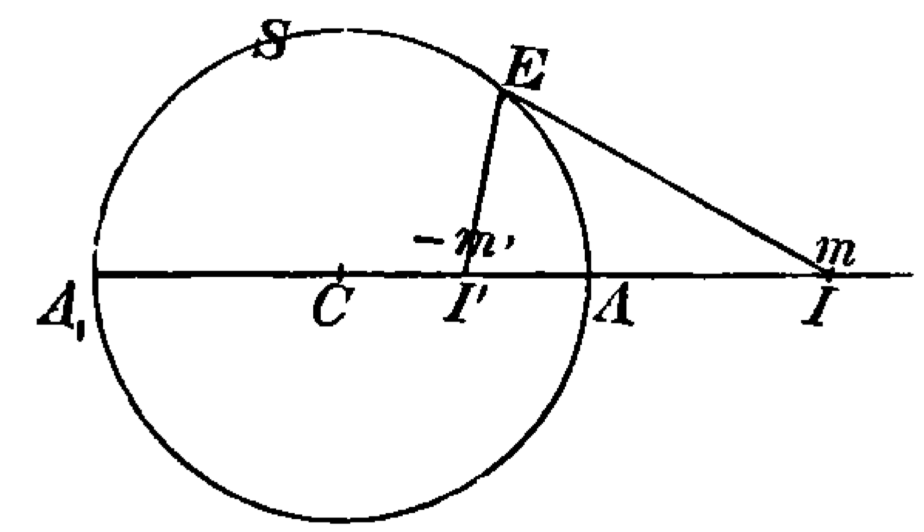
matter be distributed over this surface, with density at P equal to $\frac{R}{4\pi}$, its attraction on any internal point will be zero; and on any external point, will be the same as that of I and I'.

509. For each value of C greater than 4, the equipotential surface consists of two detached ovals approximating (the last three or four in the diagram, very closely) to spherical surfaces, with centres lying between the points I and I', but approximating more and more closely to these points, for larger and larger values of C.

Considering one of these ovals alone, one of the series enclosing I', for instance, and distributing matter over it according to the same law of density, $\frac{R}{4\pi}$, we have a shell of matter which exerts (§ 507) on external points the same force as I'; and on internal points a force equal and opposite to that of I.

Electric images.

510. As an example of exceedingly great importance in the theory of electricity, let M consist of a positive mass, m, concentrated at a point I, and a negative mass, $-m'$, at I'; and let S be a spherical surface cutting II', and II' produced in points A, $A_{,}$, such that $IA : AI' :: IA_{,} : I'A_{,} :: m : m'$. Then, by a well-known geometrical proposition, we shall have $IE : I'E :: m : m'$; and therefore



$$\frac{m}{IE} = \frac{m'}{I'E}.$$

Hence, by what we have just seen, one and the same distribution of matter over S will produce the same force as m' through all external space, and the same as m through all the space within S. And, finding the resultant of the forces $\frac{m}{IE^2}$ in EI, and $\frac{m'}{I'E^2}$ in $I'E$ produced, which, as these forces are inversely as IE to $I'E$, is (§ 256) equal to

$$\frac{m}{IE^2 . I'E} II', \text{ or } \frac{m^2 II'}{m'} \frac{1}{IE^3},$$

we conclude that the density in the shell at E is

$$\frac{m^2 II'}{4\pi m'} \cdot \frac{1}{IE^3}.$$

That the shell thus constituted does attract external points as if its mass were collected at I', and internal points as a certain mass collected at I, was proved geometrically in § 474 above.

511. If the spherical surface is given, and one of the points, I, I', for instance I, the other is found by taking $CI' = \frac{CA^2}{CI}$; and for the mass to be placed at it we have

$$m' = m\frac{I'A}{AI} = m\frac{CA}{CI} = m\frac{CI'}{CA}.$$

Hence if we have any number of particles m_1, m_2, etc., at points

Electric images.

I_1, I_2, etc., situated without S, we may find in the same way corresponding internal points I_1', I_2', etc., and masses m_1', m_2', etc.; and, by adding the expressions for the density at E given for each pair by the preceding formula, we get a spherical shell of matter which has the property of acting on all external space with the same force as $-m_1'$, $-m_2'$, etc., and on all internal points with a force equal and opposite to that of m_1, m_2, etc.

512. An infinite number of such particles may be given, constituting a continuous mass M; when of course the corresponding internal particles will constitute a continuous mass, $-M'$, of the opposite kind of matter; and the same conclusion will hold. If S is the surface of a solid or hollow metal ball connected with the earth by a fine wire, and M an external influencing body, the shell of matter we have determined is precisely the distribution of electricity on S called out by the influence of M: and the mass $-M'$, determined as above, is called the *Electric Image* of M in the ball, since the electric action through the whole space external to the ball would be unchanged if the ball were removed and $-M'$ properly placed in the space left vacant. We intend to return to this subject under Electricity.

Transformation by reciprocal radius-vectors.

513. Irrespectively of the special electric application, this method of images gives a remarkable kind of transformation which is often useful. It suggests for mere geometry what has been called the transformation by reciprocal radius-vectors; that is to say, the substitution for any set of points, or for any diagram of lines or surfaces, another obtained by drawing radii to them from a certain fixed point or origin, and measuring off lengths inversely proportional to these radii along their directions. We see in a moment by elementary geometry that any line thus obtained cuts the radius-vector through any point of it at the same angle and in the same plane as the line from which it is derived. Hence any two lines or surfaces that cut one another give two transformed lines or surfaces cutting at the same angle: and infinitely small lengths, areas, and volumes transform into others whose magnitudes are altered respectively in the ratios of the first, second, and third powers of the distances

Transformation by reciprocal radius-vectors.

of the latter from the origin, to the same powers of the distances of the former from the same. Hence the lengths, areas, and volumes in the transformed diagram, corresponding to a set of given equal infinitely small lengths, areas, and volumes, however situated, at different distances from the origin, are inversely as the squares, the fourth powers and the sixth powers of these distances. Further, it is easily proved that a straight line and a plane transform into a circle and a spherical surface, each passing through the origin; and that, generally, circles and spheres transform into circles and spheres.

514. In the theory of attraction, the transformation of masses, densities, and potentials has also to be considered. Thus, according to the foundation of the method (§ 512), equal masses, of infinitely small dimensions at different distances from the origin, transform into masses inversely as these distances, or directly as the transformed distances: and, therefore, equal densities of lines, of surfaces, and of solids, given at any stated distances from the origin, transform into densities directly as the first, the third, and the fifth powers of those distances; or inversely as the same powers of the distances, from the origin, of the corresponding points in the transformed system.

General summary of ratios.

515. The statements of the last two sections, so far as proportions alone are concerned, are most conveniently expressed thus:—

Let P be any point whatever of a geometrical diagram, or of a distribution of matter, O one particular point ("the origin"), and a one particular length (the radius of the "reflecting sphere"). In OP take a point P', corresponding to P, and for any mass m, in any infinitely small part of the given distribution, place a mass m'; fulfilling the conditions

$$OP' = \frac{a^2}{OP}, \quad m' = \frac{a}{OP} m = \frac{OP'}{a} m.$$

Then if L, A, V, $\rho(L)$, $\rho(A)$, $\rho(V)$ denote an infinitely small length, area, volume, linear-density, surface-density, volume-density in the given distribution, infinitely near to P, or anywhere at the same distance, r, from O as P, and if the corresponding elements in the transformed diagram or dis-

General summary of ratios. tribution be denoted in the same way with the addition of accents, we have

$$L' = \frac{a^4}{r^2} L = \frac{r'^2}{a^2} L\,;\ A' = \frac{a^4}{r^4} A = \frac{r'^4}{a^4} A\,;\ V' = \frac{a^6}{r^6} V = \frac{r'^6}{a^6} V,$$

$$\rho'(L) = \frac{a}{r'} \rho(L) = \frac{r}{a} \rho(L)\,;\ \rho'(A) = \frac{a^3}{r'^3} \rho(A) = \frac{r^3}{a^3} \rho(A)\,;$$

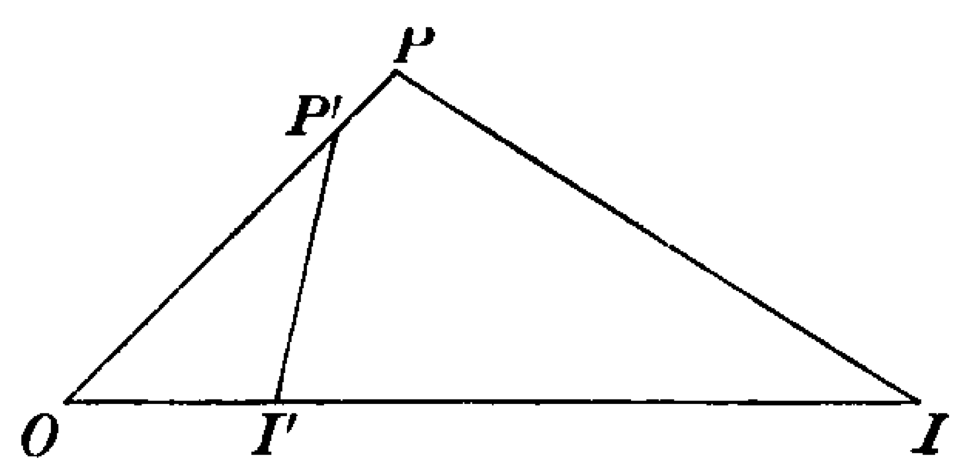
$$\rho'(V) = \frac{a^5}{r'^5} \rho(V) = \frac{r^5}{a^5} \rho(V).$$

The usefulness of this transformation in the theory of electricity, and of attraction in general, depends entirely on the following theorem :—

Application to the potential. **516.** (*Theorem.*)—Let ϕ denote the potential at P due to the given distribution, and ϕ' the potential at P' due to the transformed distribution: then shall

$$\phi' = \frac{r}{a}\phi = \frac{a}{r'}\phi.$$

Let a mass m collected at I be any part of the given distribution, and let m' at I' be the corresponding part in the transformed distribution. We have



$$a^2 = OI'.OI = OP'.OP,$$

and therefore

$$OI : OP :: OP' : OI';$$

which shows that the triangles IPO, $P'I'O$ are similar, so that

$$IP : P'I' :: \sqrt{OI.OP} : \sqrt{OP'.OI'} :: OI.OP : a^2.$$

We have besides

$$m : m' :: OI : a,$$

and therefore

$$\frac{m}{IP} : \frac{m'}{I'P'} :: a : OP.$$

Hence each term of ϕ bears to the corresponding term of ϕ' the same ratio; and therefore the sum, ϕ, must be to the sum, ϕ', in that ratio, as was to be proved.

517. As an example, let the given distribution be confined to a spherical surface, and let O be its centre and a its radius. The transformed distribution is the same. But the space within it becomes transformed into the space without it. Hence if ϕ be the potential due to any spherical shell at a point P, within it, the potential due to the same shell at the point P' in OP produced till $OP' = \frac{a^2}{OP}$, is equal to $\frac{a}{OP'}\phi$ (which is an elementary proposition in the spherical harmonic treatment of potentials, as we shall see presently). Thus, for instance, let the distribution be uniform. Then, as we know there is no force on an interior point, ϕ must be constant; and therefore the potential at P', any external point, is inversely proportional to its distance from the centre. Any distribution on a spherical shell.

Or let the given distribution be a uniform shell, S, and let O be any eccentric or any external point. The transformed distribution becomes (§§ 513, 514) a spherical shell, S', with density varying inversely as the cube of the distance from O. If O is within S, it is also enclosed by S', and the whole space within S transforms into the whole space without S'. Hence (§ 516) the potential of S' at any point without it is inversely as the distance from O, and is therefore that of a certain quantity of matter collected at O. Or if O is external to S, and consequently also external to S', the space within S transforms into the space within S'. Hence the potential of S' at any point within it is the same as that of a certain quantity of matter collected at O, which is now a point external to it. Thus, without taking advantage of the general theorems (§§ 499, 506), we fall back on the same results as we inferred from them in § 510, and as we proved synthetically earlier (§§ 471, 474, 475). It may be remarked that those synthetical demonstrations consist merely of transformations of Newton's demonstration, that attractions balance on a point within a uniform shell. Thus the first of them (§ 471) is the image of Newton's in a concentric spherical surface; and the second is its image in a spherical surface having its centre external to the shell, or internal but eccentric, according as the first or the second diagram is used. Uniform shell eccentrically reflected.

Uniform solid sphere eccentrically reflected.

518. We shall give just one other application of the theorem of § 516 at present, but much use of it will be made later, in the theory of Electricity.

Let the given distribution of matter be a uniform solid sphere, B, and let O be external to it. The transformed system will be a solid sphere, B', with density varying inversely as the fifth power of the distance from O, a point external to it. The potential of B is the same throughout external space as that due to its mass, m, collected at its centre, C. Hence the potential of B' through space external to it is the same as that of the corresponding quantity of matter collected at C', the transformed position of C. This quantity is of course equal to the mass of B'. And it is easily proved that C' is the position of the image of O in the spherical surface of B'. We conclude that a solid sphere with density varying inversely as the fifth power of the distance from an external point, O, attracts any external point as if its mass were condensed at the image of O in its external surface. It is easy to verify this for points of the axis by direct integration, and thence the general conclusion follows according to § 490.

Second investigation of attraction of ellipsoid.

519. One other application of Green's great theorem of § 503, showing us a way to find the potential and the resultant force at any point within or without an elliptic homoeoid, from which we are led to a second very interesting solution of the problem of finding the attraction of an ellipsoid differing greatly from that of § 494, we shall now give.

An elliptic homoeoid exercises no force on internal points.

Elliptic homoeoid exerts zero force on internal point:

To prove this, let the infinitely thin spherical shell of § 462, imagined as bounded by concentric spherical surfaces, be distorted (§§ 158, 160) by simple extensions and compressions in three rectangular directions, so as to become an elliptic homoeoid. In this distorted form, the volumes of all parts are diminished or increased in the proportion of the volume of the ellipsoid to the volume of the sphere; and (§ 158) the ratio of the lines HP, PK is unaltered. Hence the elements IH, KL, still attract P equally; and therefore, as in § 462, we conclude that the resultant force on an internal point is zero.

It follows immediately that the attraction on any point in the hollow space within a homoeoid not infinitely thin is zero. This proposition is due originally to Newton.

Theorem due to Newton.

Distribution of electricity on ellipsoidal conductor.

520. In passing it may be remarked that the distribution of electricity on an ellipsoidal conductor, undisturbed by electric influence, is thus proved to be in simple proportion to the thickness of a homoeoid coincident with its surface, and therefore (§ 494, foot-note) directly proportional to the perpendicular from the centre to the tangent plane.

Force external to an elliptic homoeoid found.

521. From § 519 and § 478 it follows that the resultant force on an external point anywhere infinitely near the homoeoid is perpendicular to the surface, and is equal to $4\pi t$, if t denote the thickness of the shell in that neighbourhood (its density being taken as unity). It follows also from § 519 that the potential is constant throughout the interior of the homoeoid and over its surface. Hence the distance from this surface to another equipotential infinitely near it outside is inversely proportional to t: and therefore (§ 494) this second surface is ellipsoidal and confocal with the first. By supposing the proper distribution of matter (§ 505) placed on this second surface to produce over it, and through its interior, its uniform potential, we see in the same way that the third equipotential infinitely near it outside is ellipsoidal and confocal with it; and similarly again that a fourth equipotential is an ellipsoidal surface confocal with the third, and so on. Thus we conclude that the equipotentials external to the original homoeoid are the whole series of external confocal ellipsoidal surfaces.

Digression. Second proof of Maclaurin's theorem.

522 From this theorem it follows immediately that any two confocal homoeoids of equal masses produce the same attraction on all points external to both. And from this (as pointed out by Chasles, *Journal de l'École Polytechnique*, 25[th] Cahier, Paris, 1837) follows immediately Maclaurin's theorem thus:—Consider two thick homoeoids having the outer surfaces confocal, and also their inner surfaces confocal. Divide one of them into an infinite number of similar homoeoids; and divide the other in a corresponding manner, so that each of its homoeoidal parts shall be confocal with the corresponding

Digression. Second proof of Maclaurin's theorem. one of the first. These two thick homoeoids produce the same force on any point external to both. Now let the hollow of one of them, and therefore also the hollow of the other, become infinitely small; we have two solid confocal ellipsoids, and it is proved that they exert the same force on all points external to both.

523. A beautiful geometric proof of the theorem of § 521 due to Chasles, is given below, § 532. The proof given in § 521 is from Thomson's "Electrostatics and Magnetism" (§ 812, reprinted from *Camb. Math. Jour.*, Feb. 1842). The theorem itself is due to Poisson, who proved (in the *Connaissance des Temps* for 1837, published in 1834*) that the resultant force of a homoeoid on an external point is in the direction of the interior axis of the tangential elliptic cone through the attracted point circumscribed about the homoeoid; for it is a known geometrical proposition, easily proved, that the three axes of the tangential cone are normal to the three confocal surfaces, ellipsoid, hyperboloid of one sheet, and hyperboloid of two sheets, through its vertex.

524. The magnitude of the resultant force is equal to $4\pi\tau$, where τ denotes the thickness of the confocal homoeoid equal in bulk to the given homoeoid.

Magnitude and direction of attraction of elliptic homoeoid on external point, expressed analytically

To express the magnitude and direction symbolically, let abc be the semi-axes of the given homoeoid, and $\alpha\beta\gamma$ those of the confocal one through P the attracted point; and let p, t and ϖ, τ be the perpendiculars from the centre to the tangent planes, and the thicknesses, at any point of the given homoeoid, and at the point P of the other. The volumes of the two homoeoids are respectively

$$4\pi abc\, t/p, \text{ and } 4\pi\alpha\beta\gamma\tau/\varpi;$$

hence

$$4\pi\tau = 4\pi\frac{abc}{\alpha\beta\gamma}\frac{t}{p}\varpi \quad\ldots\ldots(1),$$

and therefore the resultant force is

$$4\pi\frac{abc}{\alpha\beta\gamma}\frac{t}{p}\varpi \quad\ldots\ldots(2).$$

* See Todhunter's *History of the Mathematical Theories of Attraction and the Figure of the Earth*, Vol. II. Articles 1301—1415.

Magnitude and direction of attraction of elliptic homoeoid on external point, expressed analytically.

Supposing the rectangular co-ordinates of the attracted point xyz given; to find $\alpha\beta\gamma$ we have

$$\alpha^2 = a^2+\lambda;\quad \beta^2 = b^2+\lambda;\quad \gamma^2 = c^2+\lambda \quad\ldots\ldots(3),$$

where λ is the positive root of the equation

$$\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}+\frac{z^2}{c^2+\lambda}=1 \quad\ldots\ldots(4),$$

these equations expressing the condition that the two ellipsoidal surfaces are confocal.

To complete the analytical expression remark that

$$\frac{\varpi x}{\alpha^2},\ \frac{\varpi y}{\beta^2},\ \frac{\varpi z}{\gamma^2} \quad\ldots\ldots(5)$$

are the direction-cosines of the line of the resultant force.

Potential of an elliptic homoeoid at any point external or internal found.

525. To find the potential at any point remark that the difference of potentials at two of the external equipotential surfaces infinitely little distant from one another is (§ 486) equal to the product of the resultant force at any point into the distance between the two equipotentials in its neighbourhood. Hence, taking the potential as zero at an infinite distance (§ 485), we find by summation (a single integration) the potential at any point external to the given homoeoid. Now let

$$x\pm\tfrac{1}{2}dx,\quad y\pm\tfrac{1}{2}dy,\quad z\pm\tfrac{1}{2}dz$$

be the co-ordinates of the two points infinitely near one another, on two confocal surfaces. The distance between the two surfaces in the neighbourhood of this point is

$$\frac{\varpi x}{a^2+\lambda}dx+\frac{\varpi y}{b^2+\lambda}dy+\frac{\varpi z}{c^2+\lambda}dz \quad\ldots\ldots(6).$$

Let now the squares of the semi-axes of these surfaces be

$$a^2+\lambda\pm\tfrac{1}{2}d\lambda;\quad b^2+\lambda\pm\tfrac{1}{2}d\lambda;\quad c^2+\lambda\pm\tfrac{1}{2}d\lambda.$$

Now by differentiation of (4) we have

$$2\left(\frac{xdx}{a^2+\lambda}+\frac{ydy}{b^2+\lambda}+\frac{zdz}{c^2+\lambda}\right)$$

$$=\left\{\frac{x^2}{(a^2+\lambda)^2}+\frac{y^2}{(b^2+\lambda)^2}+\frac{z^2}{(c^2+\lambda)^2}\right\}d\lambda=\frac{d\lambda}{\varpi^2} \quad\ldots\ldots(7).$$

Hence (6) becomes $\dfrac{d\lambda}{2\varpi}$.

Potential of an elliptic homoeoid at any point external or internal found.

Hence, and by § 525 above, and by (2) of § 524 we have

$$dv = -2\pi \frac{abc}{\alpha\beta\gamma}\frac{t}{p}\,d\lambda \quad\text{.......................(8).}$$

Hence, and by (3) of § 524,

$$v = -2\pi\frac{abct}{p}\int_\infty \frac{d\lambda}{(a^2+\lambda)^{\frac{1}{2}}(b^2+\lambda)^{\frac{1}{2}}(c^2+\lambda)^{\frac{1}{2}}} \quad\text{..........(9),}$$

where ∞ denotes that the constant is so assigned as to render the value of the integral zero when $\lambda=\infty$.

Synthesis of concentric homoeoids.

526. Having now found the potential of an elliptic homoeoid, and its resultant force at any point external or internal, we can, by simple integration, find the potential and the resultant force of a homogeneous ellipsoid, or of a heterogeneous ellipsoid with, for its surfaces of equal density, similar concentric ellipsoidal surfaces. To do this we have only to divide the ellipsoid into elliptic homoeoids, and find the potential of each by (9), and the potential of the whole by summation; and again find the rectangular components of the force of each by (2) and (5); and from this by summation* the rectangular components of the required resultant.

Let abc be the semi-axes of the whole ellipsoid. Let $\theta a, \theta b, \theta c$, be the semi-axes of the middle surface of one of the interior homoeoids; and

$$(\theta\pm\tfrac{1}{2}d\theta)\,a,\quad(\theta\pm\tfrac{1}{2}d\theta)\,b,\quad(\theta\pm\tfrac{1}{2}d\theta)\,c$$

those of its outer and inner bounding surfaces. From the general definition of a homoeoid, elliptic or not, it follows immediately that $t/p=d\theta/\theta$. Let now ρ, a given function of θ, be the density of the ellipsoid in the homoeoidal stratum corresponding to θ. Hence by (9) remembering that the density there was taken as unity, and putting θa, θb, θc in place of a, b, c, we find for the potential of the homoeoid $\theta\pm\frac{1}{2}d\theta$ the following expression,

$$-2\pi abc\,\theta^2\rho d\theta\int_\infty^\lambda\frac{d\zeta}{(\theta^2a^2+\zeta)^{\frac{1}{2}}(\theta^2b^2+\zeta)^{\frac{1}{2}}(\theta^2c^2+\zeta)^{\frac{1}{2}}} \quad\text{........(10),}$$

* Chasles, "Nouvelle solution du problème de l'attraction d'un ellipsoïde hétérogène sur un point extérieur" (Liouville's *Journal*, Dec. 1840). Also W. Thomson, "On the Uniform Motion of Heat in Solid Bodies, and its connection with the Mathematical Theory of Electricity, Electrostatics and Magnetism," § 21—24. (Reprinted from *Cambridge Mathematical Journal*, Feb. 1842.)

Synthesis of concentric homoeoids.

where ζ is introduced as the variable of the definite integration, because λ is presently to be made a function of θ. Hence if V denote the potential of the whole ellipsoid, we have

$$V=-2\pi abc\int_0^1\theta^2\rho d\theta\int_\infty^\lambda\frac{d\zeta}{(\theta^2a^2+\zeta)^{\frac{1}{2}}(\theta^2b^2+\zeta)^{\frac{1}{2}}(\theta^2c^2+\zeta)^{\frac{1}{2}}} \quad\text{......(11),}$$

where λ is a function of θ given by the equation

$$\frac{x^2}{\theta^2a^2+\lambda}+\frac{y^2}{\theta^2b^2+\lambda}+\frac{z^2}{\theta^2c^2+\lambda}=1 \quad\text{.............(12).}$$

The expression (11) is simplified by introducing, instead of θ or λ, another variable λ/θ^2. Calling this u, so that

$$\lambda=\theta^2u \quad\text{.................................(13),}$$

we have by (12)

$$\theta^2=\frac{x^2}{a^2+u}+\frac{y^2}{b^2+u}+\frac{z^2}{c^2+u} \quad\text{.................(14).}$$

Potential of heterogeneous ellipsoid

By differentiation of (12) we have

$$\frac{d\lambda}{d(\theta^2)}\left[\frac{x^2}{(a^2+u)^2}+\frac{y^2}{(b^2+u)^2}+\frac{z^2}{(c^2+u)^2}\right]=-\left[\frac{a^2x^2}{(a^2+u)^2}+\frac{b^2y^2}{(b^2+u)^2}+\frac{c^2z^2}{(c^2+u)^2}\right].$$

And from (13) $du=\dfrac{1}{\theta^2}\left[\dfrac{d\lambda}{d(\theta^2)}-u\right]d(\theta^2)$.

Whence, on using (14), we find

$$-2\theta d\theta=\left[\frac{x^2}{(a^2+u)^2}+\frac{y^2}{(b^2+u)^2}+\frac{z^2}{(c^2+u)^2}\right]du.$$

Then changing the variable of integration in the function under the second integral sign in (11) from ζ to ζ/θ^2, and writing u for ζ/θ^2, we find by means of these transformations,

$$V=\pi abc\int_\infty^q\rho du\left\{\frac{x^2}{(a^2+u)^2}+\frac{y^2}{(b^2+u)^2}+\frac{z^2}{(c^2+u)^2}\right\}\int_\infty\frac{du}{(a^2+u)^{\frac{1}{2}}(b^2+u)^{\frac{1}{2}}(c^2+u)^{\frac{1}{2}}} \quad\text{.....................(15),}$$

where q is the positive root of the equation

$$\frac{x^2}{a^2+q}+\frac{y^2}{b^2+q}+\frac{z^2}{c^2+q}=1 \quad\text{..............(16).}$$

For the case of uniform density in which we may put $\rho=1$, this becomes simplified by integration by parts, thus:

$$\int_\infty^q du\,\frac{1}{(C+u)^2}\int_\infty f(u)\,du=-\frac{1}{C+q}\int_\infty^q f(u)\,du+\int_\infty^q\frac{du}{C+u}f(u)$$
$$=\frac{1}{C+q}\int_q^\infty f(u)\,du-\int_q^\infty\frac{du}{C+u}f(u).$$

Putting for C successively a^2, b^2, c^2, using the result properly in (15), and taking account of (16), and putting

$$\tfrac{4}{3}\pi\, abc = M \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(17),$$

we find

Potential of homogeneous ellipsoid.

$$V = \frac{3M}{4}\int_q^\infty \left(1 - \frac{x^2}{a^2+u} - \frac{y^2}{b^2+u} - \frac{z^2}{c^2+u}\right)\frac{du}{(a^2+u)^{\frac{1}{2}}(b^2+u)^{\frac{1}{2}}(c^2+u)^{\frac{1}{2}}} \quad \ldots\ldots\ldots(18),$$

which agrees with § 494 above.

Attraction of heterogeneous ellipsoid.

Just as we have found (15), we find from (2), (5), (13), and (14), the following expression for the x-components of the resultant force and the symmetricals for the y- and z-components:

$$X = \frac{3Mx}{2}\int_q^\infty \frac{\rho\, du}{(a^2+u)^{\frac{3}{2}}(b^2+u)^{\frac{1}{2}}(c^2+u)^{\frac{1}{2}}} \quad \ldots\ldots\ldots\ldots\ldots(19),$$

where ρ, a function of θ, is reduced to a function of u by (14).

For the case of a homogeneous ellipsoid ($\rho = 1$), these results become (20) and (21) of § 494. As there they were for external points deduced by aid of Maclaurin's theorem from the attraction of an ellipsoid on a point at its surface, so now when proved otherwise they contain a proof of Maclaurin's theorem. This we see in a moment by putting $u = w + q$ in the integrals, which makes the limits $w = 0$ and $w = \infty$.

527. In the case of a homogeneous ellipsoid of revolution the integrals expressing the potential and the force-components (which for a homogeneous ellipsoid, in general, are elliptic integrals) are reduced to algebraic and trigonometrical forms, thus: let $b = c$ and $z = 0$.

We have

$$V = \frac{3M}{4}\int_q^\infty \frac{du}{(b^2+u)(a^2+u)^{\frac{1}{2}}} - \tfrac{1}{2}(Xx + Yy) \quad \ldots\ldots\ldots(20),$$

$$\left.\begin{aligned} X &= \frac{3M}{2}\, x\int_q^\infty \frac{du}{(b^2+u)(a^2+u)^{\frac{3}{2}}} \\ Y &= \frac{3M}{2}\, y\int_q^\infty \frac{du}{(b^2+u)^2(a^2+u)^{\frac{1}{2}}} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(21).$$

To reduce these put

$$b^2 + u = \frac{b^2 - a^2}{\xi^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(22):$$

which reduces the three integrals to $2/(b^2-a^2)^{\frac{1}{2}} . \int d\xi/(1-\xi^2)^{\frac{1}{2}}$, $2/(b^2-a^2)^{\frac{3}{2}} . \int \xi^2 d\xi/(1-\xi^2)^{\frac{3}{2}}$, and $2/(b^2-a^2)^{\frac{3}{2}} . \int \xi^2 d\xi/(1-\xi^2)^{\frac{1}{2}}$; and makes the limits in each of them

$$\xi = 0 \text{ to } \xi = \sqrt{\frac{b^2-a^2}{b^2+q}}.$$

We thus find

Potential and attraction of homogeneous ellipsoid of revolution:

$$V = \frac{3M}{2(b^2-a^2)^{\frac{1}{2}}}\tan^{-1}\sqrt{\frac{b^2-a^2}{a^2+q}} - \tfrac{1}{2}(Xx + Yy) \quad \ldots\ldots\ldots\ldots\ldots(23),$$

oblate:

$$\left.\begin{aligned} X &= \frac{3Mx}{(b^2-a^2)^{\frac{3}{2}}}\left\{\sqrt{\frac{b^2-a^2}{a^2+q}} - \tan^{-1}\sqrt{\frac{b^2-a^2}{a^2+q}}\right\} \\ Y &= \frac{3My}{2(b^2-a^2)^{\frac{3}{2}}}\left\{\tan^{-1}\sqrt{\frac{b^2-a^2}{a^2+q}} - \frac{(b^2-a^2)^{\frac{1}{2}}(a^2+q)^{\frac{1}{2}}}{b^2+q}\right\} \end{aligned}\right\} \ldots(24),$$

where, for any external point, q is the positive root of the equation

$$\frac{x^2}{a^2+q} + \frac{y^2}{b^2+q} = 1 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(25),$$

x and y denoting the co-ordinates of the attracted point respectively along and perpendicular to the axis of revolution, and for any internal point or for points on the surface $q = 0$.

Formulas (23) and (24) realized for the case of $a > b$ become

prolate.

$$V = \frac{3M}{2(a^2-b^2)^{\frac{1}{2}}}\log\frac{\sqrt{(a^2-b^2)} + \sqrt{(a^2+q)}}{\sqrt{(b^2+q)}} - \tfrac{1}{2}(Xx + Yy) \ldots\ldots(26),$$

$$\left.\begin{aligned} X &= \frac{3Mx}{(a^2-b^2)^{\frac{3}{2}}}\left\{\log\frac{\sqrt{(a^2-b^2)} + \sqrt{(a^2+q)}}{\sqrt{(b^2+q)}} - \sqrt{\frac{a^2-b^2}{a^2+q}}\right\} \\ Y &= \frac{3My}{2(a^2-b^2)^{\frac{3}{2}}}\left\{\frac{(a^2-b^2)^{\frac{1}{2}}(a^2+q)^{\frac{1}{2}}}{b^2+q} - \log\frac{\sqrt{(a^2-b^2)} + \sqrt{(a^2+q)}}{\sqrt{(b^2+q)}}\right\} \end{aligned}\right\} (27).$$

The structure of these expressions (23), (24), (26), (27), is elucidated, and calculation of results from them is facilitated by taking

$$f = \sqrt{\frac{b^2-a^2}{a^2+q}}, \text{ and } \sqrt{(b^2-a^2)} = r \ldots\ldots\ldots\ldots(28),$$

and again

$$e = \sqrt{\frac{a^2-b^2}{a^2+q}}, \text{ and } \sqrt{(a^2-b^2)} = s \ldots\ldots\ldots\ldots(29);$$

prolate. which reduces them to the following alternative forms:—

$$V = \frac{3M}{2r}\tan^{-1}f - \tfrac{1}{2}(Xx+Yy) = \frac{3M}{2s}\log\sqrt{\frac{1+e}{1-e}} - \tfrac{1}{2}(Xx+Yy) \ldots (30),$$

$$\left.\begin{aligned} X &= \frac{3Mx}{r^3}(f - \tan^{-1}f) = \frac{3Mx}{s^3}\left(e - \log\sqrt{\frac{1+e}{1-e}}\right) \\ Y &= \frac{3My}{2r^3}\left(\tan^{-1}f - \frac{f}{1+f^2}\right) = -\frac{3My}{2s^3}\left(\frac{e}{1-e^2} - \log\sqrt{\frac{1+e}{1-e}}\right)\end{aligned}\right\}\ldots\ldots(31).$$

Then, for determining f or e, in the case of an external point, (25) becomes

$$f^2\left(x^2 + \frac{y^2}{1+f^2}\right) = r^2, \text{ and } e^2\left(x^2+\frac{y^2}{1-e^2}\right) = s^2 \ldots\ldots\ldots(32).$$

In the case of an internal point we have

$$f = \sqrt{\frac{b^2-a^2}{a^2}},\quad e = \sqrt{\frac{a^2-b^2}{a^2}} \ldots\ldots\ldots\ldots\ldots\ldots(33).$$

528. The investigation of the attraction of an ellipsoid which was most popular in England 40 to 50 years ago resembled that of § 494 above, in finding the attraction of an internal point by direct integration, substantially the same as that of § 494, and deducing from the result the attraction of an external point by a special theorem.

Third investigation of the attraction of an ellipsoid.

But the theorem then popularly used for the purpose was not Maclaurin's theorem, which was little known, strange to say, in England at that time; it was Ivory's theorem, much less beautiful and simple and directly suitable for the purpose than Maclaurin's, but still a very remarkable theorem, curiously different from Maclaurin's, and in one respect more important and comprehensive, because, as was shown by Poisson, it is not confined to the Newtonian Law of Attraction, but holds for force varying as any function of the distance. Before enunciating Ivory's theorem, take his following definition:—

Corresponding points on confocal ellipsoids defined.

529. Corresponding points on two confocal ellipsoids are any two points which coincide when either ellipsoid is deformed by a pure strain so as to coincide with the other.

Digression; orthogonal trajectory of confocal ellipsoids is traced by any point of a confocally distorted solid ellipsoid:

In connection with this definition, it is interesting to remark that each point on the surface of the changing ellipsoid describes an orthogonal trajectory of the intermediate series of confocal ellipsoids if the distortion specified in the definition is produced continuously in such a manner that the surface of the ellipsoid is always confocal with its original figure.

proof.

To prove this proposition, which however is not necessary for our present purpose, let abc be the semi-axes of the ellipsoid in one configuration, and $\sqrt{(a^2+h)}$, $\sqrt{(b^2+h)}$, $\sqrt{(c^2+h)}$ in another. If xyz be the co-ordinates of any point P on the surface in the first configuration, its co-ordinates in the second configuration will be

$$x\frac{\sqrt{(a^2+h)}}{a},\quad y\frac{\sqrt{(b^2+h)}}{b},\quad z\frac{\sqrt{(c^2+h)}}{c} \ldots\ldots\ldots\ldots(32).$$

When h is infinitely small the differences of the co-ordinates of these points are

$$\tfrac{1}{2}h\frac{x}{a^2},\quad \tfrac{1}{2}h\frac{y}{b^2},\quad \tfrac{1}{2}h\frac{z}{c^2}.$$

Hence the direction-cosines of the line joining them are proportional to x/a^2, y/b^2, z/c^2, and therefore it coincides with the normal to the two infinitely nearly coincident surfaces.

Ivory's Lemma on corresponding points.

530. The property of corresponding points (essential for Ivory's theorem, and for Chasles', § 532 below) is this:—

If P, P' be any two points on one ellipsoid, and Q, Q' the corresponding points on any confocal ellipsoid, PQ' is equal to $P'Q$.

To prove this, let xyz be the co-ordinates of P, and $x'y'z'$ those of P'. Taking (32) as the co-ordinates of Q, we find

$$P'Q^2 = \left(x' - x\sqrt{\frac{a^2+h}{a^2}}\right)^2 + \left(y'-y\sqrt{\frac{b^2+h}{b^2}}\right)^2 + \left(z'-z\sqrt{\frac{c^2+h}{c^2}}\right)$$

$$= x'^2 - 2xx'\sqrt{\frac{a^2+h}{a^2}} + x^2\left(1+\frac{h}{a^2}\right) + \&c.$$

Now because (x, y, z) is on the ellipsoidal surface (a, b, c), we have

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} = 1.$$

Hence the preceding becomes

$$P'Q^2 = x'^2+y'^2+z'^2 - 2\left(xx'\sqrt{\frac{a^2+h}{a^2}} + yy'\sqrt{\frac{b^2+h}{b^2}} + zz'\sqrt{\frac{c^2+h}{c^2}}\right) + x^2+y^2+z^2+h.$$

This is symmetrical in respect to xyz and $x'y'z'$, and so the proposition is proved.

Ivory's theorem,

531. The following is Ivory's Theorem:—Let P' and P be corresponding points on the surfaces of two homogeneous confocal ellipsoids (a, b, c) (a', b', c'); the x-component of the attraction of the ellipsoid abc on the point P is to the x-component of the attraction of the ellipsoid $a'b'c'$ on the point P' as bc is to $b'c'$.

proved.

Let x, y, z be the co-ordinates of P, the attracted point;
" ξ, η, ζ " co-ordinates of any point of the mass;
" D " distance between the two points;
" $F(D)\, d\xi d\eta d\zeta$ be the attraction of the elemental mass $d\xi d\eta d\zeta$ at (ξ, η, ζ), on (x, y, z);

Let X be the x-component of the attraction of the whole ellipsoid (a, b, c) on (x, y, z).

We have

$$X = \iiint d\xi d\eta d\zeta F(D) \frac{x-\xi}{D} = \iiint d\xi d\eta d\zeta F(D) \times \left(-\frac{dD}{d\xi}\right)$$

$$= \iint d\eta d\zeta \int -F(D)\, dD.$$

Now $F(D)$ being any function of D, let

$$\int F(D)\, dD = -\psi(D);$$

and let E, G be the positive and negative ends of the bar $d\eta d\zeta$ of the ellipsoid, that is to say, the points on the positive and negative sides of the plane yoz in which the surface of the ellipsoid is cut by the line parallel to ox, having $\eta\zeta$ for its other co-ordinates. The proper limits being assigned to the D-integration in the formula for X above being assigned, we find

$$X = \iint d\eta d\zeta \{\psi(EP) - \psi(GP)\}.$$

Now let $E'G'$ be points on a confocal ellipsoidal surface (a', b', c') through P, corresponding to E and G on the surface of the given ellipsoid (a, b, c); and let P' be the point on the first ellipsoidal surface corresponding to P on the second. The y- z- co-ordinates common to $E'G'$ are respectively $b'/b \,.\, \eta$ and $c'/c \,.\, \zeta$;

and by lemma $EP = E'P'$ and $GP = G'P'$. Hence if we change from $\eta\zeta$, as variables for the double integration in the preceding formula for X, to $\eta'\zeta'$, we find

$$X = \frac{bc}{b'c'} \iint d\eta' d\zeta' \{\psi(E'P') - \psi(G'P')\},$$

which is Ivory's theorem.

532. Two confocal homoeoids of equal masses being given, the potential of the first at any point, P, of the surface of the second, is equal to that of the second at the corresponding point, P', on the surface of the first.

Chasles' comparison between the potentials of two confocal homoeoids.

Let E be any element of the first and E' the corresponding element of the second. The mass of each element bears to the mass of the whole homoeoid the same ratio as the mass of the corresponding element of a uniform spherical shell, from which either homoeoid may be derived, bears to the whole mass of the spherical shell. Hence the mass of E is equal to the mass of E'; and by Ivory's lemma (§ 530) $PE = P'E$. Hence the proposition is true for the parts of the potential due to the corresponding elements, and therefore it is due for the entire shells.

This beautiful proposition is due to Chasles. It holds, whatever be the law of force. From it, for the case of the inverse square of the distance, and from Newton's Theorem for this case that the force is zero within an elliptic homoeoid, or, which is the same, that the potential is constant through the interior, it follows that the external equipotential surfaces of an elliptic homoeoid are confocal ellipsoids, and therefore that the attraction on an external point is normal to a confocal ellipsoid passing through the point; which is the same conclusion as that of § 521 above.

Proof of Poisson's theorem regarding attraction of elliptic homoeoid.

533. An ingenious application of Ivory's theorem, by Duhamel, must not be omitted here. Concentric spheres are a particular case of confocal ellipsoids, and therefore the attraction of any sphere on a point on the surface of an internal concentric sphere, is to that of the latter upon a point in the surface of the former as the squares of the radii of the spheres. Now *if the law of attraction be such that a homogeneous spherical*

Law of attraction when a uniform spherical shell exerts no action on an internal point.

Law of attraction when a uniform spherical shell exerts no action on an internal point.

shell of uniform thickness exerts no attraction on an internal point, the action of the larger sphere on the internal point is reduced to that of the smaller. Hence the smaller sphere attracts points on its surface and points external to it, with forces inversely as the squares of their distances from its centre. Hence *the law of force is the inverse square of the distance,* as is easily seen by making the smaller sphere less and less till it becomes a mere particle. This theorem is due originally to Cavendish.

Cavendish's theorem.

Centre of gravity.

534. (*Definition.*) If the action of terrestrial or other gravity on a rigid body is reducible to a single force in a line passing always through one point fixed relatively to the body, whatever be its position relatively to the earth or other attracting mass, that point is called its *centre of gravity,* and the body is called a *centrobaric body.*

Centrobaric bodies, proved possible by Green.

One of the most startling results of Green's wonderful theory of the potential is its establishment of the existence of centrobaric bodies; and the discovery of their properties is not the least curious and interesting among its very various applications.

Properties of centrobaric bodies.

534 *a.* If a body (B) is centrobaric relatively to any one attracting mass (A), it is centrobaric relatively to every other: and it attracts all matter external to itself as if its own mass were collected in its centre of gravity *.

Let O be any point so distant from B that a spherical surface described from it as centre, and not containing any part of B, is large enough entirely to contain A. Let A be placed within any such spherical surface and made to rotate about any axis, OK, through O. It will always attract B in a line through G, the centre of gravity of B. Hence if every particle of its mass be uniformly distributed over the circumference of the circle that it describes in this rotation, the mass, thus obtained, will also attract B in a line through G. And this will be the case however this mass is rotated round O; since before obtaining it we might have rotated A and OK in any way round O, holding them fixed relatively to one another. We have therefore found a body, A', symmetrical about an axis, OK, relatively to which B is necessarily centrobaric. Now, O being kept fixed, let OK, carrying A' with it, be put successively into an infinite number, n, of positions uniformly distributed round O; that is to say, so that there are equal numbers of positions of OK in all equal solid angles round O: and let $\frac{1}{n}$ part of the mass of A' be left in each of the positions into which it was thus necessarily carried. B will experience from all this distribution of matter, still a resultant force through G. But this distribution, being symmetrical all round O, consists of uniform concentric shells, and (§ 471) the mass of each of these shells might be collected at O without changing its attraction on any particle of B, and therefore without changing its resultant attraction on B. Hence B is centrobaric relatively to a mass collected at O; this being any point whatever not nearer than within a certain limiting distance from B (according to the condition stated above). That is to say, any point placed beyond this distance is attracted by B in a line through G; and hence, beyond this distance, the equipotential surfaces of B are spherical with G for common centre. B therefore attracts points beyond this distance as if its mass were collected at G: and it follows (§ 497) that it does so also through the whole space external to itself. Hence it attracts any group of points, or any mass whatever, external to it, as if its own mass were collected at G.

* Thomson, *Proc. R. S. E.*, Feb. 1864.

Properties of centrobaric bodies.

534 *b.* Hence §§ 497, 492 show that—

(1) *The centre of gravity of a centrobaric body necessarily lies in its interior;* or in other words, *can only be reached from external space by a path cutting through some of its mass.* And

(2) *No centrobaric body can consist of parts isolated from one another, each in space external to all:* in other words, *the outer boundary of every centrobaric body is a single closed surface.*

Thus we see, by (1), that no symmetrical ring, or hollow cylinder with open ends, can have a centre of gravity; for its

Properties of centrobaric bodies. centre of gravity, if it had one, would be in its axis, and therefore external to its mass.

534 *c*. *If any mass whatever, M, and any single surface, S, completely enclosing it be given, a distribution of any given amount, M', of matter on this surface may be found which shall make the whole centrobaric with its centre of gravity in any given position (G) within that surface.*

The condition here to be fulfilled is to distribute M' over S, so as by it to produce the potential

$$\frac{M+M'}{EG} - V,$$

any point, E, of S; V denoting the potential of M at this point. The possibility and singleness of the solution of this problem were proved above (§ 499). It is to be remarked, however, that if M' be not given in sufficient amount, an extra quantity must be taken, but neutralized by an equal quantity of negative matter, to constitute the required distribution on S.

The case in which there is no given body M to begin with is important; and yields the following :—

Centrobaric shell. **534 *d*.** *A given quantity of matter may be distributed in one way, but in only one way, over any given closed surface, so as to constitute a centrobaric body with its centre of gravity at any given point within it.*

Thus we have already seen that the condition is fulfilled by making the density inversely as the cube of the distance from the given point, if the surface be spherical. From what was proved in §§ 501, 506 above, it appears also that a centrobaric shell may be made of either half of the lemniscate in the diagram of § 508, or of any of the ovals within it, by distributing matter with density proportional to the resultant force of m at I and m' at I'; and that the one of these points which is within it is its centre of gravity. And generally, by drawing the equipotential surfaces relatively to a mass m collected at a point I, and any other distribution of matter whatever not surrounding this point; and by taking one of these surfaces which encloses I but no other part of the mass, we learn, by

Green's general theorem, and the special proposition of § 506, how to distribute matter over it so as to make it a centrobaric shell with I for centre of gravity. **Centrobaric shell.**

534 *e*. Under *hydrokinetics* the same problem will be solved for a cube, or a rectangular parallelepiped in general, in terms of converging series; and under *electricity* (in a subsequent volume) it will be solved in finite algebraic terms for the surface of a lens bounded by two spherical surfaces cutting one another at any sub-multiple of two right angles, and for either part obtained by dividing this surface in two by a third spherical surface cutting each of its sides at right angles.

534 *f*. *Matter may be distributed in an infinite number of ways throughout a given closed space, to constitute a centrobaric body with its centre of gravity at any given point within it.* **Centrobaric solid.**

For by an infinite number of surfaces, each enclosing the given point, the whole space between this point and the given closed surface may be divided into infinitely thin shells; and matter may be distributed on each of these so as to make it centrobaric with its centre of gravity at the given point. Both the forms of these shells and the quantities of matter distributed on them, may be arbitrarily varied in an infinite variety of ways.

Thus, for example, if the given closed surface be the pointed oval constituted by either half of the lemniscate of the diagram of § 508, and if the given point be the point I within it, a centrobaric solid may be built up of the interior ovals with matter distributed over them to make them centrobaric shells as above (§ 534*d*). From what was proved in § 518, we see that a solid sphere, with its density varying inversely as the fifth power of the distance from an external point, is centrobaric, and that its centre of gravity is the *image* (§ 512) of this point relatively to its surface. **Properties of centrobaric bodies.**

534 *g*. The centre of gravity of a centrobaric body composed of true gravitating matter is its centre of inertia. For a centrobaric body, if attracted only by another infinitely distant body, or by matter so distributed round itself as to produce (§ 499) **The centre of gravity (if it exist) is the centre of inertia.**

The centre of gravity (if it exist) is the centre of inertia.

uniform force in parallel lines throughout the space occupied by it, experiences (§ 534*a*) a resultant force always through its centre of gravity. But in this case this force is the resultant of parallel forces on all the particles of the body, which (see *Properties of Matter*, below) are rigorously proportional to their masses: and in § 561 it is proved that the resultant of such a system of parallel forces passes through the point defined in § 230, as the centre of inertia.

A centrobaric body is kinetically symmetrical about its centre of gravity.

535. The moments of inertia of a centrobaric body are equal round all axes through its centre of inertia. In other words (§ 285), all these axes are principal axes, and the body is kinetically symmetrical round its centre of inertia.

Let it be placed with its centre of inertia at a point O (origin of co-ordinates), within a closed surface having matter so distributed over it (§ 499) as to have xyz [which satisfies $\nabla^2(xyz)=0$] for potential at any point (x, y, z) within it. The resultant action on the body is (§ 534*a*) the same as if it were collected at O; that is to say, zero: or, in other words, the forces on its different parts must balance. Hence (§ 551, I., below) if ρ be the density of the body at (x, y, z)

$$\iiint yz\rho dxdydz = 0, \qquad \iiint zx\rho dxdydz = 0, \qquad \iiint xy\rho dxdydz = 0.$$

Hence OX, OY, OZ are principal axes; and this, however the body is turned, only provided its centre of gravity is kept at O.

To prove this otherwise, let V denote the potential of the given body at (x, y, z); u any function of x, y, z; and ϖ the triple integral

$$\iiint\left(\frac{du}{dx}\frac{dV}{dx}+\frac{du}{dy}\frac{dV}{dy}+\frac{du}{dz}\frac{dV}{dz}\right)dxdydz,$$

extended through the interior of a spherical surface, S, enclosing all of the given body, and having for centre its centre of gravity. Then, as in Chap. I. App. A, we have

$$\varpi = \iint \partial u V d\sigma - \iiint V\nabla^2 u dxdydz$$
$$= \iint \partial V u d\sigma - \iiint u\nabla^2 V dxdydz.$$

Properties of centrobaric bodies.

But if m be the whole mass of the given body, and a the radius of S, we have, over the whole surface of S,

$$V = \frac{m}{a}, \text{ and } \partial V = -\frac{m}{a^2}.$$

Also [§ 491 *c*] $\qquad \nabla^2 V = -4\pi\rho,$

vanishing of course for all points not belonging to the mass of the given body. Hence from the preceding we have

$$4\pi\iiint u\rho dxdydz = \frac{m}{a^2}\iint(a\partial u + u)\,d\sigma - \iiint V\nabla^2 u dxdydz.$$

Let now u be any function fulfilling $\nabla^2 u = 0$ through the whole space within S; so that, by § 492, we have $\iint \partial u d\sigma = 0$, and by § 496, $\iint u d\sigma = 4\pi a^2 u_0$, if u_0 denote the value of u at the centre of S. Hence

$$\iiint u\rho dxdydz = mu_0.$$

Let, for instance, $u = yz$. We have $u_0 = 0$, and therefore

$$\iiint yz\rho dxdydz = 0,$$

as we found above. Or let $u = (x^2+y^2)-(x^2+z^2)$, which gives $u_0 = 0$; and consequently proves that

$$\iiint (x^2+z^2)\,\rho dxdydz = \iiint (x^2+y^2)\,\rho dxdydz,$$

or the moment of inertia round OY is equal to that round OX, verifying the conclusion inferred from the other result.

Origin of spherical harmonic analysis of Legendre and Laplace.

536. The *spherical harmonic analysis*, which forms the subject of an Appendix to Chapter I., had its origin in the theory of attraction, treated with a view especially to the figure of the earth; having been first invented by Legendre and Laplace for the sake of expressing in converging series the attraction of a body of nearly spherical figure. It is also perfectly appropriate for expressing the potential, or the attraction, of an infinitely thin spherical shell, with matter distributed over it according to any arbitrary law. This we shall take first, being the simpler application.

Origin of spherical harmonic analysis of Legendre and Laplace.

Let x, y, z be the co-ordinates of P, the point in question, reckoned from O the centre, as origin of co-ordinates: ρ and ρ' the values of the density of the spherical surface at points E and E', of which the former is the point in which it is cut by OP, or this line produced: $d\sigma'$ an element of the surface at E', a its radius. Then, V being the potential at P, we have

$$V = \iint \frac{\rho' d\sigma'}{E'P} \qquad \ldots\ldots(1).$$

But, by B (48)

$$\left.\begin{aligned} \frac{1}{E'P} &= \frac{1}{a}\left\{1 + \sum_1^\infty Q_i \left(\frac{r}{a}\right)^i\right\} \text{ when } P \text{ is internal,} \\ \text{and} \quad &= \frac{1}{r}\left\{1 + \sum_1^\infty Q_i \left(\frac{a}{r}\right)^i\right\} \quad ,, \quad ,, \text{ external,} \end{aligned}\right\} \ldots\ldots(2)$$

where Q_i is the biaxal surface harmonic of (E, E'). Hence, if

$$\rho' = S_0 + S_1 + S_2 + \&c. \qquad \ldots\ldots(3)$$

be the harmonic expansion for ρ, we have, according to B (52),

$$\left.\begin{aligned} V &= 4\pi a \left\{\sum_0^\infty \frac{S_i}{2i+1}\left(\frac{r}{a}\right)^i\right\} \text{ when } P \text{ is internal,} \\ \text{and} \quad &= \frac{4\pi a^2}{r}\left\{\sum_0^\infty \frac{S_i}{2i+1}\left(\frac{a}{r}\right)^i\right\} \quad ,, \quad ,, \text{ external,} \end{aligned}\right\} \ldots\ldots(4).$$

If, for instance, $\rho = S_i$, we have

$$V = \frac{4\pi r^i}{a^{i-1}} \frac{S_i}{2i+1} \text{ inside,}$$

and

$$V = \frac{4\pi a^{i+2}}{r^{i+1}} \frac{S_i}{2i+1} \text{ outside.}$$

Thus we conclude that

Application of spherical harmonic analysis.

537. A spherical harmonic distribution of density on a spherical surface produces a similar and similarly placed spherical harmonic distribution of potential over every concentric spherical surface through space, external and internal; and so also consequently of radial component force. But the amount of the latter differs, of course (§ 478), by $4\pi\rho$, for points infinitely near one another outside and inside the surface, if ρ

denote the density of the distribution on the surface between them.

Application of spherical harmonic analysis.

If R denote the radial component of the force, we have

$$\left.\begin{aligned} R &= -\frac{dV}{dr} = -\frac{4\pi r^{i-1}}{a^{i-1}} \frac{iS_i}{2i+1} \text{ inside,} \\ \text{and} \quad &= \frac{4\pi a^{i+2}}{r^{i+2}} \frac{(i+1)S_i}{2i+1} \text{ outside,} \end{aligned}\right\} \ldots\ldots(5).$$

Hence, if $r = a$, we have

$$R\text{ (outside)} - R\text{ (inside)} = 4\pi S_i = 4\pi\rho.$$

538. The potential is of course a solid harmonic through space, both internal and external; and is of positive degree in the internal, and of negative in the external space. The expression for the radial component of the force, in each division of space, is reduced to the same form by multiplying it by the distance from the centre.

539. The harmonic development gives an expression in converging series, for the potential of any distribution of matter through space, which is useful in some applications.

Let x, y, z be the co-ordinates of P, the attracted point, and x', y', z' those of P' any point of the given mass. Then, if ρ' be the density of the matter at P', and V the potential at P, we have

$$V = \iiint \frac{\rho' dx' dy' dz'}{[(x-x')^2 + (y-y')^2 + (z-z')^2]^{\frac{1}{2}}} \qquad \ldots\ldots(6).$$

The most convenient view we can take as to the space through which the integration is to be extended is to regard it as infinite in all directions, and to suppose ρ' to be a discontinuous function of x', y', z', vanishing through all space unoccupied by matter.

Now by App. B. (u) we have

$$\left.\begin{aligned} \frac{1}{[(x-x')^2 + (y-y')^2 + (z-z')^2]^{\frac{1}{2}}} &= \frac{1}{r'}\left\{1 + \sum_1^\infty Q_i \left(\frac{r}{r'}\right)^i\right\} \text{ when } r' > r \\ \text{and} \qquad &= 1\left\{1 + \sum_1^\infty Q_i \left(\frac{r'}{r}\right)\right\} \quad ,, \quad r' < r \end{aligned}\right\} \ldots(7).$$

Application of spherical harmonic analysis.

Substituting this in (6) we have

$$V = (\iiint) \frac{\rho' dx' dy' dz'}{r'} + \frac{1}{r} [\iiint] \rho' dx' dy' dz'$$
$$+ \sum_1^\infty \left\{ r^i (\iiint) Q_i \frac{\rho' dx' dy' dz'}{r'^{i+1}} + \frac{1}{r^{i+1}} [\iiint] Q_i r'^i \rho' dx' dy' dz' \right\} \ldots (8),$$

where $(\iiint)$ denotes integration through all the space external to the spherical surface of radius r, and $[\iiint]$ integration through the interior space.

Potential of a distant body.

This formula is useful for expressing the attraction of a mass of any figure on a distant point in a single converging series. Thus when OP is greater than the greatest distance of any part of the body from O, the first series disappears, and the expression becomes a single converging series, in ascending powers of $\frac{1}{r}$:—

$$V = \frac{1}{r} \{ \iiint \rho' dx' dy' dz' + \Sigma \frac{1}{r^i} \iiint Q_i r'^i \rho' dx' dy' dz' \} \ldots\ldots\ldots (9).$$

If we use the notation of B. (u) (53), this becomes

$$V = \frac{1}{r} \left\{ \iiint \rho' dx' dy' dz' + \sum_1^\infty r^{-2i} \iiint \rho' H_i [(x, y, z), (x', y', z')] dx' dy' dz' \right\} \ldots (10),$$

and we have, by App. B. (v') and (w),

$$H_i[(x, y, z), (x', y', z')] = \frac{1.3.5\ldots(2i-1)}{1.2.3\ldots i} [\cos^i\theta - \frac{i(i-1)}{2.(2i-1)} \cos^{i-2}\theta + \frac{i(i-1)(i-2)(i-3)}{2.4.(2i-1)(2i-3)} \cos^{i-4}\theta - \text{etc.}] r^i r'^i \;(11)$$

where $\qquad \cos\theta = \frac{xx' + yy' + zz'}{rr'}.$

From this we find

$$H_1 = xx' + yy' + zz'; \; H_2 = \tfrac{3}{2}[(xx' + yy' + zz')^2 - \tfrac{1}{3}(x^2 + y^2 + z^2)(x'^2 + y'^2 + z'^2)];$$

and so on.

Let now M denote the mass of the body; and let O be taken at its centre of gravity. We shall have

$$\iiint \rho' dx' dy' dz' = M; \text{ and } \iiint \rho' H_1 dx' dy' dz' = 0.$$

Further, let OX, OY, OZ be taken as principal axes (§§ 281, 282), so that $\qquad \iiint \rho' y' z' dx' dy' dz' = 0$, etc.,

and let A, B, C be the moments of inertia round these axes. This will give

$$\iiint H_2 \rho' dx' dy' dz' = \tfrac{1}{2}\{(3x^2 - r^2)\iiint \rho' x'^2 dx' dy' dz' + \text{etc.}\} = \tfrac{1}{2}\{(3x^2 - r^2)[\tfrac{1}{2}(A + B + C) - A] + \text{etc.}\}$$
$$= \tfrac{1}{2}\} A(r^2 - 3x^2) + B(r^2 - 3y^2) C + (r^2 - 3z^2)\} = \tfrac{1}{2}\{(B + C - 2A)x^2 + (C + A - 2B)y^2 + (A + B - 2C)z^2\}.$$

Potential of a distant body.

Hence neglecting terms of the third and higher orders of small quantities $\left(\text{powers of } \frac{r'}{r}\right)$, we have the following approximate expression for the potential:—

$$V = \frac{M}{r} + \frac{1}{2r^5}\{(B + C - 2A)x^2 + (C + A - 2B)y^2 + (A + B - 2C)z^2\} \ldots (12).$$

As one example of the usefulness of this result, we may mention the investigation of the disturbance in the moon's motion produced by the non-sphericity of the earth, and of the reaction of the same disturbing force on the earth, causing *lunar nutation and precession*, which will be explained later.

Differentiating, and retaining only terms of the first and second degrees of approximation, we have for the components of the mutual force between the body and a unit particle at (x, y, z),

$$\left. \begin{array}{c} X = \frac{Mx}{r^3} - \frac{(B + C - 2A)x}{r^5} + \frac{5}{2}\frac{x}{r^7}[(B + C - 2A)x^2 + (C + A - 2B)y^2 + (A + B - 2C)z^2] \\ Y = \text{etc.}, \qquad Z = \text{etc.} \end{array} \right\} (13);$$

whence

$$Zy - Yz = 3\frac{(C - B)yz}{r^5}, \; Xz - Zx = 3\frac{(A - C)zx}{r^5}, \; Yx - Xy = 3\frac{(B - A)xy}{r^5} \ldots (14).$$

Comparing these with Chap. IX. below, we conclude that

Attraction of a particle on a distant body.

540. The attraction of a distant particle, P, on a rigid body if transferred (according to Poinsot's method explained below, § 555) to the centre of inertia, I, of the latter, gives a couple approximately equal and opposite to that which constitutes the resultant effect of centrifugal force, if the body rotates with a certain angular velocity about IP. The square of this angular velocity is inversely as the cube of the distance of P, irrespectively of its direction; being numerically equal to three times the reciprocal of the cube of this distance, if the unit of mass is such as to exercise the proper kinetic unit (§ 225) force on another equal mass at unit distance. The general tendency of the gravitation couple is to bring the principal axis of least moment of inertia into line with the attracting point. The expressions for its components round the principal axes will be used in Chap. IX. (§ 825) for the investigation of the phenomena of precession and nutation produced, in virtue of

Attraction of a particle on a distant body.

the earth's non-sphericity, by the attractions of the sun and moon. They are available to estimate the retardation produced by tidal friction against the earth's rotation, according to the principle explained above (§ 276).

541. It appears from what we have seen that the amount of the gravitation couple is inversely as the cube of the distance between the centre of inertia and the external attracting point: and therefore that the shortest distance of the line of the resultant force from the centre of inertia varies inversely as the distance of the attracting point. We thus see *how* to a first approximation every rigid body is centrobaric relatively to a distant attracting point.

Principle of the approximation used in the common theory of the centre of gravity.

542. The real meaning and value of the spherical harmonic method for a solid mass will be best understood by considering the following application:—

Let $$\rho = F(r)\, S_i \qquad\qquad (15)$$

where $F(r)$ denotes any function of r, and S_i a surface spherical harmonic function of order i, with coefficients independent of r. Substituting accordingly for ρ' in (8), and attending to B. (52) and (16), we find

$$V = \frac{4\pi S_i}{2i+1}\left\{r^i \int_r^\infty r'^{-i+1} F(r')\,dr' + r^{-i-1}\int_0^r r'^{i+2} F(r')\,dr'\right\} \quad (16).$$

Potential of solid sphere with harmonic distribution of density.

543. As an example, let it be required to find the potential of a solid sphere of radius a, having matter distributed through it according to solid harmonic function V_i.

That is to say, let

$$\rho = V = r^i S, \text{ when } r < a,$$

and $$\rho' = 0 \qquad ,, \quad r > a.$$

Hence in the preceding formula $F(r) = r^i$ from $r = 0$ to $r = a$, and $F(r) = 0$, when $r > a$; and it becomes

$$\left.\begin{aligned} V &= 4\pi V_i \left\{\frac{a^2}{2(2i+1)} - \frac{r^2}{2(2i+3)}\right\} \text{ when } P \text{ is internal,} \\ \text{and} \quad &= \frac{4\pi}{(2i+1)(2i+3)} \frac{a^{2i+3} V_i}{r^{2i+1}} \qquad ,, \quad ,, \text{ external.} \end{aligned}\right\} (17).$$

This result may also be obtained by the aid of the algebraical

formula B. (12) thus, on the same principle as the potential of a uniform spherical shell was found in § 491 (d).

Potential of solid sphere with harmonic distribution of density.

We have by § 491 (c)

$$\left.\begin{aligned} \nabla^2 V &= -4\pi V_i, \text{ when } r < a, \\ \text{and} \quad &= 0 \qquad ,, \quad r > a. \end{aligned}\right\} \qquad (18).$$

But by taking $m = 2$ in B. (12) we have

$$\nabla^2 (r^2 V_i) = 2(2i+3) V_i,$$

and therefore the solution of the equation

$$\nabla^2 V = -4\pi V_i$$

is $$V = -4\pi \frac{r^2 V_i}{2(2i+3)} + U \qquad (19),$$

where U is any function whatever satisfying the equation

$$\nabla^2 U = 0$$

through the whole interior of the sphere. By choosing U and the external values of V so as to make the values of V equal to one another for points infinitely near one another outside and inside the bounding surface, to fulfil the same condition for $\frac{dV}{dr}$, and to make V vanish when $r = \infty$, and when $r = 0$, we find

$$U = 4\pi V_i \frac{a^2}{2(2i+1)},$$

and obtain the expression of (17) for V external. For in the first place, V external and U must clearly be $A\frac{V_i}{r^{2i+1}}$, and BV_i, where A and B are constants: and the two conditions give the equations to determine them.

544. From App. B. (52) it follows immediately that any function of x, y, z whatever may be expressed, through the whole of space, in a series of surface harmonic functions, each having its coefficients functions of the distance (r) from the origin. Hence (16), with S_i placed under the sign of integration for r', gives the harmonic development of the potential of any mass whatever; being the result of the triple integrations indicated in (8) of § 539, when the mass is specified by means of a harmonic series expressing the density.

Potential of any mass, in harmonic series.

Application to figure of the earth.

545. The most important application of the harmonic development for solid spheres hitherto made is for investigating, in the Theory of the Figure of the Earth, the attraction of a finite mass consisting of approximately spherical layers of matter equally dense through each, but varying in density from layer to layer. The result of the general analytical method explained above, when worked out in detail for this case, is to exhibit the potential as the sum of two parts, of which the first and chief is the potential due to a solid sphere, A, and the second to a spherical shell, B. The sphere, A, is obtained by reducing the given spheroid to a spherical figure by cutting away all the matter lying outside the proper mean spherical surface, and filling the space vacant inside it where the original spheroid lies within it, without altering the density anywhere. The shell, B, is a spherical surface loaded with equal quantities of positive and negative matter, so as to compensate for the transference of matter by which the given spheroid was changed into A. The analytical expression of all this may be written down immediately from the preceding formulæ (§§ 536, 537); but we reserve it until, under hydrostatics and hydrokinetics, we shall be occupied with the theory of the Figure of the Earth, and of the vibrations of liquid globes.

Case of the potential symmetrical about an axis.

546. The analytical method of spherical harmonics is very valuable for several practical problems of electricity, magnetism, and electro-magnetism, in which distributions of force symmetrical round an axis occur: especially in this; that if the force (or potential) at every point through some finite length along the axes be given, it enables us immediately to deduce converging series for calculating the force for points through some finite space not in the axes. (See § 498.)

O being any conveniently chosen point of reference, in the axis of symmetry, let us have, in series converging for a portion AB of the axis,

$$U = a_0 + \frac{b_0}{r} + a_1 r + \frac{b_1}{r^2} + a_2 r^2 + \frac{b_2}{r^3} + \text{etc.} \dots\dots\dots\dots (a),$$

where U is the potential at a point, Q, in the axis, specified by

$OQ = r$. Then if V be the potential at any point P, specified by $OP = r$ and $QOP = \theta$, and, as in App. B. (47), Q_1, Q_2, ... denote the axial surface harmonics of θ, of the successive integral orders, we must have, for all values of r for which the series converges,

Case of the potential symmetrical about an axis.

$$V = a_0 + \frac{b_0}{r} + \left(a_1 r + \frac{b_1}{r^2}\right) Q_1 + \left(a_2 r^2 + \frac{b_2}{r^3}\right) Q_2 + \text{etc.} \dots\dots\dots (b),$$

provided P can be reached from Q and all points of AB within some finite distance from it however small, without passing through any of the matter to which the force in question is due, or any space for which the series does not converge. For throughout this space (§ 498) $V - V'$ must vanish, if V' be the value of the sum of the series; since $V - V'$ is [App. B. (g)] a potential function, and it vanishes for a finite portion of the axis containing Q.

The series (b) is of course convergent for all values of r which make (a) convergent, since the ultimate ratio $Q_{i+1} \div Q_i$ for infinitely great values of i, is unity, as we see from any of the expressions for these functions in App. B.

In general, that is to say unless O be a singular point, the series for U consists, according to Maclaurin's theorem, of ascending integral powers of r only, provided r does not exceed a certain limit. In certain classes of cases there are singular points, such that if O be taken at one of them, U will be expressed in a series of powers of r with fractional indices, convergent and real for all finite positive values of r not exceeding a certain limit. The expression for the potential in the neighbourhood of O in any such case, in terms of solid spherical harmonics relatively to O as centre, will contain harmonics [App. B. (a)] of fractional degrees.

Examples. (I.) Potential of circular ring;

Examples—(I.) The potential of a circular ring of radius a, and linear density ρ, at a point in the axis, distant by r from the centre:—

$$U = \frac{2\pi a \rho}{(a^2 + r^2)^{\frac{1}{2}}} \dots\dots\dots\dots\dots\dots\dots\dots (1).$$

Hence $\quad U = 2\pi\rho \left(1 - \frac{1}{2}\frac{r^2}{a^2} + \frac{1.3}{2.4}\frac{r^4}{a^4} - \text{etc.}\right)$ when $r < a$ .... (2),

and $\quad U = \frac{2\pi a\rho}{r}\left(1 - \frac{1}{2}\frac{a^2}{r^2} + \frac{1.3}{2.4}\frac{a^4}{r^4} - \text{etc.}\right)$ when $r > a$ ...(3),

Potential symmetrical about an axis.

Potential symmetrical about an axis.

from which we have

$$V = 2\pi\rho\left(1 - \tfrac{1}{2}\frac{r^2}{a^2}Q_2 + \frac{1.3}{2.4}\frac{r^4}{a^4}Q_4 - \text{etc.}\right) \text{ when } r < a\,..(4),$$

and $$V = 2\pi\rho\left(\frac{a}{r} - \tfrac{1}{2}\frac{a^3}{r^3}Q_2 + \frac{1.3}{2.4}\frac{a^5}{r^5}Q_4 - \text{etc.}\right) \text{ when } r > a\,..(5).$$

(II.) of circular disc.

(II.) Multiplying (1) by da, and integrating with reference to a from $a = 0$ as lower limit, and now calling U the potential of a circular disc of uniform surface density ρ, and radius a, at a point in its axis, we find

$$U = 2\pi\rho\,\{(a^2+r^2)^{\frac{1}{2}} - r\},$$

r being positive.

Hence, expanding first in ascending, and secondly in descending powers of r, for the cases of $r < a$ and $r > a$, we find

$$V = 2\pi\rho\left\{-rQ_1 + a + \tfrac{1}{2}\frac{r^2}{a}Q_2 - \frac{1.1}{2.4}\frac{r^4}{a^3}Q_4 + \frac{1.1.3}{2.4.6}\frac{r^6}{a^5}Q_6 - \text{etc.}\right\} \text{ when } r < a,$$

and $$V = 2\pi\rho\left\{\tfrac{1}{2}\frac{a^2}{r} - \frac{1.1}{2.4}\frac{a^4}{r^3}Q_2 + \frac{1.1.3}{2.4.6}\frac{a^6}{r^5}Q_4 - \text{etc.}\right\} \text{ when } r > a.$$

It must be remarked that the first of these expressions is only continuous from $\theta = 0$ to $\theta = \tfrac{1}{2}\pi$; and that from $\theta = \tfrac{1}{2}\pi$ to $\theta = \pi$ the first term of it must be made

$$+2\pi\rho r Q_1, \text{ instead of } -2\pi\rho r Q_1.$$

(III.) Again, taking $\frac{-d}{dr}$ of the expression for U in (II.), and now calling U the potential of a disc of infinitely small thickness c with positive and negative matter of surface density $\frac{\rho}{c}$ on its two sides, we have

$$U = 2\pi\rho\left\{1 - \frac{r}{(a^2+r^2)^{\frac{1}{2}}}\right\},$$

[obtainable also from § 477 (e), by integrating with reference to x, putting r for x, and ρ for ρc]. Hence for this case

Potential in the neighbourhood of a circular galvanometer coil.

$$V = 2\pi\rho\left(1 - \frac{r}{a}Q_1 + \tfrac{1}{2}\frac{r^3}{a^3}Q_3 - \frac{1.3}{2.4}\frac{r^5}{a^5}Q_5 + \text{etc.}\right) \text{ when } r < a,$$

and $$V = 2\pi\rho\left(\tfrac{1}{2}\frac{a^2}{r^2}Q_1 - \frac{1.3}{2.4}\frac{a^4}{r^4}Q_3 + \text{etc.}\right) \text{ when } r > a.$$

The first of these expressions also is discontinuous; and when θ

is $> \tfrac{1}{2}\pi$ and $< \pi$, its first term must be taken as $-2\pi\rho$ instead of $2\pi\rho$.

Exhaustion of potential energy.

547. If two systems, or distributions of matter, M and M', given in spaces each finite, but infinitely far asunder, be allowed to approach one another, a certain amount of work is obtained by mutual gravitation: and their mutual potential energy loses, or as we may say *suffers exhaustion*, to this amount: which amount will (§ 486) be the same by whatever paths the changes of position are effected, provided the relative initial positions and the relative final positions of all the particles are given. Hence if m_1, m_2,... be particles of M; m'_1, m'_2,... particles of M'; v'_1, v'_2,... the potentials due to M' at the points occupied by m_1, m_2,...; v_1, v_2,... those due to M at the points occupied by m'_1, m'_2,...; and E the exhaustion of mutual potential energy between the two systems in any actual configurations; we have

$$E = \Sigma m v' = \Sigma m' v.$$

This may be otherwise written, if ρ denote a discontinuous function, expressing the density at any point, (x, y, z) of the mass M, and vanishing at all points not occupied by matter of this distribution, and if ρ' be taken to specify similarly the other mass M'. Thus we have

$$E = \iiint \rho v'\, dx\,dy\,dz = \iiint \rho' v\, dx\,dy\,dz,$$

the integrals being extended through all space. The equality of the second and third members here is verified by remarking that

$$v = \iiint \frac{{}_{,}\rho\, d{}_{,}x\, d{}_{,}y\, d{}_{,}z}{D},$$

if D denote the distance between (x, y, z) and $({}_{,}x, {}_{,}y, {}_{,}z)$, the latter being any point of space, and ${}_{,}\rho$ the value of ρ at it. A corresponding expression of course gives v': and thus we find one sextuple integral to express identically the second and third members, or the value of E, as follows:—

$$E = \iiiint\!\!\iint \frac{{}_{,}\rho\rho'\, d{}_{,}x\, d{}_{,}y\, d{}_{,}z\, dx\,dy\,dz}{D}.$$

Green's method.

548. It is remarkable that it was on the consideration of an analytical formula which, when properly interpreted with reference to two masses, has precisely the same signification as

Green's method. the preceding expressions for E, that Green founded his whole structure of general theorems regarding attraction.

In App. A. (a) let a be constant, and let U, U' be the potentials at (x, y, z) of two finite masses, M, M', finitely distant from one another: so that if ρ and ρ' denote the densities of M and M' respectively at the point (x, y, z), we have [§ 491 (c)]

$$\nabla^2 U = -4\pi\rho, \quad \nabla^2 U' = -4\pi\rho'.$$

It must be remembered that ρ vanishes at every point not forming part of the mass M: and so for ρ' and M'. In the present merely abstract investigation the two masses may, in part or in whole, jointly occupy the same space: or they may be merely imagined subdivisions of the density of one real mass. Then, supposing S to be infinitely distant in all directions, and observing that $U\partial U'$ and $U'\partial U$ are small quantities of the order of the inverse cube of the distance of any point of S from M and M', whereas the whole area of S over which the surface integrals of App. A. (a) (1) are taken as infinitely great, only of the order of the square of the same distance, we have

$$\iint dS U'\partial U = 0, \text{ and } \iint dS U\partial U' = 0.$$

Hence (a) (1) becomes

$$\iiint\left(\frac{dU}{dx}\frac{dU'}{dx} + \frac{dU}{dy}\frac{dU'}{dy} + \frac{dU}{dz}\frac{dU'}{dz}\right)dxdydz = 4\pi\iiint\rho U' dxdydz = 4\pi\iiint\rho' U dxdydz;$$

showing that the first member divided by 4π is equal to the exhaustion of potential energy accompanying the approach of the two masses from an infinite mutual distance to the relative position which they actually occupy.

Without supposing S infinite, we see that the second member of (a) (1), divided by 4π, is the direct expression for the exhaustion of mutual energy between M' and a distribution consisting of the part of M within S and a distribution over S, of density $\frac{1}{4\pi}\partial U'$; and the third member the corresponding expression for M and derivations from M'.

Exhaustion of potential energy, in allowing **549.** If, instead of two distributions, M and M', two particles, m_1, m_2 alone be given; the exhaustion of mutual

potential energy in allowing them to come together from infinity, to any distance $D(1, 2)$ asunder, is **condensation of diffused matter.**

$$\frac{m_1 m_2}{D(1, 2)}.$$

If now a third particle m_3 be allowed to come into their neighbourhood, there is a further exhaustion of potential energy amounting to

$$\frac{m_1 m_3}{D(1, 3)} + \frac{m_2 m_3}{D(2, 3)}.$$

By considering any number of particles coming thus necessarily into position in a group, we find for the whole exhaustion of potential energy

$$E = \Sigma\Sigma\frac{mm'}{D}$$

where m, m' denote the masses of any two of the particles, D the distance between them, and $\Sigma\Sigma$ the sum of the expressions for all the pairs, each pair taken only once. If v denote the potential at the point occupied by m, of all the other masses, the expression becomes a simple sum, with as many terms as there are masses, which we may write thus— **Exhaustion of potential energy.**

$$E = \tfrac{1}{2}\Sigma mv;$$

the factor $\frac{1}{2}$ being necessary, because Σmv takes each such term as $\frac{m_1 m_2}{D(1, 2)}$ twice over. If the particles form an ultimately continuous mass, with density ρ at any point (x, y, z), we have only to write the sum as an integral; and thus we have

$$E = \tfrac{1}{2}\iiint \rho v\, dx\, dy\, dz$$

as the exhaustion of potential energy of gravitation accompanying the condensation of a quantity of matter from a state of infinite diffusion (that is to say, a state in which the density is everywhere infinitely small) to its actual condition in any finite body.

An important analytical transformation of this expression is suggested by the preceding interpretation of App. A. (a); by

Exhaustion of potential energy.

which we find*

$$E = \frac{1}{8\pi}\iiint\left(\frac{dv^2}{dx^2} + \frac{dv^2}{dy^2} + \frac{dv^2}{dz^2}\right) dx\,dy\,dz,$$

or

$$E = \frac{1}{8\pi}\iiint R^2 dx\,dy\,dz,$$

if R denote the resultant force at (x, y, z), the integration being extended through all space.

Detailed interpretations in connexion with the theory of energy, of the remainder of App. A., with a constant, and of its more general propositions and formulæ not involving this restriction, especially of the minimum problems with which it deals, are of importance with reference to the dynamics of incompressible fluids, and to the physical theory of the propagation of electric and magnetic force through space occupied by homogeneous or heterogeneous matter; and we intend to return to it when we shall be specially occupied with these subjects.

Gauss's method

550. The beautiful and instructive manner in which Gauss independently proved Green's theorems is more immediately and easily interpretable in terms of energy, according to the commonly-accepted idea of forces acting simply between particles at a distance without any assistance or influence of interposed matter. Thus, to prove that a given quantity, Q, of matter is distributable in one and only one way over a given single finite surface S (whether a closed or an open shell), so as to produce equal potential over the whole of this surface, he shows (1) that the integral

$$\iiiint \frac{\rho\rho' d\sigma d\sigma'}{PP'}$$

has a minimum value, subject to the condition

$$\iint \rho\, d\sigma = Q,$$

where ρ is a function of the position of a point, P, on S, ρ' its value at P', and $d\sigma$ and $d\sigma'$ elements of S at these points: and (2) that this minimum is produced by only one determinate distribution of values of ρ. By what we have just seen (§ 549) the first of these integrals is double the potential energy of a

* Nichol's *Encyclopædia*, 2d Ed. 1860. Magnetism, Dynamical Relations of.

Gauss's method.

distribution over S of an infinite number of infinitely small mutually repelling particles: and hence this minimum problem is (§ 292) merely an analytical statement of the problem to find how these particles must be distributed to be in stable equilibrium.

Equilibrium of repelling particles enclosed in a rigid smooth surface.

Similarly, Gauss's second minimum problem, of which the preceding is a particular case, and which is, to find ρ so as to make

$$\iint (\tfrac{1}{2} v - \Omega)\rho\, d\sigma$$

a minimum, subject to

$$\iint \rho\, d\sigma = Q,$$

where Ω is any given arbitrary function of the position of P, and

$$v = \iint \frac{\rho' d\sigma'}{PP'},$$

is merely an analytical statement of the question:—how must a given quantity of repelling particles confined to a surface S be distributed so as to make the whole potential energy due to their mutual forces, and to the forces exerted on them by a given fixed attracting or repelling body (of which Ω is the potential at P), be a minimum? In other words (§ 292), to find how the movable particles will place themselves, under the influence of the acting forces*.

* Gauss's investigations here referred to will be found in Vol. V. of his collected works, p. 197, in a paper entitled "Allgemeine Lehrsätze auf die im verkehrten Verhältnisse des Quadrats der Entfernung wirkenden Anziehungs- und Abstossungs-Kräfte;" originally published in 1839.