NATURAL PHILOSOPHY.

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TREATISE

ON

NATURAL PHILOSOPHY

BY

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AND

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PART I.

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PREFACE.

Les causes primordiales ne nous sont point connues; mais elles sont assujetties à des lois simples et constantes, que l'on peut découvrir par l'observation, et dont l'étude est l'objet de la philosophie naturelle.—FOURIER.

THE term Natural Philosophy was used by NEWTON, and is still used in British Universities, to denote the investigation of laws in the material world, and the deduction of results not directly observed. Observation, classification, and description of phenomena necessarily precede Natural Philosophy in every department of natural science. The earlier stage is, in some branches, commonly called Natural History; and it might with equal propriety be so called in all others.

Our object is twofold: to give a tolerably complete account of what is now known of Natural Philosophy, in language adapted to the non-mathematical reader; and to furnish, to those who have the privilege which high mathematical acquirements confer, a connected outline of the analytical processes by which the greater part of that knowledge has been extended into regions as yet unexplored by experiment.

We commence with a chapter on *Motion*, a subject totally independent of the existence of *Matter* and *Force*. In this we are naturally led to the consideration of the curvature and tortuosity of curves, the curvature of surfaces, distortions or strains, and various other purely geometrical subjects.

The *Laws of Motion*, the *Law of Gravitation and of Electric and Magnetic Attractions, Hooke's Law*, and other fundamental principles derived directly from experiment, lead by mathematical processes to interesting and useful results, for the full testing of which our most delicate experimental methods are as yet totally insufficient. A large part of the present volume is devoted to these deductions; which, though not immediately proved by experiment, are as certainly true as the elementary laws from which mathematical analysis has evolved them.

The analytical processes which we have employed are, as a rule, such as lead most directly to the results aimed at, and are therefore in great part unsuited to the general reader.

We adopt the suggestion of AMPÈRE, and use the term *Kinematics* for the purely geometrical science of motion in the abstract. Keeping in view the proprieties of language, and following the example of the most logical writers, we employ the term *Dynamics* in its true sense as the science which treats of the action of *force*, whether it maintains relative rest, or produces acceleration of relative motion. The two corresponding divisions of Dynamics are thus conveniently entitled *Statics* and *Kinetics*.

One object which we have constantly kept in view is the grand principle of the *Conservation of Energy*. According to modern experimental results, especially those of JOULE, Energy is as real and as indestructible as Matter. It is satisfactory to find that NEWTON anticipated, so far as the state of experimental science in his time permitted him, this magnificent modern generalization.

We desire it to be remarked that in much of our work, where we may appear to have rashly and needlessly interfered with methods and systems of proof in the present day generally accepted, we take the position of Restorers, and not of Innovators.

In our introductory chapter on Kinematics, the consideration of Harmonic Motion naturally leads us to *Fourier's Theorem*,

one of the most important of all analytical results as regards usefulness in physical science. In the Appendices to that chapter we have introduced an extension of *Green's Theorem*, and a treatise on the remarkable functions known as *Laplace's Coefficients*. There can be but one opinion as to the beauty and utility of this analysis of Laplace; but the manner in which it has been hitherto presented has seemed repulsive to the ablest mathematicians, and difficult to ordinary mathematical students. In the simplified and symmetrical form in which we give it, it will be found quite within the reach of readers moderately familiar with modern mathematical methods.

In the second chapter we give NEWTON'S Laws of Motion in his own words, and with some of his own comments—every attempt that has yet been made to supersede them having ended in utter failure. Perhaps nothing so simple, and at the same time so comprehensive, has ever been given as the foundation of a system in any of the sciences. The dynamical use of the *Generalized Cöordinates* of LAGRANGE, and the *Varying Action* of HAMILTON, with kindred matter, complete the chapter.

The third chapter, "Experience," treats briefly of Observation and Experiment as the basis of Natural Philosophy.

The fourth chapter deals with the fundamental Units, and the chief Instruments used for the measurement of Time, Space, and Force.

Thus closes the First Division of the work, which is strictly preliminary, and to which we have limited the present issue.

This new edition has been thoroughly revised, and very considerably extended. The more important additions are to be found in the Appendices to the first chapter, especially that devoted to *Laplace's Coefficients;* also at the end of the second chapter, where a very full investigation of the "*cycloidal motion*" of systems is now given; and in Appendix B′, which describes a number of continuous calculating machines invented and constructed since the publication of our first edition. A great improvement has been made in the treatment of *Lagrange's Generalized Equations of Motion.*

We believe that the mathematical reader will especially profit by a perusal of the large type portion of this volume; as he will thus be forced to think out for himself what he has been too often accustomed to reach by a mere mechanical application of analysis. Nothing can be more fatal to progress than a too confident reliance on mathematical symbols; for the student is only too apt to take the easier course, and consider the *formula* and not the *fact* as the physical reality.

In issuing this new edition, of a work which has been for several years out of print, we recognise with legitimate satisfaction the very great improvement which has recently taken place in the more elementary works on Dynamics published in this country, and which we cannot but attribute, in great part, to our having effectually recalled to its deserved position Newton's system of elementary definitions, and Laws of Motion.

We are much indebted to Mr BURNSIDE and Prof. CHRYSTAL for the pains they have taken in reading proofs and verifying formulas; and we confidently hope that few erratums of serious consequence will now be found in the work.

W. THOMSON.
P. G. TAIT.

NOTE TO NEW IMPRESSION, 1912

A few slight additions and corrections have been made by Sir GEORGE DARWIN and Prof. H. LAMB, but, substantially, the work remains as last passed by the authors. The additions can be identified by the initials attached in brackets.

1912

CONTENTS.

DIVISION I.—PRELIMINARY.

CHAPTER I.--KINEMATICS.

CHAPTER II.—DYNAMICAL LAWS AND PRINCIPLES.

DIVISION I.

PRELIMINARY.

CHAPTER I.—KINEMATICS.

1. THERE are many properties of motion, displacement, and deformation, which may be considered altogether independently of such physical ideas as force, mass, elasticity, temperature, magnetism, electricity. The preliminary consideration of such properties in the abstract is of very great use for Natural Philosophy, and we devote to it, accordingly, the whole of this our first chapter; which will form, as it were, the Geometry of our subject, embracing what can be observed or concluded with regard to actual motions, as long as the *cause* is not sought.

2. In this category we shall take up first the free motion of a point, then the motion of a point attached to an inextensible cord, then the motions and displacements of rigid systems—and finally, the deformations of surfaces and of solid or fluid bodies. Incidentally, we shall be led to introduce a good deal of elementary geometrical matter connected with the curvature of lines and surfaces.

3. When a point moves from one position to another it must evidently describe a *continuous* line, which may be curved or straight, or even made up of portions of curved and straight lines meeting each other at any angles. If the motion be that of a *material particle*, however, there cannot generally be any such abrupt changes of direction, since (as we shall afterwards see) this would imply the action of an *infinite* force, except in the case in which the velocity becomes zero at the angle. It is useful to consider at the outset various theorems connected **Motion of a point.**

Motion of point.

with the geometrical notion of the path described by a moving point, and these we shall now take up, deferring the consideration of Velocity to a future section, as being more closely connected with physical ideas.

4. The *direction* of motion of a moving point is at each instant the tangent drawn to its path, if the path be a curve, or the path itself if a straight line.

Curvature of a plane curve.

5. If the path be not straight the direction of motion changes from point to point, and the *rate* of this change, per unit of length of the curve $\left(\frac{d\theta}{ds}\text{ according to the notation below}\right)$, is called the *curvature*. To exemplify this, suppose two tangents drawn to a circle, and radii to the points of contact. The angle between the tangents is the change of direction required, and the rate of change is to be measured by the relation between this angle and the length of the circular arc. Let I be the angle, c the arc, and ρ the radius. We see at once that (as the angle between the radii is equal to the angle between the tangents)

$$\rho I = c,$$

and therefore $\frac{I}{c} = \frac{1}{\rho}$. Hence the curvature of a circle is inversely as its radius, and, measured in terms of the proper unit of curvature, is simply the reciprocal of the radius.

6. Any small portion of a curve may be approximately taken as a circular arc, the approximation being closer and closer to the truth, as the assumed arc is smaller. The curvature is then the reciprocal of the radius of this circle.

If $\delta\theta$ be the angle between two tangents at points of a curve distant by an arc δs, the definition of curvature gives us at once as its measure, the limit of $\frac{\delta\theta}{\delta s}$ when δs is diminished without limit; or, according to the notation of the differential calculus, $\frac{d\theta}{ds}$. But we have

$$\tan\theta = \frac{dy}{dx},$$

if, the curve being a plane curve, we refer it to two rectangular

axes OX, OY, according to the Cartesian method, and if θ denote the inclination of its tangent, at any point x, y, to OX. Hence

Curvature of a plane curve

$$\theta = \tan^{-1}\frac{dy}{dx};$$

and, by differentiation with reference to any independent variable t, we have

$$d\theta = \frac{d\left(\frac{dy}{dx}\right)}{1+\left(\frac{dy}{dx}\right)^2} = \frac{dx\,d^2y - dy\,d^2x}{dx^2+dy^2}.$$

Also, $$ds = (dx^2+dy^2)^{\frac{1}{2}}.$$

Hence, if ρ denote the radius of curvature, so that

$$\frac{1}{\rho} = \frac{d\theta}{ds} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

we conclude $$\frac{1}{\rho} = \frac{dx\,d^2y - dy\,d^2x}{(dx^2+dy^2)^{\frac{3}{2}}} \quad\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

Although it is generally convenient, in kinematical and kinetic formulæ, to regard time as the independent variable, and all the changing geometrical elements as functions of it, there are cases in which it is useful to regard the length of the arc or path described by a point as the independent variable. On this supposition we have

$$0 = d(ds^2) = d(dx^2+dy^2) = 2(dx\,d_s^2x + dy\,d_s^2y),$$

where we denote by the suffix to the letter d, the independent variable understood in the differentiation. Hence

$$\frac{dx}{d_s^2y} = -\frac{dy}{d_s^2x} = -\frac{(dx^2+dy^2)^{\frac{1}{2}}}{\{(d_s^2y)^2+(d_s^2x)^2\}^{\frac{1}{2}}};$$

and using these, with $ds^2 = dx^2+dy^2$, to eliminate dx and dy from (2), we have

$$\frac{1}{\rho} = \frac{\{(d_s^2y)^2+(d_s^2x)^2\}^{\frac{1}{2}}}{ds^2};$$

or, according to the usual short, although not quite complete, notation,

$$\frac{1}{\rho} = \left\{\left(\frac{d^2y}{ds^2}\right)^2 + \left(\frac{d^2x}{ds^2}\right)^2\right\}^{\frac{1}{2}}.$$

Tortuous curve.

7. If all points of the curve lie in one plane, it is called a *plane curve*, and in the same way we speak of a *plane* polygon or broken line. If various points of the line do not lie in one plane, we have in one case what is called a *curve of double*

Tortuous curve.

curvature, in the other a *gauche polygon*. The term 'curve of double curvature' is very bad, and, though in very general use, is, we hope, not ineradicable. The fact is, that there are not two curvatures, but only a curvature (as above defined), of which the plane is continuously changing, or twisting, round the tangent line; thus exhibiting a torsion. The course of such a curve is, in common language, well called 'tortuous;' and the measure of the corresponding property is conveniently called *Tortuosity.*

8. The nature of this will be best understood by considering the curve as a polygon whose sides are indefinitely small. Any two consecutive sides, of course, lie in a plane—and in that plane the curvature is measured as above, but in a curve which is not plane the third side of the polygon will not be in the same plane with the first two, and, therefore, the new plane in which the curvature is to be measured is different from the old one. The plane of the curvature on each side of any point of a tortuous curve is sometimes called the *Osculating Plane* of the curve at that point. As two successive positions of it contain the second side of the polygon above mentioned, it is evident that the osculating plane passes from one position to the next by revolving about the tangent to the curve.

Curvature and tortuosity.

9. Thus, as we proceed along such a curve, the curvature in general varies; and, at the same time, the plane in which the curvature lies is turning about the tangent to the curve. The tortuosity is therefore to be measured by the rate at which the osculating plane turns about the tangent, per unit length of the curve.

To express the radius of curvature, the direction cosines of the osculating plane, and the tortuosity, of a curve not in one plane, in terms of Cartesian triple co-ordinates, let, as before, $\delta\theta$ be the angle between the tangents at two points at a distance δs from one another along the curve, and let $\delta\phi$ be the angle between the osculating planes at these points. Thus, denoting by ρ the radius of curvature, and τ the tortuosity, we have

$$\frac{1}{\rho}=\frac{d\theta}{ds},$$

$$\tau=\frac{d\phi}{ds},$$

according to the regular notation for the limiting values of $\frac{\delta\theta}{\delta s}$, and $\frac{\delta\phi}{\delta s}$, when δs is diminished without limit. Let OL, OL' be lines drawn through any fixed point O parallel to any two successive positions of a moving line PT, each in the directions indicated by the order of the letters. Draw OS perpendicular to their plane in the direction from O, such that OL, OL', OS lie in the same relative order in space as the positive axes of co-ordinates, OX, OY, OZ. Let OQ bisect LOL', and let OR bisect the angle between OL' and LO produced through O.

Curvature and tortuosity.

Let the direction cosines of

OL	be	$a, b, c;$
OL'	,,	$a', b', c';$
OQ	,,	$l, m, n;$
OR	,,	$\alpha, \beta, \gamma;$
OS	,,	$\lambda, \mu, \nu:$

and let $\delta\theta$ denote the angle LOL'. We have, by the elements of analytical geometry,

$$\cos\delta\theta = aa'+bb'+cc' \quad\ldots\ldots(3);$$

$$l=\frac{\frac{1}{2}(a+a')}{\cos\frac{1}{2}\delta\theta},\quad m=\frac{\frac{1}{2}(b+b')}{\cos\frac{1}{2}\delta\theta},\quad n=\frac{\frac{1}{2}(c+c')}{\cos\frac{1}{2}\delta\theta}\quad\ldots\ldots(4);$$

$$\alpha=\frac{a'-a}{2\sin\frac{1}{2}\delta\theta},\quad \beta=\frac{b'-b}{2\sin\frac{1}{2}\delta\theta},\quad \gamma=\frac{c'-c}{2\sin\frac{1}{2}\delta\theta}\quad\ldots\ldots(5);$$

$$\lambda=\frac{bc'-b'c}{\sin\delta\theta},\quad \mu=\frac{ca'-c'a}{\sin\delta\theta},\quad \nu=\frac{ab'-a'b}{\sin\delta\theta}\quad\ldots\ldots(6).$$

Now let the two successive positions of PT be tangents to a curve at points separated by an arc of length δs. We have

$$\frac{1}{\rho}=\frac{\delta\theta}{\delta s}=\frac{2\sin\frac{1}{2}\delta\theta}{\delta s}=\frac{\sin\delta\theta}{\delta s}\quad\ldots\ldots(7)$$

when δs is infinitely small; and in the same limit

$$l=\frac{dx}{ds},\qquad m=\frac{dy}{ds},\qquad n=\frac{dz}{ds};$$

$$a'-a=d\frac{dx}{ds},\quad b'-b=d\frac{dy}{ds},\quad c'-c=d\frac{dz}{ds}\quad\ldots\ldots(8);$$

$$bc'-b'c=\frac{dy}{ds}\,d\frac{dz}{ds}-\frac{dz}{ds}\,d\frac{dy}{ds},\ \&c.\quad\ldots\ldots(9);$$

Curvature and tortuosity.

and α, β, γ become the direction cosines of the normal, PC, drawn towards the centre of curvature, C; and λ, μ, ν those of the perpendicular to the osculating plane drawn in the direction relatively to PT and PC, corresponding to that of OZ relatively to OX and OY. Then, using (8) and (9), with (7), in (5) and (6) respectively, we have

$$\alpha = \frac{d\frac{dx}{ds}}{\rho^{-1}ds}, \qquad \beta = \frac{d\frac{dy}{ds}}{\rho^{-1}ds}, \qquad \gamma = \frac{d\frac{dz}{ds}}{\rho^{-1}ds} \quad\ldots\ldots\ldots(10);$$

$$\lambda = \frac{\frac{dy}{ds}d\frac{dz}{ds} - \frac{dz}{ds}d\frac{dy}{ds}}{\rho^{-1}ds}, \quad \mu = \frac{\frac{dz}{ds}d\frac{dx}{ds} - \frac{dx}{ds}d\frac{dz}{ds}}{\rho^{-1}ds}, \quad \nu = \frac{\frac{dx}{ds}d\frac{dy}{ds} - \frac{dy}{ds}d\frac{dx}{ds}}{\rho^{-1}ds} \quad (11).$$

The simplest expression for the curvature, with choice of independent variable left arbitrary, is the following, taken from (10):

$$\frac{1}{\rho} = \frac{\sqrt{\left\{\left(d\frac{dx}{ds}\right)^2 + \left(d\frac{dy}{ds}\right)^2 + \left(d\frac{dz}{ds}\right)^2\right\}}}{ds} \quad\ldots\ldots\ldots(12).$$

This, modified by differentiation, and application of the formula

$$ds\,d^2s = dx\,d^2x + dy\,d^2y + dz\,d^2z \;\ldots\ldots\ldots\ldots(13),$$

becomes

$$\frac{1}{\rho} = \frac{\sqrt{\{(d^2x)^2 + (d^2y)^2 + (d^2z)^2 - (d^2s)^2\}}}{ds^2} \;\ldots\ldots\ldots\ldots(14).$$

Another formula for $\frac{1}{\rho}$ is obtained immediately from equations (11); but these equations may be put into the following simpler form, by differentiation, &c.,

$$\lambda = \frac{dy\,d^2z - dz\,d^2y}{\rho^{-1}ds^3}, \quad \mu = \frac{dz\,d^2x - dx\,d^2z}{\rho^{-1}ds^3}, \quad \nu = \frac{dx\,d^2y - dy\,d^2x}{\rho^{-1}ds^3} \quad (15);$$

from which we find

$$\rho^{-1} = \frac{\{(dy\,d^2z - dz\,d^2y)^2 + (dz\,d^2x - dx\,d^2z)^2 + (dx\,d^2y - dy\,d^2x)^2\}^{\frac{1}{2}}}{ds^3} \quad (16).$$

Each of these several expressions for the curvature, and for the directions of the relative lines, we shall find has its own special significance in the kinetics of a particle, and the statics of a flexible cord.

To find the tortuosity, $\frac{d\phi}{ds}$, we have only to apply the general equation above, with λ, μ, ν substituted for l, m, n, and $\frac{1}{\tau}\frac{d\lambda}{ds}$, $\frac{1}{\tau}\frac{d\mu}{ds}$, $\frac{1}{\tau}\frac{d\nu}{ds}$ for α, β, γ. Thus we have $\tau^2 = \left(\frac{d\lambda}{ds}\right)^2 + \left(\frac{d\mu}{ds}\right)^2 + \left(\frac{d\nu}{ds}\right)$,

$$\text{or} \quad \tau = \left\{\left(\mu\frac{d\nu}{ds} - \nu\frac{d\mu}{ds}\right)^2 + \left(\nu\frac{d\lambda}{ds} - \lambda\frac{d\nu}{ds}\right)^2 + \left(\lambda\frac{d\mu}{ds} - \mu\frac{d\lambda}{ds}\right)^2\right\}^{\frac{1}{2}},$$

Curvature and tortuosity

where λ, μ, ν, denote the direction cosines of the osculating plane, given by the preceding formulæ.

Integral curvature of a curve (compare § 136).

10. The *integral curvature*, or *whole change of direction* of an arc of a plane curve, is the angle through which the tangent has turned as we pass from one extremity to the other. The *average curvature* of any portion is its whole curvature divided by its length. Suppose a line, drawn from a fixed point, to move so as always to be parallel to the direction of motion of a point describing the curve: the angle through which this turns during the motion of the point exhibits what we have thus defined as the integral curvature. In estimating this, we must of course take the enlarged modern meaning of an angle, including angles greater than two right angles, and also negative angles. Thus the integral curvature of any closed curve, whether everywhere concave to the interior or not, is four right angles, provided it does not cut itself. That of a Lemniscate, or figure of 8, is *zero*. That of the Epicycloid is eight right angles; and so on.

11. The definition in last section may evidently be extended to a plane polygon, and the integral change of direction, or the angle between the first and last sides, is then the sum of its exterior angles, all the sides being produced each in the direction in which the moving point describes it while passing round the figure. This is true whether the polygon be closed or not. If closed, then, as long as it is not crossed, this sum is four right angles,—an extension of the result in Euclid, where all *re-entrant* polygons are excluded. In the case of the star-shaped figure, it is ten right angles, wanting the sum of the five acute angles of the figure; that is, eight right angles.

12. The *integral curvature* and the *average curvature* of a curve which is not plane, may be defined as follows:—Let successive lines be drawn from a fixed point, parallel to tangents at successive points of the curve. These lines will form a conical surface. Suppose this to be cut by a sphere of unit radius having its centre at the fixed point. The *length* of the

Integral curvature of a curve (compare § 136).

curve of intersection measures the *integral curvature* of the given curve. The *average curvature* is, as in the case of a plane curve, the integral curvature divided by the length of the curve. For a tortuous curve approximately plane, the integral curvature thus defined, approximates (not to the integral curvature according to the proper definition, § 10, for a plane curve, but) to the sum of the integral curvatures of all the parts of an approximately coincident plane curve, each taken as positive. Consider, for examples, varieties of James Bernouilli's plane elastic curve, § 611, and approximately coincident tortuous curves of fine steel piano-forte wire. Take particularly the plane lemniscate and an approximately coincident tortuous closed curve.

13. Two consecutive tangents lie in the osculating plane. This plane is therefore parallel to the tangent plane to the cone described in the preceding section. Thus the tortuosity may be measured by the help of the spherical curve which we have just used for defining integral curvature. We cannot as yet complete the explanation, as it depends on the theory of rolling, which will be treated afterwards (§§ 110—137). But it is enough at present to remark, that if a plane roll on the sphere, along the spherical curve, turning always round an instantaneous axis tangential to the sphere, the integral curvature of the curve of contact or trace of the rolling on the plane, is a proper measure of the *whole torsion*, or integral of tortuosity. From this and § 12 it follows that the curvature of this plane curve at any point, or, which is the same, the projection of the curvature of the spherical curve on a tangent plane of the spherical surface, is equal to the tortuosity divided by the curvature of the given curve.

Let $\frac{1}{\rho}$ be the curvature and τ the tortuosity of the given curve, and ds an element of its length. Then $\int\frac{ds}{\rho}$ and $\int\tau ds$, each integral extended over any stated length, l, of the curve, are respectively the integral curvature and the integral tortuosity. The mean curvature and the mean tortuosity are respectively

$$\frac{1}{l}\int\frac{ds}{\rho} \text{ and } \frac{1}{l}\int\tau ds.$$

Infinite tortuosity will be easily understood, by considering a helix, of inclination α, described on a right circular cylinder of radius r. The curvature in a circular section being $\frac{1}{r}$, that of the helix is, of course, $\frac{\cos^2\alpha}{r}$. The tortuosity is $\frac{\sin\alpha\cos\alpha}{r}$, or $\tan\alpha\times$ curvature. Hence, if $\alpha=\frac{\pi}{4}$ the curvature and tortuosity are equal.

Integral curvature of a curve (compare § 136).

Let the curvature be denoted by $\frac{1}{\rho}$, so that $\cos^2\alpha=\frac{r}{\rho}$. Let ρ remain finite, and let r diminish without limit. The *step* of the helix being $2\pi r\tan\alpha=2\pi\sqrt{r}\left(1-\frac{r}{\rho}\right)^{\frac{1}{2}}$, is, in the limit, $2\pi\sqrt{\rho r}$, which is infinitely small. Thus the motion of a point in the curve, though infinitely nearly in a straight line (the path being always at the infinitely small distance r from the fixed straight line, the axis of the cylinder), will have finite curvature $\frac{1}{\rho}$. The tortuosity, being $\frac{1}{\rho}\tan\alpha$ or $\frac{1}{\sqrt{\rho r}}\left(1-\frac{r}{\rho}\right)^{\frac{1}{2}}$, will in the limit be a mean proportional between the curvature of the circular section of the cylinder and the finite curvature of the curve.

The acceleration (or force) required to produce such a motion of a point (or material particle) will be afterwards investigated (§ 35 *d.*).

Flexible line.

14. A chain, cord, or fine wire, or a fine fibre, filament, or hair, may suggest what is not to be found among natural or artificial productions, a perfectly *flexible and inextensible line.* The elementary kinematics of this subject require no investigation. The mathematical condition to be expressed in any case of it is simply that the distance measured along the line from any one point to any other, remains constant, however the line be bent.

15. The use of a cord in mechanism presents us with many practical applications of this theory, which are in general extremely simple; although curious, and not always very easy, geometrical problems occur in connexion with it. We shall say nothing here about the theory of knots, knitting, weaving, Flexible line. plaiting, etc., but we intend to return to the subject, under vortex-motion in Hydrokinetics.

16. In the mechanical tracing of curves, a flexible and inextensible cord is often supposed. Thus, in drawing an ellipse, the focal property of the curve shows us that by fixing the ends of such a cord to the foci and keeping it stretched by a pencil, the pencil will trace the curve.

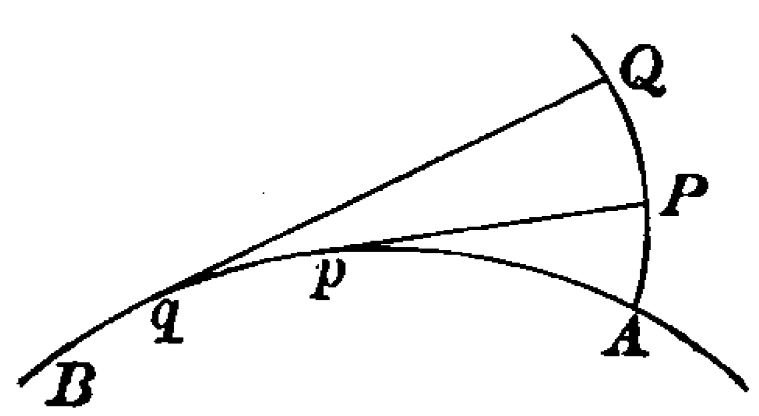
By a ruler moveable about one focus, and a string attached to a point in the ruler and to the other focus, the hyperbola may be described by the help of its analogous focal property; and so on.

Evolute. **17.** But the consideration of evolutes is of some importance in Natural Philosophy, especially in certain dynamical and optical questions, and we shall therefore devote a section or two to this application of kinematics.

Def. If a flexible and inextensible string be fixed at one point of a plane curve, and stretched along the curve, and be then unwound in the plane of the curve, its extremity will describe an *Involute* of the curve. The original curve, considered with reference to the other, is called the *Evolute.*

18. It will be observed that we speak of *an* involute, and of *the* evolute, of a curve. In fact, as will be easily seen, a curve can have but one evolute, but it has an infinite number of involutes. For all that we have to do to vary an involute, is to change the point of the curve from which the tracing point starts, or consider the involutes described by different points of the string, and these will, in general, be different curves. The following section shows that there is but one evolute.

19. Let AB be any curve, PQ a portion of an involute, pP, qQ positions of the free part of the string. It will be seen



at once that these must be tangents to the arc AB at p and q. Also (see § 90), the string at any stage, as pP, revolves about p. Hence pP is *normal* to the curve PQ. And thus the evolute of PQ is a definite curve, viz., the envelope of the normals drawn at every point of PQ,

or, which is the same thing, the locus of the centres of curva- Evolute.
ture of the curve PQ. And we may merely mention, as an obvious result of the mode of tracing, that the arc pq is equal to the difference of qQ and pP, or that the arc pA is equal to pP.

20. The rate of motion of a point, or its rate of change of Velocity.
position, is called its *Velocity.* It is greater or less as the space passed over in a given time is greater or less: and it may be *uniform, i.e.*, the same at every instant; or it may be *variable.*

Uniform velocity is measured by the space passed over in unit of time, and is, in general, expressed in feet per second; if very great, as in the case of light, it is sometimes popularly reckoned in miles per second. It is to be observed, that time is here used in the abstract sense of a uniformly increasing quantity—what in the differential calculus is called an independent variable. Its physical definition is given in the next chapter.

21. Thus a point, which moves uniformly with velocity v, describes a space of v feet each second, and therefore vt feet in t seconds, t being any number whatever. Putting s for the space described in t seconds, we have

$$s = vt.$$

Thus with unit velocity a point describes unit of space in unit of time.

22. It is well to observe here, that since, by our formula, we have generally

$$v = \frac{s}{t};$$

and since nothing has been said as to the magnitudes of s and t, we may take these as small as we choose. Thus *we get the same result whether we derive v from the space described in a million seconds, or from that described in a millionth of a second.* This idea is very useful, as it makes our results intelligible when a variable velocity has to be measured, and we find ourselves obliged to approximate to its value by considering the space described in an interval so short, that during its lapse the velocity does not sensibly alter in value.

Velocity.

23. When the point does not move uniformly, the velocity is variable, or different at different successive instants; but we define the *average* velocity during any time as the space described in that time, divided by the time, and, the less the interval is, the more nearly does the average velocity coincide with the actual velocity at any instant of the interval. Or again, we define the exact velocity at any instant as the space which the point would have described in one second, if for one second its velocity remained unchanged. That there is at every instant a definite value of the velocity of any moving body, is evident to all, and is matter of everyday conversation. Thus, a railway train, after starting, gradually increases its speed, and every one understands what is meant by saying that at a particular instant it moves at the rate of ten or of fifty miles an hour,—although, in the course of an hour, it may not have moved a mile altogether. Indeed, we may imagine, at any instant during the motion, the steam to be so adjusted as to keep the train running for some time at a perfectly uniform velocity. This would be the velocity which the train had at the instant in question. Without supposing any such definite adjustment of the driving power to be made, we can evidently obtain an approximation to this instantaneous velocity by considering the motion for so short a time, that during it the actual variation of speed may be small enough to be neglected.

24. In fact, if v be the velocity at either beginning or end, or at any instant of the interval, and s the space actually described in time t, the equation $v=\frac{s}{t}$ is more and more nearly true, as the velocity is more nearly uniform during the interval t; so that if we take the interval small enough the equation may be made as nearly exact as we choose. Thus the set of values—

Space described in one second,
Ten times the space described in the first tenth of a second,
A hundred " " " hundredth "

and so on, give nearer and nearer approximations to the velocity at the beginning of the first second. The whole foundation of

the differential calculus is, in fact, contained in this simple question, "What is the rate at which the space described increases?" *i.e.*, What is the velocity of the moving point? Newton's notation for the velocity, *i.e.* the rate at which s increases, or the *fluxion* of s, is $\dot{s}$. This notation is very convenient, as it saves the introduction of a second letter. Velocity

Let a point which has described a space s in time t proceed to describe an additional space δs in time δt, and let v_1 be the greatest, and v_2 the least, velocity which it has during the interval δt. Then, evidently,

$$\delta s < v_1 \delta t, \quad \delta s > v_2 \delta t,$$

$$i.e., \quad \frac{\delta s}{\delta t} < v_1, \quad \frac{\delta s}{\delta t} > v_2.$$

But as δt diminishes, the values of v_1 and v_2 become more and more nearly equal, and in the limit, each is equal to the velocity at time t. Hence

$$v=\frac{ds}{dt}.$$

25. The preceding definition of velocity is equally applicable whether the point move in a straight or curved line; but, since in the latter case the direction of motion continually changes, the mere amount of the velocity is not sufficient completely to describe the motion, and we must have in every such case additional data to remove the uncertainty. Resolution of velocity.

In such cases as this the method commonly employed, whether we deal with velocities, or as we shall do farther on with accelerations and forces, consists mainly in studying, not the velocity, acceleration, or force, *directly*, but its components parallel to any three assumed directions at right angles to each other. Thus, for a train moving up an incline in a NE direction, we may have given the whole velocity and the steepness of the incline, or we may express the same ideas thus—the train is moving simultaneously northward, eastward, and upward—and the motion as to amount and direction will be completely known if we know separately the northward, eastward, and upward velocities—these being called the *components* of the whole velocity in the three mutually perpendicular directions N, E, and up.

Resolution of velocity.

In general the velocity of a point at x, y, z, is (as we have seen) $\frac{ds}{dt}$, or, which is the same, $\left\{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2\right\}^{\frac{1}{2}}$.

Now denoting by u the rate at which x increases, or the velocity parallel to the axis of x, and so by v, w, for the other two; we have $u = \frac{dx}{dt}$, $v = \frac{dy}{dt}$, $w = \frac{dz}{dt}$. Hence, calling α, β, γ the angles which the direction of motion makes with the axes, and putting $q = \frac{ds}{dt}$, we have

$$\cos\alpha = \frac{dx}{ds} = \frac{\frac{dx}{dt}}{\frac{ds}{dt}} = \frac{u}{q}.$$

Hence $u = q\cos\alpha$, and therefore

26. A velocity in any direction may be resolved in, and perpendicular to, any other direction. The first component is found by multiplying the velocity by the cosine of the angle between the two directions—the second by using as factor the sine of the same angle. Or, it may be resolved into components in any three rectangular directions, each component being formed by multiplying the whole velocity by the cosine of the angle between its direction and that of the component.

It is useful to remark that if the axes of x, y, z are not rectangular, $\frac{dx}{dt}$, $\frac{dy}{dt}$, $\frac{dz}{dt}$ will still be the velocities parallel to the axes, but we shall no longer have

$$\left(\frac{ds}{dt}\right)^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2.$$

We leave as an exercise for the student the determination of the correct expression for the whole velocity in terms of its components.

If we resolve the velocity along a line whose inclinations to the axes are λ, μ, ν, and which makes an angle θ with the direction of motion, we find the two expressions below (which must of course be equal) according as we resolve q directly or by its components, u, v, w,

$$q\cos\theta = u\cos\lambda + v\cos\mu + w\cos\nu.$$

Resolution of velocity.

Substitute in this equation the values of u, v, w already given, § 25, and we have the well-known geometrical theorem for the angle between two straight lines which make given angles with the axes,

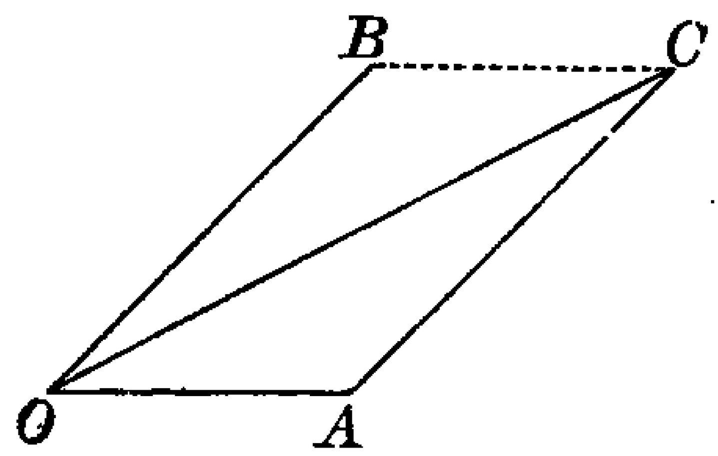
$$\cos\theta = \cos\alpha\cos\lambda + \cos\beta\cos\mu + \cos\gamma\cos\nu.$$

From the above expression we see at once that

Composition of velocities.

27. The velocity resolved in any direction is the sum of the components (in that direction) of the three rectangular components of the whole velocity. And, if we consider motion in one plane, this is still true, only we have but *two* rectangular components. These propositions are virtually equivalent to the following obvious geometrical construction:—

To compound any two velocities as OA, OB in the figure; from A draw AC parallel and equal to OB. Join OC:—then OC is the resultant velocity in magnitude and direction.



OC is evidently the diagonal of the parallelogram two of whose sides are OA, OB.

Hence the resultant of velocities represented by the sides of any closed polygon whatever, whether in one plane or not, taken all in the same order, is zero.

Hence also the resultant of velocities represented by all the sides of a polygon but one, taken in order, is represented by that one taken in the opposite direction.

When there are two velocities or three velocities in two or in three rectangular directions, the resultant is the square root of the sum of their squares—and the cosines of the inclination of its direction to the given directions are the ratios of the components to the resultant.

It is easy to see that as δs in the limit may be resolved into δr and $r\delta\theta$, where r and θ are polar co-ordinates of a plane curve, $\frac{dr}{dt}$ and $r\frac{d\theta}{dt}$ are the resolved parts of the velocity along, and perpendicular to, the radius vector. We may obtain the same result thus, $x = r\cos\theta$, $y = r\sin\theta$.

Composition of velocities.

Hence $\frac{dx}{dt} = \frac{dr}{dt}\cos\theta - r\sin\theta\frac{d\theta}{dt}$, $\frac{dy}{dt} = \frac{dr}{dt}\sin\theta + r\cos\theta\frac{d\theta}{dt}$.

But by § 26 the whole velocity along r is $\frac{dx}{dt}\cos\theta + \frac{dy}{dt}\sin\theta$, *i.e.*, by the above values, $\frac{dr}{dt}$. Similarly the transverse velocity is

$$\frac{dy}{dt}\cos\theta - \frac{dx}{dt}\sin\theta, \text{ or } r\frac{d\theta}{dt}.$$

Acceleration.

28. The velocity of a point is said to be accelerated or retarded according as it increases or diminishes, but the word *acceleration* is generally used in either sense, on the understanding that we may regard its quantity as either positive or negative. Acceleration of velocity may of course be either uniform or variable. It is said to be uniform when the velocity receives equal increments in equal times, and is then measured by the actual increase of velocity per unit of time. If we choose as the unit of acceleration that which adds a unit of velocity per unit of time to the velocity of a point, an acceleration measured by α will add α units of velocity in unit of time—and, therefore, αt units of velocity in t units of time. Hence if V be the change in the velocity during the interval t,

$$V = \alpha t, \text{ or } \alpha = \frac{V}{t}.$$

29. Acceleration is variable when the point's velocity does not receive equal increments in successive equal periods of time. It is then measured by the increment of velocity, which would have been generated in a unit of time had the acceleration remained throughout that interval the same as at its commencement. The *average* acceleration during any time is the whole velocity gained during that time, divided by the time. In Newton's notation $\dot{v}$ is used to express the acceleration in the direction of motion; and, if $v = \dot{s}$, as in § 24, we have

$$\alpha = \dot{v} = \ddot{s}.$$

Let v be the velocity at time t, δv its change in the interval δt, α_1 and α_2 the greatest and least values of the acceleration during the interval δt. Then, evidently,

$$\delta v < \alpha_1 \delta t, \quad \delta v > \alpha_2 \delta t,$$

Acceleration.

$$\text{or} \quad \frac{\delta v}{\delta t} < \alpha_1, \quad \frac{\delta v}{\delta t} > \alpha_2.$$

As δt is taken smaller and smaller, the values of α_1 and α_2 approximate infinitely to each other, and to that of α the required acceleration at time t. Hence

$$\frac{dv}{dt} = \alpha.$$

It is useful to observe that we may also write (by changing the independent variable)

$$\alpha = \frac{dv}{ds}\frac{ds}{dt} = v\frac{dv}{ds}.$$

Since $v = \frac{ds}{dt}$, we have $\alpha = \frac{d^2s}{dt^2}$, and it is evident from similar reasoning that the component accelerations parallel to the axes are $\frac{d^2x}{dt^2}$, $\frac{d^2y}{dt^2}$, $\frac{d^2z}{dt^2}$. But it is to be carefully observed that $\frac{d^2s}{dt^2}$ is *not* generally the resultant of the three component accelerations, but is so only when either the curvature of the path, or the velocity is zero; for [§ 9 (14)] we have

$$\left(\frac{d^2s}{dt^2}\right)^2 = \left(\frac{d^2x}{dt^2}\right)^2 + \left(\frac{d^2y}{dt^2}\right)^2 + \left(\frac{d^2z}{dt^2}\right)^2 - \left(\frac{1}{\rho}\frac{ds^2}{dt^2}\right)^2.$$

The direction cosines of the tangent to the path at any point x, y, z are

$$\frac{1}{v}\frac{dx}{dt}, \quad \frac{1}{v}\frac{dy}{dt}, \quad \frac{1}{v}\frac{dz}{dt}.$$

Those of the line of resultant acceleration are

$$\frac{1}{f}\frac{d^2x}{dt^2}, \quad \frac{1}{f}\frac{d^2y}{dt^2}, \quad \frac{1}{f}\frac{d^2z}{dt^2},$$

where, for brevity, we denote by f the resultant acceleration. Hence the direction cosines of the plane of these two lines are

$$\frac{dyd^2z - dzd^2y}{\{(dyd^2z - dzd^2y)^2 + (dzd^2x - dxd^2z)^2 + (dxd^2y - dyd^2x)^2\}^{\frac{1}{2}}}, \text{ etc.}$$

These (§ 9) show that this plane is the osculating plane of the curve. Again, if θ denote the angle between the two lines, we have

$$\sin\theta = \frac{\{(dyd^2z - dzd^2y)^2 + (dzd^2x - dxd^2z)^2 + (dxd^2y - dyd^2x)^2\}^{\frac{1}{2}}}{vfdt^3},$$

Accelera-tion.

or, according to the expression for the curvature (§ 9),

$$\sin\theta = \frac{ds^3}{\rho v f dt^3} = \frac{v^2}{f\rho}.$$

Hence $$f\sin\theta = \frac{v^2}{\rho}.$$

Again, $$\cos\theta = \frac{1}{vf}\left(\frac{dx}{dt}\frac{d^2x}{dt^2} + \frac{dy}{dt}\frac{d^2y}{dt^2} + \frac{dz}{dt}\frac{d^2z}{dt^2}\right) = \frac{ds\,d^2s}{vfdt^3} = \frac{d^2s}{fdt^2}.$$

Hence $$f\cos\theta = \frac{d^2s}{dt^2},$$ and therefore

Resolution and composition of accelerations.

30. The whole acceleration in any direction is the sum of the components (in that direction) of the accelerations parallel to any three rectangular axes—each component acceleration being found by the same rule as component velocities, that is, by multiplying by the cosine of the angle between the direction of the acceleration and the line along which it is to be resolved.

31. When a point moves in a curve the whole acceleration may be resolved into two parts, one in the direction of the motion and equal to the acceleration of the velocity—the other towards the centre of curvature (perpendicular therefore to the direction of motion), whose magnitude is proportional to the square of the velocity and also to the curvature of the path. The former of these changes the velocity, the other affects only the form of the path, or the direction of motion. Hence if a moving point be subject to an acceleration, constant or not, whose direction is continually perpendicular to the direction of motion, the velocity will not be altered—and the only effect of the acceleration will be to make the point move in a curve whose curvature is proportional to the acceleration at each instant.

32. In other words, if a point move in a curve, whether with a uniform or a varying velocity, its change of direction is to be regarded as constituting an acceleration towards the centre of curvature, equal in amount to the square of the velocity divided by the radius of curvature. The whole acceleration will, in every case, be the resultant of the acceleration,

thus measuring change of direction, and the acceleration of actual velocity along the curve.

Resolution and composition of accelerations.

We may take another mode of resolving acceleration for a plane curve, which is sometimes useful; along, and perpendicular to, the radius-vector. By a method similar to that employed in § 27, we easily find for the component along the radius-vector

$$\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2,$$

and for that perpendicular to the radius-vector

$$\frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right).$$

Determination of the motion from given velocity or acceleration.

33. If for any case of motion of a point we have given the whole velocity and its direction, or simply the components of the velocity in three rectangular directions, at any *time*, or, as is most commonly the case, for any *position*, the determination of the form of the path described, and of other circumstances of the motion, is a question of pure mathematics, and in all cases is capable, if not of an exact solution, at all events of a solution to any degree of approximation that may be desired.

The same is true if the total acceleration and its direction at every instant, or simply its rectangular components, be given, provided the velocity and direction of motion, as well as the position, of the point at any one instant, be given.

For we have in the first case

$$\frac{dx}{dt} = u = q\cos\alpha, \text{ etc.},$$

three simultaneous equations which can contain only x, y, z, and t, and which therefore suffice when integrated to determine x, y, and z in terms of t. By eliminating t among these equations, we obtain two equations among x, y, and z—each of which represents a surface on which lies the path described, and whose intersection therefore completely determines it.

In the second case we have

$$\frac{d^2x}{dt^2} = \alpha, \quad \frac{d^2y}{dt^2} = \beta, \quad \frac{d^2z}{dt^2} = \gamma;$$

to which equations the same remarks apply, except that here each has to be twice integrated.

Determination of the motion from given velocity or acceleration.

The arbitrary constants introduced by integration are determined at once if we know the co-ordinates, and the components of the velocity, of the point at a given epoch.

Examples of velocity.

34. From the principles already laid down, a great many interesting results may be deduced, of which we enunciate a few of the most important.

a. If the velocity of a moving point be uniform, and if its direction revolve uniformly in a plane, the path described is a circle.

Let a be the velocity, and α the angle through which its direction turns in unit of time; then, by properly choosing the axes, we have

$$\frac{dx}{dt} = -a \sin \alpha t, \quad \frac{dy}{dt} = a \cos \alpha t,$$

whence $$(x-A)^2 + (y-B)^2 = \frac{a^2}{\alpha^2}.$$

b. If a point moves in a plane, and if its component velocity parallel to each of two rectangular axes is proportional to its distance from that axis, the path is an ellipse or hyperbola whose principal diameters coincide with those axes; and the acceleration is directed to or from the origin at every instant.

$$\frac{dx}{dt} = \mu y, \quad \frac{dy}{dt} = \nu x.$$

Hence $\dfrac{d^2x}{dt^2} = \mu\nu x$, $\dfrac{d^2y}{dt^2} = \mu\nu y$, and the whole acceleration is towards or from O.

Also $\dfrac{dy}{dx} = \dfrac{\nu}{\mu}\dfrac{x}{y}$, from which $\mu y^2 - \nu x^2 = C$, an ellipse or hyperbola referred to its principal axes. (Compare § 65.)

c. When the velocity is uniform, but in direction revolving uniformly in a right circular cone, the motion of the point is in a circular helix whose axis is parallel to that of the cone.

Examples of acceleration.

35. *a.* When a point moves uniformly in a circle of radius R, with velocity V, the whole acceleration is directed towards the centre, and has the constant value $\dfrac{V^2}{R}$. See § 31.

Examples of acceleration.

b. With uniform acceleration in the direction of motion, a point describes spaces proportional to the squares of the times elapsed since the commencement of the motion.

In this case the space described in any interval is that which would be described in the same time by a point moving uniformly with a velocity equal to that at the middle of the interval. In other words, the average velocity (when the acceleration is uniform) is, during any interval, the arithmetical mean of the initial and final velocities. This is the case of a stone falling vertically.

For if the acceleration be parallel to x, we have

$$\frac{d^2x}{dt^2} = a, \text{ therefore } \frac{dx}{dt} = v = at, \text{ and } x = \tfrac{1}{2}at^2.$$

And we may write the equation (§ 29) $v\dfrac{dv}{dx} = a$, whence $\dfrac{v^2}{2} = ax$.

If at time $t=0$ the velocity was V, these equations become at once

$$v = V + at, \quad x = Vt + \tfrac{1}{2}at^2, \text{ and } \frac{v^2}{2} = \frac{V^2}{2} + ax.$$

$$\begin{aligned} \text{And initial velocity} &= V, \\ \text{final} \quad \text{,,} \quad &= V + at; \\ \text{Arithmetical mean} &= V + \tfrac{1}{2}at, \\ &= \frac{x}{t}, \end{aligned}$$

whence the second part of the above statement.

c. When there is uniform acceleration in a constant direction, the path described is a parabola, whose axis is parallel to that direction. This is the case of a projectile moving in vacuum.

For if the axis of y be parallel to the acceleration a, and if the plane of xy be that of motion at any time,

$$\frac{d^2z}{dt^2} = 0, \quad \frac{dz}{dt} = 0, \quad z = 0,$$

and therefore the motion is wholly in the plane of xy.

Then $$\frac{d^2x}{dt^2} = 0, \quad \frac{d^2y}{dt^2} = a;$$

Examples of acceleration.

and by integration

$$x = Ut + a, \quad y = \tfrac{1}{2}at^2 + Vt + b,$$

where U, V, a, b are constants.

The elimination of t gives the equation of a parabola of which the axis is parallel to y, parameter $\frac{2U^2}{a}$, and vertex the point whose co-ordinates are

$$x = a - \frac{UV}{a}, \quad y = b - \frac{V^2}{2a}.$$

d. As an illustration of acceleration in a tortuous curve, we take the case of § 13, or of § 34, *c*.

Let a point move in a circle of radius r with uniform angular velocity ω (about the centre), and let this circle move perpendicular to its plane with velocity V. The point describes a helix on a cylinder of radius r, and the inclination a is given by

$$\tan a = \frac{V}{r\omega}.$$

The curvature of the path is $\frac{1}{r}\frac{r^2\omega^2}{V^2 + r^2\omega^2}$ or $\frac{r\omega^2}{V^2 + r^2\omega^2}$, and the tortuosity $\frac{\omega}{V}\frac{V^2}{V^2 + r^2\omega^2} = \frac{V\omega}{V^2 + r^2\omega^2}$.

The acceleration is $r\omega^2$, directed perpendicularly towards the axis of the cylinder.—Call this A.

$$\text{Curvature} = \frac{A}{V^2 + Ar} = \frac{A}{V^2 + \dfrac{A^2}{\omega^2}}.$$

$$\text{Tortuosity} = \frac{V}{\sqrt{Ar}}\frac{A}{V^2 + Ar} = \frac{V\omega}{V^2 + \dfrac{A^2}{\omega^2}}.$$

Let A be finite, r indefinitely small, and therefore ω indefinitely great.

$$\text{Curvature (in the limit)} = \frac{A}{V^2}.$$

$$\text{Tortuosity (} \quad ,, \quad \text{)} = \frac{\omega}{V}.$$

Thus, if we have a material particle moving in the manner specified, and if we consider the force (see Chap. II.) required to produce the acceleration, we find that a finite force perpendicular to

the line of motion, in a direction revolving with an infinitely great angular velocity, maintains constant infinitely small deflection (in a direction opposite to its own) from the line of undisturbed motion, *finite* curvature, and infinite tortuosity.

Examples of acceleration.

e. When the acceleration is perpendicular to a given plane and proportional to the distance from it, the path is a plane curve, which is the harmonic curve if the acceleration be *towards* the plane, and a more or less fore-shortened catenary (§ 580) if *from* the plane.

As in case *c*, $\frac{d^2z}{dt^2} = 0$, $\frac{dz}{dt} = 0$, and $z = 0$, if the axis of z be perpendicular to the acceleration and to the direction of motion at any instant. Also, if we choose the origin *in* the plane,

$$\frac{d^2x}{dt^2} = 0, \quad \frac{d^2y}{dt^2} = \mu y.$$

Hence $$\frac{dx}{dt} = \text{const.} = a \text{ (suppose)},$$

and $$\frac{d^2y}{dx^2} = \frac{\mu}{a^2}y = \mp \frac{y}{b^2}.$$

This gives, if μ is negative,

$$y = P\cos\left(\frac{x}{b} + Q\right), \text{ the harmonic curve, or curve of sines.}$$

If μ be positive, $\quad y = P\epsilon^{\frac{x}{b}} + Q\epsilon^{-\frac{x}{b}}$;

and by shifting the origin along the axis of x this can be put in the form

$$y = R\,(\epsilon^{\frac{x}{b}} + \epsilon^{-\frac{x}{b}}):$$

which is the catenary if $2R = b$; otherwise it is the catenary stretched or fore-shortened in the direction of y.

Acceleration directed to a fixed centre.

36. [Compare §§ 233—236 below.] *a.* When the acceleration is directed to a fixed point, the path is in a plane passing through that point; and in this plane the areas traced out by the radius-vector are proportional to the times employed. This includes the case of a satellite or planet revolving about its primary.

Evidently there is no acceleration perpendicular to the plane containing the fixed and moving points and the direction

Acceleration directed to a fixed centre.

of motion of the second at any instant; and, there being no velocity perpendicular to this plane at starting, there is therefore none throughout the motion; thus the point moves in the plane. And had there been no acceleration, the point would have described a straight line with uniform velocity, so that in this case the areas described by the radius-vector would have been proportional to the times. Also, the area actually described in any instant depends on the length of the radius-vector and the velocity perpendicular to it, and is shown below to be unaffected by an acceleration parallel to the radius-vector. Hence the second part of the proposition.

We have $\frac{d^2x}{dt^2} = P\frac{x}{r}, \quad \frac{d^2y}{dt^2} = P\frac{y}{r}, \quad \frac{d^2z}{dt^2} = P\frac{z}{r},$

the fixed point being the origin, and P being some function of x, y, z; in *nature* a function of r only.

Hence $$x\frac{d^2y}{dt^2} - y\frac{d^2x}{dt^2} = 0, \text{ etc.},$$

which give on integration

$$y\frac{dz}{dt} - z\frac{dy}{dt} = C_1, \quad z\frac{dx}{dt} - x\frac{dz}{dt} = C_2, \quad x\frac{dy}{dt} - y\frac{dx}{dt} = C_3.$$

Hence at once $C_1x + C_2y + C_3z = 0$, or the motion is in a plane through the origin. Take this as the plane of xy, then we have only the one equation

$$x\frac{dy}{dt} - y\frac{dx}{dt} = C_3 = h \text{ (suppose)}.$$

In polar co-ordinates this is

$$h = r^2\frac{d\theta}{dt} = 2\frac{dA}{dt}$$

if A be the area intercepted by the curve, a fixed radius-vector, and the radius-vector of the moving point. Hence the area increases uniformly with the time.

b. In the same case the velocity at any point is inversely as the perpendicular from the fixed point upon the tangent to the path, the momentary direction of motion.

For evidently the product of this perpendicular and the velocity gives double the area described in one second about the fixed point.

Acceleration directed to a fixed centre.

Or thus—if p be the perpendicular on the tangent,

$$p = x\frac{dy}{ds} - y\frac{dx}{ds},$$

and therefore $$p\frac{ds}{dt} = x\frac{dy}{dt} - y\frac{dx}{dt} = h.$$

If we refer the motion to co-ordinates in its own plane, we have only the equations

$$\frac{d^2x}{dt^2} = \frac{Px}{r}, \quad \frac{d^2y}{dt^2} = \frac{Py}{r},$$

whence, as before, $$r^2\frac{d\theta}{dt} = h.$$

If, by the help of this last equation, we eliminate t from $\frac{d^2x}{dt^2} = \frac{Px}{r}$, substituting polar for rectangular co-ordinates, we arrive at the polar differential equation of the path.

For variety, we may derive it from the formulæ of § 32.

They give $\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 = P, \quad r^2\frac{d\theta}{dt} = h.$

Putting $\frac{1}{r} = u$, we have

$$\frac{d^2\left(\frac{1}{u}\right)}{dt^2} - \frac{1}{u}\left(\frac{d\theta}{dt}\right)^2 = P, \text{ and } \frac{d\theta}{dt} = hu^2.$$

But $\frac{d\left(\frac{1}{u}\right)}{dt} = hu^2\frac{d\left(\frac{1}{u}\right)}{d\theta} = -h\frac{du}{d\theta}$, therefore $\frac{d^2\left(\frac{1}{u}\right)}{dt^2} = -h^2u^2\frac{d^2u}{d\theta^2}$.

Also $\frac{1}{u}\left(\frac{d\theta}{dt}\right)^2 = h^2u^3$, the substitution of which values gives us

$$\frac{d^2u}{d\theta^2} + u = -\frac{P}{h^2u^2} \quad \text{..........................(1)},$$

the equation required. The integral of this equation involves *two* arbitrary constants besides h, and the remaining constant belonging to the two differential equations of the second order above is to be introduced on the farther integration of

$$\frac{d\theta}{dt} = hu^2 \quad \text{..............................(2)},$$

when the value of u in terms of θ is substituted from the equation of the path.

Other examples of these principles will be met with in the chapters on Kinetics.

Hodograph.

37. If from any fixed point, lines be drawn at every instant, representing in magnitude and direction the velocity of a point describing any path in any manner, the extremities of these lines form a curve which is called the *Hodograph.* The invention of this construction is due to Sir W. R. Hamilton. One of the most beautiful of the many remarkable theorems to which it led him is that of § 38.

Since the radius-vector of the hodograph represents the velocity at each instant, it is evident (§ 27) that an elementary arc represents the velocity which must be compounded with the velocity at the beginning of the corresponding interval of time, to find the velocity at its end. Hence the velocity in the hodograph is equal to the acceleration in the path; and the tangent to the hodograph is parallel to the direction of the acceleration in the path.

If x, y, z be the co-ordinates of the moving point, ξ, η, ζ those of the corresponding point of the hodograph, then evidently

$$\xi = \frac{dx}{dt}, \quad \eta = \frac{dy}{dt}, \quad \zeta = \frac{dz}{dt},$$

and therefore

$$\frac{\frac{d\xi}{\frac{d^2x}{dt^2}}}{} = \frac{d\eta}{\frac{d^2y}{dt^2}} = \frac{d\zeta}{\frac{d^2z}{dt^2}},$$

or the tangent to the hodograph is parallel to the acceleration in the orbit. Also, if σ be the arc of the hodograph,

$$\frac{d\sigma}{dt} = \sqrt{\left(\frac{d\xi}{dt}\right)^2 + \left(\frac{d\eta}{dt}\right)^2 + \left(\frac{d\zeta}{dt}\right)^2}$$
$$= \sqrt{\left(\frac{d^2x}{dt^2}\right)^2 + \left(\frac{d^2y}{dt^2}\right)^2 + \left(\frac{d^2z}{dt^2}\right)^2},$$

or the velocity in the hodograph is equal to the rate of acceleration in the path.

Hodograph of planet or comet, deduced from Kepler's laws.

38. *The hodograph for the motion of a planet or comet is always a circle, whatever be the form and dimensions of the orbit.* In the motion of a planet or comet, the acceleration is directed towards the sun's centre. Hence (§ 36, *b*) the velocity is inversely as the perpendicular from that point upon the tangent to the orbit. The orbit we assume to be a conic section, whose focus is the sun's centre. But we know that the intersection of the perpendicular with the tangent lies in the circle whose diameter is the major axis, if the orbit be an ellipse or hyperbola; in the tangent at the vertex if a parabola. Measure off on the perpendicular a third proportional to its own length and any constant line; this portion will thus represent the velocity in magnitude and in a direction perpendicular to its own—so that the locus of the new points in each perpendicular will be the hodograph turned through a right angle. But we see by geometry* that the locus of these points is always a circle. Hence the proposition. The hodograph surrounds its origin if the orbit be an ellipse, passes through it if a parabola, and the origin is without the hodograph if the orbit is a hyperbola.

For a projectile unresisted by the air, it will be shewn in Kinetics that we have the equations (assumed in § 35, *c*)

$$\frac{d^2x}{dt^2} = 0, \quad \frac{d^2y}{dt^2} = -g,$$

if the axis of y be taken vertically upwards.

Hence for the hodograph

$$\frac{d\xi}{dt} = 0, \quad \frac{d\eta}{dt} = -g,$$

or $\xi = C$, $\eta = C' - gt$, and the hodograph is a vertical straight line along which the describing point moves uniformly.

Hodograph for planet or comet, deduced from Newton's law of force.

For the case of a planet or comet, instead of assuming as above that the orbit is a conic with the sun in one focus, assume (Newton's deduction from that and the law of areas) that the acceleration is in the direction of the radius-vector, and varies inversely as the square of the distance. We have obviously

$$\frac{d^2x}{dt^2} = \frac{\mu x}{r^3}, \quad \frac{d^2y}{dt^2} = \frac{\mu y}{r^3},$$

where $r^2 = x^2 + y^2$.

Hence, as in § 36, $x\dfrac{dy}{dt} - y\dfrac{dx}{dt} = h$ ....(1),

* See our smaller work, § 51.

Hodograph for planet or comet, deduced from Newton's law of force.

and therefore

$$\frac{d^2x}{dt^2} = \frac{\mu x}{h} \frac{x\dfrac{dy}{dt} - y\dfrac{dx}{dt}}{r^3},$$

$$= \frac{\mu}{h} \frac{(x^2+y^2)\dfrac{dy}{dt} - y\left(x\dfrac{dx}{dt} + y\dfrac{dy}{dt}\right)}{r^3}, \quad = \frac{\mu}{h} \frac{r^2\dfrac{dy}{dt} - yr\dfrac{dr}{dt}}{r^3}.$$

Hence
$$\frac{dx}{dt} + A = \frac{\mu}{h}\frac{y}{r} \quad\ldots\ldots\ldots\ldots(2).$$

Similarly
$$\frac{dy}{dt} + B = -\frac{\mu}{h}\frac{x}{r} \quad\ldots\ldots\ldots\ldots(3).$$

Hence for the hodograph

$$(\xi + A)^2 + (\eta + B)^2 = \frac{\mu^2}{h^2},$$

the circle as before stated.

We may merely mention that the equation of the orbit will be found at once by eliminating $\dfrac{dx}{dt}$ and $\dfrac{dy}{dt}$ among the three first integrals (1), (2), (3) above. We thus get

$$-h + Ay - Bx = \frac{\mu}{h} r,$$

a conic section of which the origin is a focus.

Applications of the hodograph.

39. The intensity of heat and light emanating from a point, or from an uniformly radiating spherical surface, diminishes with increasing distance according to the same law as gravitation. Hence the amount of heat and light, which a planet receives from the sun during any interval, is proportional to the time integral of the acceleration during that interval, *i.e.* (§ 37) to the corresponding arc of the hodograph. From this it is easy to see, for example, that if a comet move in a parabola, the amount of heat it receives from the sun in any interval is proportional to the angle through which its direction of motion turns during that interval. There is a corresponding theorem for a planet moving in an ellipse, but somewhat more complicated.

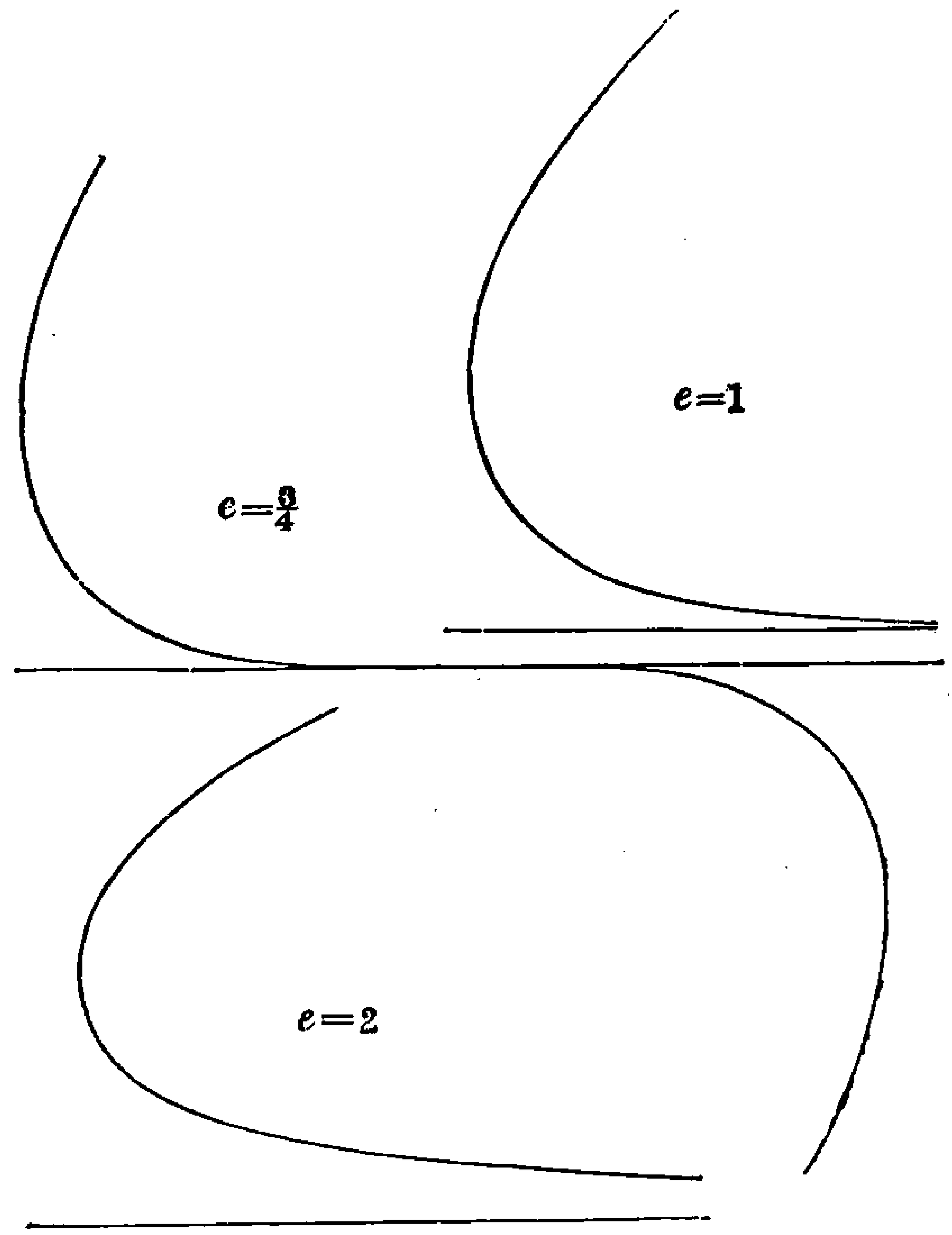
Curves of pursuit.

40. If two points move, each with a definite uniform velocity, one in a given curve, the other at every instant directing its course towards the first describes a path which is called a

Curves of pursuit.

Curve of Pursuit. The idea is said to have been suggested by the old rule of steering a privateer always directly for the vessel pursued. (Bouguer, *Mém. de l'Acad.* 1732.) It is the curve described by a dog running to its master.

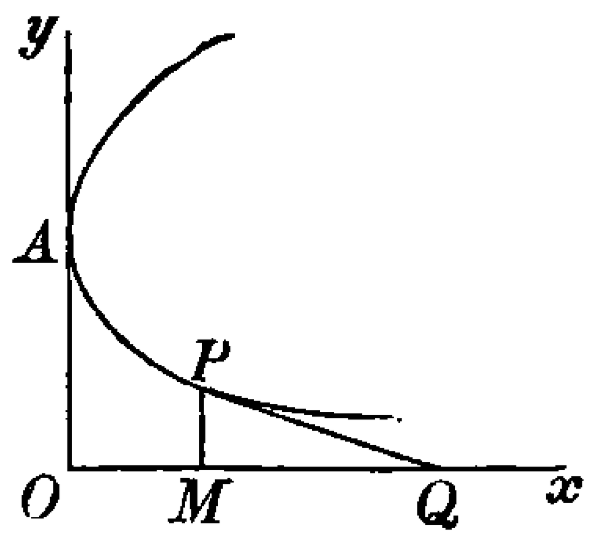
The simplest cases are those in which the first point moves in a straight line, and of these there are three, for the velocity of the first point may be greater than, equal to, or less than, that of the second. The figures in the text below represent the curves in these cases, the velocities of the pursuer being $\frac{4}{3}$, 1, and $\frac{1}{2}$ of those of the pursued, respectively. In the second and third cases the second point can never overtake the first, and consequently the line of motion of the first is an asymptote. In the first case the second point overtakes the first, and the curve at that point touches the line of motion of the first. The remainder of the curve satisfies a modified form of statement of the original question, and is called the *Curve of Flight.*



Curves of pursuit.

We will merely form the differential equation of the curve, and give its integrated form, leaving the work to the student.

Suppose Ox to be the line of motion of the first point, whose velocity is v, AP the curve of pursuit, in which the velocity is u, then the tangent at P always passes through Q, the instantaneous position of the first point. It will be evident, on a moment's consideration, that the curve AP must have a tangent perpendicular to Ox. Take this as the axis of y, and let $OA=a$. Then, if $OQ=\xi$, $AP=s$, and if x, y be the coordinates of P, we have



$$\frac{AP}{u}=\frac{OQ}{v},$$

because A, O and P, Q are pairs of simultaneous positions of the two points.

This gives $$\frac{v}{u}s=es=x-y\frac{dx}{dy}.$$

From this we find, unless $e=1$,

$$2\left(x+\frac{ae}{e^2-1}\right)=\frac{y^{e+1}}{a^e(e+1)}+\frac{a^e}{y^{e-1}(e-1)};$$

and if $e=1$, $$2\left(x+\frac{a}{4}\right)=\frac{y^2}{2a}-a\log_e\frac{y}{a},$$

the only case in which we do not get an algebraic curve. The axis of x is easily seen to be an asymptote if $e \not< 1$.

Angular velocity

41. When a point moves in any manner, the line joining it with a fixed point generally changes its direction. If, for simplicity, we consider the motion as confined to a plane passing through the fixed point, the angle which the joining line makes with a fixed line in the plane is continually altering, and its rate of alteration at any instant is called the *Angular Velocity* of the first point about the second. If uniform, it is of course measured by the angle described in unit of time; if variable, by the angle which would have been described in unit of time if the angular velocity at the instant in question were maintained constant for so long. In this respect, the process is precisely similar to that which we have already explained for the measurement of velocity and acceleration.

Unit of angular velocity is that of a point which describes, or would describe, unit angle about a fixed point in unit of time. The usual unit angle is (as explained in treatises on plane trigonometry) that which subtends at the centre of a circle an arc whose length is equal to the radius; being an angle of Angular velocity.

$$\frac{180^\circ}{\pi}=57^\circ.29578\ldots=57^\circ\,17'\,44''.8 \text{ nearly}.$$

For brevity we shall call this angle a radian.

42. The rate of increase or diminution of the angular velocity when variable is called the *angular acceleration*, and is measured in the same way and by the same unit. Angular acceleration.

By methods precisely similar to those employed for linear velocity and acceleration we see that if θ be the angle-vector of a point moving in a plane—the

Angular velocity is $\omega=\dfrac{d\theta}{dt}$, and the

Angular acceleration is $\dfrac{d\omega}{dt}=\dfrac{d^2\theta}{dt^2}=\omega\dfrac{d\omega}{d\theta}$.

Since (§ 27) $r\dfrac{d\theta}{dt}$ is the velocity perpendicular to the radius-vector, we see that

The angular velocity of a point in a plane is found by dividing the velocity perpendicular to the radius-vector by the length of the radius-vector.

43. When one point describes uniformly a circle about another, the time of describing a complete circumference being T, we have the angle 2π described uniformly in T; and, therefore, the angular velocity is $\dfrac{2\pi}{T}$. Even when the angular velocity is not uniform, as in a planet's motion, it is useful to introduce the quantity $\dfrac{2\pi}{T}$, which is then called the *mean* angular velocity. Angular velocity.

When a point moves uniformly in a straight line its angular velocity evidently diminishes as it recedes from the point about which the angles are measured.

Angular velocity.

The polar equation of a straight line is

$$r = a \sec \theta.$$

But the length of the line between the limiting angles 0 and θ is $a \tan \theta$, and this increases with uniform velocity v. Hence

$$v = \frac{d}{dt}(a \tan \theta) = a \sec^2 \theta \frac{d\theta}{dt} = \frac{r^2}{a} \frac{d\theta}{dt}.$$

Hence $\dfrac{d\theta}{dt} = \dfrac{av}{r^2}$, and is therefore inversely as the square of the radius-vector.

Similarly for the angular acceleration, we have by a second differentiation,

$$\frac{d^2\theta}{dt^2} + 2 \tan \theta \left(\frac{d\theta}{dt}\right)^2 = 0,$$

i.e., $\dfrac{d^2\theta}{dt^2} = -\dfrac{2av^2}{r^3}\left(1 - \dfrac{a^2}{r^2}\right)^{\frac{1}{2}}$, and ultimately varies inversely as the third power of the radius-vector.

Angular velocity of a plane.

44. We may also talk of the angular velocity of a moving plane with respect to a fixed one, as the rate of increase of the angle contained by them—but unless their line of intersection remain fixed, or at all events parallel to itself, a somewhat more laboured statement is required to give definite information. This will be supplied in a subsequent section.

Relative motion.

45. All motion that we are, or can be, acquainted with, is *Relative* merely. We can calculate from astronomical data for any instant the direction in which, and the velocity with which we are moving on account of the earth's diurnal rotation. We may compound this with the similarly calculable velocity of the earth in its orbit. This resultant again we may compound with the (roughly known) velocity of the sun relatively to the so-called fixed stars; but, even if all these elements were accurately known, it could not be said that we had attained any idea of an *absolute* velocity; for it is only the sun's relative motion among the stars that we can observe; and, in all probability, sun and stars are moving on (possibly with very great rapidity) relatively to other bodies in space. We must therefore consider how, from the actual motions of a set of points, we may find their relative motions with regard to any one of them; and how, having given the relative motions of all but one with regard to the latter, and the actual motion of the latter, we may find the actual motions of all. The question is very easily answered. Consider for a moment a number of passengers walking on the deck of a steamer. Their relative motions with regard to the deck are what we immediately observe, but if we compound with these the velocity of the steamer itself we get evidently their actual motion relatively to the earth. Again, in order to get the relative motion of all with regard to the deck, we *abstract our ideas from* the motion of the steamer altogether—that is, we alter the velocity of each by compounding it with the actual velocity of the vessel taken in a reversed direction.

Relative motion.

Hence to find the relative motions of any set of points with regard to one of their number, imagine, impressed upon each in composition with its own velocity, a velocity equal and opposite to the velocity of that one; it will be reduced to rest, and the motions of the others will be the same with regard to it as before.

Thus, to take a very simple example, two trains are running in opposite directions, say north and south, one with a velocity of fifty, the other of thirty, miles an hour. The relative velocity of the second with regard to the first is to be found by impressing on both a southward velocity of fifty miles an hour; the effect of this being to bring the first to rest, and to give the second a southward velocity of eighty miles an hour, which is the required relative motion.

Or, given one train moving north at the rate of thirty miles an hour, and another moving west at the rate of forty miles an hour. The motion of the second relatively to the first is at the rate of fifty miles an hour, in a south-westerly direction inclined to the due west direction at an angle of $\tan^{-1}\frac{3}{4}$. It is needless to multiply such examples, as they must occur to every one.

46. Exactly the same remarks apply to relative as compared with absolute acceleration, as indeed we may see at once, since accelerations are in all cases resolved and compounded by the same law as velocities.

Relative motion.

If x, y, z, and x', y', z', be the co-ordinates of two points referred to axes regarded as fixed; and ξ, η, ζ their relative co-ordinates—we have

$$\xi = x' - x, \quad \eta = y' - y, \quad \zeta = z' - z.$$

and, differentiating,

$$\frac{d\xi}{dt} = \frac{dx'}{dt} - \frac{dx}{dt}, \text{ etc.,}$$

which give the relative, in terms of the absolute, velocities; and

$$\frac{d^2\xi}{dt^2} = \frac{d^2x'}{dt^2} - \frac{d^2x}{dt^2}, \text{ etc.,}$$

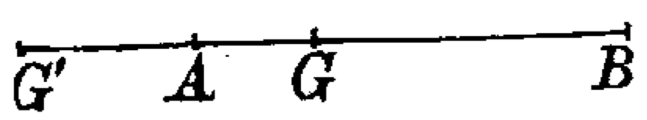
proving our assertion about relative and absolute accelerations.

The corresponding expressions in polar co-ordinates in a plane are somewhat complicated, and by no means convenient. The student can easily write them down for himself.

47. The following proposition in relative motion is of considerable importance:—

Any two moving points describe similar paths relatively to each other, or relatively to any point which divides in a constant ratio the line joining them.

Let A and B be any simultaneous positions of the points. Take G or G' in AB such that the ratio

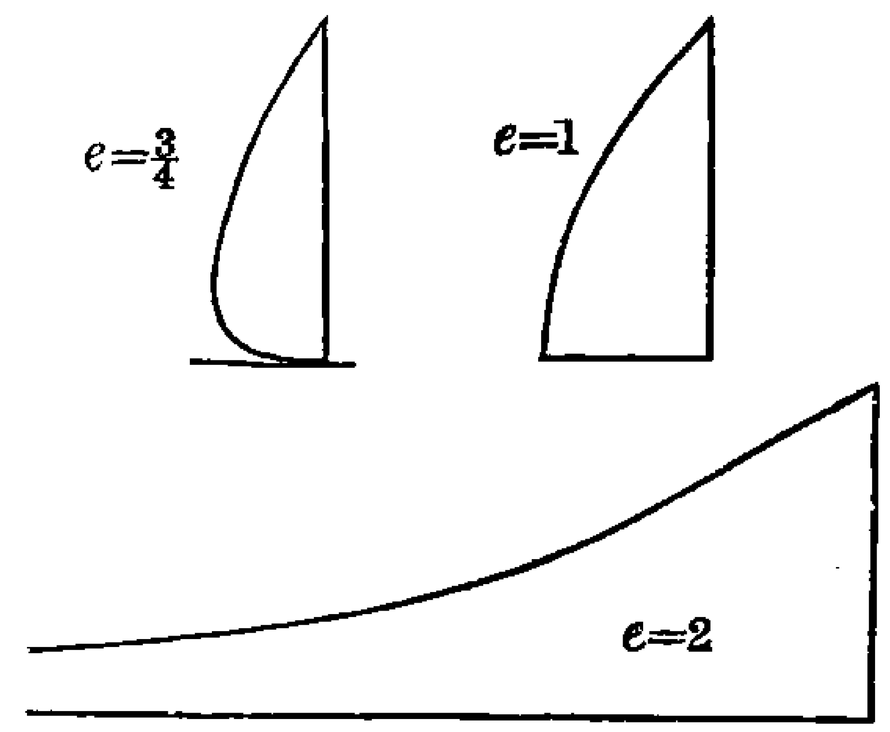


$\frac{GA}{GB}$ or $\frac{G'A}{G'B}$ has a constant value. Then as the form of the relative path depends only upon the *length* and *direction* of the line joining the two points at any instant, it is obvious that these will be the same for A with regard to B, as for B with regard to A, saving only the inversion of the direction of the joining line. Hence B's path about A, is A's about B turned through two right angles. And with regard to G and G' it is evident that the directions remain the same, while the lengths are altered in a given ratio; but this is the definition of similar curves.

48. As a good example of relative motion, let us consider that of the two points involved in our definition of the curve of pursuit, § 40. Since, to find the relative position and motion of the pursuer with regard to the pursued, we must impress on both a velocity equal and opposite to that of the latter, we see

Relative motion.

at once that the problem becomes the same as the following. A boat crossing a stream is impelled by the oars with uniform velocity relatively to the water, and always towards a fixed point in the opposite bank; but it is also carried down stream at a uniform rate; determine the path described and the time of crossing. Here, as in the former problem, there are three cases, figured below. In the first, the boat, moving faster than the current, reaches the desired point; in the second, the velocities of boat and stream being equal, the boat gets across only after an infinite time—describing a parabola—but does not land at the desired point, which is indeed the focus of the parabola, while the landing point is the vertex. In the third case, its proper velocity being less than that of the water, it never reaches the other bank, and is carried indefinitely down stream. The comparison of the figures in § 40 with those in the present section cannot fail to be instructive. They are drawn to the same scale, and for the same relative velocities. The horizontal lines represent the farther bank of the river, and the vertical lines the path of the boat if there were no current.



We leave the solution of this question as an exercise, merely noting that the equation of the curve is

$$\frac{y^{1+e}}{a^e} = \sqrt{x^2 + y^2} - x,$$

in one or other of the three cases, according as e is $>$, $=$, or <1.

When $e = 1$ this becomes

$$y^2 = a^2 - 2ax, \text{ the parabola.}$$

The time of crossing is

$$\frac{a}{u(1-e^2)},$$

which is finite only for $e < 1$, because of course a negative value is inadmissible.

Relative motion.

49. Another excellent example of the transformation of relative into absolute motion is afforded by the family of cycloids. We shall in a future section consider their mechanical description, by the *rolling* of a circle on a fixed straight line or circle. In the mean time, we take a different form of enunciation, which, however, leads to precisely the same result.

Find the actual path of a point which revolves uniformly in a circle about another point—the latter moving uniformly in a straight line or circle in the same plane.

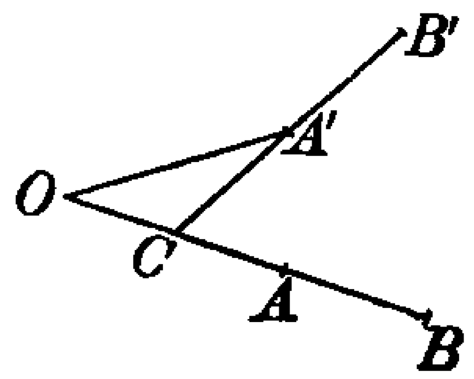
Take the former case first: let a be the radius of the relative circular orbit, and ω the angular velocity in it, v being the velocity of its centre along the straight line.

The relative co-ordinates of the point in the circle are $a\cos\omega t$ and $a\sin\omega t$, and the actual co-ordinates of the centre are vt and 0. Hence for the actual path

$$\xi = vt + a\cos\omega t, \quad \eta = a\sin\omega t.$$

Hence $\xi = \frac{v}{\omega}\sin^{-1}\frac{\eta}{a} + \sqrt{a^2-\eta^2}$, an equation which, by giving different values to v and ω, may be made to represent the cycloid itself, or either form of trochoid. See § 92.

For the epicycloids, let b be the radius of the circle which B describes about A, ω_1 the angular velocity; a the radius of A's path, ω the angular velocity.



Also at time $t=0$, let B be in the radius OA of A's path. Then at time t, if A', B' be the positions, we see at once that

$$\angle AOA' = \omega t, \quad \angle B'CA = \omega_1 t.$$

Hence, taking OA as axis of x,

$$x = a\cos\omega t + b\cos\omega_1 t, \quad y = a\sin\omega t + b\sin\omega_1 t,$$

which, by the elimination of t, give an algebraic equation between x and y whenever ω and ω_1 are commensurable.

Thus, for $\omega_1 = 2\omega$, suppose $\omega t = \theta$, and we have

$$x = a\cos\theta + b\cos 2\theta, \quad y = a\sin\theta + b\sin 2\theta,$$

or, by an easy reduction,

$$(x^2+y^2-b^2)^2 = a^2\{(x+b)^2+y^2\}.$$

Relative motion.

Put $x-b$ for x, *i.e.*, change the origin to a distance AB to the left of O, the equation becomes

$$a^2(x^2+y^2) = (x^2+y^2-2bx)^2,$$

or, in polar co-ordinates,

$$a^2 = (r-2b\cos\theta)^2, \quad r = a+2b\cos\theta,$$

and when $2b=a$, $r = a(1+\cos\theta)$, the cardioid. (See § 94.)

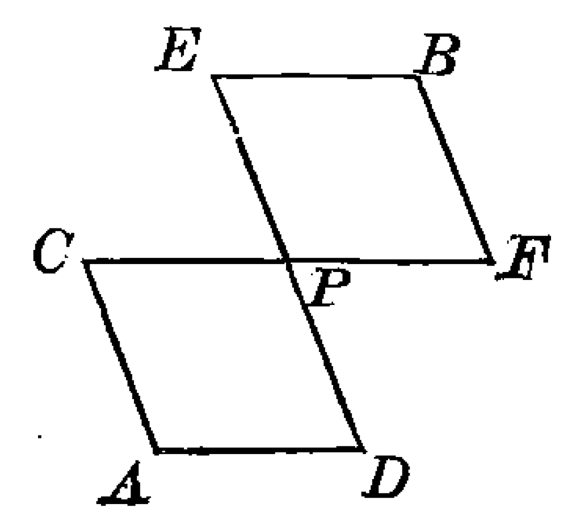
Resultant motion.

50. As an additional illustration of this part of our subject, we may define as follows:—

If one point A executes any motion whatever with reference to a second point B; if B executes any other motion with reference to a third point C; and so on—the first is said to execute, with reference to the last, a movement which is the resultant of these several movements.

The relative position, velocity, and acceleration are in such a case the geometrical resultants of the various components combined according to preceding rules.

51. The following practical methods of effecting such a combination in the simple case of the movements of two points are useful in scientific illustrations and in certain mechanical arrangements. Let two moving points be joined by an elastic string; the middle point of this string will evidently execute a movement which is *half* the resultant of the motions of the two points. But for drawing, or engraving, or for other mechanical applications, the following method is preferable:—



CF and ED are rods of equal length moving freely round a pivot at P, which passes through the middle point of each—CA, AD, EB, and BF, are rods of half the length of the two former, and so pivoted to them as to form a pair of equal rhombi CD, EF, whose angles can be altered at will. Whatever motions, whether in a plane, or in space of three dimensions, be given to A and B, P will evidently be subjected to half their resultant.

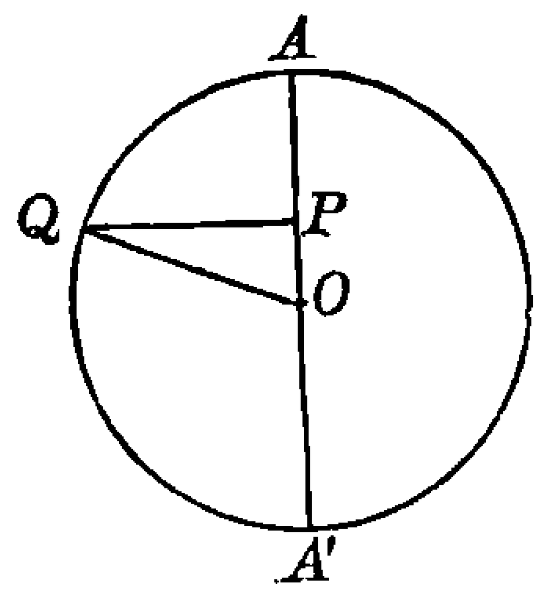
Harmonic motion.

52. Amongst the most important classes of motions which we have to consider in Natural Philosophy, there is one, namely, *Harmonic Motion*, which is of such immense use, not only in

Harmonic motion. ordinary kinetics, but in the theories of sound, light, heat, etc., that we make no apology for entering here into considerable detail regarding it.

Simple harmonic motion.

53. *Def.* When a point Q moves uniformly in a circle, the perpendicular QP drawn from its position at any instant to a fixed diameter AA' of the circle, intersects the diameter in a point P, whose position changes by a *simple harmonic motion.*



Thus, if a planet or satellite, or one of the constituents of a double star, supposed to move uniformly in a circular orbit about its primary, be viewed from a very distant position in the plane of its orbit, it will appear to move backwards and forwards in a straight line, with a simple harmonic motion. This is nearly the case with such bodies as the satellites of Jupiter when seen from the earth.

Physically, the interest of such motions consists in the fact of their being approximately those of the simplest vibrations of sounding bodies, such as a tuning-fork or pianoforte wire; whence their name; and of the various media in which waves of sound, light, heat, etc., are propagated.

54. The *Amplitude* of a simple harmonic motion is the range on one side or the other of the middle point of the course, *i.e.*, OA or OA' in the figure.

An arc of the circle referred to, measured from any fixed point to the uniformly moving point Q, is the *Argument* of the harmonic motion.

The distance of a point, performing a simple harmonic motion, from the middle of its course or range, is a *simple harmonic function of the time.* The *argument* of this function is what we have defined as the argument of the motion.

The *Epoch* in a simple harmonic motion is the interval of time which elapses from the era of reckoning till the moving point first comes to its greatest elongation in the direction reckoned as positive, from its mean position or the middle of its range. Epoch in angular measure is the angle described on the circle of reference in the period of time defined as the epoch.

Simple harmonic motion. The *Period* of a simple harmonic motion is the time which elapses from any instant until the moving point again moves in the same direction through the same position.

The *Phase* of a simple harmonic motion at any instant is the fraction of the whole period which has elapsed since the moving point last passed through its middle position in the positive direction.

Simple harmonic motion in mechanism

55. Those common kinds of mechanism, for producing rectilineal from circular motion, or *vice versa,* in which a crank moving in a circle works in a straight slot belonging to a body which can only move in a straight line, fulfil strictly the definition of a simple harmonic motion in the part of which the motion is rectilineal, if the motion of the rotating part is uniform.

The motion of the treadle in a spinning-wheel approximates to the same condition when the wheel moves uniformly; the approximation being the closer, the smaller is the angular motion of the treadle and of the connecting string. It is also approximated to more or less closely in the motion of the piston of a steam-engine connected, by any of the several methods in use, with the crank, provided always the rotatory motion of the crank be uniform.

Velocity in S. H. motion.

56. The velocity of a point executing a simple harmonic motion is a simple harmonic function of the time, a quarter of a period earlier in phase than the displacement, and having its maximum value equal to the velocity in the circular motion by which the given function is defined.

For, in the fig. of § 53, if V be the velocity in the circle, it may be represented by OQ in a direction perpendicular to its own, and therefore by OP and PQ in directions perpendicular to those lines. That is, the velocity of P in the simple harmonic motion is $\frac{V}{OQ} PQ$; which, when P is at O, becomes V.

Acceleration in S. H. motion.

57. The acceleration of a point executing a simple harmonic motion is at any time simply proportional to the displacement from the middle point, but in opposite direction, or always towards the middle point. Its maximum value is that with which a velocity equal to that of the circular motion would

Acceleration in S. H. motion.

be acquired in the time in which an arc equal to the radius is described.

For, in the fig. of § 53, the acceleration of Q (by § 35, a) is $\frac{V^2}{QO}$ along QO. Supposing, for a moment, QO to represent the magnitude of this acceleration, we may resolve it in QP, PO. The acceleration of P is therefore represented on the same scale by PO. Its magnitude is therefore $\frac{V^2}{QO}\cdot\frac{PO}{QO}=\frac{V^2}{QO^2}PO$, which is proportional to PO, and has at A its maximum value, $\frac{V^2}{QO}$, an acceleration under which the velocity V would be acquired in the time $\frac{QO}{V}$ as stated.

Let a be the amplitude, ϵ the epoch, and T the period, of a simple harmonic motion. Then if s be the displacement from middle position at time t, we have

$$s=a\cos\left(\frac{2\pi t}{T}-\epsilon\right).$$

Hence, for velocity, we have

$$v=\frac{ds}{dt}=-\frac{2\pi a}{T}\sin\left(\frac{2\pi t}{T}-\epsilon\right).$$

Hence V, the maximum value, is $\frac{2\pi a}{T}$, as above stated (§ 56).

Again, for acceleration,

$$\frac{dv}{dt}=-\frac{4\pi^2 a}{T^2}\cos\left(\frac{2\pi t}{T}-\epsilon\right)=-\frac{4\pi^2}{T^2}s. \quad \text{(See § 57.)}$$

Lastly, for the maximum value of the acceleration,

$$\frac{4\pi^2 a}{T^2}=\frac{V}{\frac{T}{2\pi}},$$

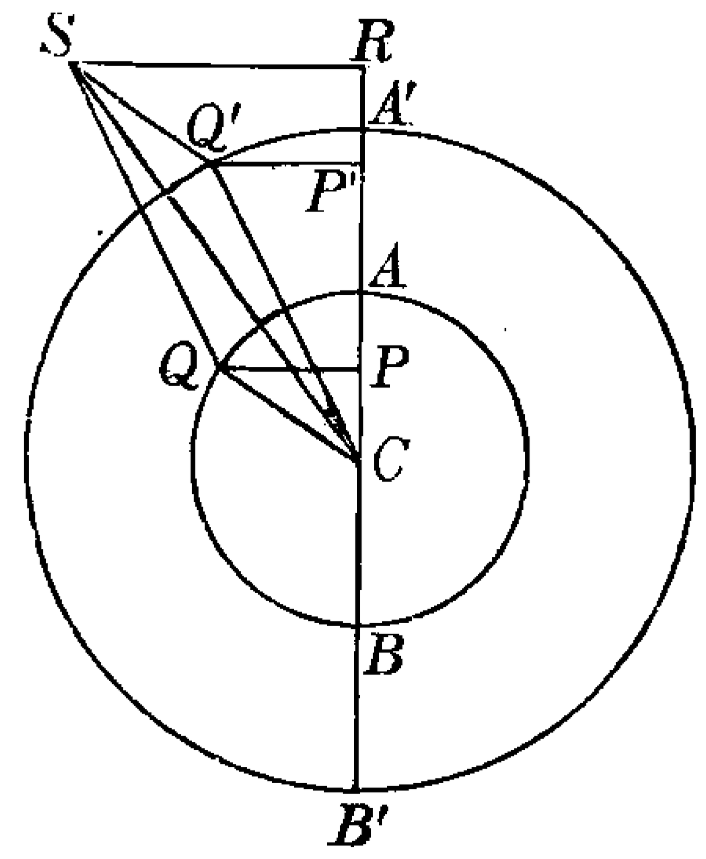
where, it may be remarked, $\frac{T}{2\pi}$ is the time of describing an arc equal to radius in the relative circular motion.

Composition of S. H. M. in one line.

58. Any two simple harmonic motions in one line, and of one period, give, when compounded, a single simple harmonic motion; of the same period; of amplitude equal to the diagonal of a parallelogram described on lengths equal to their amplitudes measured on lines meeting at an angle equal to their difference

Composition of S. H. M. in one line.

of epochs; and of epoch differing from their epochs by angles equal to those which this diagonal makes with the two sides of the parallelogram. Let P and P' be two points executing simple harmonic motions of one period, and in one line $B'BCAA'$. Let Q and Q' be the uniformly moving points in the relative circles. On CQ and CQ' describe a parallelogram $SQCQ'$; and through S draw SR perpendicular to $B'A'$ produced. We have obviously $P'R=CP$ (being projections of the equal and parallel lines $Q'S$, CQ, on CR). Hence $CR=CP+CP'$; and therefore the point R executes the resultant of the motions P and P'. But CS, the diagonal of the parallelogram, is constant, and therefore the resultant motion is simple harmonic, of amplitude CS, and of epoch exceeding that of the motion of P, and falling short of that of the motion of P', by the angles QCS and SCQ' respectively.



This geometrical construction has been usefully applied by the tidal committee of the British Association for a mechanical tide-indicator (compare § 60, below). An arm CQ' turning round C carries an arm $Q'S$ turning round Q'. Toothed wheels, one of them fixed with its axis through C, and the others pivoted on a framework carried by CQ', are so arranged that $Q'S$ turns very approximately at the rate of once round in 12 mean lunar hours, if CQ' be turned uniformly at the rate of once round in 12 mean solar hours. Days and half-days are marked by a counter geared to CQ'. The distance of S from a fixed line through C shows the deviation from mean sea-level due to the sum of mean solar and mean lunar tides for the time of day and year marked by CQ' and the counter.

An analytical proof of the same proposition is useful, being as follows:—

$$a\cos\left(\frac{2\pi t}{T}-\epsilon\right)+a'\cos\left(\frac{2\pi t}{T}-\epsilon'\right)$$

$$=(a\cos\epsilon+a'\cos\epsilon')\cos\frac{2\pi t}{T}+(a\sin\epsilon+a'\sin\epsilon')\sin\frac{2\pi t}{T}=r\cos\left(\frac{2\pi t}{T}-\theta\right),$$

Composition of S. H. M. in one line.

where $$r = \{(a\cos\epsilon + a'\cos\epsilon')^2 + (a\sin\epsilon + a'\sin\epsilon')^2\}^{\frac{1}{2}}$$
$$= \{a^2 + a'^2 + 2aa'\cos(\epsilon - \epsilon')\}^{\frac{1}{2}},$$

and $$\tan\theta = \frac{a\sin\epsilon + a'\sin\epsilon'}{a\cos\epsilon + a'\cos\epsilon'}.$$

59. The construction described in the preceding section exhibits the resultant of two simple harmonic motions, whether of the same period or not. Only, if they are not of the same period, the diagonal of the parallelogram will not be constant, but will diminish from a maximum value, the sum of the component amplitudes, which it has at the instant when the phases of the component motions agree; to a minimum, the difference of those amplitudes, which is its value when the phases differ by half a period. Its direction, which always must be nearer to the greater than to the less of the two radii constituting the sides of the parallelogram, will oscillate on each side of the greater radius to a maximum deviation amounting on either side to the angle whose sine is the less radius divided by the greater, and reached when the less radius deviates more than this by a quarter circumference from the greater. The full period of this oscillation is the time in which either radius gains a full turn on the other. The resultant motion is therefore not simple harmonic, but is, as it were, simple harmonic with periodically increasing and diminishing amplitude, and with periodical acceleration and retardation of phase. This view is particularly appropriate for the case in which the periods of the two component motions are nearly equal, but the amplitude of one of them much greater than that of the other.

To express the resultant motion, let s be the displacement at time t; and let a be the greater of the two component half-amplitudes.

$$s = a\cos(nt - \epsilon) + a'\cos(n't - \epsilon')$$
$$= a\cos(nt - \epsilon) + a'\cos(nt - \epsilon + \phi)$$
$$= (a + a'\cos\phi)\cos(nt - \epsilon) - a'\sin\phi\sin(nt - \epsilon),$$

if $$\phi = (n't - \epsilon') - (nt - \epsilon);$$

or, finally, $$s = r\cos(nt - \epsilon + \theta),$$

Composition of S. H. M. in one line.

if $$r = (a^2 + 2aa'\cos\phi + a'^2)^{\frac{1}{2}}$$

and $$\tan\theta = \frac{a\sin\phi}{a + a'\cos\phi}.$$

The maximum value of $\tan\theta$ in the last of these equations is found by making $\phi = \frac{\pi}{2} + \sin^{-1}\frac{a'}{a}$, and is equal to $\frac{a'}{(a^2 - a'^2)^{\frac{1}{2}}}$, and hence the maximum value of θ itself is $\sin^{-1}\frac{a'}{a}$. The geometrical methods indicated above (§ 58) lead to this conclusion by the following very simple construction.

To find the time and the amount of the maximum acceleration or retardation of phase, let CA be the greater half-amplitude. From A as centre, with the less half-amplitude as radius, describe a circle. CB touching this circle is the generating radius of the most deviated resultant. Hence CBA is a right angle; and
$$\sin BCA = \frac{AB}{CA}.$$

Examples of composition of S. H. M in one line.

60. A most interesting application of this case of the composition of harmonic motions is to the lunar and solar tides; which, except in tidal rivers, or long channels, or deep bays, follow each very nearly the simple harmonic law, and produce, as the actual result, a variation of level equal to the sum of variations that would be produced by the two causes separately.

The amount of the lunar equilibrium-tide (§ 812) is about 2·1 times that of the solar. Hence, if the actual tides conformed to the equilibrium theory, the spring tides would be 3·1, and the neap tides only 1·1, each reckoned in terms of the solar tide; and at spring and neap tides the hour of high water is that of the lunar tide alone. The greatest deviation of the actual tide from the phases (high, low, or mean water) of the lunar tide alone, would be about ·95 of a lunar hour, that is, ·98 of a solar hour (being the same part of 12 lunar hours that $28^0\ 26'$, or the angle whose sine is $\frac{1}{2{\cdot}1}$, is of 360^0). This maximum deviation would be in advance or in arrear according as the crown of the solar tide precedes or follows the crown of the lunar tide; and it would be exactly reached when the interval of phase between

Examples of composition of S. H. M. in one line. the two component tides is 3·95 lunar hours. That is to say, there would be maximum advance of the time of high water $4\frac{1}{2}$ days after, and maximum retardation the same number of days before, spring tides (compare § 811).

61. We may consider next the case of equal amplitudes in the two given motions. If their periods are equal, their resultant is a simple harmonic motion, whose phase is at every instant the mean of their phases, and whose amplitude is equal to twice the amplitude of either multiplied by the cosine of half the difference of their phases. The resultant is of course nothing when their phases differ by half the period, and is a motion of double amplitude and of phase the same as theirs when they are of the same phase.

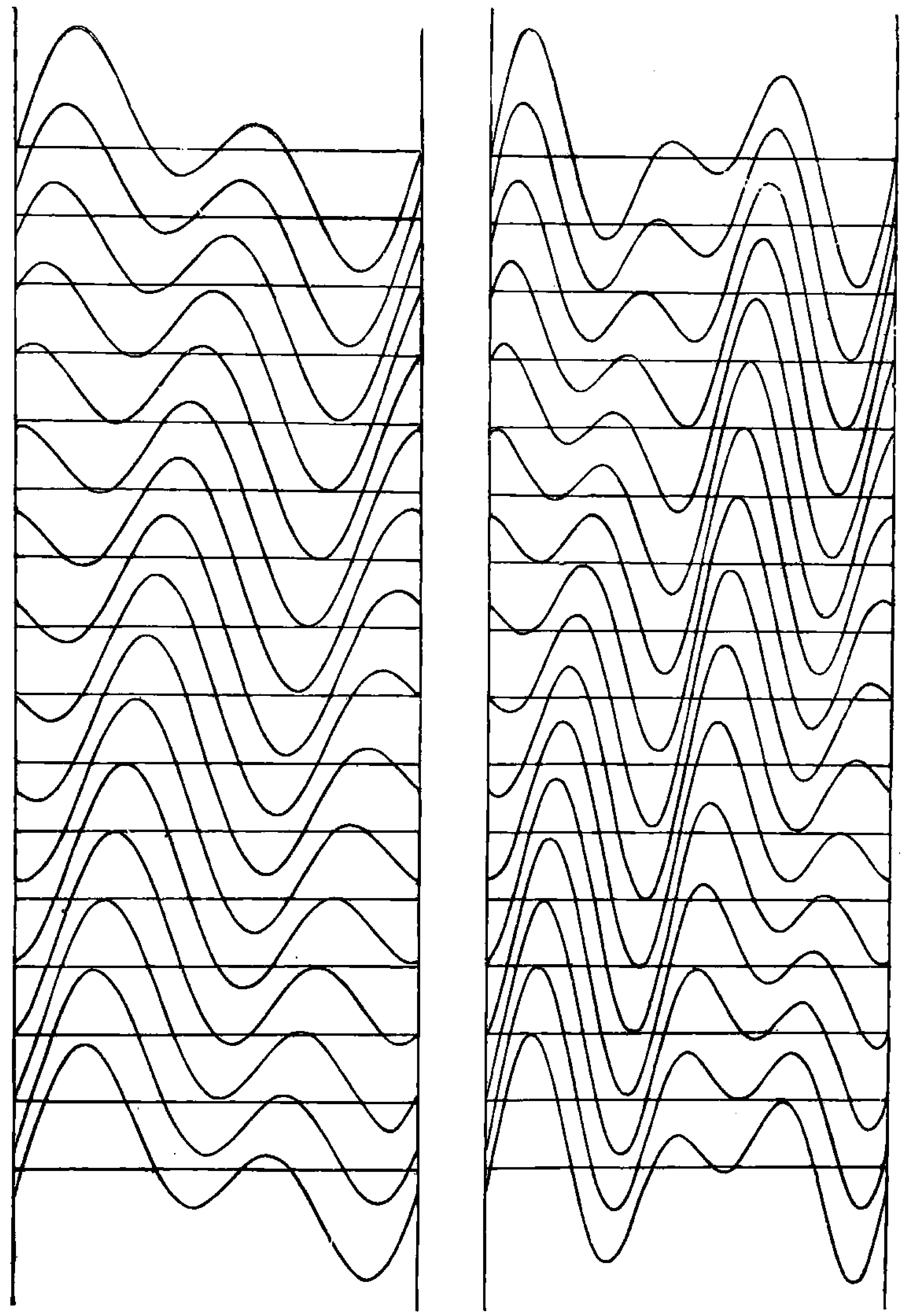
When their periods are very nearly, but not quite, equal (their amplitudes being still supposed equal), the motion passes very slowly from the former (zero, or no motion at all) to the latter, and back, in a time equal to that in which the faster has gone once oftener through its period than the slower has.

In practice we meet with many excellent examples of this case, which will, however, be more conveniently treated of when we come to apply kinetic principles to various subjects in acoustics, physical optics, and practical mechanics; such as the sympathy of pendulums or tuning-forks, the rolling of a turret ship at sea, the marching of troops over a suspension bridge, etc.

Mechanism for compounding S. H. motions in one line.

62. If any number of pulleys be so placed that a cord passing from a fixed point half round each of them has its free parts all in parallel lines, and if their centres be moved with simple harmonic motions of any ranges and any periods in lines parallel to those lines, the unattached end of the cord moves with a complex harmonic motion equal to twice the sum of the given simple harmonic motions. This is the principle of Sir W. Thomson's tide-predicting machine, constructed by the British Association, and ordered to be placed in South Kensington Museum, availably for general use in calculating beforehand for any port or other place on the sea for which the simple harmonic constituents of the tide have been determined by the "harmonic analysis" applied to

previous observations*. We may exhibit, graphically, any case of single or compound simple harmonic motion in one line by curves in which the abscissæ represent intervals of time, and the **Graphical representation of harmonic motions in one line.**



* See British Association Tidal Committee's Report, 1868, 1872, 1875: or *Lecture on Tides*, by Sir W. Thomson, "Popular Lectures and Addresses," vol. III. p. 178.

Graphical representation of harmonic motions in one line.

ordinates the corresponding distances of the moving point from its mean position. In the case of a single simple harmonic motion, the corresponding curve would be that described by the point P in § 53, if, while Q maintained its uniform circular motion, the circle were to move with uniform velocity in any direction perpendicular to OA. This construction gives the harmonic curve, or curves of sines, in which the ordinates are proportional to the sines of the abscissæ, the straight line in which O moves being the axis of abscissæ. It is the simplest possible form assumed by a vibrating string. When the harmonic motion is complex, but in one line, as is the case for any point in a violin-, harp-, or pianoforte-string (differing, as these do, from one another in their motions on account of the different modes of excitation used), a similar construction may be made. Investigation regarding complex harmonic functions has led to results of the highest importance, having their most general expression in *Fourier's Theorem*, to which we will presently devote several pages. We give, on page 45, graphic representations of the composition of two simple harmonic motions in one line, of *equal* amplitudes and of periods which are as 1 : 2 and as 2 : 3, for differences of epoch corresponding to 0, 1, 2, etc., sixteenths of a circumference. In each case the epoch of the component of greater period is a quarter of its own period. In the first, second, third, etc., of each series respectively, the epoch of the component of shorter period is less than a quarter-period by 0, 1, 2, etc., sixteenths of the period. The successive horizontal lines are the axes of abscissæ of the successive curves; the vertical line to the left of each series being the common axis of ordinates. In each of the first set the graver motion goes through one complete period, in the second it goes through two periods.

1 : 2 (Octave)	2 : 3 (Fifth)
$y=\sin x+\sin\left(2x+\frac{n\pi}{8}\right).$	$y=\sin 2x+\sin\left(3x+\frac{n\pi}{8}\right).$

Both, from $x=0$ to $x=2\pi$; and for $n=0, 1, 2......15$, in succession.

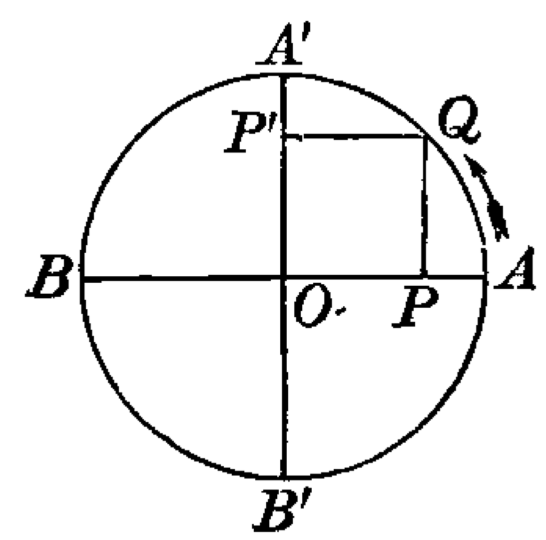
These, and similar cases, when the periodic times are not commensurable, will be again treated of under Acoustics.

S. H. motions in different directions.

63. We have next to consider the composition of simple harmonic motions in different directions. In the first place, we see that any number of simple harmonic motions of one period, and of the same phase, superimposed, produce a single simple harmonic motion of the same phase. For, the displacement at any instant being, according to the principle of the composition of motions, the geometrical resultant (see above, § 50) of the displacements due to the component motions separately, these component displacements, in the case supposed, all vary in simple proportion to one another, and are in constant directions. Hence the resultant displacement will vary in simple proportion to each of them, and will be in a constant direction.

But if, while their periods are the same, the phases of the several component motions do not agree, the resultant motion will generally be elliptic, with equal areas described in equal times by the radius-vector from the centre; although in particular cases it may be uniform circular, or, on the other hand, rectilineal and simple harmonic.

64. To prove this, we may first consider the case in which we have two equal simple harmonic motions given, and these in perpendicular lines, and differing in phase by a quarter period. Their resultant is a uniform circular motion. For, let BA, $B'A'$ be their ranges; and from O, their common middle point, as centre, describe a circle through $AA'BB'$. The given motion of P in BA will be (§ 53) defined by the motion of a point Q round the circumference of this circle; and the same point, if moving in the direction indicated by the arrow, will give a simple harmonic motion of P', in $B'A'$, a quarter of a period behind that of the motion of P in BA. But, since $A'OA$, QPO, and $QP'O$ are right angles, the figure $QP'OP$ is a parallelogram, and therefore Q is in the position of the displacement compounded of OP and OP'. Hence two equal simple harmonic motions in perpendicular lines, of phases differing by a quarter period, are equivalent to a uniform circular motion of radius equal to the maximum displacement of either singly, and in the direction from the positive end of the range of



S. H. motions in different directions.

the component in advance of the other towards the positive end of the range of this latter.

65. Now, orthogonal projections of simple harmonic motions are clearly simple harmonic with unchanged phase. Hence, if we project the case of § 64 on any plane, we get motion in an ellipse, of which the projections of the two component ranges are conjugate diameters, and in which the radius-vector from the centre describes equal areas (being the projections of the areas described by the radius of the circle) in equal times. But the plane and position of the circle of which this projection is taken may clearly be found so as to fulfil the condition of having the projections of the ranges coincident with any two given mutually bisecting lines. Hence any two given simple harmonic motions, equal or unequal in range, and oblique or at right angles to one another in direction, provided only they differ by a quarter period in phase, produce elliptic motion, having their ranges for conjugate axes, and describing, by the radius-vector from the centre, equal areas in equal times (compare § 34, *b*).

66. Returning to the composition of any number of simple harmonic motions of one period, in lines in all directions and of all phases: each component simple harmonic motion may be determinately resolved into two in the same line, differing in phase by a quarter period, and one of them having any given epoch. We may therefore reduce the given motions to two sets, differing in phase by a quarter period, those of one set agreeing in phase with any one of the given, or with any other simple harmonic motion we please to choose (*i.e.*, having their epoch anything we please).

All of each set may (§ 58) be compounded into one simple harmonic motion of the same phase, of determinate amplitude, in a determinate line; and thus the whole system is reduced to two simple fully determined harmonic motions differing from one another in phase by a quarter period.

Now the resultant of two simple harmonic motions, one a quarter of a period in advance of the other, in different lines, has been proved (§ 65) to be motion in an ellipse of which the ranges of the component motions are conjugate axes, and in which equal

S. H. motions in different directions.

areas are described by the radius-vector from the centre in equal times. Hence the general proposition of § 63.

Let
$$\left.\begin{aligned} x_1 &= l_1 a_1 \cos(\omega t - \epsilon_1),\\ y_1 &= m_1 a_1 \cos(\omega t - \epsilon_1),\\ z_1 &= n_1 a_1 \cos(\omega t - \epsilon_1),\end{aligned}\right\} \qquad (1)$$

be the Cartesian specification of the first of the given motions; and so with varied suffixes for the others;

l, m, n denoting the direction cosines,
a „ „ half amplitude,
ϵ „ „ epoch,

the proper suffix being attached to each letter to apply it to each case, and ω denoting the common relative angular velocity. The resultant motion, specified by x, y, z without suffixes, is

$$x = \Sigma l_1 a_1 \cos(\omega t - \epsilon_1) = \cos\omega t \Sigma l_1 a_1 \cos\epsilon_1 + \sin\omega t \Sigma l_1 a_1 \sin\epsilon_1,$$
$$y = \text{etc.}; \quad z = \text{etc.};$$

or, as we may write for brevity,

$$\left.\begin{aligned} x &= P\cos\omega t + P'\sin\omega t,\\ y &= Q\cos\omega t + Q'\sin\omega t,\\ z &= R\cos\omega t + R'\sin\omega t,\end{aligned}\right\} \qquad (2)$$

where
$$\left.\begin{aligned} P &= \Sigma\, l_1 a_1 \cos\epsilon_1, & P' &= \Sigma\, l_1 a_1 \sin\epsilon_1,\\ Q &= \Sigma m_1 a_1 \cos\epsilon_1, & Q' &= \Sigma m_1 a_1 \sin\epsilon_1,\\ R &= \Sigma\, n_1 a_1 \cos\epsilon_1, & R' &= \Sigma\, n_1 a_1 \sin\epsilon_1.\end{aligned}\right\} \qquad (3)$$

The resultant motion thus specified, in terms of six component simple harmonic motions, may be reduced to two by compounding P, Q, R, and P', Q', R', in the elementary way. Thus if

$$\left.\begin{aligned} &\zeta = (P^2 + Q^2 + R^2)^{\frac{1}{2}},\\ &\lambda = \frac{P}{\zeta}, \quad \mu = \frac{Q}{\zeta}, \quad \nu = \frac{R}{\zeta},\\ &\zeta' = (P'^2 + Q'^2 + R'^2)^{\frac{1}{2}},\\ &\lambda' = \frac{P'}{\zeta'}, \quad \mu' = \frac{Q'}{\zeta'}, \quad \nu' = \frac{R'}{\zeta'},\end{aligned}\right\} \qquad (4)$$

the required motion will be the resultant of $\zeta\cos\omega t$ in the line (λ, μ, ν), and $\zeta'\sin\omega t$ in the line (λ', μ', ν'). It is therefore motion in an ellipse, of which 2ζ and $2\zeta'$ in those directions are

S. H. motions in different directions.

conjugate diameters; with radius-vector from centre tracing equal areas in equal times; and of period $\frac{2\pi}{\omega}$.

H. motions of different kinds and in different lines.

67. We must next take the case of the composition of simple harmonic motions of *different* periods and in different lines. In general, whether these lines be in one plane or not, the line of motion returns into itself if the periods are commensurable; and if not, not. This is evident without proof.

If a be the amplitude, ϵ the epoch, and n the angular velocity in the relative circular motion, for a component in a line whose direction cosines are λ, μ, ν—and if ξ, η, ζ be the co-ordinates in the resultant motion,

$$\xi = \Sigma . \lambda_1 a_1 \cos(n_1 t - \epsilon_1), \quad \eta = \Sigma . \mu_1 a_1 \cos(n_1 t - \epsilon_1), \quad \zeta = \Sigma . \nu_1 a_1 \cos(n_1 t - \epsilon_1).$$

Now it is evident that at time $t+T$ the values of ξ, η, ζ will recur as soon as $n_1 T$, $n_2 T$, etc., are multiples of 2π, that is, when T is the least common multiple of $\frac{2\pi}{n_1}$, $\frac{2\pi}{n_2}$, etc.

If there be such a common multiple, the trigonometrical functions may be eliminated, and the equations (or equation, if the motion is in one plane) to the path are algebraic. If not, they are transcendental.

68. From the above we see generally that the composition of any number of simple harmonic motions in any directions and of any periods, may be effected by compounding, according to previously explained methods, their resolved parts in each of any three rectangular directions, and then compounding the final resultants in these directions.

S. H. motions in two rectangular directions.

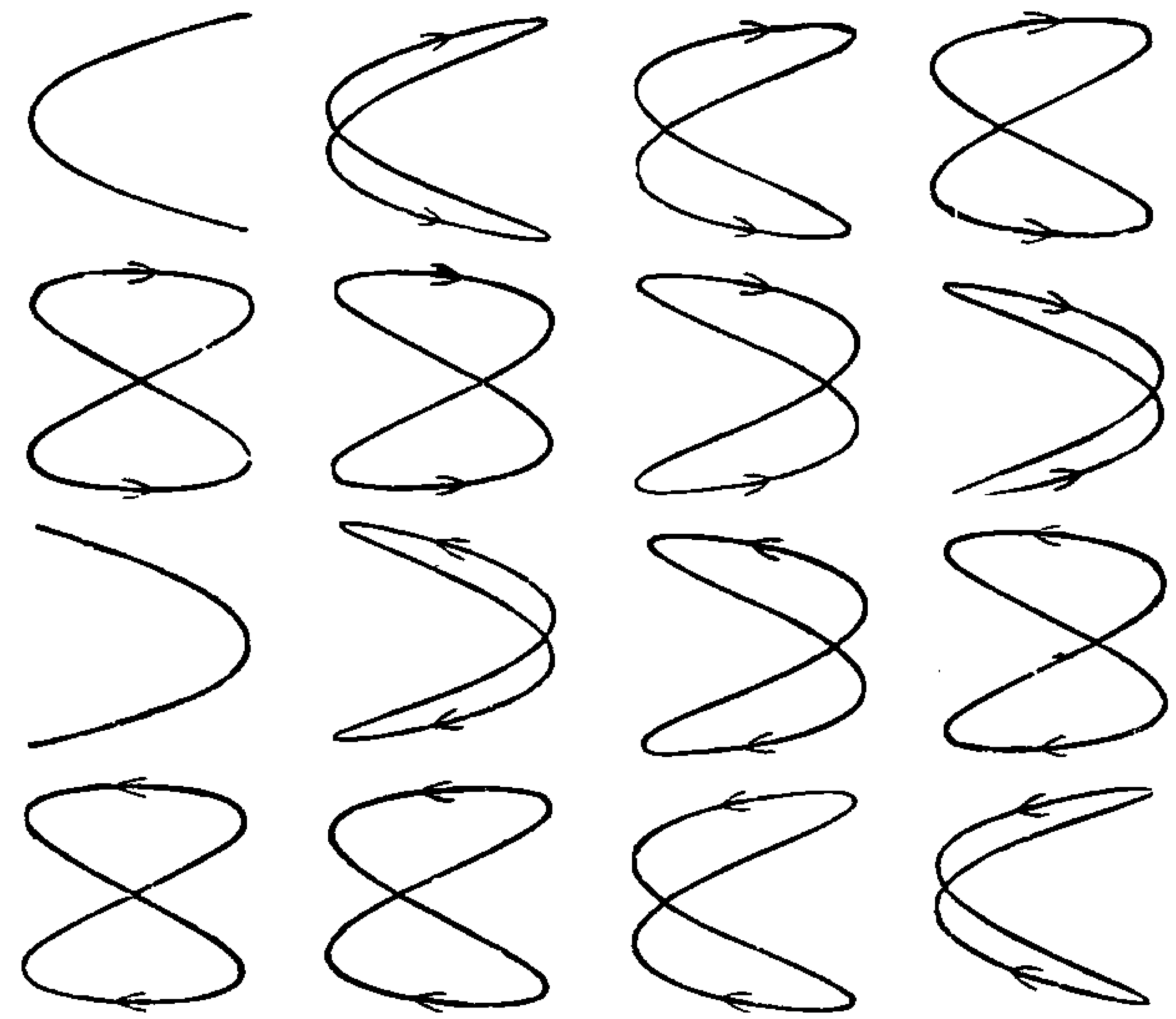
69. By far the most interesting case, and the simplest, is that of *two* simple harmonic motions of any periods, whose directions must of course be in one plane.

Mechanical methods of obtaining such combinations will be afterwards described, as well as cases of their occurrence in Optics and Acoustics.

We may suppose, for simplicity, the two component motions to take place in perpendicular directions. Also, as we can only have a re-entering curve when their periods are commensurable, it will be advisable to commence with such a case.

S. H. motions in two rectangular directions

The following figures represent the paths produced by the



combination of simple harmonic motions of *equal* amplitude in two rectangular directions, the periods of the components being as 1 : 2, and the epochs differing successively by 0, 1, 2, etc., sixteenths of a circumference.

In the case of epochs equal, or differing by a multiple of π, the curve is a portion of a parabola, and is gone over twice in opposite directions by the moving point in each complete period.

For the case figured above,

$$x = a\cos(2nt - \epsilon), \quad y = a\cos nt.$$

Hence $$x = a\{\cos 2nt \cos\epsilon + \sin 2nt \sin\epsilon\}$$

$$= a\left\{\left(\frac{2y^2}{a^2} - 1\right)\cos\epsilon + 2\frac{y}{a}\sqrt{1 - \frac{y^2}{a^2}}\sin\epsilon\right\},$$

which for any given value of ϵ is the equation of the corresponding curve. Thus for $\epsilon = 0$,

$$\frac{x}{a} = \frac{2y^2}{a^2} - 1, \text{ or } y^2 = \frac{a}{2}(x+a), \text{ the parabola as above.}$$

S. H. motions in two rectangular directions.

For $\epsilon = \frac{\pi}{2}$ we have $\frac{x}{a} = 2\frac{y}{a}\sqrt{1-\frac{y^2}{a^2}}$, or $a^2x^2 = 4y^2(a^2-y^2)$, the equation of the 5th and 13th of the above curves.

In general

$$x = a\cos(nt+\epsilon), \quad y = a\cos(n_1t+\epsilon_1),$$

from which t is to be eliminated to find the Cartesian equation of the curve.

Composition of two uniform circular motions.

70. Another very important case is that of two groups of two simple harmonic motions in one plane, such that the resultant of each group is uniform circular motion.

If their periods are equal, we have a case belonging to those already treated (§ 63), and conclude that the resultant is, in general, motion in an ellipse, equal areas being described in equal times about the centre. As particular cases we may have simple harmonic, or uniform circular, motion. (Compare § 91.)

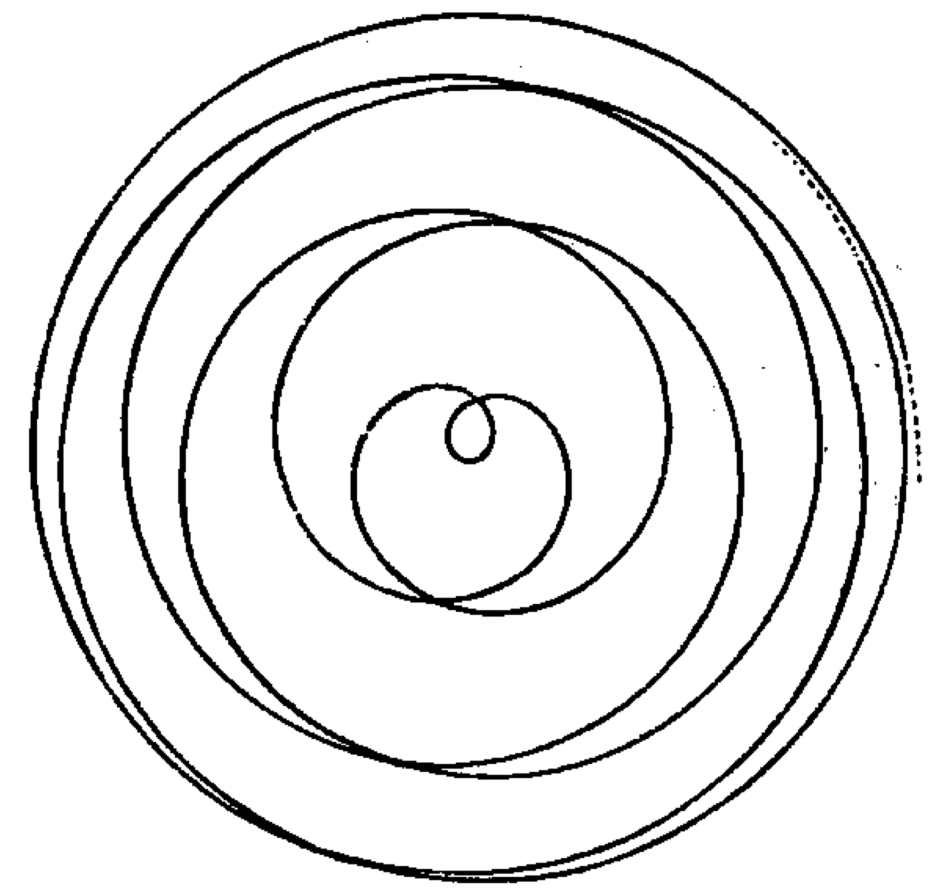
If the circular motions are in the *same* direction, the resultant is evidently circular motion in the same direction. This is the case of the motion of S in § 58, and requires no further comment, as its amplitude, epoch, etc., are seen at once from the figure.

71. If the periods of the two are very nearly equal, the resultant motion will be at any moment very nearly the circular motion given by the preceding construction. Or we may regard it as rigorously a motion in a circle with a varying radius decreasing from a maximum value, the sum of the radii of the two component motions, to a minimum, their difference, and increasing again, alternately; the direction of the resultant radius oscillating on each side of that of the greater component (as in corresponding case, § 59, above). Hence the angular velocity of the resultant motion is periodically variable. In the case of equal radii, next considered, it is constant.

72. When the radii of the two component motions are equal, we have the very interesting and important case figured below. Here the resultant radius bisects the angle between the component radii. The resultant angular velocity is the arithmetical mean of its components. We will explain in a future section

(§ 94) how this epitrochoid is traced by the rolling of one circle

Composition of two uniform circular motions.

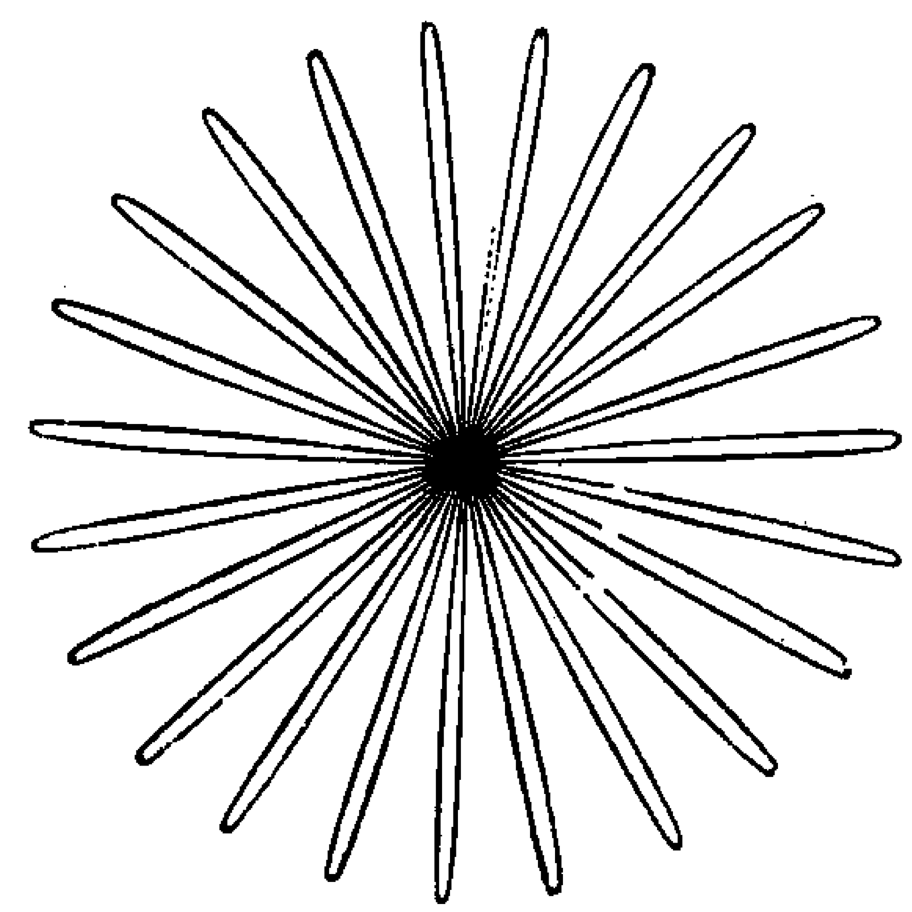


on another. (The particular case above delineated is that of a non-reëntrant curve.)

73. Let the uniform circular motions be in *opposite* directions; then, if the periods are equal, we may easily see, as before, § 66, that the resultant is in general elliptic motion, including the particular cases of uniform circular, and simple harmonic, motion.

If the periods are very nearly equal, the resultant will be easily found, as in the case of § 59.

74. If the radii of the component motions are equal, we have cases of *very* great importance in modern physics, one of which is figured below (like the preceding, a non-reëntrant curve).



Composition of two uniform circular motions.

This is intimately connected with the explanation of two sets of important phenomena,—the rotation of the plane of polarization of light, by quartz and certain fluids on the one hand, and by transparent bodies under magnetic forces on the other. It is a case of the hypotrochoid, and its corresponding mode of description will be described in a future section. It will also appear in kinetics as the path of a pendulum-bob which contains a gyroscope in rapid rotation.

Fourier's Theorem.

75. Before leaving for a time the subject of the composition of harmonic motions, we must, as promised in § 62, devote some pages to the consideration of Fourier's Theorem, which is not only one of the most beautiful results of modern analysis, but may be said to furnish an indispensable instrument in the treatment of nearly every recondite question in modern physics. To mention only sonorous vibrations, the propagation of electric signals along a telegraph wire, and the conduction of heat by the earth's crust, as subjects in their generality intractable without it, is to give but a feeble idea of its importance. The following seems to be the most intelligible form in which it can be presented to the general reader:—

THEOREM.—*A complex harmonic function, with a constant term added, is the proper expression, in mathematical language, for any arbitrary periodic function; and consequently can express any function whatever between definite values of the variable.*

76. Any arbitrary periodic function whatever being given, the amplitudes and epochs of the terms of a complex harmonic function which shall be equal to it for every value of the independent variable, may be investigated by the "method of indeterminate coefficients."

Assume equation (14) below. Multiply both members first by $\cos\dfrac{2i\pi\xi}{p}\,d\xi$ and integrate from 0 to p: then multiply by $\sin\dfrac{2i\pi\xi}{p}\,d\xi$ and integrate between same limits. Thus instantly you find (13).

Fourier's Theorem

This investigation is sufficient as a solution of the problem, —to find a complex harmonic function expressing a given arbitrary periodic function,—when once we are assured that the problem is possible; and when we have this assurance, it proves that the resolution is determinate; that is to say, that no other complex harmonic function than the one we have found can satisfy the conditions.

For description of an integrating machine by which the coefficients A_i, B_i in the Fourier expression (14) for any given arbitrary function may be obtained with exceedingly little labour, and with all the accuracy practically needed for the harmonic analysis of tidal and meteorological observations, see Proceedings of the Royal Society, Feb. 1876, or Chap. v. below.

77. The full theory of the expression investigated in § 76 will be made more intelligible by an investigation from a different point of view.

Let $F(x)$ be any periodic function, of period p. That is to say, let $F(x)$ be any function fulfilling the condition

$$F(x+ip)=F(x) \quad\text{.............................}\quad (1),$$

where i denotes any positive or negative integer. Consider the integral

$$\int_{c'}^{c}\frac{F(x)\,dx}{a^2+x^2},$$

where a, c, c' denote any three given quantities. Its value is less than $F(z)\displaystyle\int_{c'}^{c}\frac{dx}{a^2+x^2}$, and greater than $F(z')\displaystyle\int_{c'}^{c}\frac{dx}{a^2+x^2}$, if z and z' denote the values of x, either equal to or intermediate between the limits c and c', for which $F(x)$ is greatest and least respectively. But

$$\int_{c'}^{c}\frac{dx}{a^2+x^2}=\frac{1}{a}\left(\tan^{-1}\frac{c}{a}-\tan^{-1}\frac{c'}{a}\right) \quad\text{....................}\quad (2),$$

and therefore

$$\left.\begin{aligned}\int_{c'}^{c}\frac{F(x)\,a\,dx}{a^2+x^2} &< F(z)\left(\tan^{-1}\frac{c}{a}-\tan^{-1}\frac{c'}{a}\right),\\ \text{and}\qquad\qquad ,, \qquad &> F(z')\left(\tan^{-1}\frac{c}{a}-\tan^{-1}\frac{c'}{a}\right).\end{aligned}\right\} \quad\text{...........}\quad (3)$$

Fourier's Theorem.

Hence if A be the greatest of all the values of $F(x)$, and B the least,

$$\left.\begin{aligned} \int_c^\infty \frac{F(x)\,adx}{a^2+x^2} &< A\left(\frac{\pi}{2}-\tan^{-1}\frac{c}{a}\right), \\ \text{and} \qquad \text{,,} \qquad &> B\left(\frac{\pi}{2}-\tan^{-1}\frac{c}{a}\right). \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots(4)$$

Also, similarly,

$$\left.\begin{aligned} \int_{-\infty}^{c'} \frac{F(x)\,adx}{a^2+x^2} &< A\left(\tan^{-1}\frac{c'}{a}+\frac{\pi}{2}\right), \\ \text{and} \qquad \text{,,} \qquad &> B\left(\tan^{-1}\frac{c'}{a}+\frac{\pi}{2}\right). \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots(5)$$

Adding the first members of (3), (4), and (5), and comparing with the corresponding sums of the second members, we find

$$\left.\begin{aligned} \int_{-\infty}^{\infty} \frac{F(x)\,adx}{a^2+x^2} &< F(z)\left(\tan^{-1}\frac{c}{a}-\tan^{-1}\frac{c'}{a}\right)+A\left(\pi-\tan^{-1}\frac{c}{a}+\tan^{-1}\frac{c'}{a}\right), \\ \text{and} \quad \text{,,} \quad &> F(z')\left(\tan^{-1}\frac{c}{a}-\tan^{-1}\frac{c'}{a}\right)+B\left(\pi-\tan^{-1}\frac{c}{a}+\tan^{-1}\frac{c'}{a}\right). \end{aligned}\right\}(6)$$

But, by (1),

$$\int_{-\infty}^{\infty}\frac{F(x)\,dx}{a^2+x^2}=\int_0^p F(x)\,dx\left\{\Sigma_{i=-\infty}^{i=\infty}\left(\frac{1}{a^2+(x+ip)^2}\right)\right\} \ldots\ldots\ldots\ldots(7).$$

Now if we denote $\sqrt{-1}$ by v,

$$\frac{1}{a^2+(x+ip)^2}=\frac{1}{2av}\left(\frac{1}{x+ip-av}-\frac{1}{x+ip+av}\right),$$

and therefore, taking the terms corresponding to positive and equal negative values of i together, and the terms for $i=0$ separately, we have

$$\Sigma_{i=-\infty}^{i=\infty}\left(\frac{1}{a^2+(x+ip)^2}\right)=\frac{1}{2av}\left\{\frac{1}{x-av}-2\Sigma_{i=1}^{i=\infty}\frac{x-av}{i^2p^2-(x-av)^2}\right.$$
$$\left.-\frac{1}{x+av}+2\Sigma_{i=1}^{i=\infty}\frac{x+av}{i^2p^2-(x+av)^2}\right\}$$

$$=\frac{\pi}{2apv}\left\{\cot\frac{\pi(x-av)}{p}-\cot\frac{\pi(x+av)}{p}\right\}$$

$$=\frac{\dfrac{\pi}{2apv}\sin\dfrac{2\pi av}{p}}{\cos^2\dfrac{\pi av}{p}-\cos^2\dfrac{\pi x}{p}}=\frac{\dfrac{\pi}{apv}\sin\dfrac{2\pi av}{p}}{\cos\dfrac{2\pi av}{p}-\cos\dfrac{2\pi x}{p}}$$

$$=\frac{\pi}{ap}\,\frac{\epsilon^{\frac{2\pi a}{p}}-\epsilon^{-\frac{2\pi a}{p}}}{\epsilon^{\frac{2\pi a}{p}}-2\cos\dfrac{2\pi x}{p}+\epsilon^{-\frac{2\pi a}{p}}}.$$

Fourier's Theorem.

Hence,

$$\int_{-\infty}^{\infty}\frac{F(x)\,dx}{a^2+x^2}=\frac{\pi}{ap}\left(\epsilon^{\frac{2\pi a}{p}}-\epsilon^{-\frac{2\pi a}{p}}\right)\int_0^p\frac{F(x)\,dx}{\epsilon^{\frac{2\pi a}{p}}-2\cos\dfrac{2\pi x}{p}+\epsilon^{-\frac{2\pi a}{p}}} \ldots\ldots(8).$$

Next, denoting temporarily, for brevity, $\epsilon^{\frac{2\pi xv}{p}}$ by ζ, and putting

$$\epsilon^{-\frac{2\pi a}{p}}=e \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(9),$$

we have

$$\frac{1}{\epsilon^{\frac{2\pi a}{p}}-2\cos\dfrac{2\pi x}{p}+\epsilon^{-\frac{2\pi a}{p}}}=\frac{e}{1-e(\zeta+\zeta^{-1})+e^2}$$

$$=\frac{e}{1-e^2}\left(\frac{1}{1-e\zeta}+\frac{1}{1-e\zeta^{-1}}-1\right)$$

$$=\frac{e}{1-e^2}\{1+e(\zeta+\zeta^{-1})+e^2(\zeta^2+\zeta^{-2})+e^3(\zeta^3+\zeta^{-3})+\text{etc.}\}$$

$$=\frac{e}{1-e^2}\left(1+2e\cos\frac{2\pi x}{p}+2e^2\cos\frac{4\pi x}{p}+2e^3\cos\frac{6\pi x}{p}+\text{etc.}\right).$$

Hence, according to (8) and (9),

$$\int_{-\infty}^{\infty}\frac{F(x)\,dx}{a^2+x^2}=\frac{\pi}{ap}\int_0^p F(x)\,dx\left(1+2e\cos\frac{2\pi x}{p}+2e^2\cos\frac{4\pi x}{p}+\text{etc.}\right)\ldots(10).$$

Hence, by (6), we infer that

$$F(z)\left(\tan^{-1}\frac{c}{a}-\tan^{-1}\frac{c'}{a}\right)+A\left(\pi-\tan^{-1}\frac{c}{a}+\tan^{-1}\frac{c'}{a}\right)>$$

and

$$F(z')\left(\tan^{-1}\frac{c}{a}-\tan^{-1}\frac{c'}{a}\right)+B\left(\pi-\tan^{-1}\frac{c}{a}+\tan^{-1}\frac{c'}{a}\right)<$$

$$\frac{\pi}{p}\int_0^p F(x)\,dx\left(1+2e\cos\frac{2\pi x}{p}+\text{etc.}\right).$$

Now let $\quad c'=-c,\quad$ and $\quad x=\xi'-\xi$,

ξ' being a variable, and ξ constant, so far as the integration is concerned; and let

$$F(x)=\phi(x+\xi)=\phi(\xi'),$$

and therefore

$$F(z)=\phi(\xi+z),$$
$$F(z')=\phi(\xi+z').$$

Fourier's Theorem.

The preceding pair of inequalities becomes

$$\left.\begin{aligned} &\phi(\xi+z)\,.\,2\tan^{-1}\frac{c}{a}+A\left(\pi-2\tan^{-1}\frac{c}{a}\right)> \\ \text{and}\quad &\phi(\xi+z')\,.\,2\tan^{-1}\frac{c}{a}+B\left(\pi-2\tan^{-1}\frac{c}{a}\right)< \\ &\frac{\pi}{p}\left\{\int_0^p\phi(\xi')\,d\xi'+2\Sigma_{i=1}^{i=\infty}e^i\int_0^p\phi(\xi')\,d\xi'\cos\frac{2i\pi(\xi'-\xi)}{p}\right\}, \end{aligned}\right\}\ \ldots\ldots (11)$$

where ϕ denotes any periodic function whatever, of period p.

Now let c be a very small fraction of p. In the limit, where c is infinitely small, the greatest and least values of $\phi(\xi')$ for values of ξ' between $\xi+c$ and $\xi-c$ will be infinitely nearly equal to one another and to $\phi(\xi)$; that is to say,

$$\phi(\xi+z)=\phi(\xi+z')=\phi(\xi).$$

Next, let a be an infinitely small fraction of c. In the limit

$$\tan^{-1}\frac{c}{a}=\frac{\pi}{2},$$

and

$$e=\epsilon^{-\frac{2\pi a}{p}}=1.$$

Hence the comparison (11) becomes in the limit an equation which, if we divide both members by π, gives

$$\phi(\xi)=\frac{1}{p}\left\{\int_0^p\phi(\xi')\,d\xi'+2\Sigma_{i=1}^{i=\infty}\int_0^p\phi(\xi')\,d\xi'\cos\frac{2i\pi(\xi'-\xi)}{p}\right\}\ldots(12).$$

This is the celebrated theorem discovered by Fourier* for the development of an arbitrary periodic function in a series of simple harmonic terms. A formula included in it as a particular case had been given previously by Lagrange†.

If, for $\cos\dfrac{2i\pi(\xi'-\xi)}{p}$, we take its value

$$\cos\frac{2i\pi\xi'}{p}\cos\frac{2i\pi\xi}{p}+\sin\frac{2i\pi\xi'}{p}\sin\frac{2i\pi\xi}{p}$$

and introduce the following notation :—

$$\left.\begin{aligned} A_0&=\frac{1}{p}\int_0^p\phi(\xi)\,d\xi, \\ A_i&=\frac{2}{p}\int_0^p\phi(\xi)\cos\frac{2i\pi\xi}{p}\,d\xi, \\ B_i&=\frac{2}{p}\int_0^p\phi(\xi)\sin\frac{2i\pi\xi}{p}\,d\xi, \end{aligned}\right\}\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots (13)$$

* *Théorie analytique de la Chaleur.* Paris, 1822.

† *Anciens Mémoires de l'Académie de Turin.*

Fourier's Theorem.

we reduce (12) to this form :—

$$\phi(\xi)=A_0+\Sigma_{i=1}^{i=\infty}A_i\cos\frac{2i\pi\xi}{p}+\Sigma_{i=1}^{i=\infty}B_i\sin\frac{2i\pi\xi}{p}\ldots\ldots\ldots(14),$$

which is the general expression of an arbitrary function in terms of a series of cosines and of sines. Or if we take

$$P_i=(A_i^2+B_i^2)^{\frac{1}{2}},\quad\text{and}\quad\tan\epsilon_i=\frac{B_i}{A_i}\ldots\ldots\ldots\ldots\ldots\ldots(15),$$

we have $$\phi(\xi)=A_0+\Sigma_{i=1}^{i=\infty}P_i\cos\left(\frac{2i\pi\xi}{p}-\epsilon_i\right)\ldots\ldots\ldots\ldots\ldots\ldots(16),$$

which is the general expression in a series of single simple harmonic terms of the successive multiple periods.

Convergency of Fourier's series.

Each of the equations and comparisons (2), (7), (8), (10), and (11) is a true arithmetical expression, and may be verified by actual calculation of the numbers, for any particular case; provided only that $F(x)$ has no infinite value in its period. Hence, with this exception, (12) or either of its equivalents, (14), (16), is a true arithmetical expression; and the series which it involves is therefore convergent. Hence we may with perfect rigour conclude that even the extreme case in which the arbitrary function experiences an abrupt finite change in its value when the independent variable, increasing continuously, passes through some particular value or values, is included in the general theorem. In such a case, if any value be given to the independent variable differing however little from one which corresponds to an abrupt change in the value of the function, the series must, as we may infer from the preceding investigation, converge and give a definite value for the function. But if exactly the critical value is assigned to the independent variable, the series cannot converge to any definite value. The consideration of the limiting values shown in the comparison (11) does away with all difficulty in understanding how the series (12) gives definite values having a finite difference for two particular values of the independent variable on the two sides of a critical value, but differing infinitely little from one another.

If the differential coefficient $\dfrac{d\phi(\xi)}{d\xi}$ is finite for every value of ξ within the period, it too is arithmetically expressible by a series of harmonic terms, which cannot be other than the series obtained by differentiating the series for $\phi(\xi)$. Hence

Convergency of Fourier's series.

$$\frac{d\phi(\xi)}{d\xi} = -\frac{2\pi}{p}\Sigma_{i=1}^{i=\infty} iP_i \sin\left(\frac{2i\pi\xi}{p} - \epsilon_i\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots(17),$$

and this series is convergent; and we may therefore conclude that the series for $\phi(\xi)$ is more convergent than a harmonic series with

$$1, \quad \tfrac{1}{2}, \quad \tfrac{1}{3}, \quad \tfrac{1}{4}, \quad \text{etc.},$$

for its coefficients. If $\frac{d^2\phi(\xi)}{d\xi^2}$ has no infinite values within the period, we may differentiate both members of (17) and still have an equation arithmetically true; and so on. We conclude that if the n^{th} differential coefficient of $\phi(\xi)$ has no infinite values, the harmonic series for $\phi(\xi)$ must converge more rapidly than a harmonic series with

$$1, \quad \frac{1}{2^n}, \quad \frac{1}{3^n}, \quad \frac{1}{4^n}, \quad \text{etc.},$$

for its coefficients.

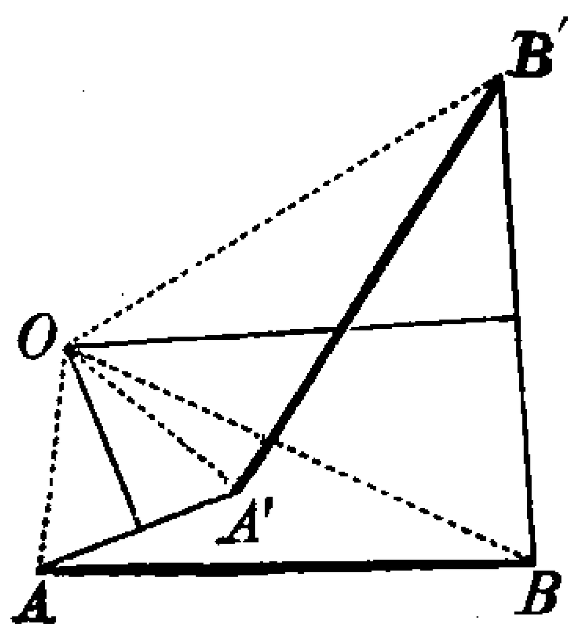
Displacement of a rigid body.

78. We now pass to the consideration of the displacement of a rigid body or group of points whose relative positions are unalterable. The simplest case we can consider is that of the motion of a plane figure in its own plane, and this, as far as kinematics is concerned, is entirely summed up in the result of the next section.

Displacements of a plane figure in its plane.

79. If a plane figure be displaced in any way in its own plane, there is always (with an exception treated in § 81) one point of it common to any two positions; that is, it may be moved from any one position to any other by rotation in its own plane about one point held fixed.

To prove this, let A, B be any two points of the plane figure in its first position, A', B' the positions of the same two after a displacement. The lines AA', BB' will not be parallel, except in one case to be presently considered. Hence the line equidistant from A and A' will meet that equidistant from B and B' in some point O. Join OA, OB, OA', OB'. Then, evidently, because $OA' = OA$, $OB' = OB$ and $A'B' = AB$, the triangles $OA'B'$ and OAB are equal and similar. Hence O is similarly situated with regard to $A'B'$ and AB, and is therefore one and



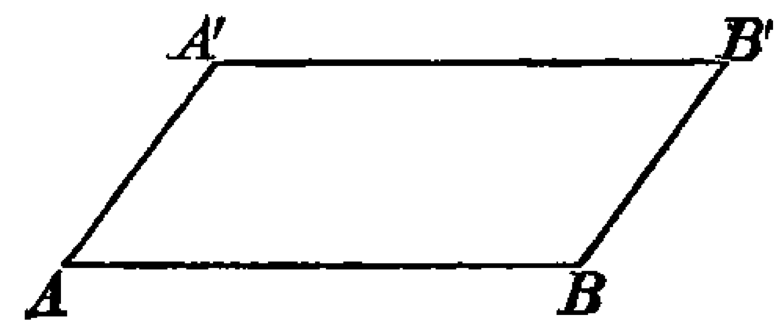
Displacements of a plane figure in its plane.

the same point of the plane figure in its two positions. If, for the sake of illustration, we actually trace the triangle OAB upon the plane, it becomes $OA'B'$ in the second position of the figure.

80. If from the equal angles $A'OB'$, AOB of these similar triangles we take the common part $A'OB$, we have the remaining angles AOA', BOB' equal, and each of them is clearly equal to the angle through which the figure must have turned round the point O to bring it from the first to the second position.

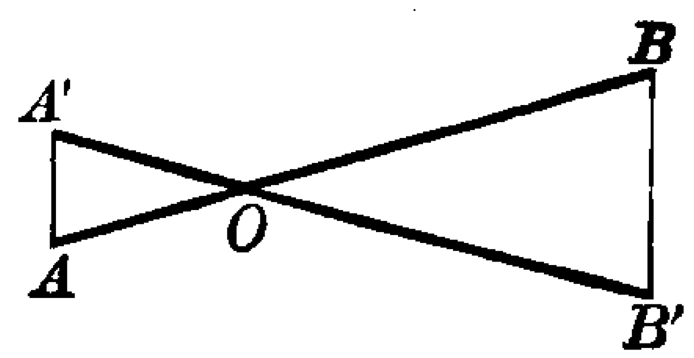
The preceding simple construction therefore enables us not only to demonstrate the general proposition, § 79, but also to determine from the two positions of one terminated line AB, $A'B'$ of the figure the common centre and the amount of the angle of rotation.

81. The lines equidistant from A and A', and from B and B', are parallel if AB is parallel to $A'B'$; and therefore the construction fails, the point O being infinitely distant, and the theorem becomes nugatory. In this case the motion is in fact a simple translation of the figure in its own plane without rotation—since, AB being parallel and equal to $A'B'$, we have AA' parallel and equal to BB'; and instead of there being one point of the figure common to both positions, the lines joining the two successive positions of all points in the figure are equal and parallel.



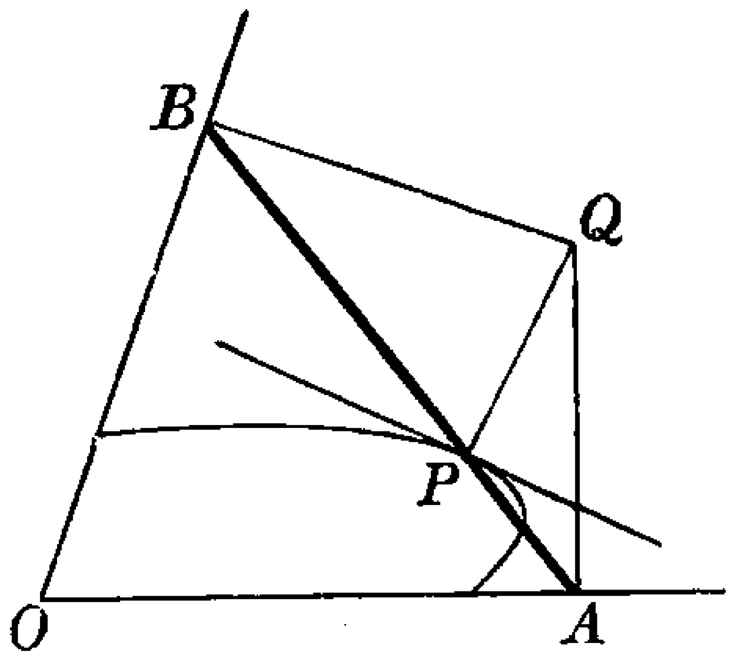
82. It is not necessary to suppose the figure to be a mere flat disc or plane—for the preceding statements apply to any one of a set of parallel planes in a rigid body, moving in any way subject to the condition that the points of any one plane in it remain always in a fixed plane in space.

83. There is yet a case in which the construction in § 79 is nugatory—that is when AA' is parallel to BB', but the lines of AB and $A'B'$ intersect. In this case, however, the point of intersection is the point O required, although the former method would not have enabled us to find it.

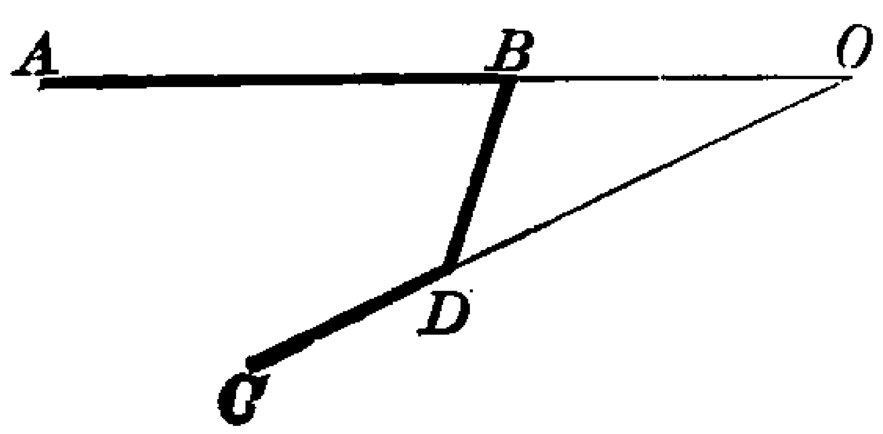


Examples of displacement in one plane.

84. Very many interesting applications of this principle may be made, of which, however, few belong strictly to our subject, and we shall therefore give only an example or two. Thus we know that if a line of given length AB move with its extremities always in two fixed lines OA, OB, any point in it as P describes an ellipse. It is required to find the direction of motion of P at any instant, *i.e.*, to draw a tangent to the ellipse. BA will pass to its next position by rotating about the point Q; found by the method of § 79 by drawing perpendiculars to OA and OB at A and B. Hence P for the instant revolves about Q, and thus its direction of motion, or the tangent to the ellipse, is perpendicular to QP. Also AB in its motion always touches a curve (called in geometry its envelop); and the same principle enables us to find the point of the envelop which lies in AB, for the motion of that point must evidently be ultimately (that is for a very small displacement) along AB, and the only point which so moves is the intersection of AB with the perpendicular to it from Q. Thus our construction would enable us to trace the envelop by points. (For more on this subject see § 91.)



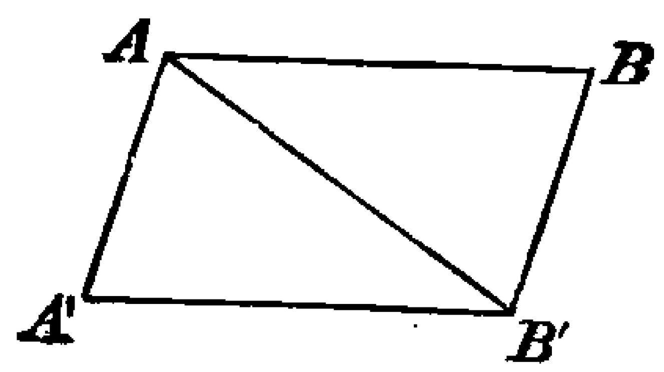
85. Again, suppose AB to be the beam of a stationary engine having a reciprocating motion about A, and by a link BD turning a crank CD about C. Determine the relation between the angular velocities of AB and CD in any position. Evidently the instantaneous direction of motion of B is transverse to AB, and of D transverse to CD—hence if AB, CD produced meet in O, the motion of BD is for an instant as if it turned about O. From this it may be easily seen that if the angular velocity of AB be ω, that of CD is $\dfrac{AB}{OB}\dfrac{OD}{CD}\omega$. A similar process is of course applicable to any combination of machinery, and we shall find it very convenient when we come to consider various dynamical problems connected with virtual velocities.



Examples of displacement in one plane.

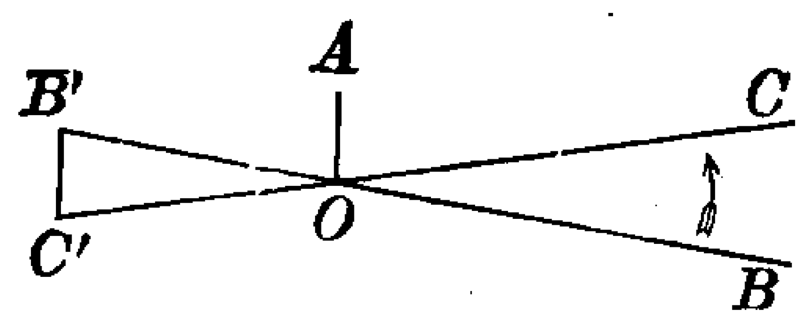
Composition of rotations about parallel axes.

86. Since in general any movement of a plane figure in its plane may be considered as a rotation about one point, it is evident that two such rotations may in general be compounded into one; and therefore, of course, the same may be done with any number of rotations. Thus let A and B be the points of the figure about which in succession the rotations are to take place. By a rotation about A, B is brought say to B', and by a rotation about B', A is brought to A'. The construction of § 79 gives us at once the point O and the amount of rotation about it which singly gives the same effect as those about A and B in succession. But there is one case of exception, viz., when the rotations about A and B are of equal amount and in opposite directions. In this case $A'B'$ is evidently parallel to AB, and therefore the compound result is a *translation* only. That is, if a body revolve in succession through equal angles, but in opposite directions, about two parallel axes, it finally takes a position to which it could have been brought by a simple translation perpendicular to the lines of the body in its initial or final position, which were successively made axes of rotation; and inclined to their plane at an angle equal to half the supplement of the common angle of rotation.



Composition of rotations and translations in one plane.

87. Hence to compound into an equivalent rotation a rotation and a translation, the latter being effected parallel to the plane of the former, we may decompose the translation into two rotations of equal amount and opposite direction, compound one of them with the given rotation by § 86, and then compound the other with the resultant rotation by the same process. Or we may adopt the following far simpler method. Let OA be the translation common to all points in the plane, and let BOC be the angle of rotation about O, BO being drawn so that OA bisects the exterior angle COB'. Take



Composition of rotations and translations in one plane.

the point B' in BO produced, such that $B'C'$, the space through which the rotation carries it, is equal and opposite to OA. This point retains its former position after the performance of the compound operation; so that a rotation and a translation in one plane can be compounded into an equal rotation about a different axis.

In general, if the origin be taken as the point about which rotation takes place in the plane of xy, and if it be through an angle θ, a point whose co-ordinates were originally x, y will have them changed to

$$\xi = x\cos\theta - y\sin\theta, \quad \eta = x\sin\theta + y\cos\theta,$$

or, if the rotation be very small,

$$\xi = x - y\theta, \quad \eta = y + x\theta.$$

Omission of the second and higher orders of small quantities.

88. In considering the composition of angular velocities about different axes, and other similar cases, we may deal with infinitely small displacements only; and it results at once from the principles of the differential calculus, that if these displacements be of the *first* order of small quantities, any point whose displacement is of the *second* order of small quantities is to be considered as rigorously at rest. Hence, for instance, if a body revolve through an angle of the first order of small quantities about an axis (belonging to the body) which during the revolution is displaced through an angle or space, also of the first order, the displacement of any point of the body is rigorously what it would have been had the axis been fixed during the rotation about it, and its own displacement made either before or after this rotation. Hence in any case of motion of a rigid system the angular velocities about a system of axes moving *with* the system are the same at any instant as those about a system fixed in space, provided only that the latter coincide at the instant in question with the moveable ones.

Superposition of small motions.

89. From similar considerations follows also the general principle of *Superposition of small motions*. It asserts that if several causes act *simultaneously* on the same particle or rigid body, and if the effect produced by each is of the first order of small quantities, the joint effect will be obtained if we consider the causes to act *successively*, each taking the point or system in the posi-

tion in which the preceding one left it. It is evident at once that this is an immediate deduction from the fact that the second order of infinitely small quantities may be with rigorous accuracy neglected. This principle is of very great use, as we shall find in the sequel; its applications are of constant occurrence.

Superposition of small motions.

A plane figure has given angular velocities about given axes perpendicular to its plane, find the resultant.

Let there be an angular velocity ω about an axis passing through the point a, b.

The consequent motion of the point x, y in the time δt is, as we have just seen (§ 87),

$$-(y-b)\,\omega\delta t \text{ parallel to } x, \quad \text{and } (x-a)\,\omega\delta t \text{ parallel to } y.$$

Hence, by the superposition of small motions, the whole motion parallel to x is

$$-(y\Sigma\omega - \Sigma b\omega)\,\delta t,$$

and that parallel to y $(x\Sigma\omega - \Sigma a\omega)\,\delta t$.

Hence the point whose co-ordinates are

$$x' = \frac{\Sigma a\omega}{\Sigma\omega} \quad \text{and} \quad y' = \frac{\Sigma b\omega}{\Sigma\omega}$$

is at rest, and the resultant axis passes through it. Any other point x, y moves through spaces

$$-(y\Sigma\omega - \Sigma b\omega)\,\delta t, \quad (x\Sigma\omega - \Sigma a\omega)\,\delta t.$$

But if the whole had turned about x', y' with velocity Ω, we should have had for the displacements of x, y,

$$-(y-y')\,\Omega\delta t, \quad (x-x')\,\Omega\delta t.$$

Comparing, we find $\Omega = \Sigma\omega$.

Hence if the sum of the angular velocities be zero, there is no rotation, and indeed the above formulæ show that there is then merely translation,

$$\Sigma(b\omega)\,\delta t \text{ parallel to } x, \quad \text{and } -\Sigma(a\omega)\,\delta t \text{ parallel to } y.$$

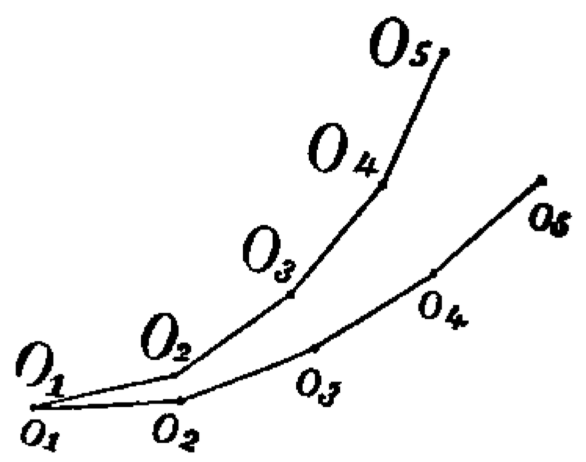
These formulæ suffice for the consideration of any problem on the subject.

Rolling of curve on curve.

90. Any motion whatever of a plane figure in its own plane might be produced by the rolling of a curve fixed to the figure upon a curve fixed in the plane.

For we may consider the whole motion as made up of successive elementary displacements, each of which corresponds, as we have seen, to an elementary rotation about some point in

Rolling of curve on curve. the plane. Let o_1, o_2, o_3, etc., be the successive points of the moving figure about which the rotations take place, O_1, O_2, O_3, etc., the positions of these points when each is the instantaneous centre of rotation. Then the figure rotates about o_1 (or O_1, which coincides with it) till o_2 coincides with O_2, then about the latter till o_3 coincides with O_3, and so on. Hence, if we join o_1, o_2, o_3, etc., in the plane of the figure, and O_1, O_2, O_3, etc., in the fixed plane, the motion will be the same as if the polygon $o_1o_2o_3$, etc., rolled upon the fixed polygon $O_1O_2O_3$, etc. By supposing the successive displacements small enough the sides of these polygons gradually diminish, and the polygons finally become continuous curves Hence the theorem.



From this it immediately follows, that any displacement of a rigid solid, which is in directions wholly perpendicular to a fixed line, may be produced by the rolling of a cylinder fixed in the solid on another cylinder fixed in space, the axes of the cylinders being parallel to the fixed line.

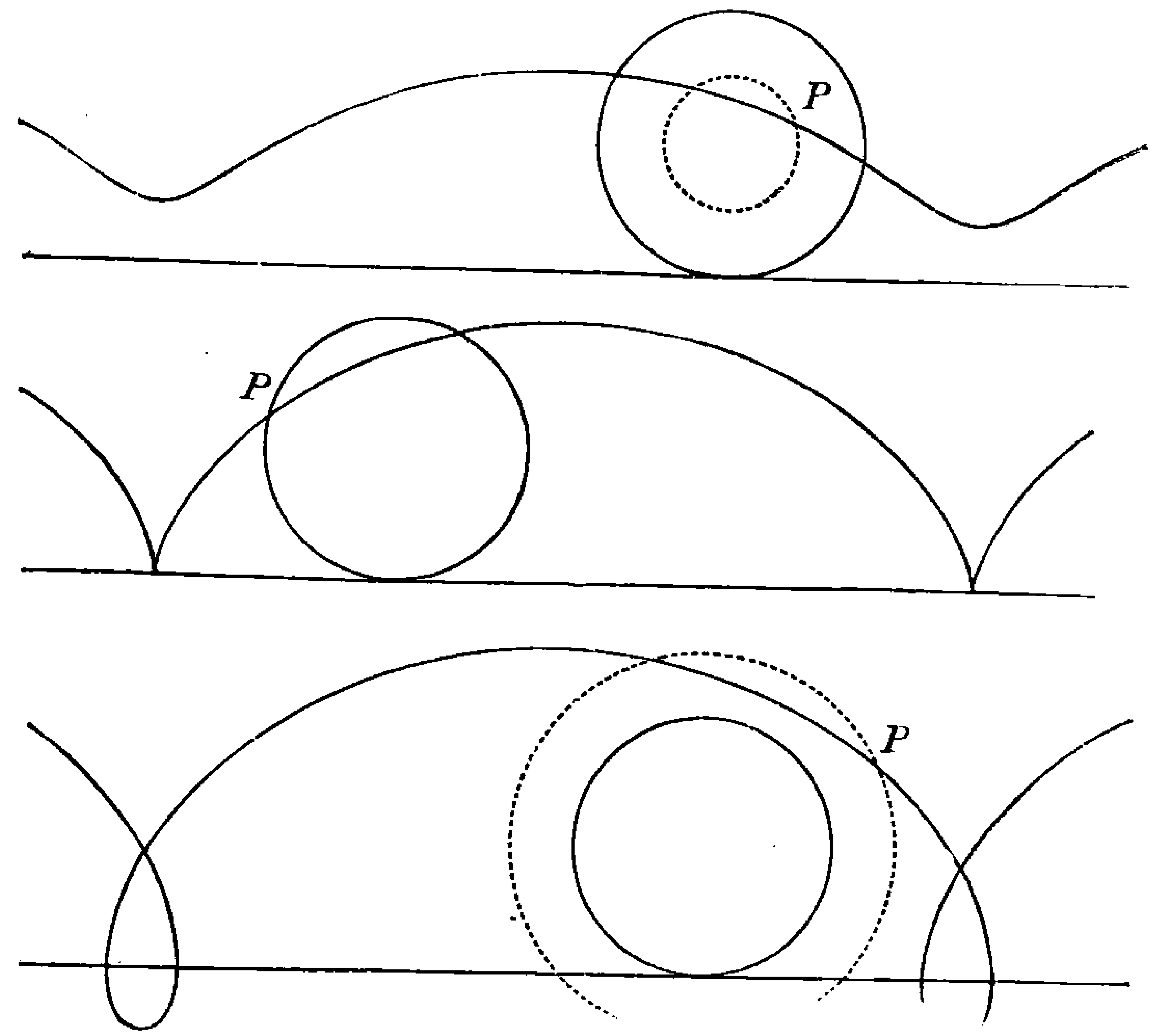
91. As an interesting example of this theorem, let us recur to the case of § 84:—A circle may evidently be circumscribed about $OBQA$; and it must be of invariable magnitude, since in it a chord of given length AB subtends a given angle O at the circumference. Also OQ is a diameter of this circle, and is therefore constant. Hence, as Q is momentarily at rest, the motion of the circle circumscribing $OBQA$ is one of internal rolling on a circle of double its diameter. Hence if a circle roll internally on another of twice its diameter, any point in its circumference describes a diameter of the fixed circle, any other point in its plane an ellipse. This is precisely the same proposition as that of § 70, although the ways of arriving at it are very different. As it presents us with a particular case of the Hypocycloid, it warns us to return to the consideration of these and kindred curves, which give good instances of kinematical theorems, but which besides are of great use in physics generally.

Cycloids and Trochoids.

92. When a circle rolls upon a straight line, a point in its circumference describes a Cycloid; an internal point describes a Prolate, an external one a Curtate, Cycloid. The two latter varieties are sometimes called Trochoids.

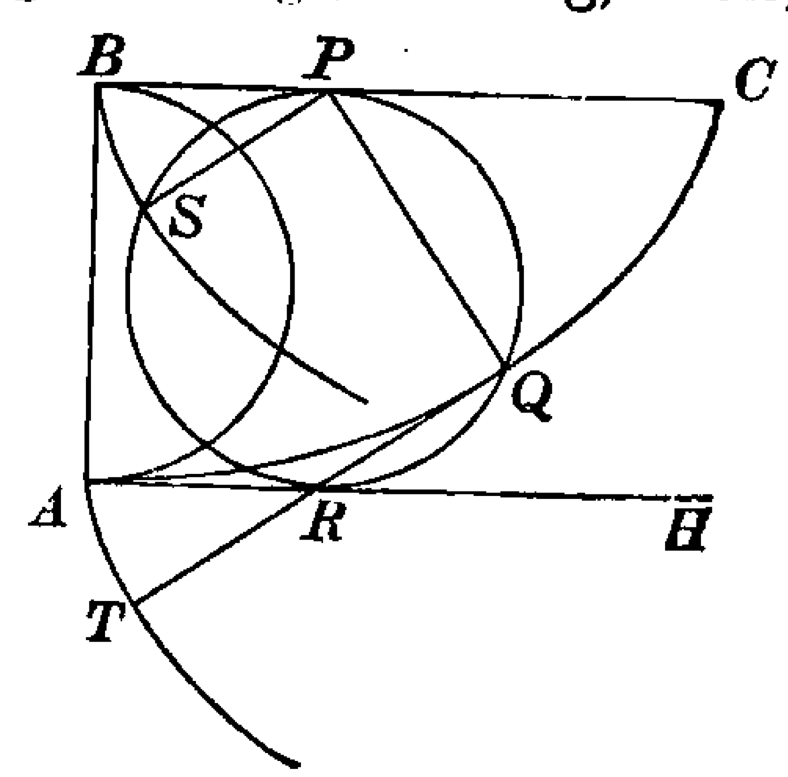
Cycloids and Trochoids.

The general form of these curves will be seen in the annexed figures; and in what follows we shall confine our remarks to the cycloid itself, as of immensely greater consequence than the others. The next section contains a simple investigation of those properties of the cycloid which are most useful in our subject.



Properties of the cycloid.

93. Let AB be a diameter of the generating (or rolling) circle, BC the line on which it rolls. The points A and B describe similar and equal cycloids, of which AQC and BS are portions. If PQR be any subsequent position of the generating circle, Q and S the new positions of A and B, $\angle QPS$ is of course a right angle. If, therefore, QR be drawn parallel to PS, PR is a diameter

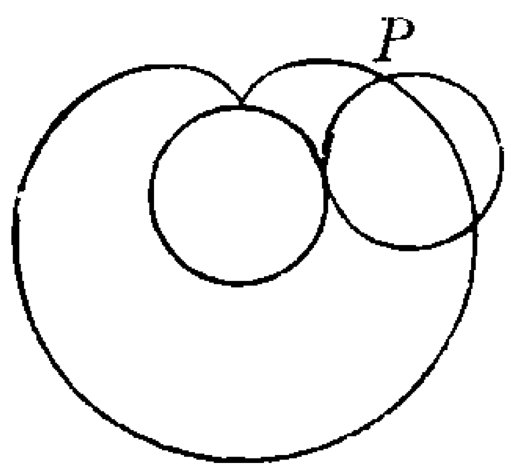


Properties of the cycloid.

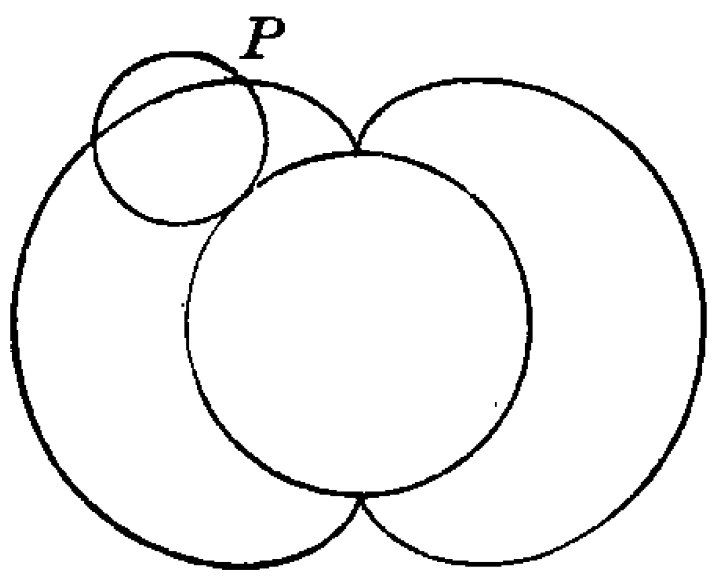
of the rolling circle. Produce QR to T, making $RT=QR=PS$. Evidently the curve AT, which is the locus of T, is similar and equal to BS, and is therefore a cycloid similar and equal to AC. But QR is perpendicular to PQ, and is therefore the instantaneous direction of motion of Q, or is the tangent to the cycloid AQC. Similarly, PS is perpendicular to the cycloid BS at S, and so is therefore TQ to AT at T. Hence (§ 19) AQC is the evolute of AT, and arc $AQ=QT=2QR$.

Epicycloids, Hypocycloids, etc.

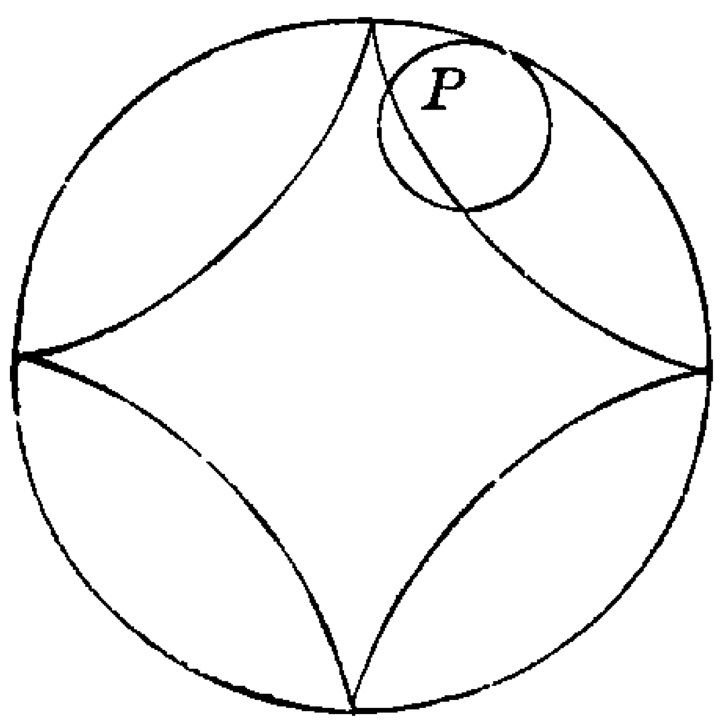
94. When the circle rolls upon another circle, the curve described by a point in its circumference is called an Epicycloid, or a Hypocycloid, as the rolling circle is without or within the fixed circle; and when the tracing point is not in the circumference, we have Epitrochoids and Hypotrochoids. Of the latter we have already met with examples, §§ 70, 91, and others will be presently mentioned. Of the former, we have in the first of the appended figures the case of a circle rolling externally on another of equal size. The curve in this case is called the Cardioid (§ 49).



In the second diagram, a circle rolls externally on another of twice its radius. The epicycloid so described is of importance in Optics, and will, with others, be referred to when we consider the subject of Caustics by reflexion.



In the third diagram, we have a hypocycloid traced by the rolling of one circle internally on another of four times its radius.



Epicycloids, Hypocycloids, etc.

The curve figured in § 72 is an epitrochoid described by a point in the plane of a large circular disc which rolls upon a circular cylinder of small diameter, so that the point passes through the axis of the cylinder.

That of § 74 is a hypotrochoid described by a point in the plane of a circle which rolls internally on another of rather more than twice its diameter, the tracing point passing through the centre of the fixed circle. Had the diameters of the circles been exactly as 1 : 2, § 72 or § 91 shows that this curve would have been reduced to a single straight line.

The general equations of this class of curves are

$$x=(a+b)\cos\theta-eb\cos\frac{a+b}{b}\theta,$$

$$y=(a+b)\sin\theta-eb\sin\frac{a+b}{b}\theta,$$

where a is the radius of the fixed, b of the rolling circle; and eb is the distance of the tracing point from the centre of the latter.

Motion about a fixed point.

95. If a rigid solid body move in any way whatever, subject only to the condition that one of its points remains fixed, there is always (without exception) one line of it through this point common to the body in any two positions. This most important theorem is due to Euler. To prove it, consider a spherical surface within the body, with its centre at the fixed point C. All points of this sphere attached to the body will move on a sphere fixed in space. Hence the construction of § 79 may be made, but with great circles instead of straight lines; and the same reasoning will apply to prove that the point O thus obtained is common to the body in its two positions. Hence every point of the body in the line OC, joining O with the fixed point, must be common to it in the two positions. Hence the body may pass from any one position to any other by rotating through a definite angle about a definite axis. Hence any position of the body may be specified by specifying the axis, and the angle, of rotation by which it may be brought to that position from a fixed position of reference, an idea due to Euler, and revived by Rodrigues.

Euler's theorem.

Rodrigues' co-ordinates.

Let OX, OY, OZ be any three fixed axes through the fixed point O round which the body turns. Let λ, μ, ν be the direction cosines, referred to these axes, of the axis OI round which the body must turn, and χ the angle through which it must turn round this axis, to bring it from some zero position to any other position. This other position, being specified by the four co-ordinates λ, μ, ν, χ (reducible, of course, to three by the relation $\lambda^2+\mu^2+\nu^2=1$), will be called for brevity $(\lambda, \mu, \nu, \chi)$. Let OA, OB, OC be three rectangular lines moving with the body, which in the "zero" position coincide respectively with OX, OY, OZ; and put

$$(XA),\ (YA),\ (ZA),\ (XB),\ (YB),\ (ZB),\ (XC),\ (YC),\ (ZC),$$

for the nine direction cosines of OA, OB, OC, each referred to OX, OY, OZ. These nine direction cosines are of course reducible to three independent co-ordinates by the well-known six relations. Let it be required now to express these nine direction cosines in terms of Rodrigues' co-ordinates λ, μ, ν, χ.

Let the lengths OX, ..., OA, ..., OI be equal, and call each unity: and describe from O as centre a spherical surface of unit radius; so that X, Y, Z, A, B, C, I shall be points on this surface. Let XA, YA, ... XB, denote arcs, and XAY, AXB, ... angles between arcs, in the spherical diagram thus obtained. We have $IA = IX = \cos^{-1}\lambda$, and $XIA=\chi$. Hence by the isosceles spherical triangle XIA,

$$\cos XA = \cos^2 IX + \sin^2 IX \cos\chi,$$

or
$$(XA) = \lambda^2 + (1-\lambda^2)\cos\chi \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

And by the spherical triangle XIB,

$$\cos XB = \cos IX \cos IB + \sin IX \sin IB \cos XIB$$
$$= \lambda\mu + \sqrt{(1-\lambda^2)(1-\mu^2)}\cos XIB \ \ldots\ldots\ (2).$$

Now $XIB = XIY + YIB = XIY + \chi$; and by the spherical triangle XIY we have

$$\cos XY = 0 = \cos IX \cos IY + \sin IX \sin IY \cos XIY$$
$$= \lambda\mu + \sqrt{(1-\lambda^2)(1-\mu^2)}\cos XIY.$$

Hence
$$\sqrt{(1-\lambda^2)(1-\mu^2)}\cos XIY = -\lambda\mu,$$

and
$$\sqrt{(1-\lambda^2)(1-\mu^2)}\sin XIY = \sqrt{(1-\lambda^2-\mu^2)} = $$

by which we have

$$\sqrt{(1-\lambda^2)(1-\mu^2)}\cos(XIY+\chi) = -\lambda\mu\cos\chi - \nu\sin\chi;$$

and using this in (2),

Rodrigues' co-ordinates.

$$\cos XB = \lambda\mu(1-\cos\chi) - \nu\sin\chi \ldots\ldots\ldots\ldots\ldots\ldots (3).$$

Similarly we find

$$\cos AY = \lambda\mu(1-\cos\chi) + \nu\sin\chi \ldots\ldots\ldots\ldots\ldots\ldots (4).$$

The other six formulæ may be written out by symmetry from (1), (3), and (4); and thus for the nine direction cosines we find

$$\left.\begin{array}{l}(XA)=\lambda^2+(1-\lambda^2)\cos\chi;\ (XB)=\lambda\mu(1-\cos\chi)-\nu\sin\chi;\ (YA)=\lambda\mu(1-\cos\chi)+\nu\sin\chi;\\ (YB)=\mu^2+(1-\mu^2)\cos\chi;\ (YC)=\mu\nu(1-\cos\chi)-\lambda\sin\chi;\ (ZB)=\mu\nu(1-\cos\chi)+\lambda\sin\chi;\\ (ZC)=\nu^2+(1-\nu^2)\cos\chi;\ (ZA)=\nu\lambda\ 1-\cos\chi)-\mu\sin\chi;\ (XC)=\nu\lambda(1-\cos\chi)+\mu\sin\chi.\end{array}\right\}(5).$$

Adding the three first equations of these three lines, and remembering that

$$\lambda^2+\mu^2+\nu^2=1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6),$$

we deduce

$$\cos\chi = \tfrac{1}{2}[(XA)+(YB)+(ZC)-1]\ldots\ldots\ldots\ldots(7);$$

and then, by the three equations separately,

$$\left.\begin{array}{l}\lambda^2=\dfrac{1+(XA)-(YB)-(ZC)}{3-(XA)-(YB)-(ZC)},\\[2ex] \mu^2=\dfrac{1-(XA)+(YB)-(ZC)}{3-(XA)-(YB)-(ZC)},\\[2ex] \nu^2=\dfrac{1-(XA)-(YB)+(ZC)}{3-(XA)-(YB)-(ZC)}.\end{array}\right\}\ldots\ldots\ldots\ldots\ (8)$$

These formulæ, (8) and (7), express, in terms of (XA), (YB), (ZC), three out of the nine direction cosines (XA), ..., the direction cosines of the axis round which the body must turn, and the cosine of the angle through which it must turn round this axis, to bring it from the zero position to the position specified by those three direction cosines.

Composition of rotations.

By aid of Euler's theorem above, successive or simultaneous rotations about any number of axes through the fixed point may be compounded into a rotation about one axis. Doing this for infinitely small rotations we find the law of composition of angular velocities.

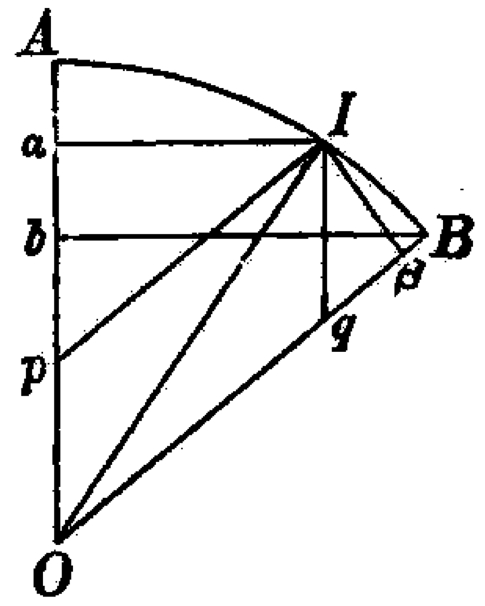
Composition of angular velocities.

Let OA, OB be two axes about which a body revolves with angular velocities ϖ, ρ respectively.

With radius unity describe the arc AB, and in it take any

Composition of angular velocities.

point I. Draw $I\alpha$, $I\beta$ perpendicular to OA, OB respectively. Let the rotations about the two axes be such that that about OB tends to *raise* I above the plane of the paper, and that about OA to depress it. In an infinitely short interval of time τ, the amounts of these displacements will be $\rho I\beta . \tau$ and $-\varpi I\alpha . \tau$. The point I, and therefore every point in the line OI, will be at rest during the interval τ if the sum of these displacements is zero, that is if $\rho . I\beta = \varpi . I\alpha$. Hence the line OI is instantaneously at rest, or the *two* rotations about OA and OB may be compounded into *one* about OI. Draw Ip, Iq, parallel to OB, OA respectively. Then, expressing in two ways the area of the parallelogram $IpOq$, we have



$$Oq . I\beta = Op . I\alpha,$$

$$Oq : Op :: \rho : \varpi.$$

Hence, if along the axes OA, OB, we measure off from O lines Op, Oq, proportional respectively to the angular velocities about these axes—the diagonal of the parallelogram of which these are contiguous sides is the resultant axis.

Again, if Bb be drawn perpendicular to OA, and if Ω be the angular velocity about OI, the whole displacement of B may evidently be represented either by $\varpi . Bb$ or $\Omega . I\beta$.
Hence

$$\Omega : \varpi :: Bb : I\beta :: \sin BOA : \sin IOB :: \sin IpO : \sin pIO,$$
$$:: OI : Op.$$

Thus it is proved that,—

Parallelogram of angular velocities.

If lengths proportional to the respective angular velocities about them be measured off on the component and resultant axes, the lines so determined will be the sides and diagonal of a parallelogram.

Composition of angular velocities about axes meeting in a point.

96. Hence the single angular velocity equivalent to three co-existent angular velocities about three mutually perpendicular axes, is determined in magnitude, and the direction of its axis is found (§ 27), as follows:—The square of the resultant angular velocity is the sum of the squares of its components,

and the ratios of the three components to the resultant are the direction cosines of the axis.

Composition of angular velocities about axes meeting in a point.

Hence simultaneous rotations about any number of axes meeting in a point may be compounded thus:—Let ω be the angular velocity about one of them whose direction cosines are l, m, n; Ω the angular velocity and λ, μ, ν the direction cosines of the resultant,

$$\lambda\Omega = \Sigma(l\omega),\ \mu\Omega = \Sigma(m\omega),\ \nu\Omega = \Sigma(n\omega),$$

whence $$\Omega^2 = \Sigma^2(l\omega) + \Sigma^2(m\omega) + \Sigma^2(n\omega),$$

and $$\lambda = \frac{\Sigma(l\omega)}{\Omega},\ \mu = \frac{\Sigma(m\omega)}{\Omega},\ \nu = \frac{\Sigma(n\omega)}{\Omega}.$$

Hence also, an angular velocity about any line may be resolved into three about any set of rectangular lines, the resolution in each case being (like that of simple velocities) effected by multiplying by the cosine of the angle between the directions.

Hence, just as in § 31 a uniform acceleration, perpendicular to the direction of motion of a point, produces a change in the *direction* of motion, but does not influence the *velocity;* so, if a body be rotating about an axis, and be subjected to an action tending to produce rotation about a perpendicular axis, the result will be a change of *direction* of the axis about which the body revolves, but no change in the *angular velocity*. On this kinematical principle is founded the dynamical explanation of the Precession of the Equinoxes (§ 107) and of some of the seemingly marvellous performances of gyroscopes and gyrostats.

The following method of treating the subject is useful in connexion with the ordinary methods of co-ordinate geometry. It contains also, as will be seen, an independent demonstration of the parallelogram of angular velocities:—

Angular velocities ϖ, ρ, σ about the axes of x, y, and z respectively, produce in time δt displacements of the point at x, y, z (§§ 87, 89),

$$(\rho z - \sigma y)\,\delta t \parallel x,\ (\sigma x - \varpi z)\,\delta t \parallel y,\ (\varpi y - \rho x)\,\delta t \parallel z.$$

Hence points for which

$$\frac{x}{\varpi} = \frac{y}{\rho} = \frac{z}{\sigma}$$

are not displaced. These are therefore the equations of the axis.

Composition of angular velocities about axes meeting in a point.

Now the perpendicular from any point x, y, z to this line is, by co-ordinate geometry,

$$\left[x^2+y^2+z^2-\frac{(\varpi x+\rho y+\sigma z)^2}{\varpi^2+\rho^2+\sigma^2}\right]^{\frac{1}{2}}$$

$$=\frac{1}{\sqrt{\varpi^2+\rho^2+\sigma^2}}\sqrt{(\rho z-\sigma y)^2+(\sigma x-\varpi z)^2+(\varpi y-\rho x)^2}$$

$$=\frac{\text{whole displacement of } x, y, z}{\sqrt{\varpi^2+\rho^2+\sigma^2}\,\delta t}.$$

The actual displacement of x, y, z is therefore the same as would have been produced in time δt by a single angular velocity, $\Omega=\sqrt{\varpi^2+\rho^2+\sigma^2}$, about the axis determined by the preceding equations.

Composition of successive finite rotations.

97. We give next a few useful theorems relating to the composition of successive *finite* rotations.

If a pyramid or cone of any form roll on a heterochirally similar* pyramid (the image in a plane mirror of the first position of the first) all round, it clearly comes back to its primitive position. This (as all rolling of cones) is conveniently exhibited by taking the intersection of each with a spherical surface. Thus we see that if a spherical polygon turns about its angular points in succession, always keeping on the spherical surface, and if the angle through which it turns about each point is twice the supplement of the angle of the polygon, or, which will come to the same thing, if it be in the other direction, but equal to twice the angle itself of the polygon, it will be brought to its original position.

The polar theorem (compare § 134, below) to this is, that a body, after successive rotations, represented by the doubles of the successive sides of a spherical polygon taken in order, is restored to its original position; which also is self-evident.

98. Another theorem is the following;—

If a pyramid rolls over all its sides on a plane, it leaves its track behind it as one plane angle, equal to the sum of the plane angles at its vertex.

* The similarity of a right-hand and a left-hand is called heterochiral: that of two right-hands, homochiral. Any object and its image in a plane mirror are heterochirally similar (Thomson, *Proc. R. S. Edinburgh*, 1873).

Composition of successive finite rotations.

Otherwise:—in a spherical surface, a spherical polygon having rolled over all its sides along a great circle, is found in the same position as if the side first lying along that circle had been simply shifted along it through an arc equal to the polygon's periphery. The polar theorem is:—if a body be made to take successive rotations, represented by the sides of a spherical polygon taken in order, it will finally be as if it had revolved about the axis through the first angular point of the polygon through an angle equal to the spherical excess (§ 134) or area of the polygon.

Motion about a fixed point. Rolling cones.

99. The investigation of § 90 also applies to this case; and it is thus easy to show that the most general motion of a spherical figure on a fixed spherical surface is obtained by the rolling of a curve fixed in the figure on a curve fixed on the sphere. Hence as at each instant the line joining C and O contains a set of points of the body which are momentarily at rest, the most general motion of a rigid body of which one point is fixed consists in the rolling of a cone fixed in the body upon a cone fixed in space—the vertices of both being at the fixed point.

Position of the body due to given rotations.

100. Given at each instant the angular velocities of the body about three rectangular axes attached to it, determine its position in space at any time.

From the given angular velocities about OA, OB, OC, we know, § 95, the position of the instantaneous axis OI with reference to the body at every instant. Hence we know the conical surface in the body which rolls on the cone fixed in space. The data are sufficient also for the determination of this other cone; and these cones being known, and the lines of them which are in contact at any given instant being determined, the position of the moving body is completely determined.

If λ, μ, ν be the direction cosines of OI referred to OA, OB, OC; ϖ, ρ, σ the angular velocities, and ω their resultant:

$$\frac{\lambda}{\varpi}=\frac{\mu}{\rho}=\frac{\nu}{\sigma}=\frac{1}{\omega},$$

by § 95. These equations, in which $\varpi, \rho, \sigma, \omega$ are given functions of t, express explicitly the position of OI relatively to OA, OB, Position of the body due to given rotations.

OC, and therefore determine the cone fixed in the body. For the cone fixed in space: if r be the radius of curvature of its intersection with the unit sphere, r' the same for the rolling cone, we find from § 105 below, that if s be the length of the arc of either spherical curve from a common initial point,

$$\omega r' = \frac{1}{r}\frac{ds}{dt}\sin(\sin^{-1}r + \sin^{-1}r') = \frac{1}{r}\frac{ds}{dt}(r\sqrt{1-r'^2} + r'\sqrt{1-r^2}),$$

which, as s, r' and ω are known in terms of t, gives r in terms of t, or of s, as we please. Hence, by a single quadrature, the "intrinsic" equation of the fixed cone.

101. An unsymmetrical system of angular co-ordinates ψ, θ, ϕ, for specifying the position of a rigid body by aid of a line OB and a plane AOB moving with it, and a line OY and a plane YOX fixed in space, which is essentially proper for many physical problems, such as the Precession of the Equinoxes and the spinning of a top, the motion of a gyroscope and its gimbals, the motion of a compass-card and of its bowl and gimbals, is convenient for many others, and has been used by the greatest mathematicians often even when symmetrical methods would have been more convenient, must now be described.

ON being the intersection of the two planes, let $YON = \psi$, and $NOB = \phi$; and let θ be the angle from the fixed plane, produced through ON, to the portion NOB of the moveable plane. (Example, θ the "obliquity of the ecliptic," ψ the longitude of the autumnal equinox reckoned from OY, a fixed line in the plane of the earth's orbit supposed fixed; ϕ the hour-angle of the autumnal equinox; B being in the earth's equator and in the meridian of Greenwich: thus ψ, θ, ϕ are angular co-ordinates of the earth.) To show the relation of this to the symmetrical system, let OA be perpendicular to OB, and draw OC perpendicular to both; OX perpendicular to OY, and draw OZ perpendicular to OY and OX; so that OA, OB, OC are three rectangular axes fixed relatively to the body, and OX, OY, OZ fixed in space. The annexed diagram shows ψ, θ, ϕ in angles and arc, and in arcs and angles, on a spherical surface of unit radius with centre at O.

To illustrate the meaning of these angular co-ordinates, suppose A, B, C initially to coincide with X, Y, Z respectively.

Position of the body due to given rotations.

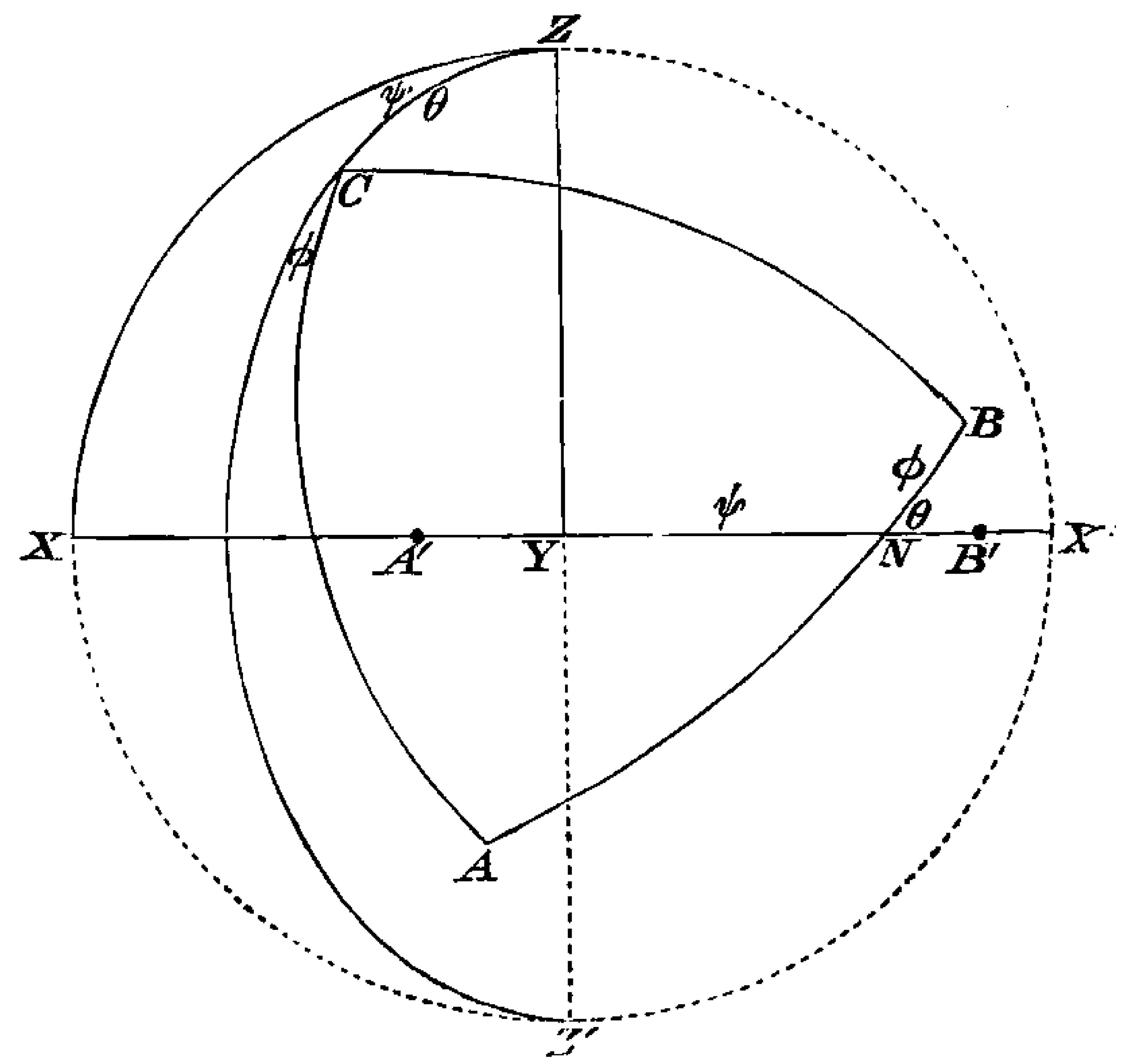
Then, to bring the body into the position specified by θ, ϕ, ψ, rotate it round OZ through an angle equal to $\psi + \phi$, thus

Letter O at centre of sphere concealed by Y.

$\widehat{XA'} = \psi + \phi$,

$\widehat{YN} = \psi$,

$\widehat{NB'} = \phi$.



bringing A and B from X and Y to A' and B' respectively; and, (taking $\widehat{YN} = \psi$,) rotate the body round ON through an angle equal to θ, thus bringing A, B, and C from the positions A', B', and Z respectively, to the positions marked A, B, C in the diagram. Or rotate first round ON through θ, so bringing C from Z to the position marked C, and then rotate round OC through $\psi + \phi$. Or, while OC is turning from OZ to the position shown on the diagram, let the body turn round OC relatively to the plane $ZCZ'O$ through an angle equal to ϕ. It will be in the position specified by these three angles.

Let $\angle XZC = \psi$, $\angle ZCA = \pi - \phi$, and $ZC = \theta$, and ϖ, ρ, σ mean the same as in § 100. By considering in succession instantaneous motions of C along and perpendicular to ZC, and the motion of AB in its own plane, we have

$$\frac{d\theta}{dt} = \varpi\sin\phi + \rho\cos\phi, \qquad \sin\theta\frac{d\psi}{dt} = \rho\sin\phi - \varpi\cos\phi,$$

and

$$\frac{d\psi}{dt}\cos\theta + \frac{d\phi}{dt} = \sigma.$$

The nine direction cosines (XA), (YB), &c., according to the notation of § 95, are given at once by the spherical triangles

Position of the body due to given rotations.

XNA, YNB, &c.; each having N for one angular point, with θ, or its supplement or its complement, for the angle at this point. Thus, by the solution in each case for the cosine of one side in terms of the cosine of the opposite angle, and the cosines and sines of the two other sides, we find

$$
\begin{aligned}
(XA) &= \cos\theta\cos\psi\cos\phi - \sin\psi\sin\phi,\\
(XB) &= -\cos\theta\cos\psi\sin\phi - \sin\psi\cos\phi,\\
(YA) &= \cos\theta\sin\psi\cos\phi + \cos\psi\sin\phi.\\[4pt]
(YB) &= -\cos\theta\sin\psi\sin\phi + \cos\psi\cos\phi,\\
(YC) &= \sin\theta\sin\psi,\\
(ZB) &= \sin\theta\sin\phi.\\[4pt]
(ZC) &= \cos\theta,\\
(ZA) &= -\sin\theta\cos\phi,\\
(XC) &= \sin\theta\cos\psi.
\end{aligned}
$$

General motion of a rigid body.

102. We shall next consider the most general possible motion of a rigid body of which no point is fixed—and first we must prove the following theorem. There is one set of parallel planes in a rigid body which are parallel to each other in any two positions of the body. The parallel lines of the body perpendicular to these planes are of course parallel to each other in the two positions.

Let C and C' be any point of the body in its first and second positions. Move the body without rotation from its second position to a third in which the point at C' in the second position shall occupy its original position C. The preceding demonstration shows that there is a line CO common to the body in its first and third positions. Hence a line $C'O'$ of the body in its second position is parallel to the same line CO in the first position. This of course clearly applies to every line of the body parallel to CO, and the planes perpendicular to these lines also remain parallel.

Let S denote a plane of the body, the two positions of which are parallel. Move the body from its first position, without rotation, in a direction perpendicular to S, till S comes into the plane of its second position. Then to get the body into its actual position, such a motion as is treated in § 79 is farther

General motion of a rigid body

required. But by § 79 this may be effected by rotation about a certain axis perpendicular to the plane S, unless the motion required belongs to the exceptional case of pure translation. Hence [this case excepted] the body may be brought from the first position to the second by translation through a determinate distance perpendicular to a given plane, and rotation through a determinate angle about a determinate axis perpendicular to that plane. This is precisely the motion of a screw in its nut.

103. In the excepted case the whole motion consists of two translations, which can of course be compounded into a single one; and thus, in this case, there is no rotation at all, or every plane of it fulfils the specified condition for S of § 102.

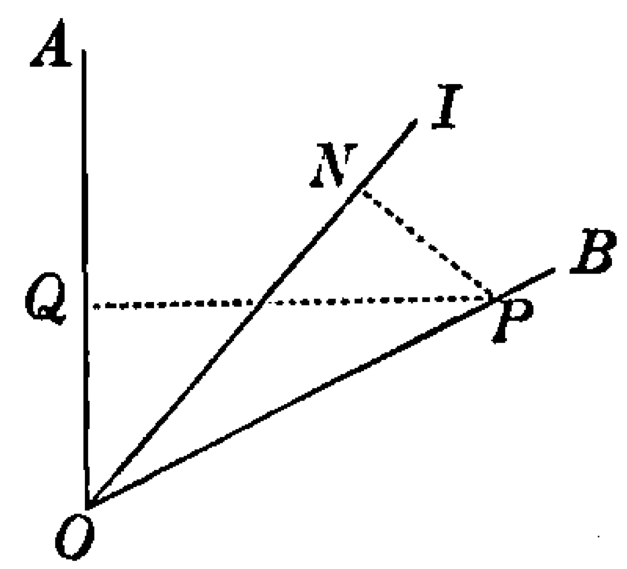
Precessional Rotation.

104. Returning to the motion of a rigid body with one point fixed, let us consider the case in which the guiding cones, § 99, are both circular. The motion in this case may be called *Precessional Rotation.*

The plane through the instantaneous axis and the axis of the fixed cone passes through the axis of the rolling cone. This plane turns round the axis of the fixed cone with an angular velocity Ω (see § 105 below), which must clearly bear a constant ratio to the angular velocity ω of the rigid body about its instantaneous axis.

105. The motion of the plane containing these axes is called the *precession* in any such case. What we have denoted by Ω is the angular velocity of the precession, or, as it is sometimes called, the rate of precession.

The angular motions ω, Ω are to one another inversely as the distances of a point in the axis of the rolling cone from the instantaneous axis and from the axis of the fixed cone.



For, let OA be the axis of the fixed cone, OB that of the rolling cone, and OI the instantaneous axis. From any point P in OB draw PN perpendicular to OI, and PQ perpendicular to OA. Then we perceive that P moves always in the circle whose centre is Q, radius PQ, and plane perpendicular to OA. Hence

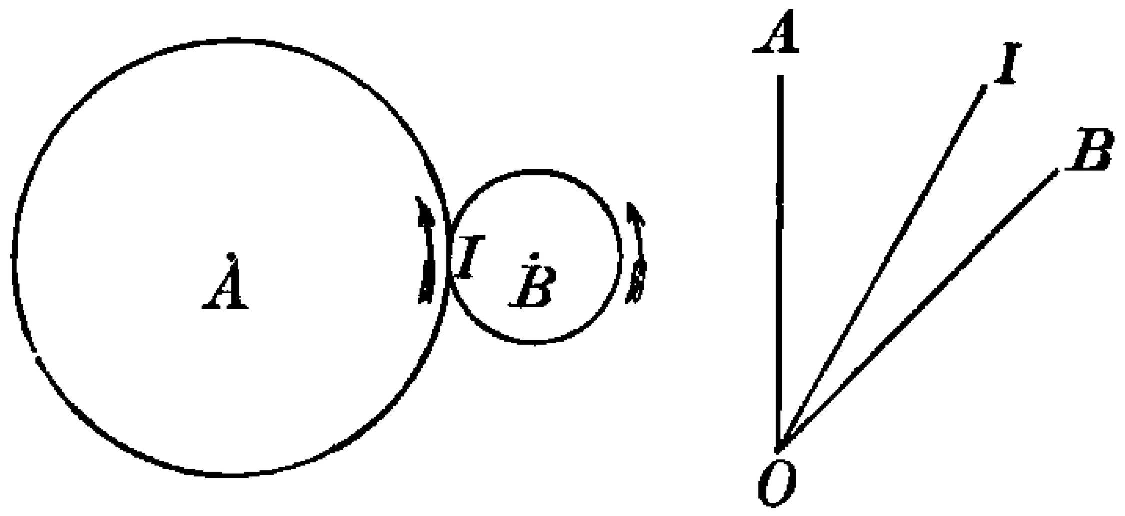
Precessional Rotation.

the actual velocity of the point P is ΩQP. But, by the principles explained above, § 99, the velocity of P is the same as that of a point moving in a circle whose centre is N, plane perpendicular to ON, and radius NP, which, as this radius revolves with angular velocity ω, is ωNP. Hence

$$\Omega \,.\, QP = \omega \,.\, NP, \text{ or } \omega : \Omega :: QP : NP.$$

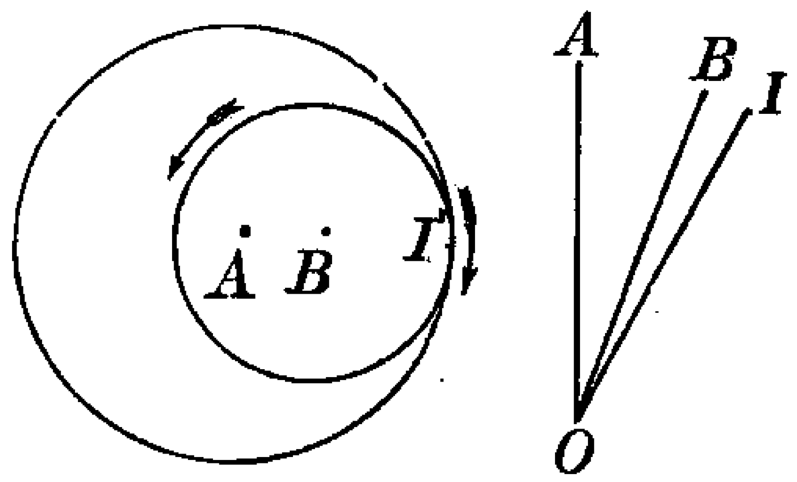
Let α be the semivertical angle of the fixed, β of the rolling, cone. Each of these may be supposed for simplicity to be acute, and their sum or difference less than a right angle—though, of course, the formulæ so obtained are (like all trigonometrical results) applicable to every possible case. We have the following three cases:—

I. Convex cone rolling on convex.



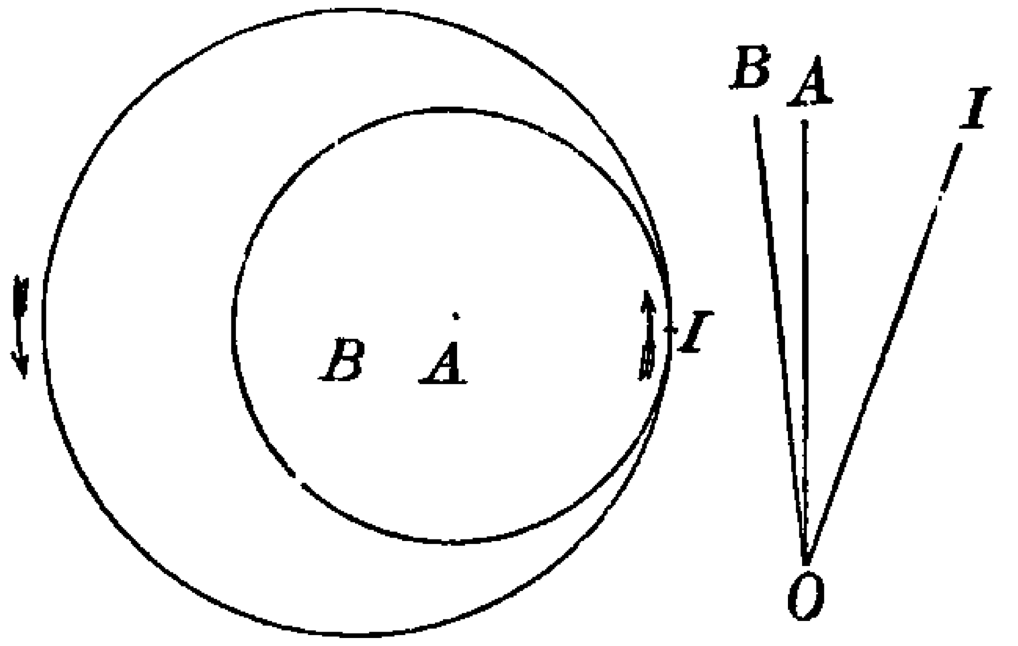
$\omega \sin \beta = \Omega \sin (\alpha + \beta)$,
where $AOI = \alpha$, $IOB = \beta$.

II. Convex cone rolling inside concave.



Let β be negative, and let $\beta' = -\beta$; then β' is positive, and we have
$-\omega \sin \beta' = \Omega \sin (\alpha - \beta')$,
where $AOI = \alpha$, $BOI = \beta'$.

III. Concave cone rolling outside convex.



In the preceding let $\beta' > \alpha$. It may then be conveniently written
$\omega \sin \beta' = \Omega \sin (\beta' - \alpha)$,
where $AOI = \alpha$, $BOI = \beta'$,
α and β' being still positive.

Cases of precessional rotation.

106. If, as illustrated by the first of these diagrams, the case is one of a convex cone rolling on a convex cone, the precessional motion, viewed on a hemispherical surface having A for its pole and O for its centre, is in a similar direction to

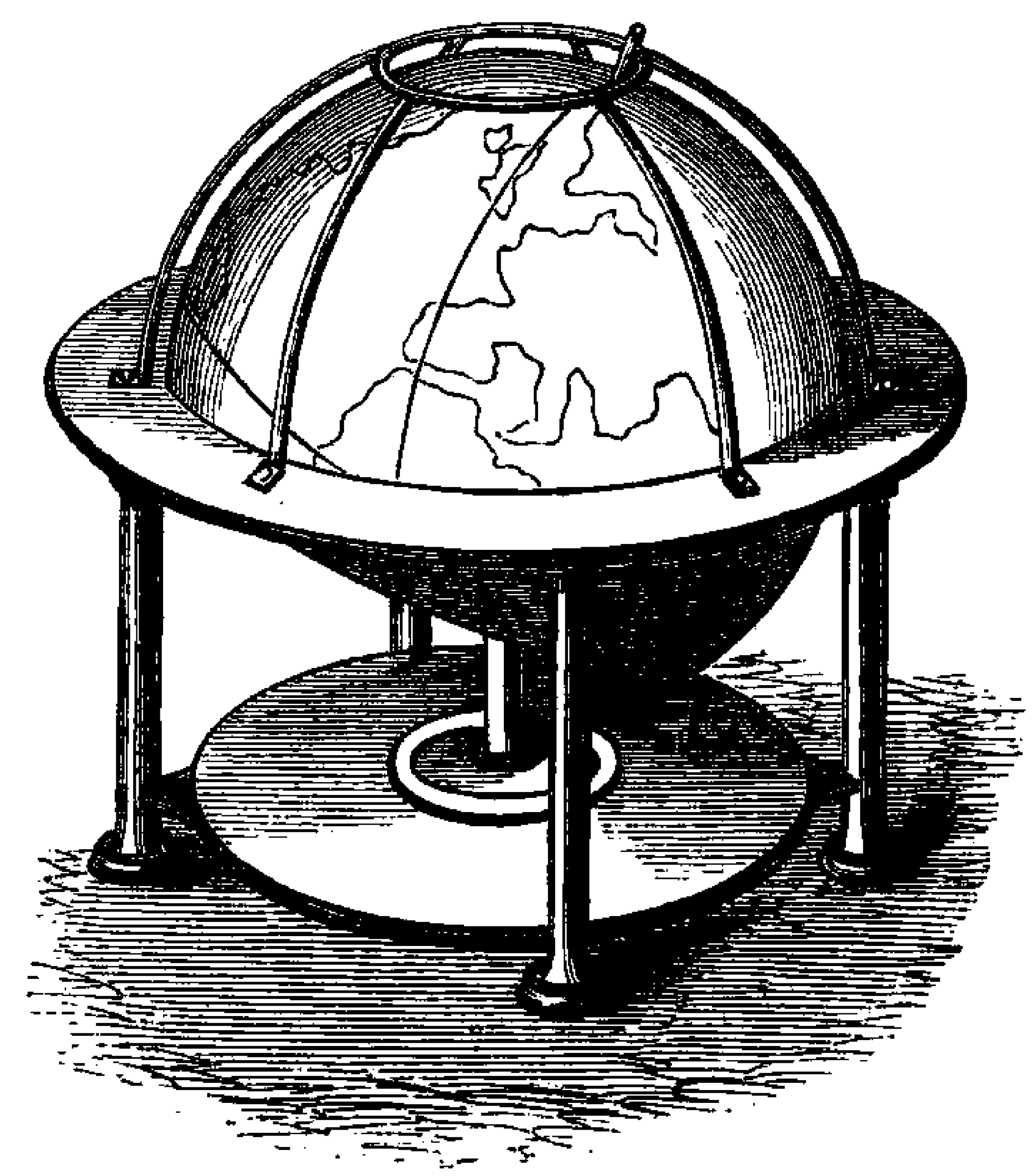
that of the angular rotation about the instantaneous axis. This we shall call *positive* precessional rotation. It is the case of a common spinning-top (peery), spinning on a very fine point which remains at rest in a hollow or hole bored by itself; not sleeping upright, nor nodding, but sweeping its axis round in a circular cone whose axis is vertical. In Case III. also we have *positive* precession. A good example of this occurs in the case of a coin spinning on a table when its plane is nearly horizontal.

Cases of precessional rotation.

107. Case II., that of a convex cone rolling inside a concave one, gives an example of *negative* precession: for when viewed as before on the hemispherical surface the direction of angular rotation of the instantaneous axis is opposite to that of the rolling cone. This is the case of a symmetrical cup (or figure of revolution) supported on a point, and stable when balanced, *i.e.*, having its centre of gravity below the pivot; when inclined and set spinning non-nutationally. For instance, if a Troughton's top be placed on its pivot in any inclined position, and then spun off with very great angular velocity about its axis of figure, the nutation will be insensible; but there will be slow precession.

Model illustrating Precessional Equinoxes.

To this case also belongs the precessional motion of the earth's axis; for which the angle $\alpha = 23^\circ\, 27'\, 28''$, the period of the rotation ω the sidereal day; that of Ω is 25,868 years. If the second diagram represent a portion of the earth's surface round the pole, the arc AI = 8,552,000 feet, and therefore the circumference of the circle in which I moves = 52,240,000 feet. Imagine this circle to be the in-



Precession of the equinoxes.

ner edge of a fixed ring in space (directionally fixed, that is to say, but having the same translational motion as the earth's centre), and imagine a circular post or pivot of radius BI to be fixed to the earth with its centre at B. This ideal pivot rolling on the inner edge of the fixed ring travels once round the 52,240,000 feet-circumference in 25,868 years, and therefore its own circumference must be 5·53 feet. Hence $BI = 0{\cdot}88$ feet; and angle BOI, or β, $= 0''{\cdot}00867$.

Free rotation of a body kinetically symmetrical about an axis.

108. Very interesting examples of Cases I. and III. are furnished by projectiles of different forms rotating about any axis. Thus the gyrations of an oval body or a rod or bar flung into the air belong to Class I. (the body having one axis of less moment of inertia than the other two, equal); and the seemingly irregular evolutions of an ill-thrown quoit belong to Class III. (the quoit having one axis of greater moment of inertia than the other two, which are equal). Case III. has therefore the following very interesting and important application.

If by a geological convulsion (or by the transference of a few million tons of matter from one part of the world to another) the earth's instantaneous axis OI (diagram III., § 105) were at any time brought to non-coincidence with its principal axis of greatest moment of inertia, which (§§ 825, 285) is an axis of approximate kinetic symmetry, the instantaneous axis will, and the fixed axis OA will, relatively to the solid, travel round the solid's axis of greatest moment of inertia in a period of about 306 days [this number being the reciprocal of the most probable value of $\frac{C-A}{C}$ (§ 828)]; and the motion is represented by the diagram of Case III. with $BI = 306 \times AI$. Thus in a very little less than a day (less by $\frac{1}{306}$ when BOI is a small angle) I revolves round A. It is OA, as has been remarked by Maxwell, that is found as the direction of the celestial pole by observations of the meridional zenith distances of stars, and this line being the resultant axis of the earth's moment of

momentum (§ 267), would remain invariable in space did no external influence such as that of the moon and sun disturb the earth's rotation. When we neglect precession and nutation, the polar distances of the stars are constant notwithstanding the ideal motion of the fixed axis which we are now considering; and the effect of this motion will be to make a periodic variation of the latitude of every place on the earth's surface having for range on each side of its mean value the angle BOA, and for its period 306 days or thereabouts. Maxwell* examined a four years series of Greenwich observations of Polaris (1851–2–3–4), and concluded that there was during those years no variation exceeding half a second of angle on each side of mean range, but that the evidence did not disprove a variation of that amount, but on the contrary gave a very slight indication of a minimum latitude of Greenwich belonging to the set of months Mar. '51, Feb. '52, Dec. '52, Nov. '53, Sept. '54.

Free rotation of a body kinetically symmetrical about an axis.

"This result, however, is to be regarded as very doubtful...... "and more observations would be required to establish the "existence of so small a variation at all.

"I therefore conclude that the earth has been for a long time "revolving about an axis very near to the axis of figure, if not "coinciding with it. The cause of this near coincidence is "either the original softness of the earth, or the present fluidity "of its interior [or the existence of water on its surface]. "The axes of the earth are so nearly equal that a considerable elevation of a tract of country might produce a "deviation of the principal axis within the limits of observation, and the only cause which would restore the uniform "motion, would be the action of a fluid which would gradually "diminish the oscillations of latitude. The permanence of "latitude essentially depends on the inequality of the earth's "axes, for if they had all been equal, any alteration in the "crust of the earth would have produced new principal axes, "and the axis of rotation would travel about those axes, alter-

* On a Dynamical Top, *Trans. R. S. E.*, 1857, p. 559.

Free rotation of a body kinetically symmetrical about an axis.

"ing the latitudes of all places, and yet not in the least altering "the position of the axis of rotation among the stars."

Perhaps by a more extensive "search and analysis of the "observations of different observatories, the nature of the "periodic variation of latitude, if it exist, may be determined. "I am not aware* of any calculations having been made to prove "its non-existence, although, on dynamical grounds, we have "every reason to look for some very small variation having the "periodic time of 325·6 days nearly" [more nearly 306 days], "a period which is clearly distinguished from any other astro- "nomical cycle, and therefore easily recognised†."

The periodic variation of the earth's instantaneous axis thus anticipated by Maxwell must, if it exists, give rise to a tide of 306 days period (§ 801). The amount of this tide at the equator would be a rise and fall amounting only to $5\frac{1}{2}$ centimetres above and below mean for a deviation of the instantaneous axis amounting to 1″ from its mean position OB, or for a deviation BI on the earth's surface amounting to 31 metres. This, although discoverable by elaborate analysis of long-continued and accurate tidal observations, would be less easily discovered than the periodic change of latitude by astronomical observations according to Maxwell's method‡.

* [Written in 1857. G. H. D.]

† Maxwell; *Transactions of the Royal Society of Edinburgh*, 20th April, 1857.

‡ Prof. Maxwell now refers us to Peters (*Recherches sur la parallaxe des étoiles fixes*, St Petersburgh Observatory Papers, Vol. I., 1853), who seems to have been the first to raise this interesting and important question. He found from the Pulkova observations of Polaris from March 11, 1842 till April 30, 1843 an angular radius of 0″·079 (probable error 0″·017), for the circle round its mean position described by the instantaneous axis, and for the time, within that interval, when the latitude of Pulkova was a maximum, Nov. 16, 1842. The period (calculated from the dynamical theory) which Peters assumed was 304 mean solar days: the rate therefore 1·201 turns per annum, or, nearly enough, 12 turns per ten years. Thus if Peters' result were genuine, and remained constant for ten years, the latitude of Pulkova would be a maximum about the 16th of Nov. again in 1852, and Pulkova being in 30° East longitude from Greenwich, the latitude of Greenwich would be a maximum $\frac{1}{12}$ of the period, or about 25 days earlier, that is to say about Oct. 22, 1852. But Maxwell's examination of observations seemed to indicate more nearly the minimum latitude of Greenwich about the same time. This discrepance is altogether in accordance with a continuation of Peters' investigation by Dr Nyrén of the Pulkova Observatory, in which, by a careful scrutiny of several series of Pulkova observations between the years 1842...1872, he concluded that there is no constancy of magnitude or phase in the deviation sought for. A similar negative conclusion was arrived at by Professor Newcomb of the United States Naval Observatory, Washington, who at our request kindly undertook an investigation of the ten-month period of latitude from the Washington Prime Vertical Observations from 1862 to 1867. His results, as did those of Peters and Nysen and Maxwell, seemed to indicate real variations of the earth's instantaneous axis amounting to possibly as much as $\frac{1}{3}''$ or $\frac{1}{4}''$ from its mean position, but altogether irregular both in amount and direction; in fact, just such as might be expected from irregular heapings up of the oceans by winds in different localities of the earth.

We intend to return to this subject and to consider cognate questions regarding irregularities of the earth as a timekeeper, and variations of its figure and of the distribution of matter within it, of the ocean on its surface, and of the atmosphere surrounding it, in §§ 267, 276, 405, 406, 830, 832, 845, 846.

Communication of angular velocity equally between inclined axes.

109. In various illustrations and arrangements of apparatus useful in Natural Philosophy, as well as in Mechanics, it is required to connect two bodies, so that when either turns about a certain axis, the other shall turn with an equal angular velocity about another axis in the same plane with the former, but inclined to it at any angle. This is accomplished in mechanism by means of equal and similar bevelled wheels, or rolling cones; when the mutual inclination of two axes is not to be varied. It is approximately accomplished by means of Hooke's joint, when the two axes are nearly in the same line, but are required to be free to vary in their mutual inclination. A chain of an infinitely great number of Hooke's joints may be imagined as constituting a perfectly flexible, untwistable cord, which, if its end-links are rigidly attached to the two bodies, connects them so as to fulfil the condition rigorously without the restriction that the two axes remain in one plane. If we imagine an infinitely short length of such a chain (still, however, having an infinitely great number of links) to have its ends attached to two bodies, it will fulfil rigorously the condition stated, and at the same time keep a definite point of one body infinitely near a definite point of the other; that is to say, it will accomplish precisely for every angle of inclination what Hooke's joint does approximately for small inclinations.

Hooke's joint.

Flexible but untwistable cord.

Universal flexure joint.

Elastic universal flexure joint.

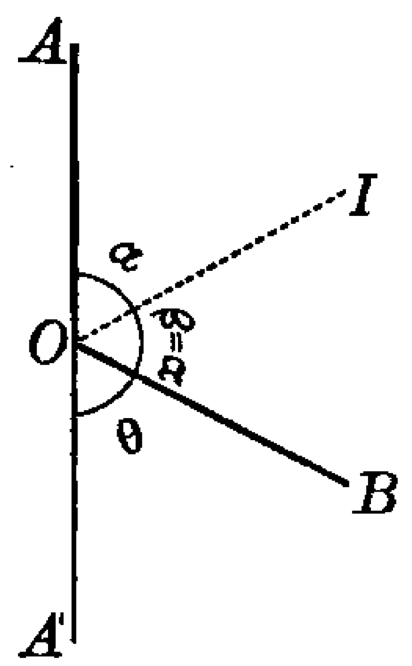
The same is dynamically accomplished with perfect accuracy for every angle, by a short, naturally straight, elastic wire of

Elastic universal flexure joint.

truly circular section, provided the forces giving rise to any resistance to equality of angular velocity between the two bodies are infinitely small. In many practical cases this mode of connexion is useful, and permits very little deviation from the conditions of a true universal flexure joint. It is used, for instance, in the suspension of the gyroscopic pendulum (§ 74) with perfect success. The dentist's tooth-mill is an interesting illustration of the elastic universal flexure joint. In it a long spiral spring of steel wire takes the place of the naturally straight wire suggested above.

Moving body attached to a fixed object by a universal flexure joint.

Of two bodies connected by a universal flexure joint, let one be held fixed. The motion of the other, as long as the angle of inclination of the axes remains constant, will be exactly that figured in Case I., § 105, above, with the angles α and β made equal. Let O be the joint; AO the axis of the fixed body; OB the axis of the moveable body. The supplement of the angle AOB is the mutual inclination of the axes; and the angle AOB itself is bisected by the instantaneous axis of the moving body. The diagram shows a case of this motion, in which the mutual inclination, θ, of the axes is acute. According to the formulæ of Case I., § 105, we have



$$\omega \sin\alpha = \Omega \sin 2\alpha,$$

or

$$\omega = 2\Omega \cos\alpha = 2\Omega \sin\frac{\theta}{2},$$

where ω is the angular velocity of the moving body about its instantaneous axis, OI, and Ω is the angular velocity of its precession; that is to say, the angular velocity of the plane through the fixed axis AA', and the moving axis OB of the moving body.

Two degrees of freedom to move enjoyed by a body thus suspended.

Besides this motion, the moving body may clearly have any angular velocity whatever about an axis through O perpendicular to the plane AOB, which, compounded with ω round OI, gives the resultant angular velocity and instantaneous axis.

Two co-ordinates, $\theta = A'OB$, and ϕ measured in a plane perpendicular to AO, from a fixed plane of reference to the plane

AOB, fully specify the position of the moveable body in this case.

General motion of one rigid body touching another.

110. Suppose a rigid body bounded by any curved surface to be touched at any point by another such body. Any motion of one on the other must be of one or more of the forms *sliding*, *rolling*, or *spinning*. The consideration of the first is so simple as to require no comment.

Any motion in which there is no slipping at the point of contact must be rolling or spinning separately, or combined.

Let one of the bodies rotate about successive instantaneous axes, all lying in the common tangent plane at the point of instantaneous contact, and each passing through this point—the other body being fixed. This motion is what we call rolling, or simple rolling, of the moveable body on the fixed.

On the other hand, let the instantaneous axis of the moving body be the common normal at the point of contact. This is pure spinning, and does not change the point of contact.

Let the moving body move, so that its instantaneous axis, still passing through the point of contact, is neither in, nor perpendicular to, the tangent plane. This motion is combined rolling and spinning.

Traces of rolling.

111. When a body rolls and spins on another body, the *trace of either on the other* is the curved or straight line along which it is successively touched. If the instantaneous axis is in the normal plane perpendicular to the traces, the rolling is called *direct*. If not direct, the rolling may be resolved into a direct rolling, and a rotation or twisting round the tangent line to the traces.

Direct rolling.

When there is *no spinning* the projections of the two traces on the common tangent plane at the point of contact of the two surfaces have equal and same-way directed curvature: or they have "contact of the second order." When there *is spinning*, the two projections still touch one another, but with contact of the first order only: their curvatures differ by a quantity equal to the angular velocity of spinning divided by the velocity of the point of contact. This last we see by noticing that the rate of change of direction along the pro-

Direct rolling.

jection of the fixed trace must be equal to the rate of change of direction along the projection of the moving trace if held fixed plus the angular velocity of the spinning.

At any instant let $2z = Ax^2 + 2Cxy + By^2$ ....(1)

and $2z' = A'x^2 + 2C'xy + B'y^2$ ....(2)

be the equations of the fixed and moveable surfaces S and S' infinitely near the point of contact O, referred to axes OX, OY in their common tangent plane, and OZ perpendicular to it: let ϖ, ρ, σ be the three components of the instantaneous angular velocity of S'; and let x, y, be co-ordinates of P, the point of contact at an infinitely small time t, later: the third co-ordinate, z, is given by (1).

Let P' be the point of S' which at this later time coincides with P. The co-ordinates of P' at the first instant are $x + \sigma yt$, $y - \sigma xt$; and the corresponding value of z' is given by (2). This point is infinitely near to (x, y, z'), and therefore at the first instant the direction cosines of the normal to S' through it differ but infinitely little from

$$-(A'x + C'y), \quad -(C'x + B'y), \quad 1.$$

But at time t the normal to S' at P' coincides with the normal to S at P, and therefore its direction cosines change from the preceding values, to

$$-(Ax + Cy), \quad -(Cx + By), \quad 1:$$

that is to say, it rotates through angles

$$(C' - C)x + (B' - B)y \text{ round } OX,$$

and $\quad -\{(A' - A)x + (C' - C)y\} \quad ,, \quad OY.$

Hence
$$\left.\begin{aligned} \varpi t &= (C' - C)x + (B' - B)y \\ \rho t &= -\{(A' - A)x + (C' - C)y\} \end{aligned}\right\} \text{..............} (3),$$

or
$$\left.\begin{aligned} \varpi &= (C' - C)\dot{x} + (B' - B)\dot{y} \\ \rho &= -\{(A' - A)\dot{x} + (C' - C)\dot{y}\} \end{aligned}\right\} \text{..................} (4),$$

if $\dot{x}$, $\dot{y}$ denote the component velocities of the point of contact.

Put $\quad q = \sqrt{(\dot{x}^2 + \dot{y}^2)}$ ....(5),

and take components of ϖ and ρ round the tangent to the traces and the perpendicular to it in the common tangent plane of the two surfaces, thus:

$$\text{(twisting component)} \ldots\ldots \frac{\dot{x}}{q}\varpi + \frac{\dot{y}}{q}\rho$$
$$= (C' - C)\frac{\dot{x}^2 - \dot{y}^2}{q} + [(B' - B) - (A' - A)]\frac{\dot{x}\dot{y}}{q} \ldots\ldots (6),$$

Direct rolling.

and

$$\text{(direct-rolling component)} \ldots\ldots \frac{\dot{y}}{q}\varpi - \frac{\dot{x}}{q}\rho$$
$$= \frac{1}{q}[(A' - A)\dot{x}^2 + 2(C' - C)\dot{x}\dot{y} + (B' - B)\dot{y}^2] \ldots\ldots (7).$$

Choose OX, OY so that $C - C' = 0$, and put $A' - A = \alpha$, $B' - B = \beta$ (6) and (7) become

$$\text{(twisting component)} \ \ldots\ldots\ldots\ldots \frac{\dot{x}}{q}\varpi + \frac{\dot{y}}{q}\rho = (\beta - \alpha)\frac{\dot{x}\dot{y}}{q} \ldots\ldots (8),$$

$$\text{(direct-rolling component)} \ldots\ldots \frac{\dot{y}}{q}\varpi - \frac{\dot{x}}{q}\rho = \frac{1}{q}(\alpha\dot{x}^2 + \beta\dot{y}^2) \ldots\ldots (9).$$

[Compare below, § 124 (2) and (1).]

And for σ, the angular velocity of spinning, the obvious proposition stated in the preceding large print gives

$$\sigma = q\left(\frac{1}{\gamma} - \frac{1}{\gamma'}\right) \ldots\ldots\ldots\ldots\ldots\ldots (10),$$

if $\frac{1}{\gamma}$ and $\frac{1}{\gamma'}$ be the curvatures of the projections on the tangent plane of the fixed and moveable traces. [Compare below, § 124 (3).]

From (1) and (2) it follows that

When one of the surfaces is a plane, and the trace on the other is a line of curvature (§ 130), the rolling is direct.

When the trace on each body is a line of curvature, the rolling is direct. *Generally*, the rolling is direct when the twists of infinitely narrow bands (§ 120) of the two surfaces, along the traces, are equal and in the same direction.

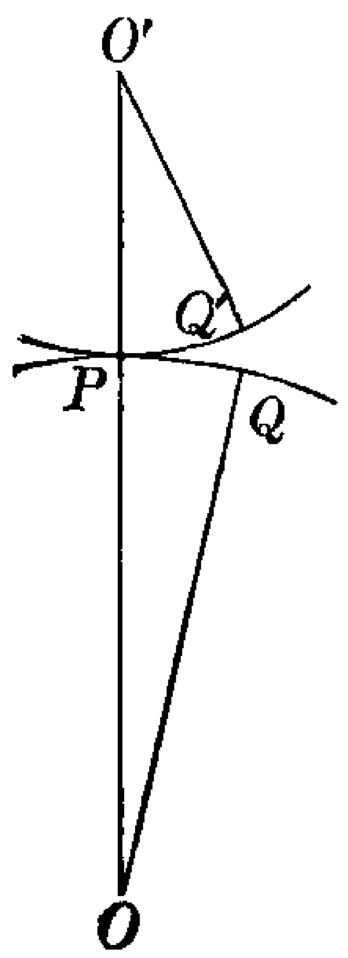
112. Imagine the traces constructed of rigid matter, and all the rest of each body removed. We may repeat the motion with these curves alone. The difference of the circumstances now supposed will only be experienced if we vary the direction of the instantaneous axis. In the former case, we can only do this by introducing more or less of spinning, and if we do so we *alter the trace* on each body. In the latter, we have always the same moveable curve rolling on the same fixed curve; and therefore a determinate line perpendicular to their common tangent for one component of the rotation; but along with this we may give arbitrarily any velocity of twisting round the common tangent. The consideration of this case is very in-

Curve rolling on curve.

structive. It may be roughly imitated in practice by two stiff wires bent into the forms of the given curves, and prevented from crossing each other by a short piece of elastic tube clasping them together.

First, let them be both plane curves, and kept in one plane. We have then *rolling*, as of one cylinder on another.

Let ρ' be the radius of curvature of the rolling, ρ of the fixed, cylinder; ω the angular velocity of the former, V the linear velocity of the point of contact. We have



$$\omega = \left(\frac{1}{\rho} + \frac{1}{\rho'}\right) V.$$

For, in the figure, suppose P to be at any time the point of contact, and Q and Q' the points which are to be in contact after an infinitely small interval t; O, O' the centres of curvature; $POQ = \theta$, $PO'Q' = \theta'$.

Then $PQ = PQ'$ = space described by point of contact. In symbols $\rho\theta = \rho'\theta' = Vt$.

Also, before $O'Q'$ and OQ can coincide in direction, the former must evidently turn through an angle $\theta + \theta'$.

Therefore $\omega t = \theta + \theta'$; and by eliminating θ and θ', and dividing by t, we get the above result.

It is to be understood, that as the radii of curvature have been considered positive here when both surfaces are convex, the negative sign must be introduced for either radius when the corresponding curve is concave.

Angular velocity of rolling in a plane.

Hence the angular velocity of the rolling curve is in this case equal to the product of the linear velocity of the point of contact by the sum or difference of the curvatures, according as the curves are both convex, or one concave and the other convex.

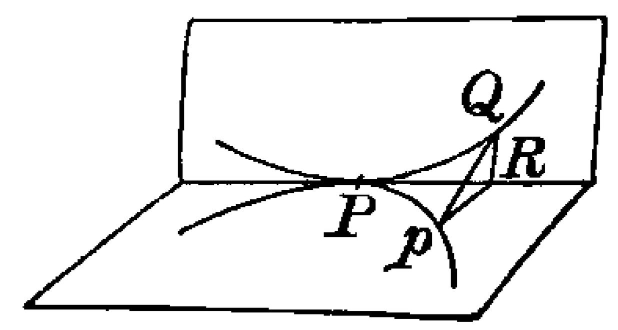
Plane curves not in same plane.

113. When the curves are both plane, but in different planes, the plane in which the rolling takes place divides the angle between the plane of one of the curves, and that of the other produced through the common tangent line, into parts whose sines are inversely as the curvatures in them respectively; and the angular velocity is equal to the linear velocity

of the point of contact multiplied by the difference of the projections of the two curvatures on this plane. The projections of the circles of the two curvatures on the plane of the common tangent and of the instantaneous axis coincide.

Plane curves not in same plane.

For, let PQ, Pp be equal arcs of the two curves as before, and let PR be taken in the common tangent (*i.e.*, the intersection of the planes of the curves) equal to each. Then QR, pR are ultimately perpendicular to PR.



Hence
$$pR = \frac{PR^2}{2\sigma},$$
$$QR = \frac{PR^2}{2\rho}.$$

Also, $\angle\, QRp = \alpha$, the angle between the planes of the curves.

We have
$$Qp^2 = \frac{PR^4}{4}\left(\frac{1}{\sigma^2} + \frac{1}{\rho^2} - \frac{2}{\sigma\rho}\cos\alpha\right).$$

Therefore if ω be the velocity of rotation as before,

$$\omega = V\sqrt{\frac{1}{\sigma^2} + \frac{1}{\rho^2} - \frac{2\cos\alpha}{\sigma\rho}}.$$

Also the instantaneous axis is evidently perpendicular, and therefore the plane of rotation parallel, to Qp. Whence the above. In the case of $\alpha = \pi$, this agrees with the result of § 112.

A good example of this is the case of a coin spinning on a table (mixed rolling and spinning motion), as its plane becomes gradually horizontal. In this case the curvatures become more and more nearly equal, and the angle between the planes of the curves smaller and smaller. Thus the resultant angular velocity becomes exceedingly small, and the motion of the point of contact very great compared with it.

Curve rolling on curve: two degrees of freedom.

114. The preceding results are, of course, applicable to tortuous as well as to plane curves; it is merely requisite to substitute the osculating plane of the former for the plane of the latter.

Curve rolling on surface: three degrees of freedom.

115. We come next to the case of a curve rolling, with or without spinning, on a surface.

It may, of course, roll on any curve traced on the surface. When this curve is given, the moving curve may, while rolling along it, revolve arbitrarily round the tangent. But the com-

Curve rolling on surface: three degrees of freedom.

ponent instantaneous axis perpendicular to the common tangent, that is, the axis of the direct rolling of one curve on the other, is determinate, § 113. If this axis does not lie in the surface, there is spinning. Hence, when the trace on the surface is given, there are two independent variables in the motion; the space traversed by the point of contact, and the inclination of the moving curve's osculating plane to the tangent plane of the fixed surface.

Trace prescribed and no spinning permitted.

116. If the trace is given, and it be prescribed as a condition that there shall be no spinning, the angular position of the rolling curve round the tangent at the point of contact is determinate. For in this case the instantaneous axis must be in the tangent plane to the surface. Hence, if we resolve the rotation into components round the tangent line, and round an axis perpendicular to it, the latter must be in the tangent plane. Thus the rolling, as of curve on curve, must be in a normal plane to the surface; and therefore (§§ 114, 113) the rolling curve must be always so situated relatively to its trace on the surface that the projections of the two curves on the tangent plane may be of coincident curvature.

Two degrees of freedom.

The curve, as it rolls on, must continually revolve about the tangent line to it at the point of contact with the surface, so as in every position to fulfil this condition.

Let α denote the inclination of the plane of curvature of the trace, to the normal to the surface at any point, α' the same for the plane of the rolling curve; $\frac{1}{\rho}$, $\frac{1}{\rho'}$ their curvatures. We reckon α as obtuse, and α' acute, when the two curves lie on opposite sides of the tangent plane. Then

$$\frac{1}{\rho'}\sin\alpha' = \frac{1}{\rho}\sin\alpha,$$

which fixes α' or the position of the rolling curve when the point of contact is given.

Angular velocity of direct rolling.

Let ω be the angular velocity of rolling about an axis perpendicular to the tangent, ϖ that of twisting about the tangent, and let V be the linear velocity of the point of contact. Then, since $\frac{1}{\rho'}\cos\alpha'$

and $-\frac{1}{\rho}\cos\alpha$ (each positive when the curves lie on opposite sides of the tangent plane) are the projections of the two curvatures on a plane through the normal to the surface containing their common tangent, we have, by § 112,

Angular velocity of direct rolling.

$$\omega = V\left(\frac{1}{\rho'}\cos\alpha' - \frac{1}{\rho}\cos\alpha\right),$$

α' being determined by the preceding equation. Let τ and τ' denote the tortuosities of the trace, and of the rolling curve, respectively. Then, first, if the curves were both plane, we see that one rolling on the other about an axis always perpendicular to their common tangent could never change the inclination of their planes. Hence, secondly, if they are both tortuous, such rolling will alter the inclination of their osculating planes by an indefinitely small amount $(\tau-\tau')\,ds$ during rolling which shifts the point of contact over an arc ds. Now α is a known function of s if the trace is given, and therefore so also is α'. But $\alpha-\alpha'$ is the inclination of the osculating planes, hence

Angular velocity round tangent.

$$V\left\{\frac{d(\alpha-\alpha')}{ds} - (\tau-\tau')\right\} = \varpi.$$

Surface on surface.

117. Next, for one surface rolling and spinning on another. First, if the trace on each is given, we have the case of § 113 or § 115, one curve rolling on another, with this farther condition, that the former must *revolve* round the tangent to the two curves so as to keep the tangent planes of the two surfaces coincident.

Both traces prescribed: one degree of freedom.

It is well to observe that when the points in contact, and the two traces, are given, the position of the moveable surface is quite determinate, being found thus:—Place it in contact with the fixed surface, the given points together, and *spin* it about the common normal till the tangent lines to the traces coincide.

Hence when both the traces are given the condition of no spinning cannot be imposed. During the rolling there must in general be spinning, such as to keep the tangents to the two traces coincident. The rolling along the trace is due to rotation round the line perpendicular to it in the tangent plane. The whole rolling is the resultant of this rotation and a rotation about the tangent line required to keep the two tangent planes coincident.

Surface on surface, both traces prescribed; one degree of freedom. In this case, then, there is but one independent variable—the space passed over by the point of contact: and when the velocity of the point of contact is given, the resultant angular velocity, and the direction of the instantaneous axis of the rolling body are determinate. We have thus a sufficiently clear view of the general character of the motion in question, but it is right that we consider it more closely, as it introduces us very naturally to an important question, the measurement of the *twist* of a rod, wire, or narrow plate, a quantity wholly distinct from the *tortuosity* of its axis (§ 7).

118. Suppose all of each surface cut away except an infinitely narrow strip, including the trace of the rolling. Then we have the rolling of one of these strips upon the other, each having at every point a definite curvature, tortuosity, and twist.

Twist. **119.** Suppose a flat bar of small section to have been bent (the requisite amount of stretching and contraction of its edges being admissible) so that its axis assumes the form of any plane or tortuous curve. If it be unbent without twisting, *i.e.*, if the curvature of each element of the bar be removed by bending it through the requisite angle in the osculating plane, and it be found untwisted when thus rendered straight, it had no *twist* in its original form. This case is, of course, included in the general theory of *twist,* which is the subject of the following sections.

Axis and transverse. **120.** A bent or straight rod of circular or any other form of section being given, a line through the centres, or any other chosen points of its sections, may be called its *axis*. Mark a line on its side all along its length, such that it shall be a straight line parallel to the axis when the rod is unbent and untwisted. A line drawn from any point of the axis perpendicular to this side line of reference, is called the *transverse* of the rod at this point.

The whole twist of any length of a straight rod is the angle between the transverses of its ends. The average twist is the integral twist divided by the length. The twist at any point is the average twist in an infinitely short length through this point; in other words, it is the rate of rotation of its transverse per unit of length along it.

The twist of a curved, plane or tortuous, rod at any point is the rate of component rotation of its transverse round its tangent line, per unit of length along it. **Twist.**

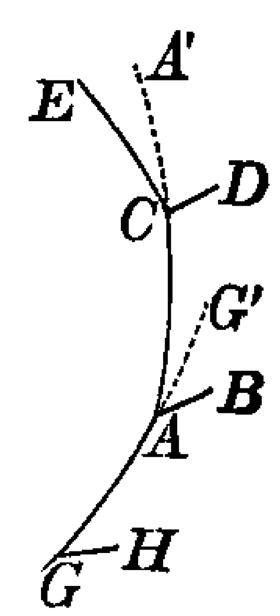
If t be the twist at any point, $\int t ds$ over any length is the integral twist in this length.

121. Integral twist in a curved rod, although readily defined, as above, in the language of the integral calculus, cannot be exhibited as the angle between any two lines readily constructible. The following considerations show how it is to be reckoned, and lead to a geometrical construction exhibiting it in a spherical diagram, for a rod bent and twisted in any manner:—

122. If the axis of the rod forms a plane curve lying in one plane, the integral twist is clearly the difference between the inclinations of the transverse at its ends to its plane. For if it be simply unbent, without altering the twist in any part, the inclination of each transverse to the plane in which its curvature lay will remain unchanged; and as the axis of the rod now has become a straight line in this plane, the mutual inclination of the transverses at any two points of it has become equal to the difference of their inclinations to the plane. **Estimation of integral twist: In a plane curve;**

123. No simple application of this rule can be made to a tortuous curve, in consequence of the change of the plane of curvature from point to point along it; but, instead, we may proceed thus:—

First, Let us suppose the plane of curvature of the axis of the wire to remain constant through finite portions of the curve, and to change abruptly by finite angles from one such portion to the next (a supposition which involves no angular points, that is to say, no infinite curvature, in the curve). Let planes parallel to the planes of curvature of three successive portions, PQ, QR, RS (not shown in the diagram), cut a spherical surface in the great circles GAG', ACA', CE. The radii of the sphere parallel to the tangents at the points Q and R of the curve where its curvature changes will cut its surface in A and C, the intersections of these circles. **In a curve consisting of plane portions in different planes.**



Let G be the point in which the radius of the sphere parallel to the tangent at P cuts the surface; and let GH, AB, CD (lines necessarily in tangent planes to the spherical surface), be parallels to the transverses of the bar drawn from the points P, Q, R of its axis. Then (§ 122) the twist from P to Q is equal to the difference of the angles HGA and BAG'; and the twist from Q to R is equal to the difference between BAC and DCA'. Hence the whole twist from P to R is equal to

Estimation of integral twist: in a curve consisting of plane portions in different planes.

$$HGA - BAG' + BAC - DCA',$$

or, which is the same thing,

$$A'CE + G'AC - (DCE - HGA).$$

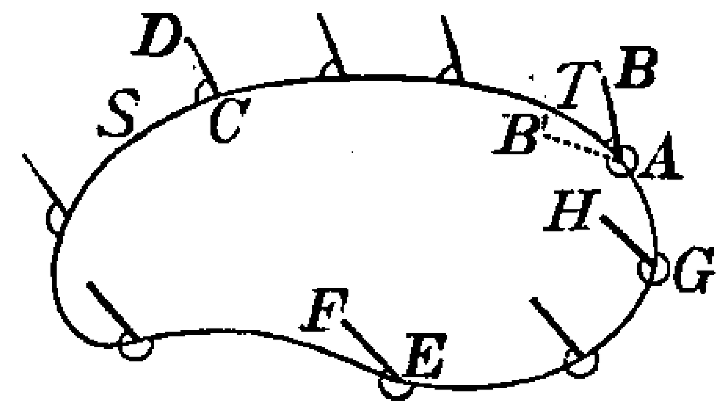
Continuing thus through any length of rod, made up of portions curved in different planes, we infer that the integral twist between any two points of it is equal to the sum of the exterior angles in the spherical diagram, wanting the excess of the inclination of the transverse at the second point to the plane of curvature at the second point above the inclination at the first point to the plane of curvature at the first point. The sum of those exterior angles is what is defined below as the "change of direction in the spherical surface" from the first to the last side of the polygon of great circles. When the polygon is closed, and the sum includes all its exterior angles, it is (§ 134) equal to 2π wanting the area enclosed if the radius of the spherical surface be unity. The construction we have made obviously holds in the limiting case, when the lengths of the plane portions are infinitely small, and is therefore applicable to a wire forming a tortuous curve with continuously varying plane of curvature, for which it gives the following conclusion:—

In a continuously tortuous curve.

Let a point move uniformly along the axis of the bar: and, parallel to the tangent at every instant, draw a radius of a sphere cutting the spherical surface in a curve, the hodograph of the moving point. From points of this hodograph draw parallels to the transverses of the corresponding points of the bar. The excess of the change of direction (§ 135) from any point to another of the hodograph, above the increase of its inclination to the transverse, is equal to the twist in the corresponding part of the bar.

Estimation of integral twist: in a continuously tortuous curve.

The annexed diagram, showing the hodograph and the parallels to the transverses, illustrates this rule. Thus, for instance, the excess of the change of direction in the spherical surface along the hodograph from A to C, above $DCS—BAT$, is equal to the twist in the bar between the points of it to which A and C correspond. Or, again, if we consider a portion of the bar from any point of it, to another point at which the tangent to its axis is parallel to the tangent at its first point, we shall have a closed curve as the spherical hodograph; and if A be the point of the hodograph corresponding to them, and AB and AB' the parallels to the transverses, the whole twist in the included part of the bar will be equal to the change of direction all round the hodograph, wanting the excess of the exterior angle $B'AT$ above the angle BAT; that is to say, the whole twist will be equal to the excess of the angle BAB' above the area enclosed by the hodograph.



The principles of twist thus developed are of vital importance in the theory of rope-making, especially the construction and the dynamics of wire ropes and submarine cables, elastic bars, and spiral springs.

Dynamics of twist in kinks.

For example: take a piece of steel pianoforte-wire carefully straightened, so that when free from stress it is straight: bend it into a circle and join the ends securely so that there can be no turning of one relatively to the other. Do this first without torsion: then twist the ring into a figure of 8, and tie the two parts together at the crossing. The area of the spherical hodograph is zero, and therefore there is one full turn (2π) of twist; which (§ 600 below) is uniformly distributed throughout the length of the wire. The form of the wire, (which is not in a plane,) will be investigated in § 610. Meantime we can see that the "torsional couples" in the normal sections farthest from the crossing give rise to forces by which the tie at the crossing is pulled in opposite directions perpendicular to the plane of the crossing. Thus if the tie is cut the wire springs back into the circular form. Now do the same thing again, Dynamics of twist in kinks.

beginning with a straight wire, but giving it one full turn (2π) of twist before bending it into the circle. The wire will stay in the 8 form without any pull on the tie. Whether the circular or the 8 form is stable or unstable depends on the relations between torsional and flexural rigidity. If the torsional rigidity is small in comparison with the flexural rigidity [as (§§ 703, 704, 705, 709) would be the case if, instead of round wire, a rod of + shaped section were used], the circular form would be stable, the 8 unstable.

Lastly, suppose any degree of twist, either more or less than 2π, to be given before bending into the circle. The circular form, which is always a figure of free equilibrium, may be stable or unstable, according as the ratio of torsional to flexural rigidity is more or less than a certain value depending on the actual degree of twist. The tortuous 8 form is not (except in the case of whole twist $= 2\pi$, when it becomes the plane elastic lemniscate of Fig. 4, § 610,) a continuous figure of free equilibrium, but involves a positive pressure of the two crossing parts on one another when the twist $> 2\pi$, and a negative pressure (or a pull on the tie) between them when twist $< 2\pi$: and with this force it is a figure of stable equilibrium.

Surface rolling on surface; both traces given.

124. Returning to the motion of one surface rolling and spinning on another, the trace on each being given, we may consider that, of each, the curvature (§ 6), the tortuosity (§ 7), and the twist reckoned according to transverses in the tangent plane of the surface, are known; and the subject is fully specified in § 117 above.

Let $\frac{1}{\rho'}$ and $\frac{1}{\rho}$ be the curvatures of the traces on the rolling and fixed surfaces respectively; α' and α the inclinations of their planes of curvature to the normal to the tangent plane, reckoned as in § 116; τ' and τ their tortuosities; t' and t their twists; and q the velocity of the point of contact. All these being known, it is required to find:—

ω the angular velocity of rotation about the transverse of the traces; that is to say, the line in the tangent plane perpendicular to their tangent line,

ϖ the angular velocity of rotation about the tangent line, and

σ „ „ of spinning.

Surface rolling on surface; both traces given.

We have

$$\omega = q\left(\frac{1}{\rho'}\cos\alpha' - \frac{1}{\rho}\cos\alpha\right) \text{ ....(1)},$$

$$\varpi = q\,(t - t') = q\left\{\frac{d\,(\alpha - \alpha')}{ds} - (\tau - \tau')\right\} \text{ ....(2)},$$

and
$$\sigma = q\left(\frac{1}{\rho'}\sin\alpha' - \frac{1}{\rho}\sin\alpha\right) \text{ ....(3)}.$$

These three formulas are respectively equivalent to (9), (8), and (10) of § 111.

Surface rolling on surface without spinning.

125. In the same case, suppose the trace on *one* only of the surfaces to be given. We may evidently impose the condition of no spinning, and then the trace on the other is determinate. This case of motion is thoroughly examined in § 137, below.

The condition is that the projections of the curvatures of the two traces on the common tangent plane must coincide.

If $\frac{1}{r'}$ and $\frac{1}{r}$ be the curvatures of the rolling and stationary surfaces in a normal section of each through the tangent line to the trace, and if α, α', ρ, ρ' have their meanings of § 124,

$\rho' = r'\cos\alpha'$, $\rho = r\cos\alpha$ (Meunier's Theorem, § 129, below).

But $\frac{1}{\rho'}\sin\alpha' = \frac{1}{\rho}\sin\alpha$, hence $\tan\alpha' = \frac{r'}{r}\tan\alpha$, the condition required.

Examples of tortuosity and twist.

126. If a straight rod with a straight line marked on one side of it be bent along any curve on a spherical surface, so that the marked line is laid in contact with the spherical surface, it acquires no twist in the operation. For if it is laid so along any finite arc of a small circle there will clearly be no twist. And no twist is produced in continuing from any point along another small circle having a common tangent with the first at this point.

If a rod be bent round a cylinder so that a line marked along one side of it may lie in contact with the cylinder, or if, what presents somewhat more readily the view now de-

Examples of tortuosity and twist.

sired, we wind a straight ribbon spirally on a cylinder, the axis of bending is parallel to that of the cylinder, and therefore oblique to the axis of the rod or ribbon. We may therefore resolve the instantaneous rotation which constitutes the bending at any instant into two components, one round a line perpendicular to the axis of the rod, which is pure bending, and the other round the axis of the rod, which is pure twist.

The twist at any point in a rod or ribbon, so wound on a circular cylinder, and constituting a uniform helix, is

$$\frac{\cos\alpha\sin\alpha}{r},$$

if r be the radius of the cylinder and α the inclination of the spiral. For if V be the velocity at which the bend proceeds along the previously straight wire or ribbon, $\frac{V\cos\alpha}{r}$ will be the angular velocity of the instantaneous rotation round the line of bending (parallel to the axis), and therefore

$$\frac{V\cos\alpha}{r}\sin\alpha \text{ and } \frac{V\cos\alpha}{r}\cos\alpha$$

are the angular velocities of twisting and of pure bending respectively.

From the latter component we may infer that the curvature of the helix is

$$\frac{\cos^2\alpha}{r},$$

a known result, which agrees with the expression used above (§ 13).

127. The hodograph in this case is a small circle of the sphere. If the specified condition as to the mode of laying on of the rod on the cylinder is fulfilled, the transverses of the spiral rod will be parallel at points along it separated by one or more whole turns. Hence the integral twist in a single turn is equal to the excess of four right angles above the spherical area enclosed by the hodograph. If α be the inclination of the spiral, $\frac{1}{2}\pi - \alpha$ will be the arc-radius of the hodograph, and therefore its area is $2\pi(1-\sin\alpha)$. Hence the integral twist in a turn of the spiral is $2\pi\sin\alpha$, which agrees with the result previously obtained (§ 126).

Curvature of surface.

128. As a preliminary to the further consideration of the rolling of one surface on another, and as useful in various parts of our subject, we may now take up a few points connected with the curvature of surfaces.

Synclastic and anticlastic surfaces.

The tangent plane at any point of a surface may or may not cut it at that point. In the former case, the surface bends away from the tangent plane partly towards one side of it, and partly towards the other, and has thus, in some of its normal sections, curvatures oppositely directed to those in others. In the latter case, the surface on every side of the point bends away from the same side of its tangent plane, and the curvatures of all normal sections are similarly directed. Thus we may divide curved surfaces into *Anticlastic* and *Synclastic*. A saddle gives a good example of the former class; a ball of the latter. Curvatures in opposite directions, with reference to the tangent plane, have of course different signs. The outer portion of an anchor-ring is synclastic, the inner anticlastic.

Curvature of oblique sections.

129. *Meunier's Theorem.*—The curvature of an oblique section of a surface is equal to that of the normal section through the same tangent line multiplied by the secant of the inclination of the planes of the sections. This is evident from the most elementary considerations regarding projections.

Principal curvatures.

130. *Euler's Theorem.*—There are at every point of a synclastic surface two normal sections, in one of which the curvature is a maximum, in the other a minimum; and these are at right angles to each other.

In an anticlastic surface there is maximum curvature (but in opposite directions) in the two normal sections whose planes bisect the angles between the lines in which the surface cuts its tangent plane. On account of the difference of sign, these may be considered as a maximum and a minimum.

Sum of curvatures in normal sections at right angles to each other.

Generally the sum of the curvatures at a point, in any two normal planes at right angles to each other, is independent of the position of these planes.

Taking the tangent plane as that of x, y, and the origin at the point of contact, and putting

$$\left(\frac{d^2z}{dx^2}\right)_0 = A,\ \left(\frac{d^2z}{dxdy}\right)_0 = B,\ \left(\frac{d^2z}{dy^2}\right)_0 = C;$$

we have $$z = \frac{1}{2}(Ax^2 + 2Bxy + Cy^2) + \text{etc.} \qquad (1)$$

The curvature of the normal section which passes through the point x, y, z is (in the limit)

$$\frac{1}{r} = \frac{2z}{x^2+y^2} = \frac{Ax^2+2Bxy+Cy^2}{x^2+y^2}.$$

If the section be inclined at an angle θ to the plane of XZ, this becomes

$$\frac{1}{r} = A\cos^2\theta + 2B\sin\theta\cos\theta + C\sin^2\theta. \qquad (2)$$

Hence, if $\frac{1}{r}$ and $\frac{1}{s}$ be curvatures in normal sections at right angles to each other,

$$\frac{1}{r} + \frac{1}{s} = A + C = \text{constant}.$$

(2) may be written

$$\frac{1}{r} = \frac{1}{2}\{A(1+\cos 2\theta) + 2B\sin 2\theta + C(1-\cos 2\theta)\}$$

$$= \frac{1}{2}\{\overline{A+C} + \overline{A-C}\cos 2\theta + 2B\sin 2\theta\},$$

or if $$\frac{1}{2}(A-C) = R\cos 2\alpha,\ B = R\sin 2\alpha,$$

that is $$R = \sqrt{\left\{\frac{1}{4}(A-C)^2 + B^2\right\}},\ \text{and}\ \tan 2\alpha = \frac{2B}{A-C},$$

we have $$\frac{1}{r} = \frac{1}{2}(A+C) + \sqrt{\left\{\frac{1}{4}(A-C)^2 + B^2\right\}}\cos 2(\theta-\alpha).$$

Principal normal sections. The maximum and minimum curvatures are therefore those in normal places at right angles to each other for which $\theta = \alpha$ and $\theta = \alpha + \frac{\pi}{2}$, and are respectively

$$\frac{1}{2}(A+C) \pm \sqrt{\left\{\frac{1}{4}(A-C)^2 + B^2\right\}}.$$

Hence their product is $AC - B^2$.

If this be positive we have a synclastic, if negative an anticlastic, surface. If it be zero we have one curvature only, and the surface is *cylindrical* at the point considered. It is demonstrated

Principal normal sections. (§ 152, below) that if this condition is fulfilled at every point, the surface is "developable" (§ 139, below).

By (1) a plane parallel to the tangent plane and very near it cuts the surface in an ellipse, hyperbola, or two parallel straight lines, in the three cases respectively. This section, whose nature informs us as to whether the curvature be synclastic, anticlastic, or cylindrical, at any point, was called by Dupin the *Indicatrix*.

Definition of Line of Curvature. A line of curvature of a surface is a line which at every point is cotangential with normal section of maximum or minimum curvature.

Shortest line between two points on a surface. **131.** Let P, p be two points of a surface infinitely near to each other, and let r be the radius of curvature of a normal section passing through them. Then the radius of curvature of an oblique section through the same points, inclined to the former at an angle α, is (§ 129) $r\cos\alpha$. Also the length along the normal section, from P to p, is less than that along the oblique section—since a given chord cuts off an arc from a circle, longer the less the radius of that circle.

If a be the length of the chord Pp, we have

$$\text{Distance } Pp \text{ along normal section} = 2r\sin^{-1}\frac{a}{2r} = a\left(1 + \frac{a^2}{24r^2}\right),$$

$$\text{,, \quad ,, \quad oblique section} = a\left(1 + \frac{a^2}{24r^2\cos^2\alpha}\right).$$

132. Hence, if the shortest possible line be drawn from one point of a surface to another, its plane of curvature is everywhere perpendicular to the surface.

Geodetic Lines. Such a curve is called a *Geodetic* line. And it is easy to see that it is the line in which a flexible and inextensible string would touch the surface if stretched between those points, the surface being supposed smooth.

133. If an infinitely narrow ribbon be laid on a surface along a geodetic line, its twist is equal to the tortuosity of its axis at each point. We have seen (§ 125) that when one body rolls on another without spinning, the projections of the traces on the common tangent plane agree in curvature at the point

Shortest line between two points on a surface.

of contact. Hence, if one of the surfaces be a plane, and the trace on the other be a geodetic line, the trace on the plane is a straight line. Conversely, if the trace on the plane be a straight line, that on the surface is a geodetic line.

And, quite generally, if the given trace be a geodetic line, the other trace is also a geodetic line.

Spherical excess.

134. The area of a spherical triangle (on a sphere of unit radius) is known to be equal to the "spherical excess," *i.e.*, the excess of the sum of its angles over two right angles, or the excess of four right angles over the sum of its exterior angles.

Area of spherical polygon.

The area of a spherical polygon whose n sides are portions of great circles—*i.e.*, geodetic lines—is to that of the hemisphere as the excess of four right angles over the sum of its exterior angles is to four right angles. (We may call this the "spherical excess" of the polygon.)

For the area of a spherical triangle is known to be equal to

$$A+B+C-\pi.$$

Divide the polygon into n such triangles, with a common vertex, the angles about which, of course, amount to 2π.

$$\begin{aligned}\text{Area} &= \text{sum of interior angles of triangles} - n\pi \\ &= 2\pi + \text{sum of interior angles of polygon} - n\pi \\ &= 2\pi - \text{sum of exterior angle of polygon.}\end{aligned}$$

Reciprocal polars on a sphere.

Given an open or closed spherical polygon, or line on the surface of a sphere composed of consecutive arcs of great circles. Take either pole of the first of these arcs, and the corresponding poles of all the others (all the poles to be on the right hand, or all on the left, of a traveller advancing along the given great circle arcs in order). Draw great circle arcs from the first of these poles to the second, the second to the third, and so on in order. Another closed or open polygon, constituting what is called the polar diagram to the given polygon, is thus obtained. The sides of the second polygon are evidently equal to the exterior angles in the first; and the exterior angles of the second are equal to the sides of the first. Hence the relation between the two diagrams is reciprocal, or each is polar to the other. The polar figure to any continuous curve on a spherical

Reciprocal polars on a sphere.

surface is the locus of the ultimate intersections of great circles equatorial to points taken infinitely near each other along it.

The area of a closed spherical figure is, consequently, according to what we have just seen, equal to the excess of 2π above the periphery of its polar, if the radius of the sphere be unity.

Integral change of direction in a surface.

135. If a point move on a surface along a figure whose sides are geodetic lines, the sum of the exterior angles of this polygon is defined to be the *integral change of the direction in the surface*.

In great circle sailing, unless a vessel sail on the equator, or on a meridian, her course, as indicated by points of the compass (true, not magnetic, for the latter change even on a meridian), perpetually changes. Yet just as we say her direction does not change if she sail in a meridian, or in the equator, so we ought to say her direction does not change if she moves in *any* great circle. Now, the great circle is the geodetic line on the sphere, and by extending these remarks to other curved surfaces, we see the connexion of the above definition with that in the case of a plane polygon (§ 10).

Change of direction in a surface, of any arc traced on it.

Note.—We cannot define integral change of direction here by any angle directly constructible from the first and last tangents to the path, as was done (§ 10) in the case of a plane curve or polygon; but from §§ 125 and 133 we have the following statement:—The whole change of direction in a curved surface, from one end to another of any arc of a curve traced on it, is equal to the change of direction from end to end of the trace of this arc on a plane by pure rolling.

Integral curvature.

136. *Def.* The excess of four right angles above the integral change of direction from one side to the same side next time in going round a closed polygon of geodetic lines on a curved surface, is the *integral curvature* of the enclosed portion of surface. This excess is zero in the case of a polygon traced on a plane. We shall presently see that this corresponds exactly to what Gauss has called the *curvatura integra*.

Curvatura integra.

Def. (Gauss.) The *curvatura integra* of any given portion of a curved surface, is the area enclosed on a spherical surface of unit radius by a straight line drawn from its centre, parallel to a normal to the surface, the normal being carried round the boundary of the given portion.

Horograph. The curve thus traced on the sphere is called the *Horograph* of the given portion of curved surface.

The *average curvature* of any portion of a curved surface is the integral curvature divided by the area. The *specific curvature* of a curved surface at any point is the average curvature of an infinitely small area of it round that point.

Change of direction round the boundary in the surface, together with area of the horograph, equals four right angles: or "Integral Curvature" equals "Curvatura Integra."

137. The excess of 2π above the change of direction, in a surface, of a point moving round any closed curve on it, is equal to the area of the horograph of the enclosed portion of surface.

Let a tangent plane roll without spinning on the surface over every point of the bounding line. (Its instantaneous axis will always lie in it, and pass through the point of contact, but will not, as we have seen, be at right angles to the given bounding curve, except when the twist of a narrow ribbon of the surface along this curve is nothing.) Considering the auxiliary sphere of unit radius, used in Gauss's definition, and the moving line through its centre, we perceive that the motion of this line is, at each instant, in a plane perpendicular to the instantaneous axis of the tangent plane to the given surface. The direction of motion of the point which cuts out the area on the spherical surface is therefore perpendicular to this instantaneous axis. Hence, if we roll a tangent plane on the spherical surface also, making it keep time with the other, the trace on this tangent plane will be a curve always perpendicular to the instantaneous axis of each tangent plane. The change of direction, in the spherical surface, of the point moving round and cutting out the area, being equal to the change of direction in its own trace on its own tangent plane (§ 135), is therefore equal to the change of direction of the instantaneous axis in the tangent plane to the given surface reckoned from a line fixed relatively to this plane. But having rolled all round, and being in position to roll round again, the instantaneous axis of the fresh start must be inclined to the trace at the same angle as in the beginning. Hence the change of direction of the instantaneous axis in either tangent plane is equal to the change of direction, in the given surface, of

Curvatura integra, and horograph.

a point going all round the boundary of the given portion of it (§ 135); to which, therefore, the change of direction, in the spherical surface, of the point going all round the spherical area is equal. But, by the well-known theorem (§ 134) of the "spherical excess," this change of direction subtracted from 2π leaves the spherical area. Hence the spherical area, called by Gauss the *curvatura integra*, is equal to 2π wanting the change of direction in going round the boundary.

Curvatura integra, and horograph

It will be perceived that when the two rollings we have considered are each complete, each tangent plane will have come back to be parallel to its original position, but any fixed line in it will have changed direction through an angle equal to the equal changes of direction just considered.

Note.—The two rolling tangent planes are at each instant parallel to one another, and a fixed line relatively to one drawn at any time parallel to a fixed line relatively to the other, remains parallel to the last-mentioned line.

If, instead of the closed curve, we have a closed polygon of geodetic lines on the given surface, the trace of the rolling of its tangent plane will be an unclosed rectilineal polygon. If each geodetic were a plane curve (which could only be if the given surface were spherical), the instantaneous axis would be always perpendicular to the particular side of this polygon which is rolled on at the instant; and, of course, the spherical area on the auxiliary sphere would be a similar polygon to the given one. But the given surface being other than spherical, there must (except in the particular case of some of the geodetics being lines of curvature) be tortuosity in every geodetic of the closed polygon; or, which is the same thing, twist in the corresponding ribbons of the surface. Hence the portion of the whole trace on the second rolling tangent plane which corresponds to any one side of the given geodetic polygon, must in general be a curve; and as there will generally be finite angles in the second rolling corresponding to (but not equal to) those in the first, the trace of the second on its tangent plane will be an unclosed polygon of curves. The trace of the same rolling on the spherical surface in which it takes place will generally be a spherical polygon, not of great circle arcs, but of other curves. The sum of the exterior angles of this polygon, and of the changes of direction from one end to the other of each of its sides, is the whole change of direction considered, and is, by the proper

Curvatura integra, and horograph.

application of the theorem of § 134, equal to 2π wanting the spherical area enclosed.

Or again, if, instead of a geodetic polygon as the given curve, we have a polygon of curves, each fulfilling the condition that the normal to the surface through any point of it is parallel to a fixed plane; one plane for the first curve, another for the second, and so on; then the figure on the auxiliary spherical surface will be a polygon of arcs of great circles; its trace on its tangent plane will be an unclosed rectilineal polygon; and the trace of the given curve on the tangent plane of the first rolling will be an unclosed polygon of curves. The sum of changes of direction in these curves, and of exterior angles in passing from one to another of them, is of course equal to the change of direction in the given surface, in going round the given polygon of curves on it. The change of direction in the other will be simply the sum of the exterior angles of the spherical polygon, or of its rectilineal trace. Remark that in this case the instantaneous axis of the first rolling, being always perpendicular to that plane to which the normals are all parallel, remains parallel to one line, fixed with reference to the tangent plane, during rolling along each curved side, and also remains parallel to a fixed line in space.

Lastly, remark that although the whole change of direction of the trace in one tangent plane is equal to that in the trace on the other, when the rolling is completed round the given circuit; the changes of direction in the two are generally unequal in any part of the circuit. They may be equal for particular parts of the circuit, viz., between those points, if any, at which the instantaneous axis is equally inclined to the direction of the trace on the first tangent plane.

Any difficulty which may have been felt in reading this Section will be removed if the following exercises on the subject be performed.

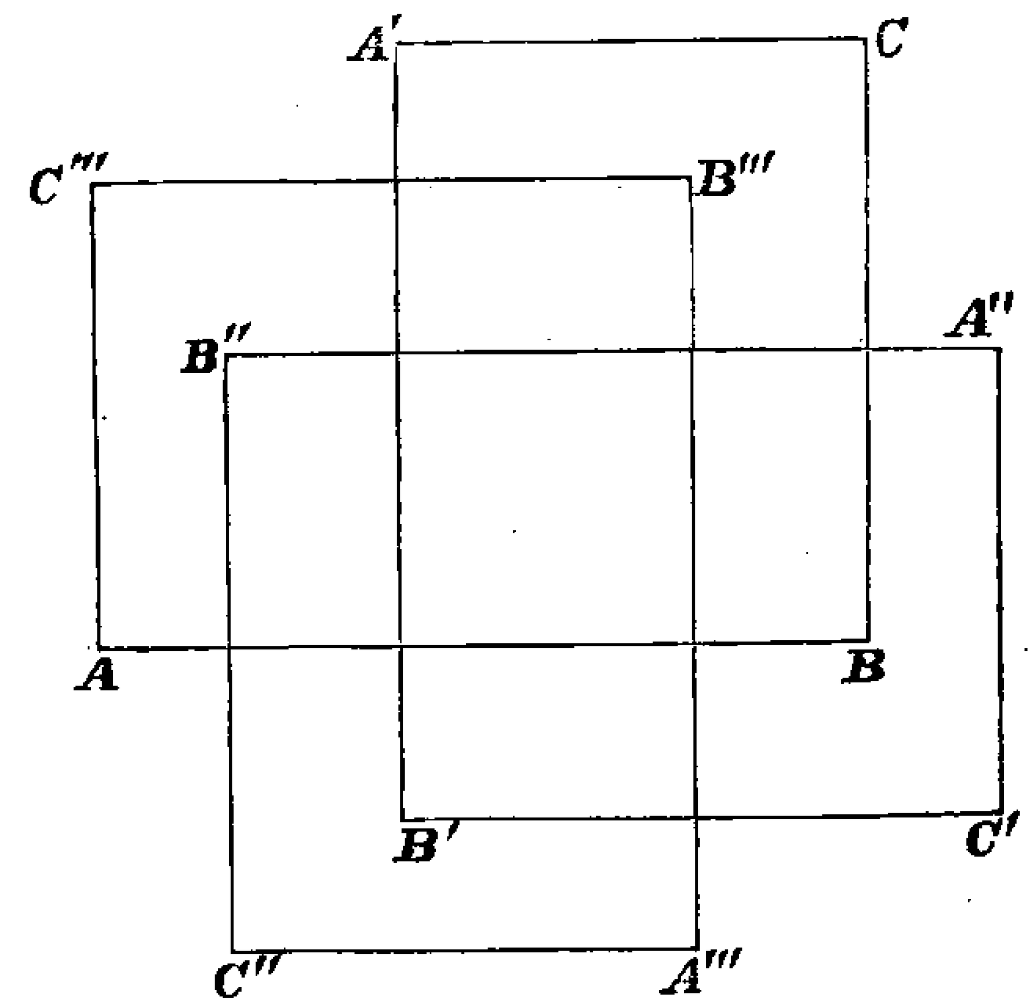
(1) Find the horograph of an infinitely small circular area of any continuous curved surface. It is an ellipse or a hyperbola according as the surface is synclastic or anticlastic (§ 128). Find the axes of the ellipse or hyperbola in either case.

(2) Find the horograph of the area cut off a synclastic surface by a plane parallel to the tangent plane at any given point of it, and infinitely near this point. Find and interpret the corresponding result for the case in which the surface is anticlastic in the neighbourhood of the given point.

Curvatura integra, and horograph.

(3) Let a tangent plane roll without spinning over the boundary of a given closed curve or geodetic polygon on any curved surface. Show that the points of the trace in the tangent plane which successively touch the same point of the given surface are at equal distances successively on the circumference of a circle, the angular values of the intermediate arcs being each $2\pi - K$ if taken in the direction in which the trace is actually described, and K if taken in the contrary direction, K being the "integral curvature" of the portion of the curved surface enclosed by the given curve or geodetic polygon. Hence if K be commensurable with π the trace on the tangent plane, however complicatedly autotomic it may be, is a finite closed curve or polygon.

(4) The trace by a tangent plane rolling successively over three principal quadrants bounding an eighth part of the circumference of an ellipsoid is represented in the accompanying diagram, the whole of which is traced when the tangent plane is



rolled four times over the stated boundary. A, B, C; A', B', C', &c. represent the points of the tangent plane touched in order by ends of the mean principal axis (A), the greatest principal axis (B), and least principal axis (C), and AB, BC, CA' are the lengths of the three principal quadrants.

Analogy between lines and surfaces as regards curvature.

138. It appears from what precedes, that the same equality or identity subsists between "whole curvature" in a plane arc and the excess of π above the angle between the terminal

Analogy between lines and surfaces as regards curvature.

tangents, as between "whole curvature" and excess of 2π above change of direction along the bounding line in the surface for any portion of a curved surface.

Or, according to Gauss, whereas the whole curvature in a plane arc is the angle between two lines parallel to the terminal normals, the whole curvature of a portion of curve surface is the solid angle of a cone formed by drawing lines from a point parallel to all normals through its boundary.

Again, average curvature in a plane curve is $\dfrac{\text{change of direction}}{\text{length}}$; and specific curvature, or, as it is commonly called, curvature, at any point of it $= \dfrac{\text{change of direction in infinitely small length}}{\text{length}}$.

Thus average curvature and specific curvature are for surfaces analogous to the corresponding terms for a plane curve.

Lastly, in a plane arc of uniform curvature, *i.e.*, in a circular arc, $\dfrac{\text{change of direction}}{\text{length}} = \dfrac{1}{\rho}$. And it is easily proved (as below) that, in a surface throughout which the specific curvature is uniform, $\dfrac{2\pi - \text{change of direction}}{\text{area}}$, or $\dfrac{\text{integral curvature}}{\text{area}}$, $= \dfrac{1}{\rho\rho'}$, where ρ and ρ' are the principal radii of curvature. Hence in a surface, whether of uniform or non-uniform specific curvature, the specific curvature at any point is equal to $\dfrac{1}{\rho\rho'}$. In geometry of three dimensions, $\rho\rho'$ (an area) is clearly analogous to ρ in a curve and plane.

Horograph.

Consider a portion S, of a surface of any curvature, bounded by a given closed curve. Let there be a spherical surface, radius r, and centre C. Draw a radius CQ, parallel to the normal at any point P of S. If this be done for every point of the boundary, the line so obtained encloses the spherical area used in Gauss's definition. Now let there be an infinitely small rectangle on S, at P, having for its sides arcs of angles ζ and ζ', on the normal sections of greatest and least curvature, and let their radii of curvature be denoted by ρ and ρ'. The lengths of these sides will be $\rho\zeta$ and $\rho'\zeta'$ respectively. Its area will therefore be $\rho\rho'\zeta\zeta'$. The corresponding figure at Q on the spherical surface will be bounded by arcs of angles equal to those, and therefore of

Area of the horograph.

lengths $r\zeta$ and $r\zeta'$ respectively, and its area will be $r^2\zeta\zeta'$. Hence if $d\sigma$ denote this area, the area of the infinitely small portion of the given surface will be $\dfrac{\rho\rho' d\sigma}{r^2}$. In a surface for which $\rho\rho'$ is constant, the area is therefore $= \dfrac{\rho\rho'}{r^2}\iint d\sigma = \rho\rho' \times$ integral curvature.

Flexible and inextensible surface.

139. A perfectly flexible but inextensible surface is suggested, although not realized, by paper, thin sheet metal, or cloth, when the surface is plane; and by sheaths of pods, seed vessels, or the like, when it is not capable of being stretched flat without tearing. The process of changing the form of a surface by bending is called "*developing*." But the term "*Developable Surface*" is commonly restricted to such inextensible surfaces as can be developed into a plane, or, in common language, "smoothed flat."

140. The geometry or kinematics of this subject is a great contrast to that of the flexible line (§ 14), and, in its merest elements, presents ideas not very easily apprehended, and subjects of investigation that have exercised, and perhaps even overtasked, the powers of some of the greatest mathematicians.

141. Some care is required to form a correct conception of what is a perfectly flexible inextensible surface. First let us consider a plane sheet of paper. It is very flexible, and we can easily form the conception from it of a sheet of ideal matter perfectly flexible. It is very inextensible; that is to say, it yields very little to any application of force tending to pull or stretch it in any direction, up to the strongest it can bear without tearing. It does, of course, stretch a little. It is easy to test that it stretches when under the influence of force, and that it contracts again when the force is removed, although not always to its original dimensions, as it may and generally does remain to some sensible extent permanently stretched. Also, flexure stretches one side and condenses the other temporarily; and, to a less extent, permanently. Under elasticity (§§ 717, 718, 719) we shall return to this. In the meantime, in considering illustrations of our kinematical propositions, it is necessary to anticipate such physical circumstances.