CHAPTER III.

EXPERIENCE.

Observation and experiment.

369. By the term Experience, in physical science, we designate, according to a suggestion of Herschel's, our means of becoming acquainted with the material universe and the laws which regulate it. In general the actions which we see ever taking place around us are *complex,* or due to the simultaneous action of many causes. When, as in astronomy, we endeavour to ascertain these causes by simply watching their effects, we *observe;* when, as in our laboratories, we interfere arbitrarily with the causes or circumstances of a phenomenon, we are said to *experiment.*

Observation.

370. For instance, supposing that we are possessed of instrumental means of measuring time and angles, we may trace out by successive observations the relative position of the sun and earth at different instants; and (the method is not susceptible of any accuracy, but is alluded to here only for the sake of illustration) from the variations in the apparent diameter of the former we may calculate the ratios of our distances from it at those instants. We have thus a set of observations involving time, angular position with reference to the sun, and ratios of distances from it; sufficient (if numerous enough) to enable us to discover the laws which connect the variations of these co-ordinates.

Similar methods may be imagined as applicable to the motion of any planet about the sun, of a satellite about its primary, or of one star about another in a binary group.

371. In general all the data of Astronomy are determined in this way, and the same may be said of such subjects as

Tides and Meteorology. Isothermal Lines, Lines of Equal Dip, Lines of Equal Intensity, Lines of Equal "Variation" (or "Declination" as it has still less happily been sometimes called), the Connexion of Solar Spots with Terrestrial Magnetism, and a host of other data and phenomena, to be explained under the proper heads in the course of the work, are thus deducible from *Observation* merely. In these cases the apparatus for the gigantic experiments is found ready arranged in Nature, and all that the philosopher has to do is to watch and measure their progress to its last details. Observation.

372. Even in the instance we have chosen above, that of the planetary motions, the observed effects are complex; because, unless possibly in the case of a double star, we have no instance of the *undisturbed* action of one heavenly body on another; but to a first approximation the motion of a planet about the sun is found to be the same as if no other bodies than these two existed; and the approximation is sufficient to indicate the probable law of mutual action, whose full confirmation is obtained when, *its* truth being assumed, the disturbing effects thus calculated are allowed for, and found to account completely for the observed deviations from the consequences of the first supposition. This may serve to give an idea of the mode of obtaining the laws of phenomena, which can only be observed in a complex form—and the method can always be directly applied when one cause is known to be pre-eminent.

373. Let us take cases of the other kind—in which the effects are so complex that we cannot deduce the causes from the observation of combinations arranged in Nature, but must endeavour to form for ourselves other combinations which may enable us to study the effects of each cause separately, or at least with only slight modification from the interference of other causes. Experiment.

374. A stone, when dropped, falls to the ground; a brick and a boulder, if dropped from the top of a cliff at the same moment, fall side by side, and reach the ground together. But a brick and a slate do not; and while the former falls in a nearly vertical direction, the latter describes a most complex

Experiment.

path. A sheet of paper or a fragment of gold leaf presents even greater irregularities than the slate. But by a slight modification of the circumstances, we gain a considerable insight into the nature of the question. The paper and gold leaf, if rolled into balls, fall nearly in a vertical line. Here, then, there are evidently at least two causes at work, one which tends to make all bodies fall, and fall vertically; and another which depends on the form and substance of the body, and tends to retard its fall and alter its course from the vertical direction. How can we study the effects of the former on all bodies without sensible complication from the latter? The effects of Wind, etc., at once point out *what* the latter cause is, the air (whose existence we may indeed suppose to have been discovered by such effects); and to study the nature of the action of the former it is necessary to get rid of the complications arising from the presence of air. Hence the necessity for *Experiment.* By means of an apparatus to be afterwards described, we remove the greater part of the air from the interior of a vessel, and in *that* we try again our experiments on the fall of bodies; and now a general law, simple in the extreme, though most important in its consequences, is at once apparent—viz., that *all* bodies, of whatever size, shape, or material, if dropped side by side at the same instant, fall side by side in a space void of air. Before experiment had thus separated the phenomena, hasty philosophers had rushed to the conclusion that some bodies possess the quality of *heaviness*, others that of *lightness*, etc. Had this state of confusion remained, the law of gravitation, vigorous though its action be throughout the universe, could never have been recognised as a general principle by the human mind.

Mere observation of lightning and its effects could never have led to the discovery of their relation to the phenomena presented by rubbed amber. A modification of the course of nature, such as the collecting of atmospheric electricity in our laboratories, was necessary. Without experiment we could never even have learned the existence of terrestrial magnetism.

Rules for the conduct of experiments.

375. When a particular agent or cause is to be studied, experiments should be arranged in such a way as to lead if possible to results depending on it alone; or, if this cannot be

done, they should be arranged so as to show differences produced by varying it.

Rules for the conduct of experiments.

376. Thus to determine the resistance of a wire against the conduction of electricity through it, we may measure the whole strength of current produced in it by electromotive force between its ends when the amount of this electromotive force *is given*, or can be ascertained. But when the wire is that of a submarine telegraph cable there is always an *unknown* and ever varying electromotive force between its ends, due to the earth (producing what is commonly called the "earth-current"), and to determine its resistance, the difference in the strength of the current produced by suddenly adding to or subtracting from the terrestrial electromotive force the electromotive force of a given voltaic battery, is to be *very quickly* measured; and this is to be done over and over again, to eliminate the effect of variation of the earth-current during the few seconds of time which must elapse before the electrostatic induction permits the current due to the battery to reach nearly enough its full strength to practically annul error on this score.

377. Endless patience and perseverance in designing and trying different methods for investigation are necessary for the advancement of science: and indeed, in discovery, he is the most likely to succeed who, not allowing himself to be disheartened by the non-success of one form of experiment, judiciously varies his methods, and thus interrogates in every conceivably useful manner the subject of his investigations.

Residual phenomena.

378. A most important remark, due to Herschel, regards what are called *residual* phenomena. When, in an experiment, all known causes being allowed for, there remain certain unexplained effects (excessively slight it may be), these must be carefully investigated, and every conceivable variation of arrangement of apparatus, etc., tried; until, if possible, we manage so to isolate the residual phenomenon as to be able to detect its cause. It is here, perhaps, that in the present state of science we may most reasonably look for extensions of our knowledge; at all events we are warranted by the recent history of Natural Philosophy in so doing. Thus, to take only

Residual phenomena. a very few instances, and to say nothing of the discovery of electricity and magnetism by the ancients, the peculiar smell observed in a room in which an electrical machine is kept in action, was long ago observed, but called the "smell of electricity," and thus left unexplained. The sagacity of Schönbein led to the discovery that this is due to the formation of Ozone, a most extraordinary body, of great chemical activity; whose nature is still uncertain, though the attention of chemists has for years been directed to it.

379. Slight anomalies in the motion of Uranus led Adams and Le Verrier to the discovery of a new planet; and the fact that the oscillations of a magnetized needle about its position of equilibrium are "damped" by placing a plate of copper below it, led Arago to his beautiful experiment showing a resistance to relative motion between a magnet and a piece of copper; which was first supposed to be due to magnetism in motion, but which soon received its correct explanation from Faraday, and has since been immensely extended, and applied to most important purposes. In fact, from this accidental remark about the oscillation of a needle was evolved the grand discovery of the Induction of Electrical Currents by magnets or by other currents.

We need not enlarge upon this point, as in the following pages the proofs of the truth and usefulness of the principle will continually recur. Our object has been not so much to give applications as principles, and to show how to attack a new combination, with the view of separating and studying in detail the various causes which generally conspire to produce observed phenomena, even those which are apparently the simplest.

Unexpected agreement or discordance of results of different trials.

380. If on repetition several times, an experiment continually gives different results, it must either have been very carelessly performed, or there must be some disturbing cause not taken account of. And, on the other hand, in cases where no very great coincidence is likely on repeated trials, an unexpected degree of agreement between the results of various trials should be regarded with the utmost suspicion, as probably due to some unnoticed peculiarity of the apparatus employed. In

either of these cases, however, careful observation cannot fail to detect the cause of the discrepancies or of the unexpected agreement, and may possibly lead to discoveries in a totally unthought-of quarter. Instances of this kind may be given without limit; one or two must suffice.

Unexpected agreement or discordance of results of different trials.

381. Thus, with a *very* good achromatic telescope a star appears to have a sensible disc. But, as it is observed that the discs of all stars appear to be of equal angular diameter, we of course suspect some common error. Limiting the aperture of the object-glass *increases* the appearance in question, which, on full investigation, is found to have nothing to do with discs at all. It is, in fact, a diffraction phenomenon, and will be explained in our chapters on Light.

382. Again, in measuring the velocity of Sound by experiments conducted at night with cannon, the results at one station were never found to agree exactly with those at the other; sometimes, indeed, the differences were very considerable. But a little consideration led to the remark, that on those nights in which the discordance was greatest a strong wind was blowing nearly from one station to the other. Allowing for the obvious effect of this, or rather eliminating it altogether, the mean velocities on different evenings were found to agree very closely.

Hypotheses.

383. It may perhaps be advisable to say a few words here about the use of hypotheses, and especially those of very different gradations of value which are promulgated in the form of Mathematical Theories of different branches of Natural Philosophy.

384. Where, as in the case of the planetary motions and disturbances, the forces concerned are thoroughly known, the mathematical theory is absolutely true, and requires only analysis to work out its remotest details. It is thus, in general, far ahead of observation, and is competent to predict effects not yet even observed—as, for instance, Lunar Inequalities due to the action of Venus upon the Earth, etc. etc., to which no amount of observation, unaided by theory, could ever have enabled us to assign the true cause. It may also, in such subjects as Geometrical Optics, be carried to developments far beyond the reach

Hypotheses. of experiment; but in this science the assumed bases of the theory are only approximate; and it fails to explain in all their peculiarities even such comparatively simple phenomena as Halos and Rainbows—though it is perfectly successful for the practical purposes of the maker of microscopes and telescopes, and has enabled really scientific instrument-makers to carry the construction of optical apparatus to a degree of perfection which merely tentative processes never could have reached.

385. Another class of mathematical theories, based to some extent on experiment, is at present useful, and has even in certain cases pointed to new and important results, which experiment has subsequently verified. Such are the Dynamical Theory of Heat, the Undulatory Theory of Light, etc. etc. In the former, which is based upon the conclusion from experiment that *heat is a form of energy*, many formulæ are at present obscure and uninterpretable, because we do not know the mechanism of the motions or distortions of the particles of bodies. Results of the theory in which these are not involved, are of course experimentally verified. The same difficulties exist in the Theory of Light. But before this obscurity can be perfectly cleared up, we must know something of the ultimate, or *molecular*, constitution of the bodies, or groups of molecules, at present known to us only in the aggregate.

Deduction of most probable result from a number of observations.

386. A third class is well represented by the Mathematical Theories of Heat (Conduction), Electricity (Statical), and Magnetism (Permanent). Although we do not know *how* Heat is propagated in bodies, nor *what* Statical Electricity or Permanent Magnetism are—the laws of their fluxes and forces are as certainly known as that of Gravitation, and can therefore like it be developed to their consequences, by the application of Mathematical Analysis. The works of Fourier*, Green†, and Poisson‡ areremarkable instances of such development. Another good example is Ampère's Theory of Electro-dynamics.

* *Théorie analytique de la Chaleur.* Paris, 1822.

† *Essay on the Application of Mathematical Analysis to the Theories of Electricity and Magnetism.* Nottingham, 1828. Reprinted in Crelle's Journal.

‡ *Mémoires sur le Magnétisme.* Mém. de l'Acad. des Sciences, 1811.

387. When the most probable result is required from a number of observations of the same quantity which do not exactly agree, we must appeal to the mathematical theory of probabilities to guide us to a method of combining the results of experience, so as to eliminate from them, as far as possible, the inaccuracies of observation. Of course it is to be understood that we do not here class as *inaccuracies of observation* any errors which may affect alike every one of a series of observations, such as the inexact determination of a zero point, or of the essential units of time and space, the personal equation of the observer, etc. The process, whatever it may be, which is to be employed in the elimination of errors, is applicable even to these, but only when *several distinct series* of observations have been made, with a change of instrument, or of observer, or of both. Deduction of most probable result from a number of observations.

388. We understand as inaccuracies of observation the whole class of errors which are as likely to lie in one direction as in another in successive trials, and which we may fairly presume would, on the average of an infinite number of repetitions, exactly balance each other in excess and defect. Moreover, we consider only errors of such a kind that their probability is the less the greater they are; so that such errors as an accidental reading of a wrong number of whole degrees on a divided circle (which, by the way, can in general be "probably" corrected by comparison with other observations) are not to be included.

389. Mathematically considered, the subject is by no means an easy one, and many high authorities have asserted that the reasoning employed by Laplace, Gauss, and others, is not well founded; although the results of their analysis have been generally accepted. As an excellent treatise on the subject has recently been published by Airy, it is not necessary for us to do more than to sketch in the most cursory manner a simple and apparently satisfactory method of arriving at what is called the *Method of Least Squares.*

390. Supposing the zero-point and the graduation of an instrument (micrometer, mural circle, thermometer, electrometer, Deduction of most probable result from a number of observations.

galvanometer, etc.) to be *absolutely* accurate, successive readings of the value of a quantity (linear distance, altitude of a star, temperature, potential, strength of an electric current, etc.) may, and in general do, continually differ. What is most probably the true value of the observed quantity?

The most probable value, in all such cases, if the observations are all equally trustworthy, will evidently be the simple mean; or if they are not equally trustworthy, the mean found by attributing *weights* to the several observations in proportion to their presumed exactness. But if several such means have been taken, or several single observations, and if these several means or observations have been differently qualified for the determination of the sought quantity (some of them being likely to give a more exact value than others), we must assign *theoretically* the best practical method of combining them.

391. Inaccuracies of observation are, in general, as likely to be in excess as in defect. They are also (as before observed) more likely to be small than great; and (practically) large errors are not to be expected at all, as such would come under the class of *avoidable mistakes*. It follows that in any one of a series of observations of the same quantity the probability of an error of magnitude x must depend upon x^2, and must be expressed by some function whose value diminishes very rapidly as x increases. The probability that the error lies between x and $x+\delta x$, where δx is very small, must also be proportional to δx.

Hence we may assume the probability of an error of any magnitude included in the range of x to $x+\delta x$ to be

$$\phi(x^2)\,\delta x.$$

Now the error must be included between $+\infty$ and $-\infty$. Hence, as a first condition,

$$\int_{-\infty}^{+\infty}\phi(x^2)\,dx = 1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

The consideration of a very simple case gives us the means of determining the form of the function ϕ involved in the preceding expression*.

* Compare Boole, *Trans. R.S.E.*, 1857. See also Tait, *Trans. R.S.E.*, 1864.

Deduction of most probable result from a number of observations.

Suppose a stone to be let fall with the object of hitting a mark on the ground. Let two horizontal lines be drawn through the mark at right angles to one another, and take them as axes of x and y respectively. The chance of the stone falling at a distance between x and $x+\delta x$ from the axis of y is $\phi(x^2)\,\delta x$.
Of its falling between y and $y+\delta y$ from the axis of x the chance is $\phi(y^2)\,\delta y$.
The chance of its falling on the elementary area $\delta x\delta y$, whose co-ordinates are x, y, is therefore (since these are independent events, and it is to be observed that this is the assumption on which the whole investigation depends*)

$$\phi(x^2)\,\phi(y^2)\,\delta x\delta y, \text{ or } a\phi(x^2)\,\phi(y^2),$$

if a denote the indefinitely small area about the point xy.

Had we taken any other set of rectangular axes with the same origin, we should have found for the same probability the expression $a\phi(x'^2)\,\phi(y'^2)$,
x', y' being the new co-ordinates of a. Hence we must have

$$\phi(x^2)\,\phi(y^2) = \phi(x'^2)\,\phi(y'^2), \text{ if } x^2+y^2 = x'^2+y'^2.$$

From this functional equation we have at once

$$\phi(x^2) = A\epsilon^{mx^2},$$

where A and m are constants. We see at once that m must be negative (as the chance of a large error is very small), and we may write for it $-\frac{1}{h^2}$, so that h will indicate the degree of delicacy or coarseness of the system of measurement employed.

Substituting in (1) we have

$$A\int_{-\infty}^{+\infty}\epsilon^{-\frac{x^2}{h^2}}\,dx = 1,$$

whence $A = \frac{1}{h\sqrt{\pi}}$, and the law of error is

Law of error.

$$\frac{1}{\sqrt{\pi}}\,\epsilon^{-\frac{x^2}{h^2}}\,\frac{\delta x}{h}.$$

The law of error, as regards *distance from the mark, without reference to the direction* of error, is evidently

$$\iint \phi(x^2)\,\phi(y^2)\,dxdy,$$

taken through the space between concentric circles whose radii are r and $r+\delta r$, and is therefore $\frac{2}{h^2}\,\epsilon^{-\frac{r^2}{h^2}}\,r\delta r$,

* [The investigation is due to Sir John Herschel (1850). The assumption in question was adversely criticised by R. L. Ellis, and has been rejected by most subsequent writers. H. L.]

Law of error.

which is of the same form as the law of error to the right or left of a line, with the additional factor r for the greater space for error at greater distances from the centre. As a verification, we see at once that

$$\frac{2}{h^2}\int_0^\infty \epsilon^{-\frac{r^2}{h^2}} r\,dr = 1$$

as was to be expected.

Probable error.

392. The *Probable Error* of an observation is a numerical quantity such that the error of the observation is as likely to exceed as to fall short of it in magnitude.

If we assume the law of error just found, and call P the probable error in one trial,

$$\int_0^P \epsilon^{-\frac{x^2}{h^2}}\,dx = \int_P^\infty \epsilon^{-\frac{x^2}{h^2}}\,dx.$$

The solution of this equation by trial and error leads to the approximate result

$$P = 0{\cdot}477\,h.$$

Probable error of a sum, difference, or multiple.

393. The probable error of any given multiple of the value of an observed quantity is evidently the same multiple of the probable error of the quantity itself.

The probable error of the sum or difference of two quantities, affected by *independent* errors, is the square root of the sum of the squares of their separate probable errors.

To prove this, let us investigate the *law* of error of

$$X \pm Y = Z$$

where the laws of error of X and Y are

$$\frac{1}{\sqrt{\pi}}\epsilon^{-\frac{x^2}{a^2}}\frac{dx}{a}, \text{ and } \frac{1}{\sqrt{\pi}}\epsilon^{-\frac{y^2}{b^2}}\frac{dy}{b},$$

respectively. The chance of an error in Z, of a magnitude included between the limits z, $z+\delta z$, is evidently

$$\frac{1}{\pi ab}\int_{-\infty}^{+\infty}\epsilon^{-\frac{x^2}{a^2}}dx\int_{z-x}^{z+\delta z-x}\epsilon^{-\frac{y^2}{b^2}}dy.$$

For, whatever value is assigned to x, the value of y is given by the limits $z-x$ and $z+\delta z-x$ [or $z+x$, $z+\delta z+x$; but the chances of $\pm x$ are the same, and both are included in the limits ($\pm\infty$) of integration with respect to x].

Probable error of a sum, difference, or multiple.

The value of the above integral becomes, by effecting the integration with respect to y,

$$\frac{\delta z}{\pi ab}\int_{-\infty}^{+\infty}\epsilon^{-\frac{x^2}{a^2}}\epsilon^{-\frac{(z-x)^2}{b^2}}dx,$$

and this is easily reduced to

$$\frac{1}{\sqrt{\pi}}\epsilon^{-\frac{z^2}{a^2+b^2}}\frac{\delta z}{\sqrt{a^2+b^2}}.$$

Thus the probable error is $0{\cdot}477\sqrt{a^2+b^2}$, whence the proposition. And the same theorem is evidently true for *any* number of quantities.

Practical application.

394. As above remarked, the principal use of this theory is in the deduction, from a large series of observations, of the values of the quantities sought in such a form as to be liable to the smallest probable error. As an instance—by the principles of physical astronomy, the place of a planet is calculated from assumed values of the elements of its orbit, and tabulated in the *Nautical Almanac*. The *observed* places do not exactly agree with the predicted places, for two reasons—first, the data for calculation are not exact (and in fact the main object of the observation is to correct their assumed values); second, each observation is in error to some unknown amount. Now the difference between the observed, and the calculated, places depends on the errors of assumed elements and of observation. The methods are applied to eliminate as far as possible the second of these, and the resulting equations give the required corrections of the elements.

Method of least square.

Thus if θ be the calculated R.A. of a planet: δa, δe, $\delta\varpi$, etc., the corrections required for the assumed elements—the true R.A. is $\theta + A\delta a + E\delta e + \Pi\delta\varpi + \text{etc.}$, where A, E, Π, etc., are approximately known. Suppose the observed R.A. to be Θ, then

$$\theta + A\delta a + E\delta e + \Pi\delta\varpi + \ldots = \Theta$$

or

$$A\delta a + E\delta e + \Pi\delta\varpi + \ldots = \Theta - \theta,$$

a known quantity, subject to error of observation. Every observation made gives us an equation of the same *form* as this, and in general the number of observations greatly exceeds that of the quantities δa, δe, $\delta\varpi$, etc., to be found. But it will be sufficient to consider the simple case where only *one* quantity is to be found.

Method of least squares.

Suppose a number of observations, of the same quantity x, lead to the following equations:—

$$x = B_1, \quad x = B_2, \quad \text{etc.},$$

and let the probable errors be $E_1, E_2, \ldots$ Multiply the terms of each equation by numbers inversely proportional to $E_1, E_2, \ldots$ This will make the probable errors of the second members of all the equations the same, e suppose. The equations have now the general form $\qquad ax = b,$

and it is required to find a system of linear factors, by which these equations, being multiplied in order and added, shall lead to a final equation giving the value of x with the probable error a minimum. Let them be f_1, f_2, etc. Then the final equation is

$$(\Sigma af)\, x = \Sigma\, (bf)$$

and therefore $\qquad P^2 (\Sigma af)^2 = e^2 \Sigma\, (f^2)$

by the theorems of § 393, if P denote the probable error of x.

Hence $\dfrac{\Sigma\,(f^2)}{(\Sigma af)^2}$ is a minimum, and its differential coefficients with respect to each separate factor f must vanish.

This gives a series of equations, whose general form is

$$f \Sigma\,(af) - a \Sigma\,(f^2) = 0,$$

which give evidently $f_1 = a_1$, $f_2 = a_2$, etc.

Hence the following rule, which may easily be seen to hold for any number of linear equations containing a smaller number of unknown quantities,

Make the probable error of the second member the same in each equation, by the employment of a proper factor; multiply each equation by the coefficient of x in it and add all, for one of the final equations; and so, with reference to y, z, etc., for the others. The probable errors of the values of x, y, etc., found from these final equations will be less than those of the values derived from any other *linear* method of combining the equations.

This process has been called the method of *Least Squares*, because the values of the unknown quantities found by it are such as to render the sum of the squares of the errors of the original equations a minimum.

That is, in the simple case taken above,

$$\Sigma\,(ax - b)^2 = \text{minimum}.$$

For it is evident that this gives, on differentiating with respect to x, $\qquad \Sigma a\,(ax - b) = 0,$

which is the law above laid down for the formation of the single equation.

395. When a series of observations of the same quantity has been made at different times, or under different circumstances, the law connecting the value of the quantity with the time, or some other variable, may be derived from the results in several ways—all more or less approximate. Two of these methods, however, are so much more extensively used than the others, that we shall devote a page or two here to a preliminary notice of them, leaving detailed instances of their application till we come to Heat, Electricity, etc. They consist in (1) a *Curve*, giving a graphic representation of the relation between the ordinate and abscissa, and (2) an *Empirical Formula* connecting the variables. Methods of representing experimental results.

396. Thus if the abscissæ represent intervals of time, and the ordinates the corresponding height of the barometer, we may construct curves which show at a glance the dependence of barometric pressure upon the time of day; and so on. Such curves may be accurately drawn by photographic processes on a sheet of sensitive paper placed behind the mercurial column, and made to move past it with a uniform horizontal velocity by clockwork. A similar process is applied to the Temperature and Electrification of the atmosphere, and to the components of terrestrial magnetism. Curves.

397. When the observations are not, as in the last section, continuous, they give us only a series of points in the curve, from which, however, we may in general approximate very closely to the result of continuous observation by drawing, *liberâ manu*, a curve passing through these points. This process, however, must be employed with great caution; because, unless the observations are sufficiently close to each other, most important fluctuations in the curve may escape notice. It is applicable, with abundant accuracy, to all cases where the quantity observed changes very slowly. Thus, for instance, weekly observations of the temperature at depths of from 6 to

Curves

24 feet underground were found by Forbes sufficient for a very accurate approximation to the law of the phenomenon.

Interpolation and empirical formulæ.

398. As an instance of the processes employed for obtaining an empirical formula, we may mention methods of *Interpolation*, to which the problem can always be reduced. Thus from sextant observations, at known intervals, of the altitude of the sun, it is a common problem of astronomy to determine at what instant the altitude is greatest, and what is that greatest altitude. The first enables us to find the true solar time at the place; and the second, by the help of the *Nautical Almanac*, gives the latitude. The differential calculus, and the calculus of finite differences, give us formulæ for any required data; and Lagrange has shown how to obtain a very useful one by elementary algebra.

By Taylor's Theorem, if $y=f(x)$, we have

$$y=f(x_0+\overline{x-x_0})=f(x_0)+(x-x_0)f'(x_0)+\frac{(x-x_0)^2}{1\,.\,2}f''(x_0)+\ldots$$
$$+\frac{(x-x_0)^n}{1\,.\,2\ldots n}f^{(n)}[x_0+\theta(x-x_0)]\ldots\ldots\ldots(1),$$

where θ is a proper fraction, and x_0 is *any* quantity whatever. This formula is useful only when the successive derived values of $f(x_0)$ diminish very rapidly.

In finite differences we have

$$f(x+h)=D^h f(x)=(1+\Delta)^h f(x)$$
$$=f(x)+h\Delta f(x)+\frac{h(h-1)}{1\,.\,2}\Delta^2 f(x)+\ \ldots\ldots\ldots\ldots\ldots(2);$$

a very useful formula when the higher differences are small.

(1) suggests the proper form for the required expression, but it is only in rare cases that $f'(x_0)$, $f''(x_0)$, etc., are derivable directly from observation. But (2) is useful, inasmuch as the successive differences, $\Delta f(x)$, $\Delta^2 f(x)$, etc., are easily calculated from the tabulated results of observation, provided these have been taken for equal successive increments of x.

If for values $x_1, x_2, \ldots x_n$ a function takes the values $y_1, y_2, y_3, \ldots y_n$, Lagrange gives for it the obvious expression

$$\left[\frac{y_1}{x-x_1}\,\frac{1}{(x_1-x_2)(x_1-x_3)\ldots(x_1-x_n)}+\frac{y_2}{x-x_2}\,\frac{1}{(x_2-x_1)(x_2-x_3)\ldots(x_2-x_n)}+\ldots\right](x-x_1)(x-x_2)\ldots(x-x_n).$$

Here it is of course assumed that the function required is a rational and integral one in x of the $n-1^{\text{th}}$ degree; and, in general, a similar limitation is in practice applied to the other formulæ above; for in order to find the complete expression for $f(x)$ in either, it is necessary to determine the values of $f'(x_0)$, $f''(x_0)$, ... in the first, or of $\Delta f(x)$, $\Delta^2 f(x)$, ... in the second. If n of the coefficients be required, so as to give the n chief terms of the general value of $f(x)$, we must have n observed simultaneous values of x and $f(x)$, and the expressions become determinate and of the $n-1^{\text{th}}$ degree in $x-x_0$ and h respectively.

Interpolation and empirical formulæ.

In practice it is usually sufficient to employ at most three terms of either of the first two series. Thus to express the length l of a rod of metal as depending on its temperature t, we may assume from (1)

$$l=l_0+A(t-t_0)+B(t-t_0)^2,$$

l_0 being the measured length at any temperature t_0.

398′. These formulæ are practically useful for calculating the probable values of any observed element, for values of the independent variable lying within the range for which observation has given values of the element. But except for values of the independent variable either actually within this range, or not far beyond it in either direction, these formulæ express functions which, in general, will differ more and more widely from the truth the further their application is pushed beyond the range of observation.

Periodic functions.

In a large class of investigations the observed element is in its nature a periodic function of the independent variable. The harmonic analysis (§ 77) is suitable for all such. When the values of the independent variable for which the element has been observed are not equidifferent the coefficients, determined according to the method of least squares, are found by a process which is necessarily very laborious; but when they are equidifferent, and especially when the difference is a submultiple of the period, the equation derived from the method of least squares becomes greatly simplified. Thus, if θ denote an angle increasing in proportion to t, the time, through four right angles in the period, T, of the phenomenon; so that

$$\theta=\frac{2\pi t}{T};$$

Periodic functions.

let $$f(\theta) = A_0 + A_1 \cos\theta + A_2 \cos 2\theta + \ldots$$
$$+ B_1 \sin\theta + B_2 \sin 2\theta + \ldots$$

where A_0, A_1, A_2, ... B_1, B_2, ... are unknown coefficients, to be determined so that $f(\theta)$ may express the most probable value of the element, not merely at times between observations, but through all time as long as the phenomenon is strictly periodic. By taking as many of these coefficients as there are of distinct data by observation, the formula is made to agree precisely with these data. But in most applications of the method, the periodically recurring part of the phenomenon is expressible by a small number of terms of the harmonic series, and the higher terms, calculated from a great number of data, express either irregularities of the phenomenon not likely to recur, or errors of observation. Thus a comparatively small number of terms may give values of the element even for the very times of observation, more probable than the values actually recorded as having been observed, if the observations are numerous but not minutely accurate.

The student may exercise himself in writing out the equations to determine five, or seven, or more of the coefficients according to the method of least squares; and reducing them by proper formulæ of analytical trigonometry to their simplest and most easily calculated forms where the values of θ for which $f(\theta)$ is given are equidifferent. He will thus see that when the difference is $\frac{2\pi}{i}$, i being any integer, and when the number of the data is i or any multiple of it, the equations contain each of them only one of the unknown quantities: so that the method of least squares affords the most probable values of the coefficients, by the easiest and most direct elimination.

CHAPTER IV.

MEASURES AND INSTRUMENTS.

Necessity of accurate measurements.

399. HAVING seen in the preceding chapter that for the investigation of the laws of nature we must carefully watch experiments, either those gigantic ones which the universe furnishes, or others devised and executed by man for specia objects—and having seen that in all such observations accurate measurements of Time, Space, Force, etc., are absolutely necessary, we may now appropriately describe a few of the more useful of the instruments employed for these purposes, and the various standards or units which are employed in them.

400. Before going into detail we may give a rapid *résumé* of the principal Standards and Instruments to be described in this chapter. As most, if not all, of them depend on physical principles to be detailed in the course of this work—we shall assume in anticipation the establishment of such principles, giving references to the future division or chapter in which the experimental demonstrations are more particularly explained. This course will entail a slight, but unavoidable, confusion—slight, because Clocks, Balances, Screws, etc., are familiar even to those who know nothing of Natural Philosophy; unavoidable, because it is in the very nature of our subject that no one part can grow alone, each requiring for its full development the utmost resources of all the others. But if one of our departments thus borrows from others, it is satisfactory to find that it more than repays by the power which its improvement affords them.

Classes of instruments.

401. We may divide our more important and fundamental instruments into four classes—

Those for measuring Time;
" " Space, linear or angular;
" " Force;
" " Mass.

Other instruments, adapted for special purposes such as the measurement of Temperature, Light, Electric Currents, etc., will come more naturally under the head of the particular physical energies to whose measurement they are applicable. Descriptions of self-recording instruments such as tide-gauges, and barometers, thermometers, electrometers, recording photographically or otherwise the continuously varying pressure, temperature, moisture, electric potential of the atmosphere, and magnetometers recording photographically the continuously varying direction and magnitude of the terrestrial magnetic force, must likewise be kept for their proper places in our work.

Calculating Machines.

Calculating Machines have also important uses in assisting physical research in a great variety of ways. They belong to two classes:—

I. Purely Arithmetical, dealing with integral numbers of units. All of this class are evolved from the primitive use of calculuses or little stones for counters (from which we derived the very names *calculation* and "The Calculus"), through such mechanism as that of the Chinese Abacus, still serving its original purpose well in infant schools, up to the Arithmometer of Thomas of Colmar and the grand but partially realized conceptions of calculating machines by Babbage.

II. Continuous Calculating Machines. As these are not only useful as auxiliaries for physical research but also involve dynamical and kinematical principles belonging properly to our subject, some of them have been described in the Appendix to this Chapter, from which dynamical illustrations will be taken in our chapters on Statics and Kinetics.

Classes of instruments.

402. We shall consider in order the more prominent fundamental instruments of the four classes, and some of their most important applications:—

Clock, Chronometer, Chronoscope, Applications to Observation and to self-registering Instruments.

Vernier and Screw-Micrometer, Cathetometer, Spherometer, Dividing Engine, Theodolite, Sextant or Circle.

Common Balance, Bifilar Balance, Torsion Balance, Pendulum, Ergometer.

Among Standards we may mention—

1. *Time.*—Day, Hour, Minute, Second, sidereal and solar.
2. *Space.*—Yard and Mètre: Radian, Degree, Minute, Second.
3. *Force.*—Weight of a Pound or Kilogramme, etc., in any particular locality (gravitation unit); poundal, or dyne (kinetic unit).
4. *Mass.* Pound, Kilogramme, etc.

403. Although without instruments it is impossible to procure or apply any standard, yet, as without the standards no instrument could give us *absolute* measure, we may consider the standards first—referring to the instruments as if we already knew their principles and applications.

Angular measure.

404. First we may notice the standards or units of angular measure:

Radian, or angle whose arc is equal to radius;

Degree, or ninetieth part of a right angle, and its successive subdivisions into sixtieths called *Minutes*, *Seconds*, *Thirds*, etc. The division of the right angle into 90 degrees is convenient because it makes the half-angle of an equilateral triangle ($\sin^{-1}\frac{1}{2}$) an integral number (30) of degrees. It has long been universally adopted by all Europe. The decimal division of the right angle, decreed by the French Republic when it successfully introduced other more sweeping changes, utterly and deservedly failed.

The division of the degree into 60 minutes and of the minute into 60 seconds is not convenient; and tables of the

Angular measure.

circular functions for degrees and hundredths of the degree are much to be desired. Meantime, when reckoning to tenths of a degree suffices for the accuracy desired, in any case the ordinary tables suffice, as 6′ is $\frac{1}{10}$ of a degree.

The decimal system is exclusively followed in reckoning by radians. The value of two right angles in this reckoning is 3·14159..., or π. Thus π radians is equal to 180°. Hence $180^0 \div \pi$ is 57°·29578..., or 57° 17′ 44″·8 is equal to one radian. In mathematical analysis, angles are uniformly reckoned in terms of the radian.

Measure of time.

405. The practical standard of time is the *Sidereal Day*, being the period, nearly constant*, of the earth's rotation about its axis (§ 247). From it is easily derived the *Mean Solar Day*, or the mean interval which elapses between successive passages of the sun across the meridian of any place. This is not so nearly as the Sidereal Day, an absolute or invariable unit:

* In our first edition it was stated in this section that Laplace had calculated from ancient observations of eclipses that the period of the earth's rotation about its axis had not altered by $\frac{1}{10000000}$ of itself since 720 B.C. In § 830 it was pointed out that this conclusion is overthrown by farther information from Physical Astronomy acquired in the interval between the printing of the two sections, in virtue of a correction which Adams had made as early as 1863 upon Laplace's dynamical investigation of an acceleration of the moon's mean motion, produced by the sun's attraction, showing that only about half of the observed acceleration of the moon's mean motion relatively to the angular velocity of the earth's rotation was accounted for by this cause. [Quoting from the first edition, § 830] "In 1859 Adams communicated to Delaunay his final result:—that at "the end of a century the moon is 5″·7 before the position she would have, "relatively to a meridian of the earth, according to the angular velocities of the "two motions, at the beginning of the century, and the acceleration of the "moon's motion truly calculated from the various disturbing causes then recog- "nized. Delaunay soon after verified this result: and about the beginning of "1866 suggested that the true explanation may be a retardation of the earth's "rotation by tidal friction. Using this hypothesis, and allowing for the conse- "quent retardation of the moon's mean motion by tidal reaction (§ 276), Adams, "in an estimate which he has communicated to us, founded on the rough as- "sumption that the parts of the earth's retardation due to solar and lunar tides "are as the squares of the respective tide-generating forces, finds 22ˢ as the "error by which the earth would in a century get behind a perfect clock rated "at the beginning of the century. If the retardation of rate giving this integral "effect were uniform (§ 35, *b*), the earth, as a timekeeper, would be going slower "by ·22 of a second per year in the middle, or ·44 of a second per year at the "end, than at the beginning of a century."

secular changes in the period of the earth's revolution about the sun affect it, though very slightly. It is divided into 24 hours, and the hour, like the degree, is subdivided into successive sixtieths, called minutes and seconds. The usual subdivision of seconds is decimal.

Measure of time.

It is well to observe that seconds and minutes of time are distinguished from those of angular measure by notation. Thus we have for time $13^h\ 43^m\ 27^s{\cdot}58$, but for angular measure 13° 43′ 27″·58.

When long periods of time are to be measured, the mean solar year, consisting of 366·242203 sidereal days, or 365·242242 mean solar days, or the century consisting of 100 such years, may be conveniently employed as the unit*.

Necessity for a perennial standard. A spring suggested.

406. The ultimate standard of accurate chronometry must (if the human race live on the earth for a few million years) be founded on the physical properties of some body of more constant character than the earth: for instance, a carefully arranged metallic spring, hermetically sealed in an exhausted glass vessel. The time of vibration of such a spring would be necessarily more constant from day to day than that of the balance-spring of the best possible chronometer, disturbed as this is by the train of mechanism with which it is connected: and it would almost certainly be more constant from age to age than the time of rotation of the earth (cooling and shrinking, as it certainly is, to an extent that must be very considerable in fifty million years).

Measure of length, founded on artificial metallic standards.

407. The British standard of length is the *Imperial Yard*, defined as the distance between two marks on a certain metallic bar, preserved in the Tower of London, when the whole has a temperature of 60° Fahrenheit. It was not directly derived from any fixed quantity in nature, although some important relations with such have been measured with great accuracy. It has been

* [In Houzeau's *Vade Mecum*, p. 482 will be found a statement that in the year 1900 the length of the interval of time between two successive passages of the sun through the vernal equinox is 365·2421933 days of mean solar time, or 366·2421933 days of sidereal time. It would seem as if the second of the two numbers in the text is intended to be the same as the first of the two numbers just quoted. In the first of the numbers in the text the rotation of the earth and its revolution round the sun are presumably given by reference to a fixed star. The difference between the numbers in the text appears to be the precession of the equinox; but it will be noticed that no epoch is given to which the results are applicable. G. H. D.]

Earth's dimensions not constant,

carefully compared with the length of a seconds pendulum vibrating at a certain station in the neighbourhood of London, so that if it should again be destroyed, as it was at the burning of the Houses of Parliament in 1834, and should all exact copies of it, of which several are preserved in various places, be also lost, it can be restored by pendulum observations. A less accurate, but still (except in the event of earthquake disturbance) a very good, means of reproducing it exists in the measured base-lines of the Ordnance Survey, and the thence calculated distances between definite stations in the British Islands, which have been ascertained in terms of it with a degree of accuracy sometimes within an inch per mile, that is to say, within about $\frac{1}{60000}$.

408. In scientific investigations, we endeavour as much as possible to keep to one unit at a time, and the foot, which is defined to be one-third part of the yard, is, for British measurement, generally the most convenient. Unfortunately the inch, or one-twelfth of a foot, must sometimes be used. The statute mile, or 1760 yards, is most unhappily often used when great lengths are considered. The British measurements of area and volume are infinitely inconvenient and wasteful of brain-energy, and of plodding labour. Their contrast with the simple, uniform, metrical system of France, Germany, and Italy, is but little creditable to English intelligence.

409. In the French metrical system the decimal division is exclusively employed. The standard, (unhappily) called the *Mètre*, was defined originally as the ten-millionth part of the length of the quadrant of the earth's meridian from the pole to the equator; but it is now defined practically by the accurate standard metres laid up in various national repositories in Europe. It is somewhat longer than the yard, as the following Table shows:

nor easily measured with great accuracy.

Measure of length.

Inch = 25·39977 millimètres.	Centimètre = ·3937043 inch.
Foot = 3·047972 decimètres.	Mètre = 3·280869 feet.
British statute mile = 1609·329 mètres.	Kilomètre = ·6213767 British statute mile.

Measure of surface.

410. The unit of superficial measure is in Britain the square yard, in France the mètre carré. Of course we may use square inches, feet, or miles, as also square millimètres, kilomètres, etc., or the *Hectare* = 10,000 square mètres.

Measure of surface.

Square inch = 6·451483 square centimètres.
" foot = 9·290135 " decimètres.
" yard = 83·61121 " decimètres.
Acre = ·4046792 of a hectare.
Square British statute mile = 258·9946 hectares.
Hectare = 2·471093 acres.

Measure of volume.

411. Similar remarks apply to the cubic measure in the two countries, and we have the following Table:—

Cubic inch = 16·38661 cubic centimètres.
" foot = 28·31606 " decimètres or *Litres.*
Gallon = 4·543808 litres.
" = 277·274 cubic inches, by Act of Parliament now repealed.
Litre = ·035315 cubic feet.

Measure of mass.

412. The British unit of mass is the Pound (defined by standards only); the French is the *Kilogramme*, defined originally as a litre of water at its temperature of maximum density; but now practically defined by existing standards.

Grain = 64·79896 milligrammes.	Gramme = 15·43235 grains.
Pound = 453·5927 grammes.	Kilogramme = 2·20462125 lbs.

Professor W. H. Miller finds (*Phil. Trans.* 1857) that the "*kilogramme des Archives*" is equal in mass to 15432·34874 grains; and the "*kilogramme type laiton,*" deposited in the Ministère de l'Intérieure in Paris, as standard for French commerce, is 15432·344 grains.

Measure of force.

413. The measurement of force, whether in terms of the weight of a stated mass in a stated locality, or in terms of the *absolute* or *kinetic* unit, has been explained in Chap. II. (See §§ 220—226). From the measures of force and length, we derive at once the measure of work or mechanical effect. That practically employed by engineers is founded on the gravitation measure of force. Neglecting the difference of gravity at London and Paris, we see from the above tables that the following relations exist between the London and the Parisian reckoning of work:—

Foot-pound = 0·13825 kilogramme-mètre.
Kilogramme-mètre = 7·2331 foot-pounds.

Clock.

414. A *Clock* is primarily an instrument which, by means of a train of wheels, records the number of vibrations executed by a pendulum; a *Chronometer* or *Watch* performs the same duty for the oscillations of a flat spiral spring—just as the train of wheel-work in a gas-metre counts the number of revolutions of the main shaft caused by the passage of the gas through the machine. As, however, it is impossible to avoid friction, resistance of air, etc., a pendulum or spring, left to itself, would not long continue its oscillations, and, while its motion continued, would perform each oscillation in less and less time as the arc of vibration diminished: a continuous supply of energy is furnished by the descent of a weight, or the uncoiling of a powerful spring. This is so applied, through the train of wheels, to the pendulum or balance-wheel by means of a mechanical contrivance called an *Escapement*, that the oscillations are maintained of nearly uniform extent, and therefore of nearly uniform duration. The construction of escapements, as well as of trains of clock-wheels, is a matter of *Mechanics*, with the details of which we are not concerned, although it may easily be made the subject of mathematical investigation. The means of avoiding errors introduced by changes of temperature, which have been carried out in *Compensation* pendulums and balances, will be more properly described in our chapters on Heat. It is to be observed that there is little inconvenience if a clock lose or gain *regularly;* that can be easily and accurately allowed for: irregular rate is fatal.

Electrically controlled clocks.

415. By means of a recent application of electricity to be afterwards described, one good clock, carefully regulated from time to time to agree with astronomical observations, may be made (without injury to its own performance) to control any number of other less-perfectly constructed clocks, so as to compel their pendulums to vibrate, beat for beat, with its own.

Chronoscope.

416. In astronomical observations, time is estimated to tenths of a second by a practised observer, who, while watching the phenomena, counts the beats of the clock. But for the *very* accurate measurement of short intervals, many instruments have been devised. Thus if a small orifice be opened in a large and

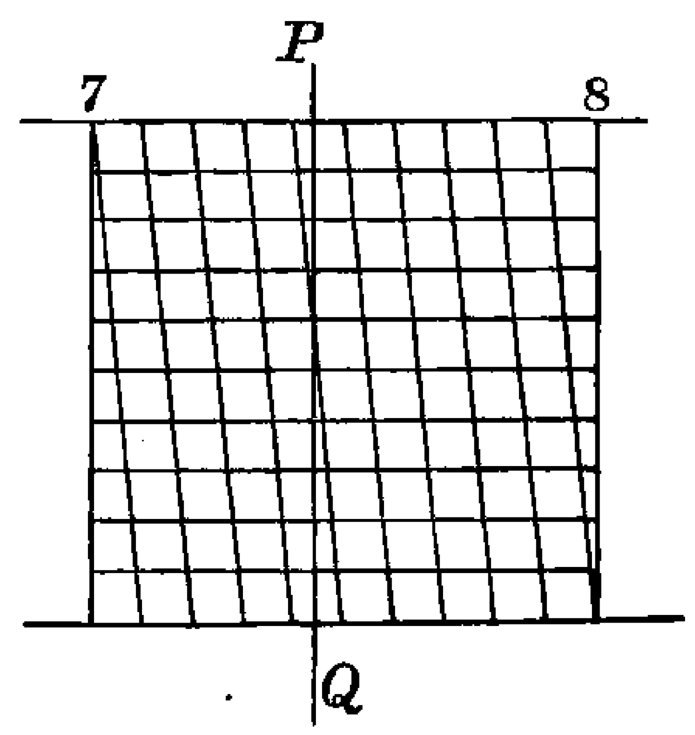
deep vessel full of mercury, and if we know by trial the weight of metal that escapes say in five minutes, a simple proportion gives the interval which elapses during the escape of any given weight. It is easy to contrive an adjustment by which a vessel may be placed under, and withdrawn from, the issuing stream at the time of occurrence of any two successive phenomena.

417. Other contrivances, called Stop-watches, Chronoscopes, etc., which can be read off at rest, started on the occurrence of any phenomenon, and stopped at the occurrence of a second, then again read off; or which allow of the making (by pressing a stud) a slight mark, on a dial revolving at a given rate, at the instant of the occurrence of each phenomenon to be noted, are common enough. But, of late, these have almost entirely given place to the Electric Chronoscope, an instrument which will be fully described later, when we shall have occasion to refer to experiments in which it has been usefully employed.

418. We now come to the measurement of space, and of angles, and for these purposes the most important instruments are the *Vernier* and the *Screw*.

Diagonal scale

419. Elementary geometry, indeed, gives us the means of dividing any straight line into any assignable number of equal parts; but in practice this is by no means an accurate or reliable method. It was formerly used in the so-called Diagonal Scale, of which the construction is evident from the diagram. The reading is effected by a sliding-piece whose edge is perpendicular to the length of the scale. Suppose that it is PQ whose position on the scale is required. This can evidently



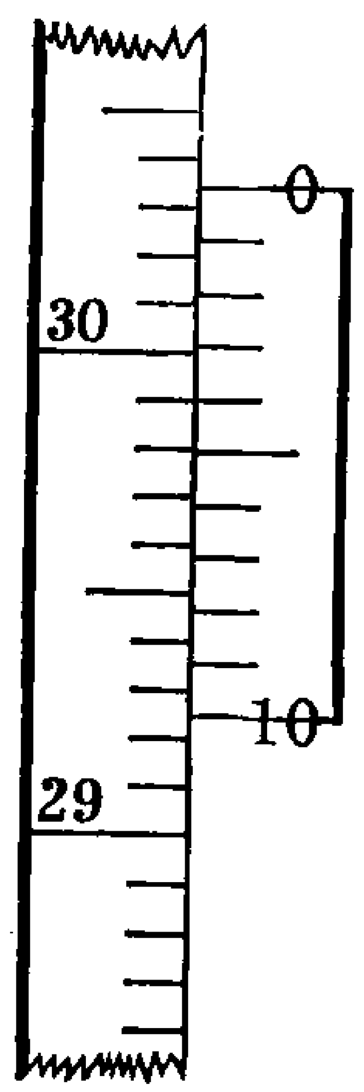
cut only *one* of the transverse lines. *Its* number gives the number of tenths of an inch [4 in the figure], and the horizontal line next above the point of intersection gives evidently the number of hundredths [in the present case 4]. Hence the reading is 7·44. As an idea of the comparative uselessness of this

Diagonal scale. method, we may mention that a quadrant of 3 feet radius which belonged to Napier of Merchiston, and is divided on the limb by this method, reads to minutes of a degree; no higher accuracy than is now attainable by the pocket sextants made by Troughton and Simms, the radius of whose arc is virtually little more than an inch. The latter instrument is read by the help of a Vernier.

Vernier. **420.** The Vernier is commonly employed for such instruments as the Barometer, Sextant, and Cathetometer, while the Screw is micrometrically applied to the more delicate instruments, such as Astronomical Circles, and Micrometers, and the Spherometer.

421. The vernier consists of a slip of metal which slides along a divided scale, the edges of the two being coincident. Hence, when it is applied to a divided circle, its edge is circular, and it moves about an axis passing through the centre of the divided limb.

In the sketch let 0, 1, 2,...10 be the divisions on the vernier, 0, 1, 2, etc., any set of consecutive divisions on the limb or scale



along whose edge it slides. If, when 0 and 0 coincide, 10 and 11 coincide also, then 10 divisions of the vernier are equal in length to 11 on the limb; and therefore each division on the vernier is $\frac{11}{10}$ths or $1\frac{1}{10}$ of a division on the limb. If, then, the vernier be moved till 1 coincides with 1, 0 will be $\frac{1}{10}$th of a division of the limb beyond 0; if 2 coincide with 2, 0 will be $\frac{2}{10}$ths beyond 0; and so on. Hence to read the vernier in any position, note first the division next to 0, and behind it on the limb. This is the *integral* number of divisions to be read. For the fractional part, see which division of the vernier is in a line with one on the limb; if it be the 4th (as in the figure), that indicates an addition to the reading of $\frac{4}{10}$ths of a division of the limb; and so on. Thus, if the figure represent a barometer scale divided into inches and tenths, the reading is $\overset{\text{in}}{30}{\cdot}34$, the zero line of the vernier being adjusted to the level of the mercury.

422. If the limb of a sextant be divided, as it usually is, to third parts of a degree, and the vernier be formed by dividing 21 of these into 20 equal parts, the instrument can be read to twentieths of divisions on the limb, that is, to minutes of arc. Vernier.

If no line on the vernier coincide with one on the limb, then since the divisions of the former are the longer there will be one of the latter included between the two lines of the vernier, and it is usual in practice to take the mean of the readings which would be given by a coincidence of either pair of bounding lines.

423. In the above sketch and description, the numbers on the scale and vernier have been supposed to run *opposite* ways. This is generally the case with British instruments. In some foreign ones the divisions run in the same direction on vernier and limb, and in that case it is easy to see that to read to tenths of a scale division we must have ten divisions of the vernier equal to *nine* of the scale.

In general, to read to the nth part of a scale division, n divisions of the vernier must equal $n+1$ or $n-1$ divisions on the limb, according as these run in opposite or similar directions.

424. The principle of the *Screw* has been already noticed (§ 102). It may be used in either of two ways, *i.e.*, the nut may be fixed, and the screw advance through it, or the screw may be prevented from moving longitudinally by a fixed collar, in which case the nut, if prevented by fixed guides from rotating, will move in the direction of the common axis. The advance in either case is evidently proportional to the angle through which the screw has turned about its axis, and this may be measured by means of a divided head fixed perpendicularly to the screw at one end, the divisions being read off by a pointer or vernier attached to the frame of the instrument. The nut carries with it either a tracing point (as in the dividing engine) or a wire, thread, or half the object-glass of a telescope (as in micrometers), the thread or wire, or the play of the tracing point, being at right angles to the axis of the screw. Screw.

425. Suppose it be required to divide a line into any number of equal parts. The line is placed parallel to the axis

Screw. of the screw with one end exactly under the tracing point, or under the fixed wire of a microscope carried by the nut, and the screw-head is read off. By turning the head, the tracing point or microscope wire is brought to the other extremity of the line; and the number of turns and fractions of a turn required for the whole line is thus ascertained. Dividing this by the number of equal parts required, we find at once the number of turns and fractional parts corresponding to *one* of the required divisions, and by giving that amount of rotation to the screw over and over again, drawing a line after each rotation, the required division is effected.

Screw-Micrometer. **426.** In the Micrometer, the movable wire carried by the nut is parallel to a fixed wire. By bringing them into optical contact the zero reading of the head is known; hence when another reading has been obtained, we have by subtraction the number of turns corresponding to the length of the object to be measured. The *absolute* value of a turn of the screw is determined by calculation from the number of threads in an inch, or by actually applying the micrometer to an object of known dimensions.

Spherometer. **427.** For the measurement of the thickness of a plate, or the curvature of a lens, the *Spherometer* is used. It consists of a screw nut rigidly fixed in the middle of a very rigid three-legged table, with its axis perpendicular to the plane of the three feet (or finely rounded ends of the legs), and an accurately cut screw working in this nut. The lower extremity of the screw is also finely rounded. The number of turns, whole or fractional, of the screw, is read off by a divided head and a pointer fixed to the stem. Suppose it be required to measure the thickness of a plate of glass. The three feet of the instrument are placed upon a nearly enough flat surface of a hard body, and the screw is gradually turned until its point touches and presses the surface. The muscular sense of touch perceives resistance to the turning of the screw when, after touching the hard body, it presses on it with a force somewhat exceeding the weight of the screw. The first effect of the contact is a diminution of resistance to the turning, due to the weight of the screw coming

to be borne on its fine pointed end instead of on the thread of the nut. The *sudden* increase of resistance at the instant when the screw commences to bear part of the weight of the nut finds the sense prepared to perceive it with remarkable delicacy on account of its contrast with the immediately preceding diminution of resistance. The screw-head is now read off, and the screw turned backwards until room is left for the insertion, beneath its point, of the plate whose thickness is to be measured. The screw is again turned until increase of resistance is again perceived; and the screw-head is again read off. The difference of the readings of the head is equal to the thickness of the plate, reckoned in the proper unit of the screw and the division of its head. Spherometer.

428. If the curvature of a lens is to be measured, the instrument is first placed, as before, on a plane surface, and the reading for the contact is taken. The same operation is repeated on the spherical surface. The difference of the screw readings is evidently the greatest thickness of the glass which would be cut off by a plane passing through the three feet. This enables us to calculate the radius of the spherical surface (the distance from foot to foot of the instrument being known).

Let a be the distance from foot to foot, l the length of screw corresponding to the difference of the two readings, R the radius of the spherical surface; we have at once $2R=\frac{a^2}{3l}+l$, or, as l is generally very small compared with a, the diameter is, very approximately, $\frac{a^2}{3l}$.

429. The *Cathetometer* is used for the accurate determination of differences of level—for instance, in measuring the height to which a fluid rises in a capillary tube above the exterior free surface. It consists of a long divided metallic stem, turning round an axis as nearly as may be parallel to its length, on a fixed tripod stand: and, attached to the stem, a spirit-level. Upon the stem slides a metallic piece bearing a telescope of which the length is approximately enough perpendicular to the axis. The telescope tube is as nearly as may be perpendicular to the length of the stem. By levelling screws in two feet of the Cathetometer.

Catheto-meter. tripod the bubble of the spirit-level is brought to one position of its glass when the stem is turned all round its axis. This secures that the axis is vertical. In using the instrument the telescope is directed in succession to the two objects whose difference of level is to be found, and in each case moved (generally by a delicate screw) up or down the stem, until a horizontal wire in the focus of its eye-piece coincides with the image of the object. The difference of readings on the vertical stem (each taken generally by aid of a vernier sliding-piece) corresponding to the two positions of the telescope gives the required difference of level.

Balance. **430.** The common *Gravity Balance* is an instrument for testing the equality of the gravity of the masses placed in the two pans. We may note here a few of the precautions adopted in the best balances to guard against the various defects to which the instrument is liable; and the chief points to be attended to in its construction to secure delicacy, and rapidity of weighing.

The balance-beam should be very stiff, and as light as possible consistently with the requisite stiffness. For this purpose it is generally formed either of tubes, or of a sort of lattice-framework. To avoid friction, the axle consists of a knife-edge, as it is called; that is, a wedge of hard steel, which, when the balance is in use, rests on horizontal plates of polished agate. A similar contrivance is applied in very delicate balances at the points of the beam from which the scale-pans are suspended. When not in use, and just before use, the beam with its knife-edge is lifted by a lever arrangement from the agate plates. While thus secured it is loaded with weights as nearly as possible equal (this can be attained by previous trial with a coarser instrument), and the accurate determination is then readily effected. The last fraction of the required weight is determined by a rider, a very small weight, generally formed of wire, which can be worked (by a lever) from the outside of the glass case in which the balance is enclosed, and which may be placed in different positions upon one arm of the beam. This arm is graduated to tenths, etc., and thus shows at once the value of the rider in any case as depending on its moment or leverage, § 232.

431. Qualities of a balance: Balance.

1. *Stability.*—For stability of the beam alone without pans and weights, its centre of gravity must be below its bearing knife-edge. For stability with the heaviest weights the line joining the points at the ends of the beam from which the pans are hung must be below the knife-edge bearing the whole.

2. *Sensibility.*—The beam should be sensibly deflected from a horizontal position by the smallest difference between the weights in the scale-pans. The definite measure of the sensibility is the angle through which the beam is deflected by a stated difference between the loads in the pans.

3. *Quickness.*—This means rapidity of oscillation, and consequently speed in the performance of a weighing. It depends mainly upon the depth of the centre of gravity of the whole below the knife-edge and the length of the beam.

In our Chapter on Statics we shall give the investigation. The sensibility and quickness will there be calculated for any given form and dimensions of the instrument.

A fine balance should turn with about a 500,000th of the greatest load which can safely be placed in either pan. In fact few measurements of any kind are correct to more than *six* significant figures.

The process of *Double Weighing*, which consists in counterpoising a mass by shot, or sand, or pieces of fine wire, and then substituting weights for it in the same pan till equilibrium is attained, is more laborious, but more accurate, than single weighing; as it eliminates all errors arising from unequal length of the arms, etc.

Correction is required for the weights of air displaced by the two bodies weighed against one another when their difference is too large to be negligible.

432. In the *Torsion-balance*, invented and used with great effect by Coulomb, a force is measured by the torsion of a glass fibre, or of a metallic wire. The fibre or wire is fixed at its upper end, or at both ends, according to circumstances. In general it carries a very light horizontal rod or needle, to the extremities of which are attached the body on Torsion-balance.

Torsion-balance.

which is exerted the force to be measured, and a counterpoise. The upper extremity of the torsion fibre is fixed to an index passing through the centre of a divided disc, so that the angle through which that extremity moves is directly measured. If, at the same time, the angle through which the needle has turned be measured, or, more simply, if the index be always turned till the needle assumes a definite position determined by marks or sights attached to the case of the instrument—we have the amount of torsion of the fibre, and it becomes a simple statical problem to determine from the latter the force to be measured; its direction, and point of application, and the dimensions of the apparatus, being known. The force of torsion as depending on the angle of torsion was found by Coulomb to follow the law of simple proportion up to the limits of perfect elasticity—as might have been expected from Hooke's Law (see *Properties of Matter*), and it only remains that we determine the amount for a particular angle in absolute measure. This determination is in general simple enough in theory; but in practice requires considerable care and nicety. The torsion-balance, however, being chiefly used for comparative, not absolute, measure, this determination is often unnecessary. More will be said about it when we come to its applications.

433. The ordinary spiral spring-balances used for roughly comparing either small or large weights or forces, are, properly speaking, only a modified form of torsion-balance*, as they act almost entirely by the torsion of the wire, and not by longitudinal extension or by flexure. Spring-balances we believe to be capable, if carefully constructed, of rivalling the ordinary balance in accuracy, while, for some applications, they far surpass it in sensibility and convenience. They measure directly *force*, not *mass;* and therefore if used for determining masses in different parts of the earth, a correction must be applied for the varying force of gravity. The correction for temperature must not be overlooked. These corrections may be avoided by the method of double weighing.

* Binet, *Journal de l'École Polytechnique*, x. 1815: and J. Thomson, *Cambridge and Dublin Math. Journal* (1848).

434. Perhaps the most delicate of all instruments for the measurement of force is the *Pendulum*. It is proved in kinetics (see Div. II.) that for any pendulum, whether oscillating about a mean vertical position under the action of gravity, or in a horizontal plane, under the action of magnetic force, or force of torsion, the square of the number of *small* oscillations in a given time is proportional to the magnitude of the force under which these oscillations take place. Pendulum.

For the estimation of the relative amounts of gravity at different places, this is by far the most perfect instrument. The method of coincidences by which this process has been rendered so excessively delicate will be described later.

435. The *Bifilar Suspension*, an arrangement for measuring small horizontal forces, or couples in horizontal planes, in terms of the weight of the suspended body, is due originally to Sir William Snow Harris, who used it in one of his electrometers, as a substitute for the simple torsion-balance of Coulomb. It was used also by Gauss in his bifilar magnetometer for measuring the horizontal component of the terrestrial magnetic force*. In this instrument the bifilar suspension is adjusted to keep a bar-magnet in a position approximately perpendicular to the magnetic meridian. The small natural augmentations and diminutions of the horizontal component are shown by small azimuthal motions of the bar. On account of some obvious mechanical and dynamical difficulties this instrument was not found very convenient for absolute determinations, but from the time of its first practical introduction by Gauss and Weber it has been in use in all Magnetic Observatories for measuring the natural variations of the horizontal magnetic component. It is now made with a much smaller magnet than the great bar weighing twenty-five pounds originally given with it by Gauss; but the bars in actual use at the present day are still enormously too large† for their duty. The weight of the

Bifilar Balance.

Bifilar Magnetometer.

* Gauss, *Resultate aus den Beobachtungen des magnetischen Vereins im Jahre* 1837. Translated in Taylor's *Scientific Memoirs*, Vol. II., Article VI.

† The suspended magnets used for determining the direction and the intensity of the horizontal magnetic force in the Dublin Magnetic Observatory,

Bifilar Magnetometer.

bar with attached mirror ought not to exceed eight grammes, so that two single silk fibres may suffice for the bearing threads. The only substantial alteration, besides the diminution of its magnitude, which has been made in the instrument since Gauss and Weber's time is the addition of photographic apparatus and clockwork for automatic record of its motions. For absolute determinations of the horizontal component force, Gauss's method of deflecting a freely suspended magnet by a magnetic bar brought into proper positions in its neighbourhood, and again making an independent set of observations to determine the period of oscillation of the same deflecting bar when suspended by a fine fibre and set to vibrate through a small horizontal angle on each side of the magnetic meridian, is the method which has been uniformly in use both in magnetic observatories and in travellers' observations with small portable apparatus since it was first invented by Gauss*.

Absolute measurement of Terrestrial Magnetic Force.

Bifilar Balance.

In the bifilar balance the two threads may be of unequal lengths, the line joining their upper fixed ends need not be horizontal, and their other ends may be attached to any two points of the suspended body: but for most purposes, and particularly for regular instruments such as electrometers and magnetometers with bifilar suspension, it is convenient to have, as nearly as may be, the two threads of equal length, their fixed ends at the same level, and their other ends attached to the suspended body symmetrically with reference to its centre of gravity (as illustrated in the last set of drawings of § 345^x). Supposing the instrument-maker to have fulfilled these conditions of symmetry as nearly as he can with reference to the four points of attachment of the threads, we have still to adjust properly the lengths of the threads. For this purpose remark that a small difference in the lengths will throw the suspended body into an unsymmetrical

as described by Dr Lloyd in his *Treatise on Magnetism* (London, 1874), are each of them 15 inches long, $\frac{7}{8}$ of an inch broad, and $\frac{1}{4}$ of an inch in thickness, and must therefore weigh about a pound each. The corresponding magnets used at the Kew Observatory are much smaller. They are each 5·4 inches long, 0·8 inch broad, and 0·1 inch thick, and therefore the weight of each is about 0·012 pound, or nearly 55 grammes.

* *Intensitas Vis Magneticae Terrestris ad Mensuram Absolutam revocata*, Commentationes Societatis Gottingensis, 1832.

Bifilar Balance.

position, in which, particularly if its centre of gravity be very low (as it is in Sir W. Thomson's Quadrant Electrometer), much more of its weight will be borne by one thread than by the other. This will diminish very much the amount of the horizontal couple required to produce a stated azimuthal deflection in the regular use of the instrument, in other words will increase its sensibility above its proper amount, that is to say, the amount which it would have if the conditions of symmetry were fully realized. Hence the proper adjustment for equalizing the lengths of the threads in a symmetrical bifilar balance, or for giving them their right difference in an unsymmetrical arrangement, in order to make the instrument as accurate as it can be, is to alter the length of one or both of the threads, until we attain to the condition of *minimum sensibility*, that is to say minimum angle of deflection under the influence of a given amount of couple.

The great merit of the bifilar balance over the simple torsion-balance of Coulomb for such applications as that to the horizontal magnetometer in the continuous work of an observatory, is the comparative smallness of the influence it experiences from changes of temperature. The torsional rigidity of iron, copper, and brass wires is diminished about $\frac{1}{2}$ per cent. with 10° elevation of temperature, while the linear expansions of the same metals are each less than $\frac{1}{50}$ per cent. with the same elevation of temperature. Hence in the unifilar torsion-balance, if iron, copper, or brass (the only metals for which the change of torsional rigidity with change of temperature has hitherto been measured) is used for the material of the bearing fibre, the sensibility is augmented $\frac{1}{2}$ per cent. by 10° elevation of temperature.

On the other hand, in the bifilar balance, if torsional rigidity does not contribute any sensible proportion to the whole directive couple (and this condition may be realized as nearly as we please by making the bearing wires long enough and making the distance between them great enough to give the requisite amount of directive couple), the sensibility of the balance is affected only by the linear expansions of the substances concerned. If the equal distances between the two pairs of points

Bifilar Balance.

of attachment, in the normal form of bifilar balance (or that in which the two threads are vertical when the suspended body is uninfluenced by horizontal force or couple), remained constant, the sensibility would be augmented with elevation of temperature in simple proportion to the linear expansions of the bearing wires; and this small influence might, if it were worth while to make the requisite mechanical arrangements, be perfectly compensated by choosing materials for the frames or bars bearing the attachments of the wires so that the proportionate augmentation of the distance between them should be just half the elongation of either wire, because the sensibility, as shown by the mathematical formula below, is simply proportional to the length of the wires and inversely proportional to the square of the distance between them. But, even without any such compensation, the temperature-error due to linear expansions of the materials of the bifilar balance is so small that in the most accurate regular use of the instrument in magnetic observatories it may be almost neglected; and at most it is less than $\frac{1}{25}$ of the error of the unifilar torsion-balance, at all events if, as is probably the case, the changes of rigidity with changes of temperature in other metals are of similar amounts to those for the three metals on which experiments have been made. In reality the chief temperature-error of the bifilar magnetometer depends on the change of the magnetic moment of the suspended magnet with change of temperature. It seems that the magnetism of a steel magnet diminishes with rise of temperature and augments with fall of temperature, but experimental information is much wanted on this subject.

The amount of the effect is very different in different bars, and it must be experimentally determined for each bar serving in a bifilar magnetometer. The amount of the change of magnetic moment in the bar which had been most used in the Dublin Magnetic Observatory was found to be ·000029 per degree Fahrenheit or at the rate of ·000052 per degree Centigrade, being about the same amount as that of the change of torsional rigidity with temperature of the three metals referred to above.

Let a be the half length of the bar between the points of attachment of the wires, θ the angle through which the bar has

been turned (in a horizontal plane) from its position of equilibrium, l the length of one of the wires, ι its inclination to the vertical.

Then $l\cos\iota$ is the difference of levels between the ends of each wire, and evidently, by the geometry of the case,

$$\tfrac{1}{2}l\sin\iota = a\sin\tfrac{1}{2}\theta.$$

Now if Q be the couple tending to turn the bar, and W its weight, the principle of mechanical effect gives

$$\begin{aligned} Qd\theta &= -Wd(l\cos\iota) \\ &= Wl\sin\iota d\iota. \end{aligned}$$

But, by the geometrical condition above,

$$l^2\sin\iota\cos\iota d\iota = a^2\sin\theta d\theta.$$

Hence $$\frac{Q}{a^2\sin\theta} = \frac{W}{l\cos\iota},$$

or $$Q = \frac{Wa^2}{l}\frac{\sin\theta}{\sqrt{1-\frac{4a^2}{l^2}\sin^2\frac{\theta}{2}}},$$

which gives the couple in terms of the deflection θ.

If the torsion of the wires be taken into account, it is sensibly equal to θ (since the greatest inclination to the vertical is small), and therefore the couple resulting from it will be $E\theta$. This must be added to the value of Q just found in order to get the whole deflecting couple.

436. Ergometers are instruments for measuring energy. *White's friction brake* measures the amount of work actually performed in any time by an engine or other "prime mover," by allowing it during the time of trial to waste all its work on friction. *Morin's ergometer* measures work without wasting any of it, in the course of its transmission from the prime mover to machines in which it is usefully employed. It consists of a simple arrangement of springs, measuring at every instant the *couple* with which the prime mover turns the shaft that transmits its work, and an integrating machine from which the work done by this couple during any time can be read off.

Ergometers.

Let L be the couple at any instant, and ϕ the whole angle through which the shaft has turned from the moment at which the reckoning commences. The integrating machine shows at any moment the value of $\int Ld\phi$, which (§ 240) is the whole work done.

Ergometers. **437.** White's friction brake consists of a lever clamped to the shaft, but not allowed to turn with it. The moment of the force required to prevent the lever from going round with the shaft, multiplied by the whole angle through which the shaft turns, measures the whole work done against the friction of the clamp. The same result is much more easily obtained by wrapping a rope or chain several times round the shaft, or round a cylinder or drum carried round by the shaft, and applying measured forces to its two ends in proper directions to keep it nearly steady while the shaft turns round without it. The difference of the moments of these two forces round the axis, multiplied by the angle through which the shaft turns, measures the whole work spent on friction against the rope. If we remove all other resistance to the shaft, and apply the proper amount of force at each end of the dynamimetric rope or chain (which is very easily done in practice), the prime mover is kept running at the proper speed for the test, and having its whole work thus wasted for the time and measured.

APPENDIX B′.

CONTINUOUS CALCULATING MACHINES.

I. TIDE-PREDICTING MACHINE.

Tide-predicting Machine. The object is to predict the tides for any port for which the tidal constituents have been found from the harmonic analysis from tide-gauge observations; not merely to predict the times and heights of high water, but the depths of water at any and every instant, showing them by a continuous curve, for a year, or for any number of years in advance.

This object requires the summation of the simple harmonic functions representing the several constituents* to be taken into account, which is performed by the machine in the following manner:—For each tidal constituent to be taken into account the machine has a shaft with an overhanging crank, which carries a pulley pivoted on a parallel axis adjustable to a greater or less distance from the shaft's axis, according to the greater or less range of the particular tidal constituent for the different ports for which the machine is to be used. The several shafts, with their axes all parallel, are geared together so that their periods are to a sufficient degree of approximation proportional to the periods of the tidal constituents. The crank on each shaft can be turned round on the shaft and clamped in any position: thus it is set to the proper position for the epoch of the particular tide which it is to produce. The axes of the several shafts are horizontal, and their vertical planes are at successive distances one from another, each equal to the diameter of one of the pulleys (the diameters of these being equal). The shafts are in two rows, an upper and a lower, and the grooves of the pulleys are all in one plane perpendicular to their axes.

Suppose, now, the axes of the pulleys to be set each at zero distance from the axis of its shaft, and let a fine wire or chain,

* See Report for 1876 of the Committee of the British Association appointed for the purpose of promoting the Extension, Improvement, and Harmonic Analysis of Tidal Observations.

Tide-predicting Machine.

with one end hanging down and carrying a weight, pass alternately over and under the pulleys in order, and vertically upwards or downwards (according as the number of pulleys is even or odd) from the last pulley to a fixed point. The weight is to be properly guided for vertical motion by a geometrical slide. Turn the machine now, and the wire will remain undisturbed with all its free parts vertical and the hanging weight unmoved. But now set the axis of any one of the pulleys to a distance $\frac{1}{2}T$ from its shaft's axis and turn the machine. If the distance of this pulley from the two on each side of it in the other row is a considerable multiple of $\frac{1}{2}T$, the hanging weight will now (if the machine is turned uniformly) move up and down with a simple harmonic motion of amplitude (or semi-range) equal to T in the period of its shaft. If, next, a second pulley is displaced to a distance $\frac{1}{2}T'$, a third to a distance $\frac{1}{2}T''$, and so on, the hanging weight will now perform a complex harmonic motion equal to the sum of the several harmonic motions, *each* in its proper period, which would be produced separately by the displacements T, T', T''. Thus, if the machine was made on a large scale, with T, T',... equal respectively to the actual semi-ranges of the several constituent tides, and if it was turned round slowly (by clockwork, for example), each shaft going once round in the actual period of the tide which it represents, the hanging weight would rise and fall exactly with the water-level as affected by the whole tidal action. This, of course, could be of no use, and is only suggested by way of illustration. The actual machine is made of such magnitude, that it can be set to give a motion to the hanging weight equal to the actual motion of the water-level reduced to any convenient scale: and provided the whole range does not exceed about 30 centimetres, the geometrical error due to the deviation from perfect parallelism in the successive free parts of the wire is not so great as to be practically objectionable. The proper order for the shafts is the order of magnitude of the constituent tides which they produce, the greatest next the hanging weight, and the least next the fixed end of the wire: this so that the greatest constituent may have only one pulley to move, the second in magnitude only two pulleys, and so on.

One machine of this kind has already been constructed for the British Association, and another (with a greater number of shafts to include a greater number of tidal constituents) is being constructed for the Indian Government. The British Association machine, which is kept available for general use, under charge of the Science and Art Department in South Kensington, has ten shafts, which taken in order, from the hanging weight, give respectively the following tidal constituents*:

Tide-predicting Machine.

1. The mean lunar semi-diurnal.
2. The mean solar semi-diurnal.
3. The larger elliptic semi-diurnal.
4. The luni-solar diurnal declinational.
5. The lunar diurnal declinational.
6. The luni-solar semi-diurnal declinational.
7. The smaller elliptic semi-diurnal.
8. The solar diurnal declinational.
9. The lunar quarter-diurnal, or first shallow-water tide of mean lunar semi-diurnal.
10. The luni-solar quarter-diurnal, shallow-water tide.

The hanging weight consists of an ink-bottle with a glass tubular pen, which marks the tide level in a continuous curve on a long band of paper, moved horizontally across the line of motion of the pen, by a vertical cylinder geared to the revolving shafts of the machine. One of the five sliding points of the geometrical slide is the point of the pen sliding on the paper stretched on the cylinder, and the couple formed by the normal pressure on this point, and on another of the five, which is about four centimetres above its level and one and a half centimetres from the paper, balances the couple due to gravity of the ink-bottle and the vertical component of the pull of the bearing wire, which is in a line about a millimetre or two farther from the paper than that in which the centre of gravity moves. Thus is ensured, notwithstanding small inequalities on the paper, a pressure of the pen on the paper very approximately constant and as small as is desired.

Hour marks are made on the curve by a small horizontal movement of the ink-bottle's lateral guides, made once an hour; a somewhat greater movement, giving a deeper notch, serves to mark the noon of every day.

The machine may be turned so rapidly as to run off a year's tides for any port in about four hours.

Each crank should carry an adjustable counterpoise, to be

* See Report for 1876 of the British Association's Tidal Committee.

Tide-predicting Machine.

adjusted so that when the crank is not vertical the pulls of the approximately vertical portions of wire acting on it through the pulley which it carries shall, as exactly as may be, balance on the axis of the shaft, and the motion of the shaft should be resisted by a slight weight hanging on a thread wrapped once round it and attached at its other end to a fixed point. This part of the design, planned to secure against "lost time" or "back lash" in the gearings, and to preserve uniformity of pressure between teeth and teeth, teeth and screws, and ends of axles and "end-plates," was not carried out in the British Association machine.

II. Machine for the Solution of Simultaneous Linear Equations*.

Equation-Solver.

Let $B_1, B_2, \ldots B_n$ be n bodies each supported on a fixed axis (in practice each is to be supported on knife-edges like the beam of a balance).

Let $P_{11}, P_{21}, P_{31}, \ldots P_{n1}$ be n pulleys each pivoted on B_1;
$P_{12}, P_{22}, P_{32}, \ldots P_{n2}$ " " B_2;
$P_{13}, P_{23}, P_{33}, \ldots P_{n3}$ " " B_3;
....

" $C_1, C_2, C_3, \ldots C_n$, be n cords passing over the pulleys;
" $D_1, P_{11}, P_{12}, P_{13}, \ldots P_{1n}, E_1$, be the course of C_1;
" $D_2, P_{21}, P_{22}, P_{23}, \ldots P_{2n}, E_2$, " " C_2;
....

" $D_1, E_1, D_2, E_2, \ldots D_n, E_n$, be fixed points;

" $l_1, l_2, l_3, \ldots l_n$ be the lengths of the cords between D_1, E_1, and $D_2, E_2, \ldots$ and D_n, E_n, along the courses stated above, when $B_1, B_2, \ldots B_n$, are in particular positions which will be called their zero positions;

" $l_1+e_1, l_2+e_2, \ldots l_n+e_n$ be their lengths between the same fixed points, when $B_1, B_2, \ldots B_n$ are turned through angles $x_1, x_2, \ldots x_n$ from their zero positions;

$$
\begin{array}{l}
(11), (12), (13), \ldots (1n),\\
(21), (22), (23), \ldots (2n),\\
(31), (32), (33), \ldots (3n),\\
\cdots\cdots\cdots\cdots\cdots\cdots
\end{array}
$$

* Sir W. Thomson, *Proceedings of the Royal Society*, Vol. XXVIII., 1878.

Equation Solver.

quantities such that

$$
\left.
\begin{array}{l}
(11)\, x_1 + (12)\, x_2 + \ldots + (1n)\, x_n = e_1\\
(21)\, x_1 + (22)\, x_2 + \ldots + (2n)\, x_n = e_2\\
(31)\, x_1 + (32)\, x_2 + \ldots + (3n)\, x_n = e_3\\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
(n1)\, x_1 + (n2)\, x_2 + \ldots + (nn)\, x_n = e_n
\end{array}
\right\} \ldots\ldots\ldots\ldots (\mathrm{I}).
$$

We shall suppose $x_1, x_2, \ldots x_n$ to be each so small that (11), (12),...(21), etc., do not vary sensibly from the values which they have when $x_1, x_2, \ldots x_n$, are each infinitely small. In practice it will be convenient to so place the axes of $B_1, B_2, \ldots B_n$, and the mountings of the pulleys on $B_1, B_2, \ldots B_n$, and the fixed points D_1, E_1, D_2, etc., that when $x_1, x_2, \ldots x_n$ are infinitely small, the straight parts of each cord and the lines of infinitesimal motion of the centres of the pulleys round which it passes shall be all parallel. Then $\frac{1}{2}(11), \frac{1}{2}(21), \ldots \frac{1}{2}(n1)$ will be simply equal to the distances of the centres of the pulleys $P_{11}, P_{21}, \ldots P_{n1}$, from the axis of B_1; $\frac{1}{2}(12), \frac{1}{2}(22) \ldots \frac{1}{2}(n2)$ the distances of $P_{12}, P_{22}, \ldots P_{n2}$ from the axis of B_2; and so on.

In practice the mountings of the pulleys are to be adjustable by proper geometrical slides, to allow any prescribed positive or negative value to be given to each of the quantities (11), (12), ... (21), etc.

Suppose this to be done, and each of the bodies $B_1, B_2, \ldots B_n$ to be placed in its zero position and held there. Attach now the cords firmly to the fixed points $D_1, D_2, \ldots D_n$ respectively; and, passing them round their proper pulleys, bring them to the other fixed points $E_1, E_2, \ldots E_n$, and pass them through infinitely small smooth rings fixed at these points. Now hold the bodies $B_1, B_2, \ldots$ each fixed, and (in practice by weights hung on their ends, outside $E_1, E_2, \ldots E_n$) pull the cords through $E_1, E_2, \ldots E_n$ with any given tensions* $T_1, T_2, \ldots T_n$. Let $G_1, G_2, \ldots G_n$ be moments round the fixed axes of $B_1, B_2, \ldots B_n$ of the forces required to hold the bodies fixed when acted on by the cords thus

* The idea of force here first introduced is not essential, indeed is not technically admissible to the purely kinematic and algebraic part of the subject proposed. But it is not merely an ideal kinematic construction of the algebraic problem that is intended; and the design of a kinematic machine, for success in practice, essentially involves dynamical considerations. In the present case some of the most important of the purely algebraic questions concerned are very interestingly illustrated by these dynamical considerations.

Equation-Solver.

stretched. The principle of "virtual velocities," just as it came from Lagrange (or the principle of "work"), gives immediately, in virtue of (I),

$$\left.\begin{array}{l} G_1 = (11)\,T_1 + (21)\,T_2 + \dots + (n1)\,T_n \\ G_2 = (12)\,T_1 + (22)\,T_2 + \dots + (n2)\,T_n \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots \\ G_n = (1n)\,T_1 + (2n)\,T_2 + \dots + (nn)\,T_n \end{array}\right\} \dots\dots\dots\dots \text{(II)}.$$

Apply and keep applied to each of the bodies, B_1, B_2, ... B_n (in practice by the weights of the pulleys, and by counter-pulling springs), such forces as shall have for their moments the values G_1, G_2 ... G_n, calculated from equations (II) with whatever values seem desirable for the tensions T_1, T_2, ... T_3. (In practice, the straight parts of the cords are to be approximately vertical, and the bodies B_1, B_2, are to be each balanced on its axis when the pulleys belonging to it are removed, and it is advisable to make the tensions each equal to half the weight of one of the pulleys with its adjustable frame.) The machine is now ready for use. To use it, pull the cords simultaneously or successively till lengths equal to e_1, e_2, ... e_n are passed through the rings E_1, E_2, ... E_n, respectively.

The *pulls* required to do this may be positive or negative; in practice, they will be infinitesimal downward or upward pressures applied by hand to the stretching weights which remain permanently hanging on the cords.

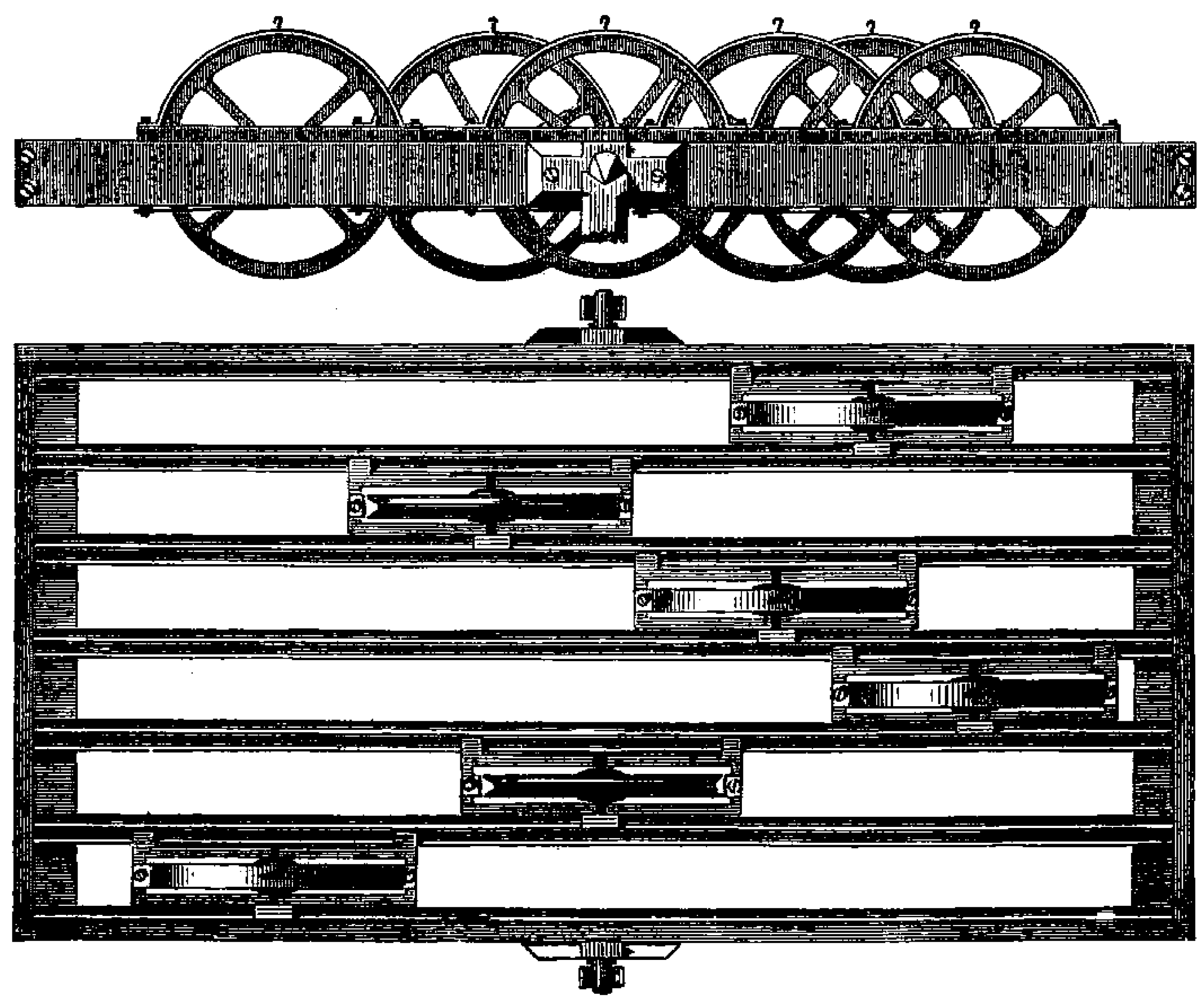
Observe the angles through which the bodies B_1, B_2, ... B_n are turned by this given movement of the cords. These angles are the required values of the unknown x_1, x_2, ... x_n, satisfying the simultaneous equations (I).

The actual construction of a practically useful machine for calculating as many as eight or ten or more of unknowns from the same number of linear equations does not promise to be either difficult or over-elaborate. A fair approximation having been found by a first application of the machine, a very moderate amount of straightforward arithmetical work (aided very advantageously by Crelle's multiplication tables) suffices to calculate the residual errors, and allow the machines (with the setting of the pulleys unchanged) to be re-applied to calculate the corrections (which may be treated decimally, for convenience): thus, 100 times the amount of the correction on each of the original unknowns may be made the new unknowns, if the magnitudes thus falling to be dealt with are convenient for the machine. There is, of course, no limit to the accuracy thus obtainable by successive approximations. The exceeding easiness of each application of the machine promises well for its real usefulness, whether for cases in which a single application suffices, or for others in which the requisite accuracy is reached after two, three, or more, of successive approximations.

Equation-Solver.

The accompanying drawings represent a machine for finding six* unknowns from six equations. Fig. 1 represents in elevation and plan one of the six bodies B_1, B_2, etc. Fig. 2 shows in elevation and plan one of the thirty-six pulleys P, with its cradle on geometrical slide (§ 198). Fig. 3 shows in front-elevation the general disposition of the instrument.

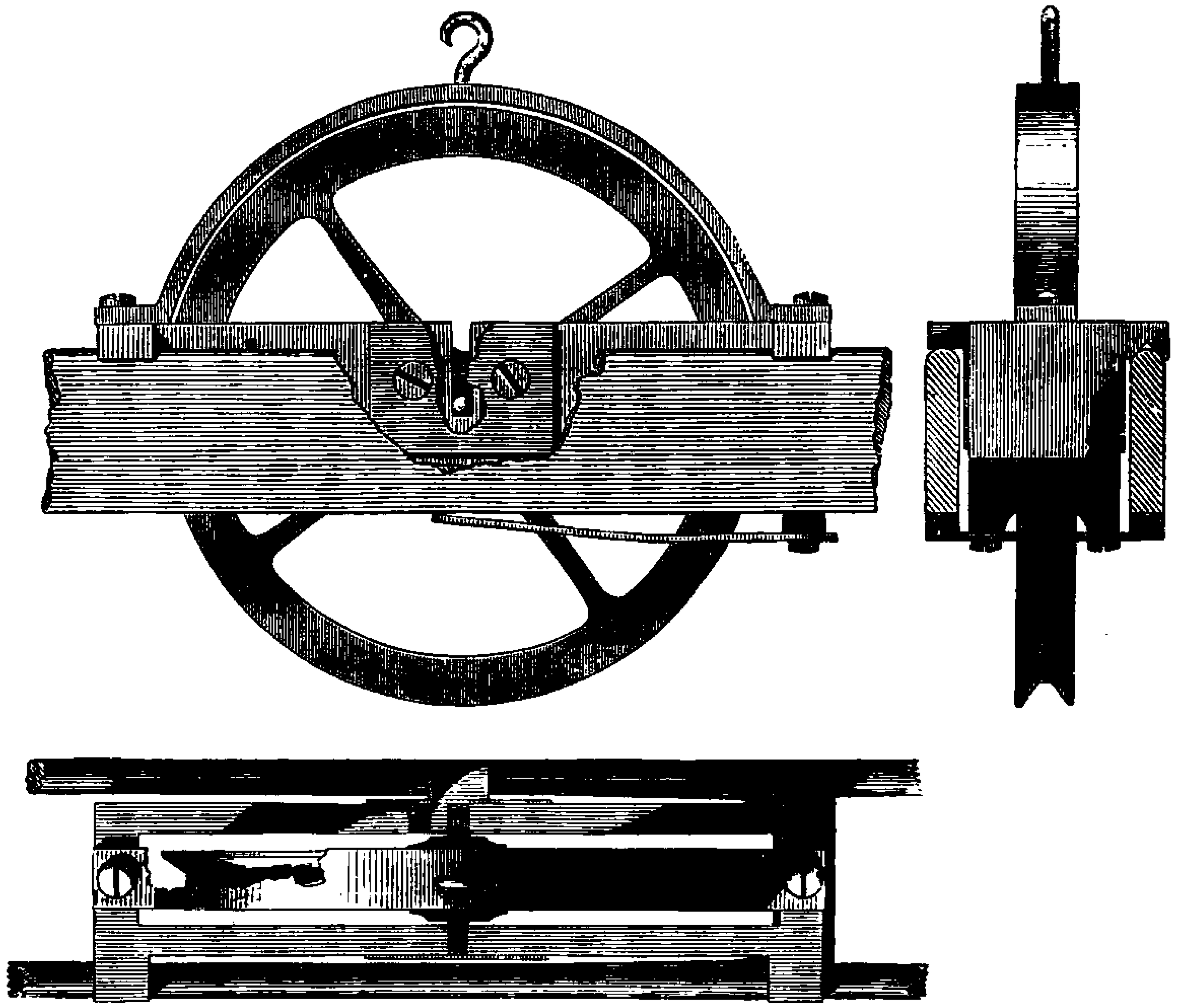
FIG. 1. One of the six moveable bodies, B.



* This number has been chosen for the first practical machine to be constructed, because a chief application of the machine may be to the calculation of the corrections on approximate values already found of the six elements of the orbit of a comet or asteroid.

Equation-Solver.

FIG. 2. One of the thirty-six pulleys, *P*, with its sliding cradle. Full Size.



Front elevation.

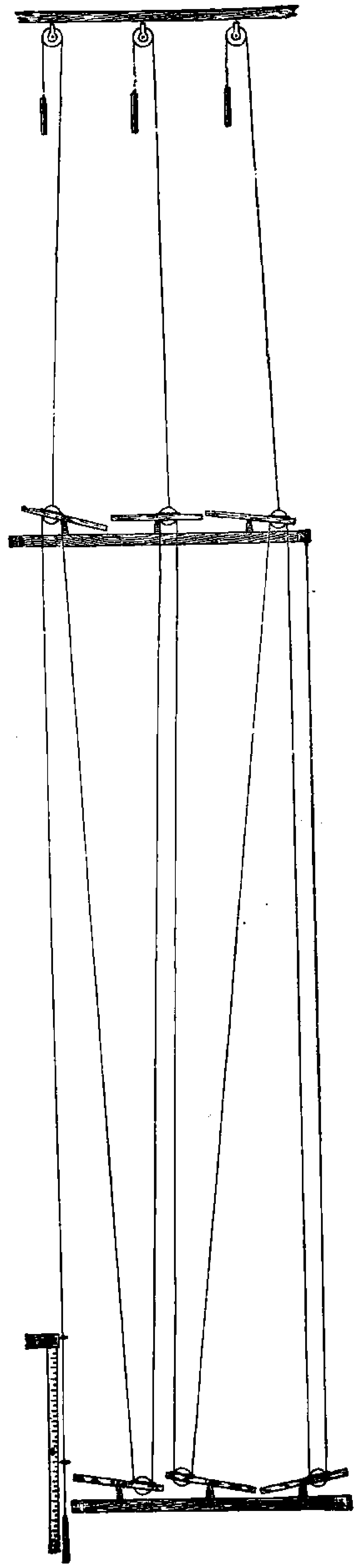
Side elevation.

Plan.

In Fig. 3 only one of the six cords, and the six pulleys over which it passes, is shown, not any of the other thirty. The three pulleys seen at the top of the sketch are three out of eighteen pivoted on immoveable bearings above the machine, for the purpose of counterpoising the weights of the pulleys *P*, with their sliding cradles. Each of the counterpoises is equal to twice the weight of one of the pulleys *P* with its sliding cradle. Thus if the bodies *B* are balanced on their knife-edges with each sliding cradle in its central position, they remain balanced when one or all of the cradles are shifted to either side; and the tension of each of the thirty-six essential cords is exactly equal to half the weight of one of the pulleys with its adjustable frame, as specified above (the deviations from exact verticality of all the free portions of the thirty-six essential cords and the eighteen counterpoising cords being neglected).

Equation-Solver.

FIG. 3. General disposition of machine.



III. An Integrating Machine having a New Kinematic Principle*.

Disk-Globe-, and Cylinder-Integrating Machine.

The kinematic principle for integrating *ydx*, which is used in the instruments well known as Morin's Dynamometer† and Sang's Planimeter‡, admirable as it is in many respects, involves one element of imperfection which cannot but prevent our contemplating it with full satisfaction. This imperfection consists in the sliding action which the edge wheel or roller is required to take in conjunction with its rolling action, which alone is desirable for exact communication of motion from the disk or cone to the edge roller.

The very ingenious, simple, and practically useful instrument well known as Amsler's Polar Planimeter, although different in its main features of principle and mode of action from the instruments just referred to, ranks along with them in involving the like imperfection of requiring to have a sidewise sliding action of its edge rolling wheel, besides the desirable rolling action on the surface which imparts to it its revolving motion—a surface

* Professor James Thomson, *Proceedings of the Royal Society*, Vol. XXIV., 1876, p. 262.

† Instruments of this kind, and any others for measuring mechanical work, may better in future be called Ergometers than Dynamometers. The name "dynamometer" has been and continues to be in common use for signifying a spring instrument for measuring *force;* but an instrument for measuring *work*, being distinct in its nature and object, ought to have a different and more suitable designation. The name "dynamometer," besides, appears to be badly formed from the Greek; and for designating an instrument for *measurement of force*, I would suggest that the name may with advantage be changed to *dynamimeter*. In respect to the mode of forming words in such cases, reference may be made to Curtius's Grammar, Dr Smith's English edition, § 354, p. 220.—J. T., 26th February, 1876.

‡ Sang's Planimeter is very clearly described and figured in a paper by its inventor, in the Transactions of the Royal Scottish Society of Arts, Vol. IV. January 12, 1852.

which in this case is not a disk or cone, but is the surface of the paper, or any other plane face, on which the map or other plane diagram to be evaluated in area is drawn.

Disk-, Globe-, and Cylinder-Integrating Machine.

Professor J. Clerk Maxwell, having seen Sang's Planimeter in the Great Exhibition of 1851, and having become convinced that the combination of slipping and rolling was a drawback on the perfection of the instrument, began to search for some arrangement by which the motion should be that of perfect rolling in every action of the instrument, corresponding to that of combined slipping and rolling in previous instruments. He succeeded in devising a new form of planimeter or integrating machine with a quite new and very beautiful principle of kinematic action depending on the mutual rolling of two equal spheres, each on the other. He described this in a paper submitted to the Royal Scottish Society of Arts in January 1855, which is published in Vol. IV. of the Transactions of that Society. In that paper he also offered a suggestion, which appears to be both interesting and important, proposing the attainment of the desired conditions of action by the mutual rolling of a cone and cylinder with their axes at right angles.

The idea of using pure rolling instead of combined rolling and slipping was communicated to me by Prof. Maxwell, when I had the pleasure of learning from himself some particulars as to the nature of his contrivance. Afterwards (some time between the years 1861 and 1864), while endeavouring to contrive means for the attainment in meteorological observatories of certain integrations in respect to the motions of the wind, and also in endeavouring to devise a planimeter more satisfactory in principle than either Sang's or Amsler's planimeter (even though, on grounds of practical simplicity and convenience, unlikely to turn out preferable to Amsler's in ordinary cases of taking areas from maps or other diagrams, but something that I hoped might possibly be attainable which, while having the merit of working by pure rolling contact, might be simpler than the instrument of Prof. Maxwell and preferable to it in mechanism), I succeeded in devising for the desired object a new kinematic method, which has ever since appeared to me likely sometime to prove valuable when occasion for its employment might be found. Now, within the last few days, this principle, on being suggested to my brother as perhaps capable of being usefully employed towards the development of tide-calculating machines

Disk-, Globe-, and Cylinder-Integrator.

which he had been devising, has been found by him to be capable of being introduced and combined in several ways to produce important results. On his advice, therefore, I now offer to the Royal Society a brief description of the new principle as devised by me.

The new principle consists primarily in the transmission of motion from a disk or cone to a cylinder by the intervention of a loose ball, which presses by its gravity on the disk and cylinder, or on the cone and cylinder, as the case may be, the pressure being sufficient to give the necessary frictional coherence at each point of rolling contact; and the axis of the disk or cone and that of the cylinder being both held fixed in position by bearings in stationary framework, and the arrangement of these axes being such that when the disk or the cone and the cylinder are kept steady, or, in other words, without rotation on their axes, the ball can roll along them in contact with both, so that the point of rolling contact between the ball and the cylinder shall traverse a straight line on the cylindric surface parallel necessarily to the axis of the cylinder—and so that, in the case of a disk being used, the point of rolling contact of the ball with the disk shall traverse a straight line passing through the centre of the disk—or that, in case of a cone being used, the line of rolling contact of the ball on the cone shall traverse a straight line on the conical surface, directed necessarily towards the vertex of the cone. It will thus readily be seen that, whether the cylinder and the disk or cone be at rest or revolving on their axes, the two lines of rolling contact of the ball, one on the cylindric surface and the other on the disk or cone, when both considered as lines traced out in space fixed relatively to the framing of the whole instrument, will be two parallel straight lines, and that the line of motion of the ball's centre will be straight and parallel to them. For facilitating explanations, the motion of the centre of the ball along its path parallel to the axis of the cylinder may be called the ball's longitudinal motion.

Now for the integration of ydx: the distance of the point of contact of the ball with the disk or cone from the centre of the disk or vertex of the cone in the ball's longitudinal motion is to represent y, while the angular space turned by the disk or cone from any initial position represents x; and then the angular space turned by the cylinder will, when multiplied by a suitable

Disk-, Globe-, and Cylinder-Integrator.

constant numerical coefficient, express the integral in terms of any required unit for its evaluation.

The longitudinal motion may be imparted to the ball by having the framing of the whole instrument so placed that the lines of longitudinal motion of the two points of contact and of the ball's centre, which are three straight lines mutually parallel, shall be inclined to the horizontal sufficiently to make the ball tend decidedly to descend along the line of its longitudinal motion, and then regulating its motion by an abutting controller, which may have at its point of contact, where it presses on the ball, a plane face perpendicular to the line of the ball's motion. Otherwise the longitudinal motion may, for some cases, preferably be imparted to the ball by having the direction of that motion horizontal, and having two controlling flat faces acting in close contact without tightness at opposite extremities of the ball's diameter, which at any moment is in the line of the ball's motion or is parallel to the axis of the cylinder.

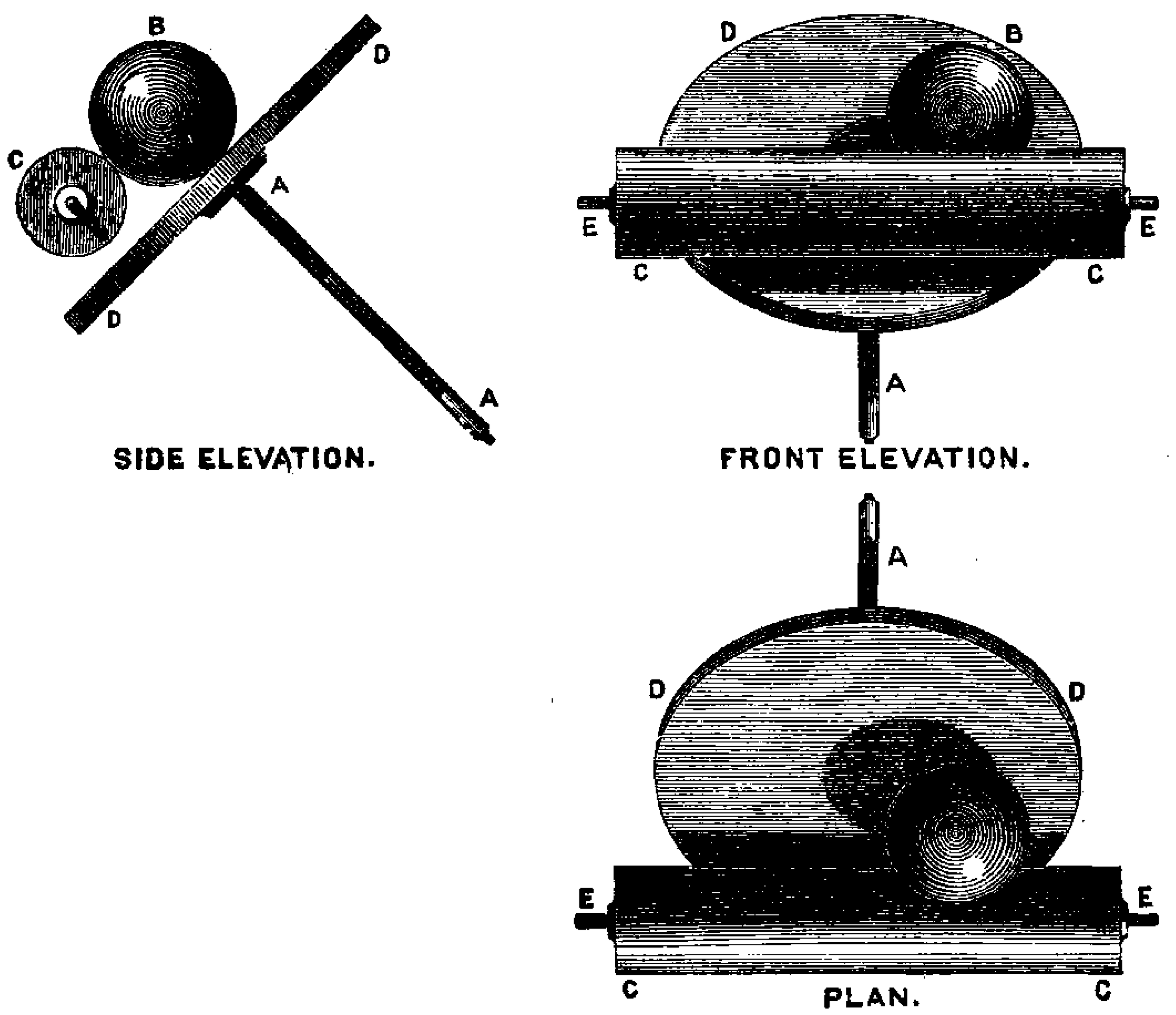
It is worthy of notice that, in the case of the disk-, ball-, and cylinder-integrator, no theoretical nor important practical fault in the action of the instrument would be involved in any deficiency of perfect exactitude in the practical accomplishment of the desired condition that the line of motion of the ball's point of contact with the disk should pass through the centre of the disk. The reason of this will be obvious enough on a little consideration.

The plane of the disk may suitably be placed inclined to the horizontal at some such angle as 45°; and the accompanying sketch, together with the model, which will be submitted to the Society by my brother, will aid towards the clear understanding of the explanations which have been given.

My brother has pointed out to me that an additional operation, important for some purposes, may be effected by arranging that the machine shall give a continuous record of the growth of the integral by introducing additional mechanisms suitable for continually describing a curve such that for each point of it the abscissa shall represent the value of x, and the ordinate shall represent the integral attained from $x=0$ forward to that value of x. This, he has pointed out, may be effected in practice by having a cylinder axised on the axis of the disk, a roll of paper covering this cylinder's surface, and a straight bar situated parallel to this cylinder's axis and resting with enough of pres-

Disk-, Globe-, and Cylinder-Integrator.

sure on the surface of the primary registering or *the indicating* cylinder (the one, namely, which is actuated by its contact with the ball) to make it have sufficient frictional coherence with that



surface, and by having this bar made to carry a pencil or other tracing point which will mark the desired curve on the secondary registering or *the recording* cylinder. As, from the nature of the apparatus, the axis of the disk and of the secondary registering or recording cylinder ought to be steeply inclined to the horizontal, and as, therefore, this bar, carrying the pencil, would have the line of its length and of its motion alike steeply inclined with that axis, it seems that, to carry out this idea, it may be advisable to have a thread attached to the bar and extending off in the line of the bar to a pulley, passing over the pulley, and having suspended at its other end a weight which will be just sufficient to counteract the tendency of the rod, in virtue of gravity, to glide down along the line of its own slope, so as to leave it perfectly free to be moved up or down by the frictional coherence between itself and the moving surface of the indicating cylinder worked directly by the ball.

IV. An Instrument for calculating $\left(\int \phi(x)\,\psi(x)\,dx\right)$, the Integral of the Product of two given Functions*.

Machine to calculate Integral of Product of two Functions.

In consequence of the recent meeting of the British Association at Bristol, I resumed an attempt to find an instrument which should supersede the heavy arithmetical labour of calculating the integrals required to analyze a function into its simple harmonic constituents according to the method of Fourier. During many years previously it had appeared to me that the object ought to be accomplished by some simple mechanical means; but it was not until recently that I succeeded in devising an instrument approaching sufficiently to simplicity to promise practically useful results. Having arrived at this stage, I described my proposed machine a few days ago to my brother Professor James Thomson, and he described to me in return a kind of mechanical integrator which had occurred to him many years ago, but of which he had never published any description. I instantly saw that it gave me a much simpler means of attaining my special object than anything I had been able to think of previously. An account of his integrator is communicated to the Royal Society along with the present paper.

To calculate $\int \phi(x)\,\psi(x)\,dx$, the rotating disk is to be displaced from a zero or initial position through an angle equal to

$$\int_0^x \phi(x)\,dx,$$

while the rolling globe is moved so as always to be at a distance from its zero position equal to $\psi(x)$. This being done, the cylinder obviously turns through an angle equal to $\int_0^x \phi(x)\,\psi(x)\,dx$, and thus solves the problem.

One way of giving the required motions to the rotating disk and rolling globe is as follows:—

* Sir W. Thomson, *Proceedings of the Royal Society*, Vol. XXIV., 1876, p. 266.

Machine to calculate Integral of Product of two Functions.

On two pieces of paper draw the curves

$$y=\int_0^x \phi(x)\,dx, \text{ and } y=\psi(x).$$

Attach these pieces of paper to the circumference of two circular cylinders, or to different parts of the circumference of one cylinder, with the axis of x in each in the direction perpendicular to the axis of the cylinder. Let the two cylinders (if there are two) be geared together so as that their circumferences shall move with equal velocities. Attached to the framework let there be, close to the circumference of each cylinder, a slide or guide-rod to guide a moveable point, moved by the hand of an operator, so as always to touch the curve on the surface of the cylinder, while the two cylinders are moved round.

Two operators will be required, as one operator could not move the two points so as to fulfil this condition—at all events unless the motion were very slow. One of these points, by proper mechanism, gives an angular motion to the rotating disk equal to its own linear motion, the other gives a linear motion equal to its own to the centre of the rolling globe.

The machine thus described is immediately applicable to calculate the values H_1, H_2, H_3, etc. of the harmonic constituents of a function $\psi(x)$ in the splendid generalization of Fourier's simple harmonic analysis, which he initiated himself in his solutions for the conduction of heat in the sphere and the cylinder, and which was worked out so ably and beautifully by Poisson*, and by Sturm and Liouville in their memorable papers on this subject published in the first volume of Liouville's *Journal des Mathématiques*. Thus if

$$\psi(x)=H_1\phi_1(x)+H_2\phi_2(x)+H_3\phi_3(x)+\text{etc.}$$

be the expression for an arbitrary function ψx, in terms of the generalized harmonic functions $\phi_1(x)$, $\phi_2(x)$, $\phi_3(x)$, etc., these functions being such that

$$\int_0^l \phi_1(x)\,\phi_2(x)\,dx=0,\ \int_0^l \phi_1(x)\,\phi_3(x)\,dx=0,\ \int_0^l \phi_2(x)\,\phi_3(x)=0,\ \text{etc.},$$

* His general demonstration of the reality of the roots of transcendental equations essential to this analysis (an exceedingly important step in advance from Fourier's position), which he first gave in the *Bulletin de la Société Philomathique* for 1828, is reproduced in his *Théorie Mathématique de la Chaleur*, § 90.

Machine to calculate Integral of Product of two Functions.

we have

$$H_1=\frac{\int_0^l \phi_1(x)\,\psi(x)\,dx}{\int_0^l \{\phi_1(x)\}^2\,dx},$$

$$H_2=\frac{\int_0^l \phi_2(x)\,\psi(x)\,dx}{\int_0^l \{\phi_2(x)\}^2\,dx},$$

etc.

In the physical applications of this theory the integrals which constitute the denominators of the formulas for H_1, H_2, etc. are always to be evaluated in finite terms by an extension of Fourier's formula for the $\int_0^X xu_i^2\,dx$ of his problem of the cylinder* made by Sturm in equation (10), § IV. of his *Mémoire sur une Classe d'Équations à différences partielles* in Liouville's *Journal*, Vol. I. (1836). The integrals in the numerators are calculated with great ease by aid of the machine worked in the manner described above

The great practical use of this machine will be to perform the simple harmonic Fourier-analysis for tidal, meteorological, and perhaps even astronomical, observations. It is the case in which

$$\phi(x)=\frac{\sin}{\cos}(nx);$$

and the integration is performed through a range equal to $\frac{2i\pi}{n}$ (i any integer) that gives this application. In this case the addition of a simple crank mechanism, to give a simple harmonic angular motion to the rotating disk in the proper period $\frac{2\pi}{n}$, when the cylinder bearing the curve $y=\psi(x)$ moves uniformly, supersedes the necessity for a cylinder with the curve $y=\phi(x)$ traced on it, and an operator keeping a point always on this curve in the manner described above. Thus one operator will be enough to carry on the process; and I believe that in the application of it to the tidal harmonic analysis he will be able in an

* Fourier's *Théorie Analytique de la Chaleur*, § 319, p. 391 (Paris, 1822).

Machine to calculate Integral of Product of two Functions.

hour or two to find by aid of the machine any one of the simple harmonic elements of a year's tides recorded in curves in the usual manner by an ordinary tide-gauge—a result which hitherto has required not less than twenty hours of calculation by skilled arithmeticians. I believe this instrument will be of great value also in determining the diurnal, semi-diurnal, ter-diurnal, and quarter-diurnal constituents of the daily variations of temperature, barometric pressure, east and west components of the velocity of the wind, north and south components of the same; also of the three components of the terrestrial magnetic force; also of the electric potential of the air at the point where the stream of water breaks into drops in atmospheric electrometers, and of other subjects of ordinary meteorological or magnetic observations; also to estimate precisely the variation of terrestrial magnetism in the eleven years sun-spot period, and of sun-spots themselves in this period; also to disprove (or prove, as the case may be) supposed relations between sun-spots and planetary positions and conjunctions; also to investigate lunar influence on the height of the barometer, and on the components of the terrestrial magnetic force, and to find if lunar influence is sensible on any other meteorological phenomena—and if so, to determine precisely its character and amount.

From the description given above it will be seen that the mechanism required for the instrument is exceedingly simple and easy. Its accuracy will depend essentially on the accuracy of the circular cylinder, of the globe, and of the plane of the rotating disk used in it. For each of the three surfaces a much less elaborate application of the method of scraping than that by which Sir Joseph Whitworth has given a true plane with such marvellous accuracy will no doubt suffice for the practical requirements of the instrument now proposed.

V. Mechanical Integration of Linear Differential Equations of the Second Order with Variable Coefficients*.

Mechanical Integration of Linear Differential Equations of Second Order.

Every linear differential equation of the second order may, as is known, be reduced to the form

$$\frac{d}{dx}\left(\frac{1}{P}\frac{du}{dx}\right) = u \quad\quad (1),$$

where P is any given function of x.

On account of the great importance of this equation in mathematical physics (vibrations of a non-uniform stretched cord, of a hanging chain, of water in a canal of non-uniform breadth and depth, of air in a pipe of non-uniform sectional area, conduction of heat along a bar of non-uniform section or non-uniform conductivity, Laplace's differential equation of the tides, etc. etc.), I have long endeavoured to obtain a means of facilitating its practical solution.

Methods of calculation such as those used by Laplace himself are exceedingly valuable, but are very laborious, too laborious unless a serious object is to be attained by calculating out results with minute accuracy. A ready means of obtaining approximate results which shall show the general character of the solutions, such as those so well worked out by Sturm†, has always seemed to me a desideratum. Therefore I have made many attempts to plan a mechanical integrator which should give solutions by successive approximations. This is clearly done now, when we have the instrument for calculating $\int\phi(x)\,\psi(x)\,dx$, founded on my brother's disk-, globe-, and cylinder-integrator, and described in a previous communication to the Royal Society; for it is easily proved‡ that if

* Sir W. Thomson, *Proceedings of the Royal Society*, Vol. XXIV., 1876, p. 269.

† *Mémoire sur les équations différentielles linéaires du second ordre*, Liouville's *Journal*, Vol. I. 1836.

‡ Cambridge Senate-House Examination, Thursday afternoon, January 22nd, 1874.

Mechanical Integration of Linear Differential Equations of Second Order.

$$\left.\begin{aligned} u_2 &= \int_0^x P\left(C - \int_0^x u_1\,dx\right)dx, \\ u_3 &= \int_0^x P\left(C - \int_0^x u_2\,dx\right)dx, \\ &\text{etc.,} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots (2)$$

where u_1 is any function of x, to begin with, as for example $u_1 = x$; then u_2, u_3, etc. are successive approximations converging to that one of the solutions of (1) which vanishes when $x = 0$.

Now let my brother's integrator be applied to find $C - \int_0^x u_1\,dx$, and let its result feed, as it were, continuously a second machine, which shall find the integral of the product of its result into $P\,dx$. The second machine will give out continuously the value of u_2. Use again the same process with u_2 instead of u_1, and then u_3, and so on.

After thus altering, as it were, u_1 into u_2 by passing it through the machine, then u_2 into u_3 by a second passage through the machine, and so on, the thing will, as it were, become refined into a solution which will be more and more nearly rigorously correct the oftener we pass it through the machine. If u_{i+1} does not sensibly differ from u_i, then each is sensibly a solution.

So far I had gone and was satisfied, feeling I had done what I wished to do for many years. But then came a pleasing surprise. Compel agreement between the function fed into the double machine and that given out by it. This is to be done by establishing a connexion which shall cause the motion of the centre of the globe of the first integrator of the double machine to be the same as that of the surface of the second integrator's cylinder. The motion of each will thus be necessarily a solution of (1). Thus I was led to a conclusion which was quite unexpected; and it seems to me very remarkable that the general differential equation of the second order with variable coefficients may be rigorously, continuously, and in a single process solved by a machine.

Take up the whole matter *ab initio:* here it is. Take two of my brother's disk-, globe-, and cylinder-integrators, and connect the fork which guides the motion of the globe of each of the integrators, by proper mechanical means, with the circumference of the other integrator's cylinder. Then move one integrator's disk through an angle $= x$, and simultaneously move the other

Mechanical Integration of Linear Differential Equations of Second Order.

integrator's disk through an angle always $= \int_0^x P\,dx$, a given function of x. The circumference of the second integrator's cylinder and the centre of the first integrator's globe move each of them through a space which satisfies the differential equation (1).

To prove this, let at any time g_1, g_2 be the displacements of the centres of the two globes from the axial lines of the disks; and let dx, $P\,dx$ be infinitesimal angles turned through by the two disks. The infinitesimal motions produced in the circumferences of two cylinders will be

$$g_1\,dx \quad \text{and} \quad g_2 P\,dx.$$

But the connexions pull the second and first globes through spaces respectively equal to those moved through by the circumferences of the first and second cylinders. Hence

$$g_1\,dx = dg_2, \quad \text{and} \quad g_2 P\,dx = dg_1;$$

and eliminating g_2,

$$\frac{d}{dx}\left(\frac{1}{P}\frac{dg_1}{dx}\right) = g_1,$$

which shows that g_1 put for u satisfies the differential equation (1).

The machine gives the complete integral of the equation with its two arbitrary constants. For, for any particular value of x, give arbitrary values G_1, G_2. [That is to say mechanically; disconnect the forks from the cylinders, shift the forks till the globes' centres are at distances G_1, G_2 from the axial lines, then connect, and move the machine.]

We have for this value of x,

$$g_1 = G_1, \quad \text{and} \quad \frac{dg_1}{dx} = G_2 P;$$

that is, we secure arbitrary values for g_1 and $\frac{dg_1}{dx}$ by the arbitrariness of the two initial positions G_1, G_2 of the globes.

VI. Mechanical Integration of the general Linear Differential Equation of any Order with Variable Coefficients*.

Mechanical Integration of General Linear Differential Equation of Any Order

Take any number i of my brother's disk-, globe-, and cylinder-integrators, and make an integrating chain of them thus:—Connect the cylinder of the first so as to give a motion equal to its own† to the fork of the second. Similarly connect the cylinder of the second with the fork of the third, and so on. Let g_1, g_2, g_3, up to g_i, be the positions‡ of the globes at any time. Let infinitesimal motions $P_1 dx$, $P_2 dx$, $P_3 dx$, ... be given simultaneously to all the disks (dx denoting an infinitesimal motion of some part of the mechanism whose displacement it is convenient to take as independent variable). The motions ($d\kappa_1$, $d\kappa_2$, ... $d\kappa_i$) of the cylinders thus produced are

$$d\kappa_1 = g_1 P \, dx, \;\; d\kappa_2 = g_2 P_2 dx, \; \ldots \; d\kappa_i = g_i P_i dx \;\ldots\ldots(1).$$

But, by the connexions between the cylinders and forks which move the globes, $d\kappa_1 = dg_2$, $d\kappa_2 = dg_3$, ... $d\kappa_{i-1} = dg_i$; and therefore

$$\left.\begin{array}{ll} & dg_2 = g_1 P_1 dx, \; dg_3 = g_2 P_2 dx, \; \ldots \; dg_i = g_{i-1} P_{i-1} dx \\ \text{and} & d\kappa_1 = g_1 P_1 dx, \; d\kappa_2 = g_2 P_2 dx, \; \ldots \; d\kappa_i = g_i P_i \, dx. \end{array}\right\} \ldots(2).$$

Hence

$$g_1 = \frac{1}{P_1}\frac{d}{dx}\frac{1}{P_2}\frac{d}{dx} \cdots \frac{1}{P_{i-1}}\frac{d}{dx}\frac{1}{P_i}\frac{d\kappa_i}{dx} \;\ldots\ldots\ldots\ldots(3).$$

Suppose, now, for the moment that we couple the last cylinder with the first fork, so that their motions shall be equal—that is to say, $\kappa_i = g_1$. Then, putting u to denote the common value of these variables, we have

$$u = \frac{1}{P_1}\frac{d}{dx}\frac{1}{P_2}\frac{d}{dx} \cdots \frac{1}{P_{i-1}}\frac{d}{dx}\frac{1}{P_i}\frac{du}{dx} \;\ldots\ldots\ldots\ldots(4).$$

* Sir W. Thomson, *Proceedings of the Royal Society*, Vol. xxiv., 1876, p. 271.

† For brevity, the motion of the circumference of the cylinder is called the cylinder's motion.

‡ For brevity, the term "position" of any one of the globes is used to denote its distance, positive or negative, from the axial line of the rotating disk on which it presses.

Mechanical Integration of General Linear Differential Equation of Any Order.

Thus an endless chain or cycle of integrators with disks moved as specified above gives to each fork a motion fulfilling a differential equation, which for the case of the fork of the ith integrator is equation (4). The differential equations of the displacements of the second fork, third fork, ... $(i-1)$th fork may of course be written out by inspection from equation (4).

This seems to me an exceedingly interesting result; but though P_1, P_2, P_3, ... P_i may be any given functions whatever of x, the differential equations so solved by the simple cycle of integrators cannot, except for the case of $i = 2$, be regarded as the general linear equation of the order i, because, so far as I know, it has not been proved for any value of i greater than 2 that the general equation, which in its usual form is as follows,

$$Q_1 \frac{d^i u}{dx^i} + Q_2 \frac{d^{i-1} u}{dx^{i-1}} + \ldots Q_i \frac{du}{dx} - u = 0 \;\ldots\ldots\ldots\ldots(5),$$

can be reduced to the form (4). The general equation of the form (5), where Q_1, Q_2, ... Q_i are any given forms of x, may be integrated mechanically by a chain of connected integrators thus:—

First take an open chain of i simple integrators as described above, and simplify the movement by taking

$$P_1 = P_2 = P_3 = \ldots = P_2 = 1,$$

so that the speeds of all the disks are equal, and dx denotes an infinitesimal angular motion of each. Then by (2) we have

$$g_i = \frac{d\kappa_i}{dx}, \;\; g_{i-1} = \frac{d^2\kappa_i}{dx^2}, \; \ldots, \;\; g_2 = \frac{d^{i-1}\kappa_i}{dx^{i-1}}, \;\; g_1 = \frac{d^i\kappa_i}{dx^i} \;\ldots\ldots(6).$$

Now establish connexions between the i forks and the ith cylinder, so that

$$Q_1 g_1 + Q_2 g_2 + \ldots + Q_{i-1} g_{i-1} + Q_i g_i = \kappa_i \;\ldots\ldots\ldots(7).$$

Putting in this for g_1, g_2, etc. their values by (6), we find an equation the same as (5), except that κ_i appears instead of u. Hence the mechanism, when moved so as to fulfil the condition (7), performs by the motion of its last cylinder an integration of the equation (5). This mechanical solution is complete; for we may give arbitrarily any initial values to κ_i, g_i, g_{i-1}, ... g_3, g_2; that is to say, to

$$u, \; \frac{du}{dx}, \; \frac{d^2u}{dx^2}, \; \ldots \; \frac{d^{i-1}u}{dx^{i-1}}.$$

Mechanical Integration of General Linear Differential Equation of Any Order.

Until it is desired actually to construct a machine for thus integrating differential equations of the third or any higher order, it is not necessary to go into details as to plans for the mechanical fulfilment of condition (7); it is enough to know that it can be fulfilled by pure mechanism working continuously in connexion with the rotating disks of the train of integrators.

Addendum.

Mechanical Integration of any Differential Equation of Any Order.

The integrator may be applied to integrate any differential equation of any order. Let there be i simple integrators; let x_1, g_1, κ_1 be the displacements of disk, globe, and cylinder of the first, and so for the others. We have

$$g_1 = \frac{d\kappa_1}{dx_1}, \qquad g_2 = \frac{d\kappa_2}{dx_2}, \text{ etc.}$$

Now by proper mechanism establish such relations between

$$x_1, g_1, \kappa_1, x_2, g_2, \text{ etc.}$$

that

$$f^{(1)}(x_1, g_1, \kappa_1, x_2, \ldots) = 0,$$
$$f^{(2)}(x_1, g_1, \kappa_1, x_2, \ldots) = 0,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots$$
$$f^{(2i-1)}(x_1, g_1, \kappa_1, x_2, \ldots) = 0$$

($2i-1$ relations).

This will leave just one degree of freedom; and thus we have $2i-1$ simultaneous equations solved. As one particular case of relations take

$$x_1 = x_2 = \ldots (i-1 \text{ relations}),$$

and $\quad g_2 = \kappa_1, \quad g_3 = \kappa_2, \quad \text{etc. } (i-1 \text{ relations});$

so that

$$g_1 = \frac{d\kappa_i}{dx^i}, \qquad g_2 = \frac{d^{i-1}\kappa_i}{dx^{i-1}}, \text{ etc.}$$

Thus one relation is still available. Let it be

$$f(x, g_1, g_2, \ldots g_i, \kappa_i) = 0.$$

Thus the machine solves the differential equation

$$f\left(x, \frac{d^iu}{dx^i}, \frac{d^{i-1}u}{dx^{i-1}}, \ldots \frac{du}{dx}, u\right) = 0 \text{ (putting } u \text{ for } \kappa_i).$$

Or again, take $2i$ double integrators. Let the disks of all be connected so as to move with the same speed, and let t be the

displacement of any one of them from any particular position. Let

Mechanical Integration of any Differential Equation of Any Order.

$$x, y, x', y', x'', y'', \ldots x^{(i-1)}, y^{(i-1)}$$

be the displacements of the second cylinders of the several double integrators. Then (the second globe-frame of each being connected to its first cylinder) the displacements of the first globe-frames will be

$$\frac{d^2x}{dt^2}, \frac{d^2y}{dt^2}, \frac{d^2x'}{dt^2}, \frac{d^2y'}{dt^2}, \text{ etc.}$$

Let now X, Y, X', Y', etc. be each a given function of

$$x, y, x', y', x'', \text{ etc.}$$

By proper mechanism make the first globe of the first double integrator-frame move so that its displacement shall be equal to X, and so on. The machine then solves the equations

$$\frac{d^2x}{dt^2} = X, \quad \frac{d^2y}{dt^2} = Y, \quad \frac{d^2x}{dt^2} = X', \text{ etc.}$$

For example, let

$$\begin{aligned} X &= (x'-x) f\{(x'-x)^2 + (y'-y)^2\} \\ &\quad + (x''-x) f\{(x''-x)^2 + (y''-y)^2\} \\ &\quad + \ldots\ldots\ldots\ldots\ldots\ldots \\ Y &= (y'-y) f\{(x'-x)^2 + (y'-y)^2\} \\ &\quad + (y''-y) f\{(x''-x)^2 + (y''-y)^2\} \\ &\quad + \ldots\ldots\ldots\ldots\ldots\ldots \\ X' &= \text{etc.}, \quad Y' = \text{etc.}, \end{aligned}$$

where f denotes any function.

Construct in (frictionless) steel the surface whose equation is

$$z = \xi f(\xi^2 + \eta^2)$$

(and repetitions of it, for practical convenience, though *one* theoretically suffices). By aid of it (used as if it were a cam, but for two independent variables) arrange that one moving auxiliary piece (an x-auxiliary I shall call it), capable of moving to and fro in a straight line, shall have displacement always equal to

$$(x'-x) f\{(x'-x)^2 + (y'-y)^2\},$$

that another (a y-auxiliary) shall have displacement always equal to

$$(y'-y) f\{(x'-x)^2 + (y'-y)^2\},$$

Mechanical Integration of any Differential Equation of Any Order.

that another (an x-auxiliary) shall have displacement equal to

$$(x'' - x) f\{(x'' - x)^2 + (y'' - y)^2\},$$

and so on.

Then connect the first globe-frame of the first double integrator, so that its displacement shall be equal to the sum of the displacements of the x-auxiliaries; that is to say, to

$$(x' - x) f\{(x' - x)^2 + (y' - y)^2\}$$
$$+ (x'' - x) f\{(x'' - x)^2 + (y'' - y)^2\}$$
$$+ \text{etc.}$$

This may be done by a cord passing over pulleys attached to the x-auxiliaries, with one end of it fixed and the other attached to the globe-frame (as in my tide-predicting machine, or in Wheatstone's alphabetic telegraph-sending instrument).

Then, to begin with, adjust the second globe-frames and the second cylinders to have their displacements equal to the initial velocity-components and initial co-ordinates of i particles free to move in one plane. Turn the machine, and the positions of the particles at time t are shown by the second cylinders of the several double integrators, supposing them to be free particles attracting or repelling one another with forces varying according to any function of the distance.

The same may clearly be done for particles moving in three dimensions of space, since the components of force on each may be mechanically constructed by aid of a cam-surface whose equation is

$$z = \xi f(\eta)$$

and taking η for the distance between any two particles, and

$$\xi = x' - x$$

or $$= y' - y$$

or $$= x'' - x, \text{ etc.}$$

Thus we have a complete mechanical integration of the problem of finding the free motions of any number of mutually influencing particles, not restricted by any of the approximate suppositions which the analytical treatment of the lunar and planetary theories requires.

VII. Harmonic Analyzer*.

Harmonic Analyzer.

This is a realization of an instrument designed rudimentarily in the author's communication to the Royal Society ("Proceedings," February 3rd, 1876), entitled "On an Instrument for Calculating ($\int \phi(x) \psi(x) dx$), the Integral of the Product of two given Functions."

It consists of five disk-, globe-, and cylinder integrators of the kind described in Professor James Thomson's paper "On an Integrating Machine having a new Kinematic Principle," of the same date, and represented in the woodcuts of Appendix B′, III.

The five disks are all in one plane, and their centres in one line. The axes of the cylinders are all in a line parallel to it. The diameters of the five cylinders are all equal, so are those of the globes; hence the centres of the globes are in a line parallel to the line of the centres of the disks, and to the line of the axes of the cylinders.

One long wooden rod, properly supported and guided, and worked by a rack and pinion, carries five forks to move the five globes and a pointer to trace the curve on the paper cylinder. The shaft of the paper cylinder carries at its two ends cranks at right angles to one another; and a toothed wheel which turns a parallel shaft, and a third shaft in line with the first, by means of three other toothed wheels. This third shaft carries at its two ends two cranks at right angles to one another.

Another toothed wheel on the shaft of the paper drum turns another parallel shaft, which, by a slightly oblique toothed wheel working on a crown wheel with slightly oblique teeth, turns one of the five disks uniformly (supposing to avoid circumlocution the paper drum to be turning uniformly). The cylinder of the integrator, of which this one is the disk, gives the continuously growing value of $\int y dx$.

Each of the four cranks gives a simple harmonic angular motion to one of the other four disks by means of a slide and crosshead, carrying a rack which works a sector attached to the disk. Hence, the cylinders moved by the disks, driven by the

* Sir W. Thomson, *Proceedings of the Royal Society*, Vol. XXVII., 1878, p. 371.

Harmonic Analyzer.

first mentioned pair of cranks, give the continuously growing values of

$$\int y \cos\frac{2\pi x}{c}\,dx, \text{ and } \int y \sin\frac{2\pi x}{c}\,dx\,;$$

where c denotes the circumference of the paper drum: and the two remaining cylinders give

$$\int y \cos\frac{2\pi\omega x}{c}\,dx, \text{ and } \int y \sin\frac{2\pi\omega x}{c}\,dx\,;$$

where ω denotes the angular velocity of the shaft carrying the second pair of shafts, that of the first being unity.

The machine, with the toothed wheels actually mounted on it when shown to the Royal Society, gave $\omega = 2$, and was therefore adopted for the meteorological application. By removal of two of the wheels and substitution of two others, which were laid on the table of the Royal Society, the value of ω becomes $\frac{39 \times 109}{40 \times 110}$* (according to factors found by Mr E. Roberts, and supplied by him to the author, for the ratio of the mean lunar to the mean solar periods relatively to the earth's rotation). Thus, the same machine can serve for analyzing out simultaneously the mean lunar and mean solar semi-diurnal tides from a tide-gauge curve. But the dimensions of the actual machine do not allow range enough of motion for the majority of tide-gauge curves, and they are perfectly sufficient and suitable for meteorological work. The machine, with the train giving $\omega = 2$, is therefore handed over to the Meteorological Office to be brought immediately into practical work by Mr Scott (as soon as a brass cylinder of proper diameter to suit the $24h$ length of his curves is substituted for the wooden model cylinder in the machine as shown to the Royal Society): and the construction of a new machine for the tidal analysis, to have eleven disk-, globe-, and cylinder-integrators in line, and four crank shafts having their axes in line with the paper drum, according to the preceding description, in proper periods to analyse a tide curve by one process for mean level, and for the two components of each of the five chief tidal constituents—that is to say,

Tidal Harmonic Analyzer.

* The actual numbers of the teeth in the two pairs of wheels constituting the train are 78 : 80 and 109 : 110.

Tidal Harmonic Analyzer.

(1) The mean solar semi-diurnal;
(2) „ „ lunar „
(3) „ „ lunar quarter-diurnal, shallow-water tide;
(4) „ „ lunar declinational diurnal;
(5) „ „ luni-solar declinational diurnal;

is to be immediately commenced. It is hoped that it may be completed without need to apply for any addition to the grant already made by the Royal Society for harmonic analyzers.

Counterpoises are applied to the crank shafts to fulfil the condition that gravity on cranks, and sliding pieces, and sectors, is in equilibrium. Error from "back lash" or "lost time" is thus prevented simply by frictional resistance against the rotation of the uniformly rotating disk and of the tertiary shafts, and by the weights of the sectors attached to the oscillating disks.

Addition, April, 1879. The machine promised in the preceding paper has now been completed with one important modification:—Two of the eleven constituent integrators, instead of being devoted, as proposed in No. 3 of the preceding schedule, to evaluate the lunar quarter-diurnal shallow-water tide, are arranged to evaluate the solar declinational diurnal tide, this being a constituent of great practical importance in all other seas than the North Atlantic, and of very great scientific interest. For the evaluation of quarter-diurnal tides, whether lunar or solar, and of semi-diurnal tides of periods the halves of those of the diurnal tides, that is to say of all tidal constituents whose periods are the halves of those of the five main constituents for which the machine is primarily designed, an extra paper-cylinder, of half the diameter of the one used in the primary application of the machine, is constructed. By putting in this secondary cylinder and repassing the tidal curve through the machine the secondary tidal constituents (corresponding to the first "overtones" or secondary harmonic constituents of musical sounds) are to be evaluated. Similarly tertiary, quaternary, etc. tides (corresponding to the second and higher overtones in musical sounds) may be evaluated by passing the curve over cylinders of one-third and of smaller sub-multiples of the diameter of the primary cylinder. These secondary and tertiary tidal constituents are only perceptible at places where the rise and fall is influenced by a large area of sea, or a considerable length of

Secondary, tertiary, quaternary, etc. tides, due to influence of shallow water,—analogous to musical overtones

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channel through which the whole amount of the rise and fall is notable in proportion to the mean depth. They are very perceptible at almost all commercial ports, except in the Mediterranean, and to them are due such curious and practically important tidal characteristics as the double high waters at Southampton and in the Solent and on the south coast of England from the Isle of Wight to Portland, and the protracted duration of high water at Havre. [The instrument has been deposited in the South Kensington Museum.]

END OF PART I.

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