Examples of derivation continued.

continued applications of (99) (100) with upper sign, is the regular "Laplace's function" growing from $C' \sin^n \theta \frac{\sin}{\cos} n\phi$, which is the case represented by $u'=C'$ in (109). But in this continuation we are only doing for the case of n an integer, part of what was done in § (n'), Example 2, where the other part, from the other part of the solution of (109) now lost, gives the other part of the complete solution of Laplace's equation subject to the limitation $i-n$ (or $i-s$) a positive integer, but not to the limitation of i an integer or n an integer.

(s') Returning to the commencement of § (r'), with s put for n, we find a complete solution growing in the form

$$\frac{Kf_i(\mu)}{(1-\mu)^s} + (-)^i \frac{K'f_i(-\mu)}{(1+\mu)^s} \dots\dots\dots\dots\dots (110);$$

which may be immediately reduced to

$$\frac{Kf_i(\mu)(1+\mu)^s + (-)^i K'f_i(-\mu)(1-\mu)^s}{(1-\mu^2)^s} \dots\dots\dots (110');$$

f_i denoting an integral algebraic function of the i^{th} degree, readily found by the proper successive applications of (99) (100). Hence, by (83) (79), we have

$$w = \frac{Kf_i(\mu)(1+\mu)^s + (-)^i K'f_i(-\mu)(1-\mu)^s}{(1-\mu^2)^{\frac{s}{2}}} \dots\dots\dots (111),$$

as the complete solution of Laplace's equation

$$\frac{d}{d\mu}\left[(1-\mu^2)\frac{dw}{d\mu}\right] + \left[\frac{-s^2}{1-\mu^2} + i(i+1)\right] w = 0 \dots\dots\dots (112),$$

Finite algebraic expression of complete solution for tesseral harmonics of *integral order*.

for the case of i an integer without any restriction as to the value of s, which may be integral or fractional, real or imaginary, with no failure except the case of s an integer and $i > s$, of which the complete treatment is included in § (m'), Example 2, above.

CHAPTER II.

DYNAMICAL LAWS AND PRINCIPLES.

Ideas of matter and force introduced.

205. In the preceding chapter we considered as a subject of pure geometry the motion of points, lines, surfaces, and volumes, whether taking place with or without change of dimensions and form; and the results we there arrived at are of course altogether independent of the idea of *matter*, and of the *forces* which matter exerts. We have heretofore assumed the *existence* merely of motion, distortion, etc.; we now come to the consideration, not of how we *might* consider such motions, etc., to be produced, but of the *actual* causes which in the material world *do* produce them. The axioms of the present chapter must therefore be considered to be due to actual experience, in the shape either of observation or experiment. How this experience is to be obtained will form the subject of a subsequent chapter.

206. We cannot do better, at all events in commencing, than follow Newton somewhat closely. Indeed the introduction to the *Principia* contains in a most lucid form the general foundations of Dynamics. The *Definitiones* and *Axiomata sive Leges Motûs*, there laid down, require only a few amplifications and additional illustrations, suggested by subsequent developments, to suit them to the present state of science, and to make a much better introduction to dynamics than we find in even some of the best modern treatises.

Matter.

207. We cannot, of course, give a definition of *Matter* which will satisfy the metaphysician, but the naturalist may be content to know matter as *that which can be perceived by the senses*, or as *that which can be acted upon by, or can exert, force.* The

Force.

latter, and indeed the former also, of these definitions involves the idea of *Force*, which, in point of fact, is a direct object of sense; probably of all our senses, and certainly of the "muscular sense." To our chapter on Properties of Matter we must refer for further discussion of the question, *What is matter?* And we shall then be in a position to discuss the question of the subjectivity of *Force*.

Mass.

Density.

208. *The Quantity of Matter* in a body, or, as we now call it, the *Mass* of a body, is proportional, according to Newton, to the *Volume* and the *Density* conjointly. In reality, the definition gives us the meaning of density rather than of mass; for it shows us that if twice the original quantity of matter, air for example, be forced into a vessel of given capacity, the density will be doubled, and so on. But it also shows us that, of matter of uniform density, the mass or quantity is proportional to the volume or space it occupies.

Let M be the mass, ρ the density, and V the volume, of a homogeneous body. Then

$$M = V\rho;$$

if we so take our units that unit of mass is that of unit volume of a body of unit density.

If the density vary from point to point of the body, we have evidently, by the above formula and the elementary notation of the integral calculus,

$$M = \iiint \rho\, dxdydz,$$

where ρ is supposed to be a known function of x, y, z, and the integration extends to the whole space occupied by the matter of the body whether this be continuous or not.

It is worthy of particular notice that, in this definition, Newton says, if there be anything which *freely* pervades the interstices of all bodies, this is *not* taken account of in estimating their Mass or Density.

Measurement of mass

209. Newton further states, that a practical measure of the mass of a body is its *Weight*. His experiments on pendulums, by which he establishes this most important result, will be described later, in our chapter on Properties of Matter.

As will be presently explained, the unit mass most convenient for British measurements is an imperial pound of matter.

210. The *Quantity of Motion*, or the *Momentum*, of a rigid body moving without rotation is proportional to its mass and velocity conjointly. The whole motion is the sum of the motions of its several parts. Thus a doubled mass, or a doubled velocity, would correspond to a double quantity of motion; and so on. Momentum.

Hence, if we take as unit of momentum the momentum of a unit of matter moving with unit velocity, the momentum of a mass M moving with velocity v is Mv.

211. *Change of Quantity of Motion*, or *Change of Momentum*, is proportional to the mass moving and the change of its velocity conjointly. Change of momentum

Change of velocity is to be understood in the general sense of § 27. Thus, in the figure of that section, if a velocity represented by OA be changed to another represented by OC, the change of velocity is represented in magnitude and direction by AC.

212. *Rate of Change of Momentum* is proportional to the mass moving and the acceleration of its velocity conjointly. Thus (§ 35, *b*) the rate of change of momentum of a falling body is constant, and in the vertical direction. Again (§ 35, *a*) the rate of change of momentum of a mass M, describing a circle of radius R, with uniform velocity V, is $\frac{MV^2}{R}$, and is directed to the centre of the circle; that is to say, it is a change of direction, not a change of speed, of the motion. Hence if the mass be compelled to keep in the circle by a cord attached to it and held fixed at the centre of the circle, the force with which the cord is stretched is equal to $\frac{MV^2}{R}$: this is called the centrifugal force of the mass M moving with velocity V in a circle of radius R. Rate of change of momentum.

Generally (§ 29), for a body of mass M moving anyhow in space there is change of momentum, at the rate, $M\frac{d^2s}{dt^2}$ in the direc-

Rate of change of momentum.

tion of motion, and $M\frac{v^2}{\rho}$ towards the centre of curvature of the path; and, if we choose, we may exhibit the whole acceleration of momentum by its three rectangular components $M\frac{d^2x}{dt^2}$, $M\frac{d^2y}{dt^2}$, $M\frac{d^2z}{dt^2}$, or, according to the Newtonian notation, $M\ddot{x}$, $M\ddot{y}$, $M\ddot{z}$.

Kinetic energy.

213. The *Vis Viva*, or *Kinetic Energy*, of a moving body is proportional to the mass and the square of the velocity, conjointly. If we adopt the same units of mass and velocity as before, there is particular advantage in defining kinetic energy as *half* the product of the mass and the square of its velocity.

214. *Rate of Change of Kinetic Energy* (when defined as above) is the product of the velocity into the component of rate of change of momentum in the direction of motion.

For $$\frac{d}{dt}\left(\frac{Mv^2}{2}\right) = v\,\frac{d(Mv)}{dt}.$$

Particle and point.

215. It is to be observed that, in what precedes, with the exception of the definition of mass, we have taken no account of the dimensions of the moving body. This is of no consequence so long as it does not rotate, and so long as its parts preserve the same relative positions amongst one another. In this case we may suppose the whole of the matter in it to be condensed in one point or particle. We thus speak of a *material particle*, as distinguished from a *geometrical point*. If the body rotate, or if its parts change their relative positions, then we cannot choose any one point by whose motions alone we may determine those of the other points. In such cases the momentum and change of momentum of the whole body in any direction are, the sums of the momenta, and of the changes of momentum, of its parts, in these directions; while the kinetic energy of the whole, being non-directional, is simply the sum of the kinetic energies of the several parts or particles.

Inertia.

216. Matter has an innate power of resisting external influences, so that every body, as far as it can, remains at rest, or moves uniformly in a straight line.

This, the *Inertia* of matter, is proportional to the quantity of

matter in the body. And it follows that some *cause* is requisite to disturb a body's uniformity of motion, or to change its direction from the natural rectilinear path. Inertia.

Force.

217. Force is any cause which tends to alter a body's natural state of rest, or of uniform motion in a straight line.

Force is wholly expended in the *Action* it produces; and the body, after the force ceases to act, retains by its inertia the direction of motion and the velocity which were given to it. Force may be of divers kinds, as pressure, or gravity, or friction, or any of the attractive or repulsive actions of electricity, magnetism, etc.

Specification of a force.

218. The three elements specifying a force, or the three elements which must be known, before a clear notion of the force under consideration can be formed, are, its place of application, its direction, and its magnitude.

Place of application.

(*a*) The place of application of a force. The first case to be considered is that in which the place of application is a point. It has been shown already in what sense the term "point" is to be taken, and, therefore, in what way a force may be imagined as acting at a point. In reality, however, the place of application of a force is always either a surface or a space of three dimensions occupied by matter. The point of the finest needle, or the edge of the sharpest knife, is still a surface, and acts by pressing over a finite area on bodies to which it may be applied. Even the most rigid substances, when brought together, do not touch at a point merely, but mould each other so as to produce a surface of application. On the other hand, gravity is a force of which the place of application is the whole matter of the body whose weight is considered; and the smallest particle of matter that has weight occupies some finite portion of space. Thus it is to be remarked, that there are two kinds of force, distinguishable by their place of application—force, whose place of application is a surface, and force, whose place of application is a solid. When a heavy body rests on the ground, or on a table, force of the second character, acting downwards, is balanced by force of the first character acting upwards.

Direction. (*b*) The second element in the specification of a force is its direction. The direction of a force is the line in which it acts. If the place of application of a force be regarded as a point, a line through that point, in the direction in which the force tends to move the body, is the direction of the force. In the case of a force distributed over a surface, it is frequently possible and convenient to assume a single point and a single line, such that a certain force acting at that point in that line would produce sensibly the same effect as is really produced.

Magnitude. (*c*) The third element in the specification of a force is its magnitude. This involves a consideration of the method followed in dynamics for measuring forces. Before measuring anything, it is necessary to have a unit of measurement, or a standard to which to refer, and a principle of numerical specification, or a mode of referring to the standard. These will be supplied presently. See also § 258, below.

Accelerative effect. **219.** The *Accelerative Effect of a Force* is proportional to the velocity which it produces in a given time, and is measured by that which is, or would be, produced in unit of time; in other words, the *rate of change of velocity* which it produces. This is simply what we have already defined as acceleration, § 28.

Measure of force. **220.** The *Measure of a Force* is the quantity of motion which it produces per unit of time.

The reader, who has been accustomed to speak of a force of so many pounds, or so many tons, may be startled when he finds that such expressions are not definite unless it be specified at what part of the earth's surface the pound, or other definite quantity of matter named, is to be weighed; for the *heaviness* or *gravity* of a given quantity of matter differs in different latitudes. But the force required to produce a stated quantity of motion in a given time is perfectly definite, and independent of locality. Thus, let W be the mass of a body, g the velocity it would acquire in falling freely for a second, and P the force of gravity upon it, measured in kinetic or absolute units. We have

$$P = Wg.$$

221. According to the system commonly followed in mathematical treatises on dynamics till fourteen years ago, when a small instalment of the first edition of the present work was issued for the use of our students, the unit of mass was g times the mass of the standard or unit weight. This definition, giving a varying and a very unnatural unit of mass, was exceedingly inconvenient. By taking the gravity of a constant mass for the unit of force it makes the unit of force greater in high than in low latitudes. In reality, standards of weight are *masses*, not *forces*. They are employed primarily in commerce for the purpose of measuring out a definite *quantity* of matter; not an amount of matter which shall be attracted by the earth with a given force. **Inconvenient system of modern treatises.** **Standards of weight are *masses*, and not primarily intended for measurement of force.**

A merchant, with a balance and a set of standard weights, would give his customers the same quantity of the same kind of matter however the earth's attraction might vary, depending as he does upon *weights* for his measurement; another, using a spring-balance, would defraud his customers in high latitudes, and himself in low, if his instrument (which depends on constant forces and not on the gravity of constant masses) were correctly adjusted in London.

It is a secondary application of our standards of weight to employ them for the measurement of *forces*, such as steam pressures, muscular power, etc. In all cases where great accuracy is required, the results obtained by such a method have to be reduced to what they would have been if the measurements of force had been made by means of a perfect spring-balance, graduated so as to indicate the forces of gravity on the standard weights in some conventional locality.

It is therefore very much simpler and better to take the imperial pound, or other national or international standard weight, as, for instance, the gramme (see the chapter on Measures and Instruments), as the unit of mass, and to derive from it, according to Newton's definition above, the unit of force. This is the method which Gauss has adopted in his great improvement (§ 223 below) of the system of measurement of forces.

Clairault's formula for the amount of gravity.

222. The formula, deduced by Clairault from observation, and a certain theory regarding the figure and density of the earth, may be employed to calculate the most probable value of the apparent force of gravity, being the resultant of true gravitation and centrifugal force, in any locality where no pendulum observation of sufficient accuracy has been made. This formula, with the two coefficients which it involves, corrected according to the best modern pendulum observations (Airy, *Encyc. Metropolitana*, *Figure of the Earth*), is as follows:—

Let G be the apparent force of gravity on a unit mass at the equator, and g that in any latitude λ; then

$$g = G\,(1 + \cdot 005133 \sin^2 \lambda).$$

The value of G, in terms of the British absolute unit, to be explained immediately, is

$$32\cdot 088.$$

According to this formula, therefore, polar gravity will be

$$g = 32\cdot 088 \times 1\cdot 005133 = 32\cdot 2527.$$

Gauss's absolute Unit of Force.

223. Gravity having failed to furnish a definite standard, independent of locality, recourse must be had to something else. The principle of measurement indicated as above by Newton, but first introduced practically by Gauss, furnishes us with what we want. According to this principle, the unit force is that force which, acting on a national standard unit of matter during the unit of time, generates the unity of velocity.

This is known as Gauss's absolute unit; absolute, because it furnishes a standard force independent of the differing amounts of gravity at different localities. It is however terrestrial and inconstant if the unit of time depends on the earth's rotation, as it does in our present system of chronometry. The period of vibration of a piece of quartz crystal of specified shape and size and at a stated temperature (a tuning-fork, or bar, as one of the bars of glass used in the "musical glasses") gives us a unit of time which is constant through all space and all time, and independent of the earth. A unit of force founded on such a unit of time would be better entitled to the designation *absolute* than is the "absolute unit" now generally adopted, which is founded on the *mean solar second*. But this depends essentially on one particular piece of matter, and is therefore liable to all the accidents, etc. which affect so-called National Standards however carefully they may be preserved, as well as to the almost insuperable practical difficulties which are experienced when we attempt to make exact copies of them. Still, in the present state of science, we are really confined to such approximations. The recent discoveries due to the Kinetic theory of gases and to Spectrum analysis (especially when it is applied to the light of the heavenly bodies) indicate to us *natural standard* pieces of matter such as atoms of hydrogen, or sodium, ready made in infinite numbers, all absolutely alike in every physical property. The time of vibration of a sodium particle corresponding to any one of its modes of vibration, is known to be absolutely independent of its position in the universe, and it will probably remain the same so long as the particle itself exists. The wavelength for that particular ray, *i.e.* the space through which light is propagated *in vacuo* during the time of one complete vibration of this period, gives a perfectly invariable unit of length; and it is possible that at some not very distant day the mass of such a sodium particle may be employed as a natural standard for the remaining fundamental unit. This, the latest improvement made upon our original suggestion of a *Perennial Spring* (First edition, § 406), is due to Clerk Maxwell*; who has also communicated to us another very important and interesting suggestion for founding the unit of time upon physical properties of a substance without the necessity of specifying any particular quantity of it. It is this, water being chosen as the substance of all others known to us which is most easily obtained in perfect purity and in perfectly definite physical condition.—Call the standard density of water the maximum density of the liquid when under the pressure of its own vapour alone. The time of revolution of an infinitesimal satellite close to the surface of a globe of water at standard density (or of any kind of matter at the same density) may be taken as the unit of time; for it is independent of the size of the globe. This has

Maxwell's two suggestions for Absolute Unit of Time.

* *Electricity and Magnetism*, 1872.

Third suggestion for Absolute Unit of Time.

suggested to us still another unit, founded, however, still upon the same physical principle. The time of the gravest simple harmonic infinitesimal vibration of a globe of liquid, water at standard density, or of other perfect liquids at the same density, may be taken as the unit of time; for the time of the simple harmonic vibration of any one of the fundamental modes of a liquid sphere is independent of the size of the sphere.

Let f be the force of gravitational attraction between two units of matter at unit distance. The force of gravity at the surface of a globe of radius r, and density ρ, is $\frac{4\pi}{3}f\rho r$. Hence if ω be the angular velocity of an infinitesimal satellite, we have, by the equilibrium of centrifugal force and gravity (§§ 212, 477),

$$\omega^2 r = \frac{4\pi}{3} f\rho r.$$

Hence
$$\omega = \sqrt{\frac{4\pi f\rho}{3}},$$

and therefore if T be the satellite's period,

$$T = 2\pi\sqrt{\frac{3}{4\pi f\rho}}$$

(which is equal to the period of a simple pendulum whose length is the globe's radius, and weighted end infinitely near the surface of the globe). And it has been proved* that if a globe of liquid be distorted infinitesimally according to a spherical harmonic of order i, and left at rest, it will perform simple harmonic oscillations in a period equal to

$$2\pi\sqrt{\left\{\frac{3}{4\pi f\rho}\cdot\frac{2i+1}{2i(i-1)}\right\}}.$$

Hence if T' denote the period of the gravest, that, namely, for which $i = 2$, we have

$$T' = T\sqrt{\frac{5}{4}}.$$

The semi-period of an infinitesimal satellite round the earth is equal, reckoned in seconds, to the square root of the number of metres in the earth's radius, the metre being very approximately the length of the seconds pendulum, whose period is two seconds. Hence taking the earth's radius as 6,370,000 metres, and its density as $5\frac{1}{2}$ times that of our standard globe,

Suggestions for Absolute Unit of Time.

$$T = 3\text{ h. } 17\text{ m.}$$
$$T' = 3\text{ h. } 40\text{ m.}$$

* "Dynamical Problems regarding Elastic Spheroidal Shells and Spheroids of Incompressible Liquid" (W. Thomson), *Phil. Trans.* Nov. 27, 1862.

224. The absolute unit depends on the unit of matter, the unit of time, and the unit of velocity; and as the unit of velocity depends on the unit of space and the unit of time, there is, in the definition, a single reference to mass and space, but a *double* reference to time; and this is a point that must be particularly attended to.

225. The unit of mass may be the British imperial pound; the unit of space the British standard foot; and, accurately enough for practical purposes for a few thousand years, the unit of time may be the mean solar second.

British absolute unit.

We accordingly define the British absolute unit force as "the force which, acting on one pound of matter for one second, generates a velocity of one foot per second." Prof. James Thomson has suggested the name "Poundal" for this unit of force.

Comparison with gravity.

226. To illustrate the reckoning of force in "absolute measure," find how many absolute units will produce, in any particular locality, the same effect as the force of gravity on a given mass. To do this, measure the effect of gravity in producing acceleration on a body unresisted in any way. The most accurate method is indirect, by means of the pendulum. The result of pendulum experiments made at Leith Fort, by Captain Kater, is, that the velocity which would be acquired by a body falling unresisted for one second is at that place 32·207 feet per second. The preceding formula gives exactly 32·2, for the latitude 55° 33′, which is approximately that of Edinburgh. The variation in the force of gravity for one degree of difference of latitude about the latitude of Edinburgh is only ·0000832 of its own amount. It is nearly the same, though somewhat more, for every degree of latitude southwards, as far as the southern limits of the British Isles. On the other hand, the variation per degree is sensibly less, as far north as the Orkney and Shetland Isles. Hence

Gravity of Unit weight or mass in terms of Kinetic Unit.

the augmentation of gravity per degree from south to north throughout the British Isles is at most about $\frac{1}{12000}$ of its whole amount in any locality. The average for the whole of Great Britain and Ireland differs certainly but little from 32·2. Our present application is, that the force of gravity at Edinburgh is 32·2 times the force which, acting on a pound for a second, would generate a velocity of one foot per second; in other words, 32·2 is the number of absolute units which measures the weight of a pound in this latitude. Thus, approximately, the poundal is equal to the gravity of about half an ounce.

227. Forces (since they involve only direction and magnitude) may be represented, as velocities are, by straight lines in their directions, and of lengths proportional to their magnitudes, respectively.

Also the laws of composition and resolution of any number of forces acting at the same point, are, as we shall show later (§ 255), the same as those which we have already proved to hold for velocities; so that with the substitution of force for velocity, §§ 26, 27, are still true.

Effective component of a force.

228. In rectangular resolution the *Component* of a force in any direction, (sometimes called the *Effective Component* in that direction,) is therefore found by multiplying the magnitude of the force by the cosine of the angle between the directions of the force and the component. The remaining component in this case is perpendicular to the other.

It is very generally convenient to resolve forces into components parallel to three lines at right angles to each other; each such resolution being effected by multiplying by the cosine of the angle concerned.

Geometrical Theorem preliminary to definition of centre of inertia.

229. The point whose distances from three planes at right angles to one another are respectively equal to the mean distances of any group of points from these planes, is at a distance from any plane whatever, equal to the mean distance of the group from the same plane. Hence of course, if it is in motion, its velocity perpendicular to that plane is the mean of the velocities of the several points, in the same direction.

Geometrical Theorem preliminary to definition of centre of inertia.

Let (x_1, y_1, z_1), etc., be the points of the group in number i; and $\bar{x}, \bar{y}, \bar{z}$ be the co-ordinates of a point at distances respectively equal to their mean distances from the planes of reference; that is to say, let

$$\bar{x} = \frac{x_1 + x_2 + \text{etc.}}{i}, \quad \bar{y} = \frac{y_1 + y_2 + \text{etc.}}{i}, \quad \bar{z} = \frac{z_1 + z_2 + \text{etc.}}{i}.$$

Thus, if p_1, p_2, etc., and p, denote the distances of the points in question from any plane at a distance a from the origin of coordinates, perpendicular to the direction (l, m, n), the sum of a and p_1 will make up the projection of the broken line x_1, y_1, z_1 on (l, m, n), and therefore

$$p_1 = lx_1 + my_1 + nz_1 - a, \text{ etc.};$$

and similarly, $\quad p = l\bar{x} + m\bar{y} + n\bar{z} - a.$

Substituting in this last the expressions for $\bar{x}, \bar{y}, \bar{z}$, we find

$$p = \frac{p_1 + p_2 + \text{etc.}}{i},$$

which is the theorem to be proved. Hence, of course,

$$\frac{dp}{dt} = \frac{1}{i}\left(\frac{dp_1}{dt} + \frac{dp_2}{dt} + \text{etc.}\right).$$

Centre of inertia.

230. The *Centre of Inertia* of a system of equal material points (whether connected with one another or not) is the point whose distance is equal to their average distance from any plane whatever (§ 229).

A group of material points of unequal masses may always be imagined as composed of a greater number of equal material points, because we may imagine the given material points divided into different numbers of very small parts. In any case in which the magnitudes of the given masses are incommensurable, we may approach as near as we please to a rigorous fulfilment of the preceding statement, by making the parts into which we divide them sufficiently small.

On this understanding the preceding definition may be applied to define the centre of inertia of a system of material points, whether given equal or not. The result is equivalent to this:—

Centre of Inertia.

The centre of inertia of any system of material points whatever (whether rigidly connected with one another, or connected in any way, or quite detached), is a point whose distance from any plane is equal to the sum of the products of each mass into its distance from the same plane divided by the sum of the masses.

We also see, from the proposition stated above, that a point whose distance from three rectangular planes fulfils this condition, must fulfil this condition also for every other plane.

The co-ordinates of the centre of inertia, of masses w_1, w_2, etc., at points (x_1, y_1, z_1), (x_2, y_2, z_2), etc., are given by the following formulæ:—

$$\bar{x} = \frac{w_1 x_1 + w_2 x_2 + \text{etc.}}{w_1 + w_2 + \text{etc.}} = \frac{\Sigma wx}{\Sigma w}, \quad \bar{y} = \frac{\Sigma wy}{\Sigma w}, \quad \bar{z} = \frac{\Sigma wz}{\Sigma w}.$$

These formulæ are perfectly general, and can easily be put into the particular shape required for any given case. Thus, suppose that, instead of a set of detached material points, we have a continuous distribution of matter through certain definite portions of space; the density at x, y, z being ρ, the elementary principles of the integral calculus give us at once

$$\bar{x} = \frac{\iiint \rho x dx dy dz}{\iiint \rho dx dy dz}, \text{ etc.},$$

where the integrals extend through all the space occupied by the mass in question, in which ρ has a value different from zero.

The Centre of Inertia or Mass is thus a perfectly definite point in every body, or group of bodies. The term *Centre of Gravity* is often very inconveniently used for it. The theory of the resultant action of gravity which will be given under Abstract Dynamics shows that, except in a definite class of distributions of matter, there is no one fixed point which can properly be called the Centre of Gravity of a rigid body. In ordinary cases of terrestrial gravitation, however, an approximate solution is available, according to which, in common parlance, the term "*Centre of Gravity*" may be used as equivalent to *Centre of Inertia;* but it must be carefully remembered that the fundamental ideas involved in the two definitions are essentially different.

The second proposition in § 229 may now evidently be stated thus:—The sum of the momenta of the parts of the system in any direction is equal to the momentum in the same direction of a mass equal to the sum of the masses moving with a velocity equal to the velocity of the centre of inertia. Centre of Inertia.

231. The *Moment* of any physical agency is the numerical measure of its importance. Thus, the moment of a force round a point or round a line, signifies the measure of its importance as regards producing or balancing rotation round that point or round that line. Moment.

232. The *Moment* of a force about a point is defined as the product of the force into its perpendicular distance from the point. It is numerically double the area of the triangle whose vertex is the point, and whose base is a line representing the force in magnitude and direction. It is often convenient to represent it by a line numerically equal to it, drawn through the vertex of the triangle perpendicular to its plane, through the front of a watch held in the plane with its centre at the point, and facing so that the force tends to turn round this point in a direction opposite to the hands. The moment of a force round any axis is the moment of its component in any plane perpendicular to the axis, round the point in which the plane is cut by the axis. Here we imagine the force resolved into two components, one parallel to the axis, which is ineffective so far as rotation round the axis is concerned; the other perpendicular to the axis (that is to say, having its line in any plane perpendicular to the axis). This latter component may be called the effective component of the force, with reference to rotation round the axis. And its moment round the axis may be defined as its moment round the nearest point of the axis, which is equivalent to the preceding definition. It is clear that the moment of a force round any axis, is equal to the area of the projection on any plane perpendicular to the axis, of the figure representing its moment round any point of the axis. Moment of a force about a point. Moment of a force about an axis.

233. The projection of an area, plane or curved, on any plane, is the area included in the projection of its bounding line. Digression on projection of areas.

Digression on projection of areas.

If we imagine an area divided into any number of parts, the projections of these parts on any plane make up the projection of the whole. But in this statement it must be understood that the areas of partial projections are to be reckoned as positive if particular sides, which, for brevity, we may call the outside of the projected area and the front of the plane of projection, face the same way, and negative if they face oppositely.

Of course if the projected surface, or any part of it, be a plane area at right angles to the plane of projection, the projection vanishes. The projections of any two shells having a common edge, on any plane, are equal, but with the same, or opposite, signs as the case may be. Hence, by taking two such shells facing opposite ways, we see that the projection of a closed surface (or a shell with evanescent edge), on any plane, is nothing.

Equal areas in one plane, or in parallel planes, have equal projections on any plane, whatever may be their figures.

Hence the projection of any plane figure, or of any shell, edged by a plane figure, on another plane, is equal to its area, multiplied by the cosine of the angle at which its plane is inclined to the plane of projection. This angle is acute or obtuse, according as the outside of the projected area, and the front of plane of projection, face on the whole towards the same parts, or oppositely. Hence lines representing, as above described, moments about a point in different planes, are to be compounded as forces are.—See an analogous theorem in § 96.

Couple.

234. A *Couple* is a pair of equal forces acting in dissimilar directions in parallel lines. The *Moment* of a couple is the sum of the moments of its forces about any point in their plane, and is therefore equal to the product of either force into the shortest distance between their directions. This distance is called the *Arm* of the couple.

The *Axis of a Couple* is a line drawn from any chosen point of reference perpendicular to the plane of the couple, of such magnitude and in such direction as to represent the magnitude of the moment, and to indicate the direction in which the couple tends to turn. The most convenient rule for fulfilling the latter condition is this:—Hold a watch with its centre at the

point of reference, and with its plane parallel to the plane of the couple. Then, according as the motion of the hands is contrary to or along with the direction in which the couple tends to turn, draw the axis of the couple through the face or through the back of the watch, *from* its centre. Thus a couple is completely represented by its axis; and couples are to be resolved and compounded by the same geometrical constructions performed with reference to their axes as forces or velocities, with reference to the lines directly representing them.

Couple.

Moment of velocity.

235. If we substitute, for the force in § 232, a velocity, we have the moment of a velocity about a point; and by introducing the mass of the moving body as a factor, we have an important element of dynamical science, the *Moment of Momentum*. The laws of composition and resolution are the same as those already explained; but for the sake of some simple applications we give an elementary investigation.

Moment of momentum.

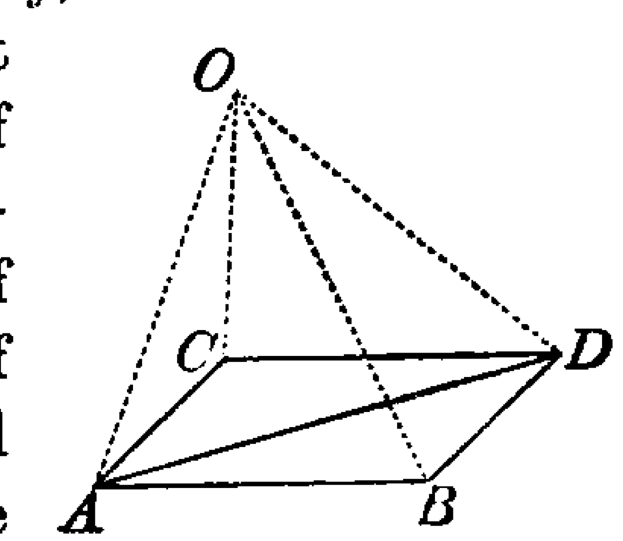
Moment of a rectilineal displacement.

The moment of a rectilineal motion is the product of its length into the distance of its line from the point.

The moment of the resultant velocity of a particle about any point in the plane of the components is equal to the algebraic sum of the moments of the components, the proper sign of each moment being determined as above, § 233. The same is of course true of moments of displacements, of moments of forces and of moments of momentum.

For two forces, motions, velocities, or momenta, in one plane, the sum of their moments proved equal to the moment of their resultant round any point in that plane.

First, consider two component motions, AB and AC, and let AD be their resultant (§ 27). Their half moments round the point O are respectively the areas OAB, OCA. Now OCA, together with half the area of the parallelogram $CABD$, is equal to OBD. Hence the sum of the two half moments together with half the area of the parallelogram, is equal to AOB together with BOD, that is to say, to the area of the whole figure $OABD$. But ABD, a part of this figure, is equal to half the area of the parallelogram; and therefore the remainder, OAD, is equal to the sum of the two half moments. But OAD is half the moment of the resultant velocity round the point O. Hence the moment of the



resultant is equal to the sum of the moments of the two components.

If there are any number of component rectilineal motions in one plane, we may compound them in order, any two taken together first, then a third, and so on; and it follows that the sum of their moments is equal to the moment of their resultant. It follows, of course, that the sum of the moments of any number of component velocities, all in one plane, into which the velocity of any point may be resolved, is equal to the moment of their resultant, round any point in their plane. It follows also, that if velocities, in different directions all in one plane, be successively given to a moving point, so that at any time its velocity is their resultant, the moment of its velocity at any time is the sum of the moments of all the velocities which have been successively given to it.

Any number of moments in one plane compounded by addition.

Cor.—If one of the components always passes through the point, its moment vanishes. This is the case of a motion in which the acceleration is directed to a fixed point, and we thus reproduce the theorem of § 36, *a*, that in this case the areas described by the radius-vector are proportional to the times; for, as we have seen, the moment of velocity is double the area traced out by the radius-vector in unit of time.

Moment round an axis.

236. The moment of the velocity of a point round any axis is the moment of the velocity of its projection on a plane perpendicular to the axis, round the point in which the plane is cut by the axis.

Moment of a whole motion, round an axis.

The moment of the whole motion of a point during any time, round any axis, is twice the area described in that time by the radius-vector of its projection on a plane perpendicular to that axis.

If we consider the conical area traced by the radius-vector drawn from any fixed point to a moving point whose motion is not confined to one plane, we see that the projection of this area on any plane through the fixed point is half of what we have just defined as the moment of the whole motion round an axis perpendicular to it through the fixed point. Of all these planes, there is one on which the projection of the area is greater

than on any other; and the projection of the conical area on any plane perpendicular to this plane, is equal to nothing, the proper interpretation of positive and negative projections being used.

Moment of a whole motion, round an axis.

If any number of moving points are given, we may similarly consider the conical surface described by the radius-vector of each drawn from one fixed point. The same statement applies to the projection of the many-sheeted conical surface, thus presented. The resultant axis of the whole motion in any finite time, round the fixed point of the motions of all the moving points, is a line through the fixed point perpendicular to the plane on which the area of the whole projection is greater than on any other plane; and the moment of the whole motion round the resultant axis, is twice the area of this projection.

Resultant axis.

The resultant axis and moment of velocity, of any number of moving points, relatively to any fixed point, are respectively the resultant axis of the whole motion during an infinitely short time, and its moment, divided by the time.

The moment of the whole motion round any axis, of the motion of any number of points during any time, is equal to the moment of the whole motion round the resultant axis through any point of the former axis, multiplied into the cosine of the angle between the two axes.

The resultant axis, relatively to any fixed point, of the whole motion of any number of moving points, and the moment of the whole motion round it, are deduced by the same elementary constructions from the resultant axes and moments of the individual points, or partial groups of points of the system, as the direction and magnitude of a resultant displacement are deduced from any given lines and magnitudes of component displacements.

Moment of momentum.

Corresponding statements apply, of course, to the moments of velocity and of momentum.

Virtual velocity.

237. If the point of application of a force be displaced through a small space, the resolved part of the displacement in the direction of the force has been called its *Virtual Velocity*.

Virtual velocity.

This is positive or negative according as the virtual velocity is in the same, or in the opposite, direction to that of the force.

The product of the force, into the virtual velocity of its point of application, has been called the *Virtual Moment* of the force. These terms we have introduced since they stand in the history and developments of the science; but, as we shall show further on, they are inferior substitutes for a far more useful set of ideas clearly laid down by Newton.

Work.

238. A force is said to *do work* if its place of application has a positive component motion in its direction; and the work done by it is measured by the product of its amount into this component motion.

Practical unit.

Thus, in lifting coals from a pit, the amount of work done is proportional to the weight of the coals lifted; that is, to the force overcome in raising them; and also to the height through which they are raised. The unit for the measurement of work adopted in practice by British engineers, is that required to overcome a force equal to the gravity of a pound through the space of a foot; and is called a *Foot-Pound.*

Scientific unit.

In purely scientific measurements, the unit of work is not the foot-pound, but the kinetic unit force (§ 225) acting through unit of space. Thus, for example, as we shall show further on, this unit is adopted in measuring the work done by an electric current, the units for electric and magnetic measurements being founded upon the kinetic unit force.

Work of a force.

If the weight be raised obliquely, as, for instance, along a smooth inclined plane, the space through which the force has to be overcome is increased in the ratio of the length to the height of the plane; but the force to be overcome is not the whole gravity of the weight, but only the component of the gravity parallel to the plane; and this is less than the gravity in the ratio of the height of the plane to its length. By multiplying these two expressions together, we find, as we might expect, that the amount of work required is unchanged by the substitution of the oblique for the vertical path.

239. Generally, for any force, the work done during an infinitely small displacement of the point of application is the virtual moment of the force (§ 237), or is the product of the resolved part of the force in the direction of the displacement into the displacement.

From this it appears, that if the motion of the point of application be always perpendicular to the direction in which a force acts, such a force does no work. Thus the mutual normal pressure between a fixed and moving body, as the tension of the cord to which a pendulum bob is attached, or the attraction of the sun on a planet if the planet describe a circle with the sun in the centre, is a case in which no work is done by the force.

Work of a couple.

240. The work done by a force, or by a couple, upon a body turning about an axis, is the product of the moment of the force or couple into the angle (in radians, or fraction of a radian) through which the body acted on turns, if the moment remains the same in all positions of the body. If the moment be variable, the statement is only valid for infinitely small displacements, but may be made accurate by employing the proper *average* moment of the force or of the couple. The proof is obvious.

If Q be the moment of the force or couple for a position of the body given by the angle θ, $Q(\theta_1-\theta_0)$ if Q is constant, or $\int_{\theta_0}^{\theta_1} Q d\theta = q(\theta_1-\theta_0)$ where q is the proper average value of Q when variable, is the work done by the couple during the rotation from θ_0 to θ_1.

Transformation of work.

241. Work done on a body by a force is always shown by a corresponding increase of vis viva, or kinetic energy, if no other forces act on the body which can do work or have work done against them. If work be done against any forces, the increase of kinetic energy is less than in the former case by the amount of work so done. In virtue of this, however, the body possesses an equivalent in the form of *Potential Energy* (§ 273), if its physical conditions are such that these forces will act equally, and in the same directions, if the motion of the system is reversed. Thus there may be no change of kinetic energy pro-

Potential energy.

Potential energy.

duced, and the work done may be wholly stored up as potential energy.

Thus a weight requires work to raise it to a height, a spring requires work to bend it, air requires work to compress it, etc.; but a raised weight, a bent spring, compressed air, etc., are *stores* of energy which can be made use of at pleasure.

Newton's Laws of Motion.

242. In what precedes we have given some of Newton's *Definitiones* nearly in his own words; others have been enunciated in a form more suitable to modern methods; and some terms have been introduced which were invented subsequent to the publication of the *Principia*. But the *Axiomata, sive Leges Motûs*, to which we now proceed, are given in Newton's own words; the two centuries which have nearly elapsed since he first gave them have not shown a necessity for any addition or modification. The first two, indeed, were discovered by Galileo, and the third, in some of its many forms, was known to Hooke, Huyghens, Wallis, Wren, and others; before the publication of the *Principia*. Of late there has been a tendency to split the second law into two, called respectively the second and third, and to ignore the third entirely, though using it *directly* in every dynamical problem; but all who have done so have been forced *indirectly* to acknowledge the completeness of Newton's system, by introducing as an axiom what is called D'Alembert's principle, which is really Newton's rejected third law in another form. Newton's own interpretation of his third law directly points out not only D'Alembert's principle, but also the modern principles of Work and Energy.

Axiom.

243. An Axiom is a proposition, the truth of which must be admitted as soon as the terms in which it is expressed are clearly understood. But, as we shall show in our chapter on "Experience," physical axioms are axiomatic to those only who have sufficient knowledge of the action of physical causes to enable them to see their truth. Without further remark we shall give Newton's Three Laws; it being remembered that, as the properties of matter *might* have been such as to render a totally different set of laws axiomatic, these laws must be considered as resting on convictions drawn from observation and experiment, *not* on intuitive perception.

244. Lex I. *Corpus omne perseverare in statu suo quiescendi vel movendi uniformiter in directum, nisi quatenus illud à viribus impressis cogitur statum suum mutare.* Newton's first law.

Every body continues in its state of rest or of uniform motion in a straight line, except in so far as it may be compelled by force to change that state.

245. The meaning of the term *Rest*, in physical science is essentially relative. Absolute rest is undefinable. If the universe of matter were finite, its centre of inertia might fairly be considered as absolutely at rest; or it might be imagined to be moving with any uniform velocity in any direction whatever through infinite space. But it is remarkable that the first law of motion enables us (§ 249, below) to explain what may be called *directional* rest. As will soon be shown, § 267, the plane in which the moment of momentum of the universe (if finite) round its centre of inertia is the greatest, which is clearly determinable from the actual motions at any instant, is fixed in direction in space. Rest.

246. We may logically convert the assertion of the first law of motion as to velocity into the following statements:—

The times during which any particular body, not compelled by force to alter the speed of its motion, passes through equal spaces, are equal. And, again—Every other body in the universe, not compelled by force to alter the speed of its motion, moves over equal spaces in successive intervals, during which the particular chosen body moves over equal spaces.

247. The first part merely expresses the convention universally adopted for the measurement of *Time*. The earth, in its rotation about its axis, presents us with a case of motion in which the condition, of not being compelled by force to alter its speed, is more nearly fulfilled than in any other which we can easily or accurately observe. And the numerical measurement of time practically rests on defining *equal intervals of time*, as *times during which the earth turns through equal* Time.

angles. This is, of course, a mere convention, and not a law of nature; and, as we now see it, is a part of Newton's first law.

Examples of the law.

248. The remainder of the law is not a convention, but a great truth of nature, which we may illustrate by referring to small and trivial cases as well as to the grandest phenomena we can conceive.

A curling-stone, projected along a horizontal surface of ice, travels equal distances, except in so far as it is retarded by friction and by the resistance of the air, in successive intervals of time during which the earth turns through equal angles. The sun moves through equal portions of interstellar space in times during which the earth turns through equal angles, except in so far as the resistance of interstellar matter, and the attraction of other bodies in the universe, alter his speed and that of the earth's rotation.

Directional fixedness.

249. If two material points be projected from one position, A, at the same instant with any velocities in any directions, and each left to move uninfluenced by force, the line joining them will be always parallel to a fixed direction. For the law asserts, as we have seen, that $AP : AP' :: AQ : AQ'$, if P, Q, and again P', Q' are simultaneous positions; and therefore PQ is parallel to $P'Q'$. Hence if four material points O, P, Q, R are all projected at one instant from one position, OP, OQ, OR are fixed directions of reference ever after. But, practically, the determination of fixed directions in space, § 267, is made to depend upon the rotation of groups of particles exerting forces on each other, and thus involves the Third Law of Motion.

The "Invariable Plane" of the solar system.

250. The whole law is singularly at variance with the tenets of the ancient philosophers who maintained that circular motion is perfect.

The last clause, "*nisi quatenus*," etc., admirably prepares for the introduction of the second law, by conveying the idea that *it is force alone which can produce a change of motion.* How, we naturally inquire, does the change of motion produced depend on the magnitude and direction of the force which produces it? And the answer is—

Newton's second law.

251. Lex II. *Mutationem motûs proportionalem esse vi motrici impressæ, et fieri secundum lineam rectam quâ vis illa imprimitur.*

Change of motion is proportional to force applied, and takes place in the direction of the straight line in which the force acts.

252. If any force generates motion, a double force will generate double motion, and so on, whether simultaneously or successively, instantaneously, or gradually applied. And this motion, if the body was moving beforehand, is either added to the previous motion if directly conspiring with it; or is subtracted if directly opposed; or is geometrically compounded with it, according to the kinematical principles already explained, if the line of previous motion and the direction of the force are inclined to each other at an angle. (This is a paraphrase of Newton's own comments on the second law.)

253. In Chapter I. we have considered change of velocity, or acceleration, as a purely geometrical element, and have seen how it may be at once inferred from the given initial and final velocities of a body. By the definition of quantity of motion (§ 210), we see that, if we multiply the change of velocity, thus geometrically determined, by the mass of the body, we have the change of motion referred to in Newton's law as the measure of the force which produces it.

It is to be particularly noticed, that in this statement there is nothing said about the actual motion of the body before it was acted on by the force: it is only the *change* of motion that concerns us. Thus the same force will produce precisely the same change of motion in a body, whether the body be at rest, or in motion with any velocity whatever.

254. Again, it is to be noticed that nothing is said as to the body being under the action of *one* force only; so that we may logically put a part of the second law in the following (apparently) amplified form:—

When any forces whatever act on a body, then, whether the body be originally at rest or moving with any velocity and in any direction, each force produces in the body the exact change of *motion which it would have produced if it had acted singly on the body originally at rest.*

Composition of forces.

255. A remarkable consequence follows immediately from this view of the second law. Since forces are measured by the changes of motion they produce, and their directions assigned by the directions in which these changes are produced; and since the changes of motion of one and the same body are in the directions of, and proportional to, the changes of velocity—a single force, measured by the resultant change of velocity, and in its direction, will be the equivalent of any number of simultaneously acting forces. Hence

The resultant of any number of forces (applied at one point) is to be found by the same geometrical process as the resultant of any number of simultaneous velocities.

256. From this follows at once (§ 27) the construction of the *Parallelogram of Forces* for finding the resultant of two forces, and the *Polygon of Forces* for the resultant of any number of forces, in lines all through one point.

The case of the equilibrium of a number of forces acting at one point, is evidently deducible at once from this; for if we introduce one other force equal and opposite to their resultant, this will produce a change of motion equal and opposite to the resultant change of motion produced by the given forces; that is to say, will produce a condition in which the point experiences no change of motion, which, as we have already seen, is the only kind of rest of which we can ever be conscious.

257. Though Newton perceived that the Parallelogram of Forces, or the fundamental principle of Statics, is essentially involved in the second law of motion, and gave a proof which is virtually the same as the preceding, subsequent writers on Statics (especially in this country) have very generally ignored the fact; and the consequence has been the introduction of various unnecessary Dynamical Axioms, more or less obvious, but in reality included in or dependent upon Newton's laws of motion. We have retained Newton's method, not only on account of its admirable simplicity, but because we believe it

contains the most philosophical foundation for the static as well as for the kinetic branch of the dynamic science.

Measurement of force and mass.

258. But the second law gives us the means of measuring force, and also of measuring the mass of a body.

For, if we consider the actions of various forces upon the same body for equal times, we evidently have changes of velocity produced which are *proportional to* the forces. The changes of velocity, then, give us in this case the means of comparing the magnitudes of different forces. Thus the velocities acquired in one second by the same mass (falling freely) at different parts of the earth's surface, give us the relative amounts of the earth's attraction at these places.

Again, if equal forces be exerted on different bodies, the changes of velocity produced in equal times must be *inversely* as the masses of the various bodies. This is approximately the case, for instance, with trains of various lengths started by the same locomotive: it is exactly realized in such cases as the action of an electrified body on a number of solid or hollow spheres of the same external diameter, and of different metals or of different thicknesses.

Again, if we find a case in which different bodies, each acted on by a force, acquire in the same time the same changes of velocity, the forces must be proportional to the masses of the bodies. This, when the resistance of the air is removed, is the case of falling bodies; and from it we conclude that the weight of a body in any given locality, or the force with which the earth attracts it, is proportional to its mass; a most important physical truth, which will be treated of more carefully in the chapter devoted to "Properties of Matter."

Translations from the kinematics of a point.

259. It appears, lastly, from this law, that every theorem of Kinematics connected with acceleration has its counterpart in Kinetics.

For instance, suppose X, Y, Z to be the components, parallel to fixed axes of x, y, z respectively, of the whole force acting on a particle of mass M. We see by § 212 that

$$M\frac{d^2x}{dt^2} = X, \quad M\frac{d^2y}{dt^2} = Y, \quad M\frac{d^2z}{dt^2} = Z;$$

or

$$M\ddot{x} = X, \quad M\ddot{y} = Y, \quad M\ddot{z} = Z.$$

Translations from the kinematics of a point.

Also, from these, we may evidently write,

$$M\ddot{s} = X\frac{dx}{ds} + Y\frac{dy}{ds} + Z\frac{dz}{ds} = X\frac{\dot{x}}{\dot{s}} + Y\frac{\dot{y}}{\dot{s}} + Z\frac{\dot{z}}{\dot{s}},$$

$$0 = X\frac{\dot{y}\ddot{z} - \dot{z}\ddot{y}}{\rho^{-1}\dot{s}^3} + Y\frac{\dot{z}\ddot{x} - \dot{x}\ddot{z}}{\rho^{-1}\dot{s}^3} + Z\frac{\dot{x}\ddot{y} - \dot{y}\ddot{x}}{\rho^{-1}\dot{s}^3},$$

$$\frac{M\dot{s}^2}{\rho} = X\frac{\dot{s}\ddot{x} - \dot{x}\ddot{s}}{\rho^{-1}\dot{s}^3} + Y\frac{\dot{s}\ddot{y} - \dot{y}\ddot{s}}{\rho^{-1}\dot{s}^3} + Z\frac{\dot{s}\ddot{z} - \dot{z}\ddot{s}}{\rho^{-1}\dot{s}^3}.$$

The second members of these equations are respectively the components of the impressed force, along the tangent (§ 9), perpendicular to the osculating plane (§ 9), and towards the centre of curvature, of the path described.

Measurement of force and mass.

260. We have, by means of the first two laws, arrived at a *definition* and a *measure* of force; and have also found how to compound, and therefore also how to resolve, forces; and also how to investigate the motion of a single particle subjected to given forces. But more is required before we can completely understand the more complex cases of motion, especially those in which we have mutual actions between or amongst two or more bodies; such as, for instance, attractions, or pressures, or transference of energy in any form. This is perfectly supplied by

Newton's third law.

261. LEX III. *Actioni contrariam semper et æqualem esse reactionem: sive corporum duorum actiones in se mutuò semper esse æquales et in partes contrarias dirigi.*

To every action there is always an equal and contrary reaction: or, the mutual actions of any two bodies are always equal and oppositely directed.

262. If one body presses or draws another, it is pressed or drawn by this other with an equal force in the opposite direction. If any one presses a stone with his finger, his finger is pressed with the same force in the opposite direction by the stone. A horse towing a boat on a canal is dragged backwards by a force equal to that which he impresses on the towing-rope forwards. By whatever amount, and in whatever direction, one body has its motion changed by impact upon another, this other body has its motion changed by the same

Newton's third law.

amount in the opposite direction; for at each instant during the impact the force between them was equal and opposite on the two. When neither of the two bodies has any rotation, whether before or after impact, the changes of velocity which they experience are inversely as their masses.

When one body attracts another from a distance, this other attracts it with an equal and opposite force. This law holds not only for the attraction of gravitation, but also, as Newton himself remarked and verified by experiment, for magnetic attractions: also for electric forces, as tested by Otto-Guericke.

263. What precedes is founded upon Newton's own comments on the third law, and the actions and reactions contemplated are simple forces. In the scholium appended, he makes the following remarkable statement, introducing another description of actions and reactions subject to his third law, the full meaning of which seems to have escaped the notice of commentators:—

Si æstimetur agentis actio ex ejus vi et velocitate conjunctim; et similiter resistentis reactio æstimetur conjunctim ex ejus partium singularum velocitatibus et viribus resistendi ab earum attritione, cohæsione, pondere, et acceleratione oriundis; erunt actio et reactio, in omni instrumentorum usu, sibi invicem semper æquales.

In a previous discussion Newton has shown what is to be understood by the velocity of a force or resistance; *i.e.*, that it is the velocity of the point of application of the force *resolved in the direction of the force.* Bearing this in mind, we may read the above statement as follows:—

If the Activity of an agent be measured by its amount and its velocity conjointly; and if, similarly, the Counter-activity of the resistance be measured by the velocities of its several parts and their several amounts conjointly, whether these arise from friction, cohesion, weight, or acceleration;—Activity and Counter-activity, in all combinations of machines, will be equal and opposite.*

Farther on (§§ 264, 293) we shall give an account of the

* We translate Newton's word "*Actio*" here by "Activity" to avoid confusion with the word "Action" so universally used in modern dynamical treatises, according to the definition of § 326 below, in relation to Maupertuis' principle of "Least Action."

splendid dynamical theory founded by D'Alembert and Lagrange on this most important remark.

D'Alembert's principle.

264. Newton, in the passage just quoted, points out that forces of resistance against acceleration are to be reckoned as reactions equal and opposite to the actions by which the acceleration is produced. Thus, if we consider any one material point of a system, its reaction against acceleration must be equal and opposite to the resultant of the forces which that point experiences, whether by the actions of other parts of the system upon it, or by the influence of matter not belonging to the system. In other words, it must be in equilibrium with these forces. Hence Newton's view amounts to this, that all the forces of the system, with the reactions against acceleration of the material points composing it, form groups of equilibrating systems for these points considered individually. Hence, by the principle of superposition of forces in equilibrium, all the forces acting on points of the system form, with the reactions against acceleration, an equilibrating set of forces on the whole system. This is the celebrated principle first explicitly stated, and very usefully applied, by D'Alembert in 1742, and still known by his name. We have seen, however, that it is very distinctly implied in Newton's own interpretation of his third law of motion. As it is usual to investigate the general equations or conditions of equilibrium, in dynamical treatises, before entering in detail on the kinetic branch of the subject, this principle is found practically most useful in showing how we may write down at once the equations of motion for any system for which the equations of equilibrium have been investigated.

Mutual forces between particles of a rigid body.

265. Every rigid body may be imagined to be divided into indefinitely small parts. Now, in whatever form we may eventually find a *physical* explanation of the origin of the forces which act between these parts, it is certain that each such small part may be considered to be held in its position relatively to the others by mutual forces in lines joining them.

266. From this we have, as immediate consequences of the second and third laws, and of the preceding theorems relating

to Centre of Inertia and Moment of Momentum, a number of important propositions such as the following:—

Motion of centre of inertia of a rigid body.

(*a*) The centre of inertia of a rigid body moving in any manner, but free from external forces, moves uniformly in a straight line.

(*b*) When any forces whatever act on the body, the motion of the centre of inertia is the same as it would have been had these forces been applied with their proper magnitudes and directions at that point itself.

Moment of momentum of a rigid body.

(*c*) Since the moment of a force acting on a particle is the same as the moment of momentum it produces in unit of time, the changes of moment of momentum in any two parts of a rigid body due to their mutual action are equal and opposite. Hence the moment of momentum of a rigid body, about any axis which is fixed in direction, and passes through a point which is either fixed in space or moves uniformly in a straight line, is unaltered by the mutual actions of the parts of the body.

(*d*) The rate of increase of moment of momentum, when the body is acted on by external forces, is the sum of the moments of these forces about the axis.

Conservation of momentum, and of moment of momentum.

267. We shall for the present take for granted, that the mutual action between two rigid bodies may in every case be imagined as composed of pairs of equal and opposite forces in straight lines. From this it follows that the sum of the quantities of motion, parallel to any fixed direction, of two rigid bodies influencing one another in any possible way, remains unchanged by their mutual action; also that the sum of the moments of momentum of all the particles of the two bodies, round any line in a fixed direction in space, and passing through any point moving uniformly in a straight line in any direction, remains constant. From the first of these propositions we infer that the centre of inertia of any number of mutually influencing bodies, if in motion, continues moving uniformly in a straight line, unless in so far as the direction or velocity of its motion is changed by forces acting mutually between them and some other matter not belonging to them; also that the centre of inertia of any body or system of bodies moves

The "Invariable Plane" is a plane through the centre of inertia, perpendicular to the resultant axis.

just as all their matter, if concentrated in a point, would move under the influence of forces equal and parallel to the forces really acting on its different parts. From the second we infer that the axis of resultant rotation through the centre of inertia of any system of bodies, or through any point either at rest or moving uniformly in a straight line, remains unchanged in direction, and the sum of moments of momenta round it remains constant if the system experiences no force from without. This principle used to be called *Conservation of Areas*, a very ill-considered designation. From this principle it follows that if by internal action such as geological upheavals or subsidences, or pressure of the winds on the water, or by evaporation and rain- or snow-fall, or by any influence not depending on the attraction of sun or moon (even though dependent on solar heat), the disposition of land and water becomes altered, the component round any fixed axis of the moment of momentum of the earth's rotation remains constant.

Terrestrial application.

Rate of doing work.

268. The foundation of the abstract theory of energy is laid by Newton in an admirably distinct and compact manner in the sentence of his scholium already quoted (§ 263), in which he points out its application to mechanics*. The *actio agentis*, as he defines it, which is evidently equivalent to the product of the effective component of the force, into the velocity of the point on which it acts, is simply, in modern English phraseology, the rate at which the agent works. The subject for measurement here is precisely the same as that for which Watt, a hundred years later, introduced the practical unit of a "*Horse-power*," or the rate at which an agent works when overcoming 33,000 times the weight of a pound through the space of a foot in a minute; that is, producing 550 foot-pounds of work per second. The unit, however, which is most generally convenient is that which Newton's definition implies, namely, the rate of doing work in which the unit of energy is produced in the unit of time.

Horse-power.

* The reader will remember that we use the word "mechanics" in its true classical sense, the science of machines, the sense in which Newton himself used it, when he dismissed the further consideration of it by saying (in the scholium referred to), *Cæterum mechanicam tractare non est hujus instituti.*

269. Looking at Newton's words (§ 263) in this light, we see that they may be logically converted into the following form:—

Energy in abstract dynamics.

Work done on any system of bodies (in Newton's statement, the parts of any machine) has its equivalent in work done against friction, molecular forces, or gravity, if there be no acceleration; but if there be acceleration, part of the work is expended in overcoming the resistance to acceleration, and the additional kinetic energy developed is equivalent to the work so spent. This is evident from § 214.

When part of the work is done against molecular forces, as in bending a spring; or against gravity, as in raising a weight; the recoil of the spring, and the fall of the weight, are capable at any future time, of reproducing the work originally expended (§ 241). But in Newton's day, and long afterwards, it was supposed that work was *absolutely lost* by friction; and, indeed, this statement is still to be found even in recent authoritative treatises. But we must defer the examination of this point till we consider in its modern form the principle of *Conservation of Energy*.

270. If a system of bodies, given either at rest or in motion, be influenced by no forces from without, the sum of the kinetic energies of all its parts is augmented in any time by an amount equal to the whole work done in that time by the mutual forces, which we may imagine as acting between its points. When the lines in which these forces act remain all unchanged in length, the forces do no work, and the sum of the kinetic energies of the whole system remains constant. If, on the other hand, one of these lines varies in length during the motion, the mutual forces in it will do work, or will consume work, according as the distance varies with or against them.

271. A limited system of bodies is said to be *dynamically conservative* (or simply *conservative*, when force is understood to be the subject), if the mutual forces between its parts always perform, or always consume, the same amount of work during any motion whatever, by which it can pass from one particular configuration to another.

Conservative system.

Foundation of the theory of energy.

272. The whole theory of energy in physical science is founded on the following proposition:—

If the mutual forces between the parts of a material system are independent of their velocities, whether relative to one another, or relative to any external matter, the system must be dynamically conservative.

Physical axiom that "the Perpetual Motion is impossible" introduced.

For if more work is done by the mutual forces on the different parts of the system in passing from one particular configuration to another, by one set of paths than by another set of paths, let the system be directed, by frictionless constraint, to pass from the first configuration to the second by one set of paths and return by the other, over and over again for ever. It will be a continual source of energy without any consumption of materials, which is impossible.

Potential energy of conservative system.

273. The *potential energy* of a conservative system, in the configuration which it has at any instant, is the amount of work required to bring it to that configuration against its mutual forces during the passage of the system from any one chosen configuration to the configuration at the time referred to. It is generally, but not always, convenient to fix the particular configuration chosen for the zero of reckoning of potential energy, so that the potential energy, in every other configuration practically considered, shall be positive.

274. The potential energy of a conservative system, at any instant, depends solely on its configuration at that instant, being, according to definition, the same at all times when the system is brought again and again to the same configuration. It is therefore, in mathematical language, said to be a function of the co-ordinates by which the positions of the different parts of the system are specified. If, for example, we have a conservative system consisting of two material points; or two rigid bodies, acting upon one another with force dependent only on the relative position of a point belonging to one of them, and a point belonging to the other; the potential energy of the system depends upon the co-ordinates of one of these points relatively to lines of reference in fixed directions through the other. It will therefore, in general, depend on three independent co-ordinates, which we may conveniently take as the distance between the two points, and two angles specifying the absolute direction of the line joining them. Thus, for example, let the bodies be two uniform metal globes, electrified with any given quantities of electricity, and placed in an insulating medium such as air, in a region of space under the influence of a vast distant electrified body. The mutual action between these two spheres will depend solely on the relative position of their centres. It will consist partly of gravitation, depending solely on the distance between their centres, and of electric force, which will depend on the distance between them, but also, in virtue of the inductive action of the distant body, will depend on the absolute direction of the line joining their centres. In our divisions devoted to gravitation and electricity respectively, we shall investigate the portions of the mutual potential energy of the two bodies depending on these two agencies separately. The former we shall find to be the product of their masses divided by the distance between their centres; the latter a somewhat complicated function of the distance between the centres and the angle which this line makes with the direction of the resultant electric force of the distant electrified body. Or again, if the system consist of two balls of soft iron, in any locality of the earth's surface, their mutual action will be partly gravitation, and partly due to the magnetism induced in them by terrestrial magnetic force. The portion of the mutual potential energy depending on the latter cause, will be a function of the distance between their centres and the inclination of this line to the direction of the terrestrial magnetic force. It will agree in mathematical expression with the potential energy of electric action in the preceding case, so far as the inclination is concerned, but the law of variation with the distance will be less easily determined.

Potential energy of conservative system.

Inevitable loss of energy of visible motions.

275. In nature the hypothetical condition of § 271 is *apparently violated* in all circumstances of motion. A material system can never be brought through any returning cycle of motion without spending more work against the mutual forces of its parts than is gained from these forces, because no relative motion can take place without meeting with frictional or

Inevitable loss of energy of visible motions.

other forms of resistance; among which are included (1) mutual friction between solids sliding upon one another; (2) resistances due to the viscosity of fluids, or imperfect elasticity of solids; (3) resistances due to the induction of electric currents; (4) resistances due to varying magnetization under the influence of imperfect magnetic retentiveness. No motion in nature can take place without meeting resistance due to some, if not to all, of these influences. It is matter of every day experience that friction and imperfect elasticity of solids impede the action of all artificial mechanisms; and that even when bodies are detached, and left to move freely in the air, as falling bodies, or as projectiles, they experience resistance owing to the viscosity of the air.

The greater masses, planets and comets, moving in a less resisting medium, show less indications of resistance*. Indeed it cannot be said that observation upon any one of these bodies, with the exception of Encke's comet, has demonstrated resistance. But the analogies of nature, and the ascertained facts of physical science, forbid us to doubt that every one of them, every star, and every body of any kind moving in any part of space, has its relative motion impeded by the air, gas, vapour, medium, or whatever we choose to call the substance occupying the space immediately round it; just as the motion of a rifle bullet is impeded by the resistance of the air.

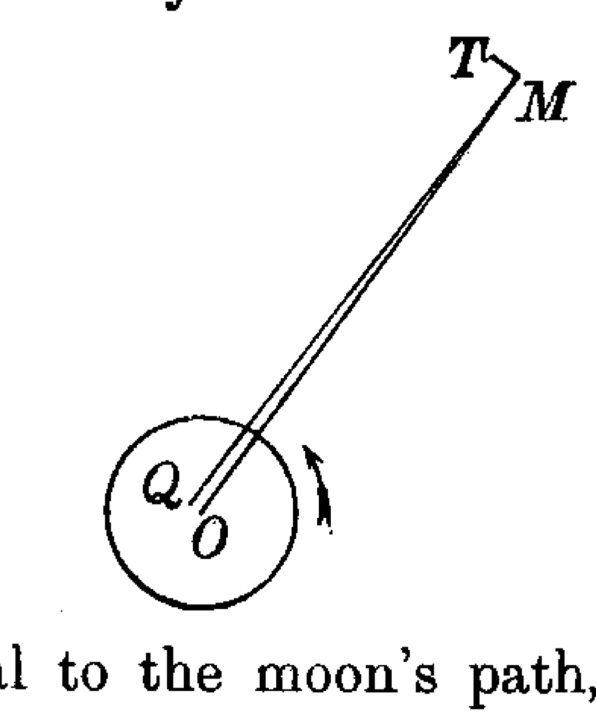
Effect of tidal friction.

276. There are also indirect resistances, owing to friction impeding the tidal motions, on all bodies (like the earth) partially or wholly covered by liquid, which, as long as these bodies move relatively to neighbouring bodies, must keep drawing off energy from their relative motions. Thus, if we consider, in the first place, the action of the moon alone, on the earth with its oceans, lakes, and rivers, we perceive that it must tend to equalize the periods of the earth's rotation about its axis, and of the revolution of the two bodies about their centre of inertia; because as long as these periods differ, the tidal action on the

* Newton, *Principia.* (Remarks on the first law of motion.) "Majora autem Planetarum et Cometarum corpora motus suos et progressivos et circulares, in spatiis minus resistentibus factos, conservant diutius."

Effect of tidal friction.

earth's surface must keep subtracting energy from their motions. To view the subject more in detail, and, at the same time, to avoid unnecessary complications, let us suppose the moon to be a uniform spherical body. The mutual action and reaction of gravitation between her mass and the earth's, will be equivalent to a single force in some line through her centre; and must be such as to impede the earth's rotation as long as this is performed in a shorter period than the moon's motion round the earth. It must therefore lie in some such direction as the line MQ in the diagram, which represents, necessarily with enormous exaggeration, its deviation, OQ, from the earth's centre. Now the actual force on the moon in the line MQ, may be regarded as consisting of a force in the line MO towards the earth's centre, sensibly equal in amount to the whole force, and a comparatively very small force in the line MT perpendicular to MO. This latter is very nearly tangential to the moon's path, and is in the direction *with* her motion. Such a force, if suddenly commencing to act, would, in the first place, increase the moon's velocity; but after a certain time she would have moved so much farther from the earth, in virtue of this acceleration, as to have lost, by moving against the earth's attraction, as much velocity as she had gained by the tangential accelerating force. The effect of a continued tangential force, acting with the motion, but so small in amount as to make only a small deviation at any moment from the circular form of the orbit, is to gradually increase the distance from the central body, and to cause as much again as its own amount of work to be done against the attraction of the central mass, by the kinetic energy of motion lost. The circumstances will be readily understood, by considering this motion round the central body in a very gradual spiral path tending outwards. Provided the law of the central force is the inverse square of the distance, the tangential component of the central force against the motion will be twice as great as the disturbing tangential force in the direction with the motion; and therefore one-half of the amount of work done



Inevitable loss of energy of visible motions. Tidal friction.

against the former, is done by the latter, and the other half by kinetic energy taken from the motion. The integral effect on the moon's motion, of the particular disturbing cause now under consideration, is most easily found by using the principle of moments of momenta. Thus we see that as much moment of momentum is gained in any time by the motions of the centres of inertia of the moon and earth relatively to their common centre of inertia, as is lost by the earth's rotation about its axis. The sum of the moments of momentum of the centres of inertia of the moon and earth as moving at present, is about 4·45 times the present moment of momentum of the earth's rotation. The average plane of the former is the ecliptic; and therefore the axes of the two momenta are inclined to one another at the average angle of $23^\circ\ 27\frac{1}{2}'$, which, as we are neglecting the sun's influence on the plane of the moon's motion, may be taken as the actual inclination of the two axes at present. The resultant, or whole moment of momentum, is therefore 5·38 times that of the earth's present rotation, and its axis is inclined $19^\circ\ 13'$ to the axis of the earth. Hence the ultimate tendency of the tides is, to reduce the earth and moon to a simple uniform rotation with this resultant moment round this resultant axis, as if they were two parts of one rigid body: in which condition the moon's distance would be increased (approximately) in the ratio 1 : 1·46, being the ratio of the square of the present moment of momentum of the centres of inertia to the square of the whole moment of momentum; and the period of revolution in the ratio 1 : 1·77, being that of the cubes of the same quantities. The distance would therefore be increased to 347,100 miles, and the period lengthened to 48·36 days. Were there no other body in the universe but the earth and the moon, these two bodies might go on moving thus for ever, in circular orbits round their common centre of inertia, and the earth rotating about its axis in the same period, so as always to turn the same face to the moon, and therefore to have all the liquids at its surface at rest relatively to the solid. But the existence of the sun would prevent any such state of things from being permanent. There would be solar tides—twice high water and twice low water—in the period of the earth's revolution relatively to the sun (that is

Inevitable loss of energy of visible motions. Tidal friction.

to say, twice in the solar day, or, which would be the same thing, the month). This could not go on without loss of energy by fluid friction. It is easy to trace the whole course of the disturbance in the earth's and moon's motions which this cause would produce*: its first effect must be to bring the moon to fall in to the earth, with compensation for loss of moment of momentum of the two round their centre of inertia in increase of its distance from the sun, and then to reduce the very rapid rotation of the compound body, Earth-and-Moon, after the collision, and farther increase its distance from the Sun till ultimately, (corresponding action on liquid matter on the Sun having *its* effect also, and it being for our illustration supposed that there are no other planets,) the two bodies shall rotate round their common centre of inertia, like parts of one rigid body. It is remarkable that the whole frictional effect of the lunar and solar tides should be, first to augment the moon's distance from the earth to a maximum, and then to diminish it, till ultimately the moon falls in to the earth: and first to diminish, after that to increase, and lastly to diminish the earth's rotational velocity. We hope to return to the subject later†, and to consider the general problem of the motion of any number of rigid bodies or material points acting on one another with mutual forces, under any actual physical law, and therefore, as we shall see, necessarily subject to loss of energy as long as any of their mutual distances vary; that is to say, until all subside into a state of motion in circles round an axis passing through their centre of inertia, like parts of one rigid body. It is probable

* The friction of these solar tides on the earth would cause the earth to rotate still slower; and then the moon's influence, tending to keep the earth rotating with always the same face towards herself, would resist this further reduction in the speed of the rotation. Thus (as explained above with reference to the moon) there would be from the sun a force opposing the earth's rotation, and from the moon a force promoting it. Hence according to the preceding explanation applied to the altered circumstances, the line of the earth's attraction on the moon passes now as before, not through the centre of inertia of the earth, but now in a line slightly *behind* it (instead of *before*, as formerly). It therefore now resists the moon's motion of revolution. The combined effect of this resistance and of the earth's attraction on the moon is, like that of a resisting medium, to cause the moon to fall in towards the earth in a spiral path with gradually increasing velocity.

† [See II. § 830 and Appendices G (*a*) and (*b*), where numerical values are given differing slightly from those used here. G. H. D.]

Inevitable loss of energy of visible motions. Tidal friction.

that the moon, in ancient times liquid or viscous in its outer layer if not throughout, was thus brought to turn always the same face to the earth.

277. We have no data in the present state of science for estimating the relative importance of tidal friction, and of the resistance of the resisting medium through which the earth and moon move; but whatever it may be, there can be but one ultimate result for such a system as that of the sun and planets, if continuing long enough under existing laws, and not disturbed by meeting with other moving masses in space. That result is the falling together of all into one mass, which, although rotating for a time, must in the end come to rest relatively to the surrounding medium.

Ultimate tendency of the solar system.

Conservation of energy.

278. The theory of energy cannot be completed until we are able to examine the physical influences which accompany loss of energy in each of the classes of resistance mentioned above, § 275. We shall then see that in every case in which energy is lost by resistance, heat is generated; and we shall learn from Joule's investigations that the quantity of heat so generated is a perfectly definite equivalent for the energy lost. Also that in no natural action is there ever a development of energy which cannot be accounted for by the disappearance of an equal amount elsewhere by means of some known physical agency. Thus we shall conclude, that if any limited portion of the material universe could be perfectly isolated, so as to be prevented from either giving energy to, or taking energy from, matter external to it, the sum of its potential and kinetic energies would be the same at all times: in other words, that every material system subject to no other forces than actions and reactions between its parts, is a dynamically conservative system, as defined above, § 271. But it is only when the inscrutably minute motions among small parts, possibly the ultimate molecules of matter, which constitute light, heat, and magnetism; and the intermolecular forces of chemical affinity; are taken into account, along with the palpable motions and measurable forces of which we become cognizant by direct observation, that we can recognise

the universally conservative character of all natural dynamic action, and perceive the bearing of the principle of reversibility on the whole class of natural actions involving resistance, which seem to violate it. In the meantime, in our studies of abstract dynamics, it will be sufficient to introduce a special reckoning for energy lost in working against, or gained from work done by, forces not belonging palpably to the conservative class.

Conservation of energy.

279. As of great importance in farther developments, we prove a few propositions intimately connected with energy.

280. The kinetic energy of any system is equal to the sum of the kinetic energies of a mass equal to the sum of the masses of the system, moving with a velocity equal to that of its centre of inertia, and of the motions of the separate parts relatively to the centre of inertia.

Kinetic energy of a system.

For if x, y, z be the co-ordinates of any particle, m, of the system; ξ, η, ζ its co-ordinates relative to the centre of inertia; and $\bar{x}, \bar{y}, \bar{z}$, the co-ordinates of the centre of inertia itself; we have for the whole kinetic energy

$$\tfrac{1}{2}\Sigma m\left\{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2\right\}=\tfrac{1}{2}\Sigma m\left\{\left(\frac{d(\bar{x}+\xi)}{dt}\right)^2+\left(\frac{d(\bar{y}+\eta)}{dt}\right)^2+\left(\frac{d(\bar{z}+\zeta)}{dt}\right)^2\right\}.$$

But by the properties of the centre of inertia, we have

$$\Sigma m\frac{d\bar{x}}{dt}\frac{d\xi}{dt}=\frac{d\bar{x}}{dt}\Sigma m\frac{d\xi}{dt}=0, \text{ etc. etc.}$$

Hence the preceding is equal to

$$\tfrac{1}{2}\Sigma m\left\{\left(\frac{d\bar{x}}{dt}\right)^2+\left(\frac{d\bar{y}}{dt}\right)^2+\left(\frac{d\bar{z}}{dt}\right)^2\right\}+\tfrac{1}{2}\Sigma m\left\{\left(\frac{d\xi}{dt}\right)^2+\left(\frac{d\eta}{dt}\right)^2+\left(\frac{d\zeta}{dt}\right)^2\right\},$$

which proves the proposition.

281. The kinetic energy of rotation of a rigid system about any axis is (§ 95) expressed by $\frac{1}{2}\Sigma mr^2\omega^2$, where m is the mass of any part, r its distance from the axis, and ω the angular velocity of rotation. It may evidently be written in the form $\frac{1}{2}\omega^2\Sigma mr^2$. The factor Σmr^2 is of very great importance in kinetic investigations, and has been called the *Moment of Inertia* of the system about the axis in question. The moment of inertia about any axis is therefore found by summing the

Moment of inertia.

Moment of inertia. products of the masses of all the particles each into the square of its distance from the axis.

Moment of momentum of a rotating rigid body. It is important to notice that the moment of momentum of any rigid system about an axis, being $\Sigma mvr = \Sigma mr^2\omega$, is the product of the angular velocity into the moment of inertia.

If we take a quantity k, such that

$$k^2\Sigma m = \Sigma mr^2$$

Radius of gyration. k is called the *Radius of Gyration* about the axis from which r is measured. The radius of gyration about any axis is therefore the distance from that axis at which, if the whole mass were placed, it would have the same moment of inertia as before. Fly-wheel. In a fly-wheel, where it is desirable to have as great a moment of inertia with as small a mass as possible, within certain limits of dimensions, the greater part of the mass is formed into a ring of the largest admissible diameter, and the radius of this ring is then approximately the radius of gyration of the whole.

Moment of inertia about any axis. A rigid body being referred to rectangular axes passing through any point, it is required to find the moment of inertia about an axis through the origin making given angles with the co-ordinate axes.

Let λ, μ, ν be its direction-cosines. Then the distance (r) of the point x, y, z from it is, by § 95,

$$r^2 = (\mu z - \nu y)^2 + (\nu x - \lambda z)^2 + (\lambda y - \mu x)^2,$$

and therefore

$$Mk^2 = \Sigma mr^2 = \Sigma m\left[\lambda^2(y^2+z^2) + \mu^2(z^2+x^2) + \nu^2(x^2+y^2) - 2\mu\nu yz - 2\nu\lambda zx - 2\lambda\mu xy\right]$$

which may be written

$$A\lambda^2 + B\mu^2 + C\nu^2 - 2\alpha\mu\nu - 2\beta\nu\lambda - 2\gamma\lambda\mu,$$

where A, B, C are the moments of inertia about the axes, and $\alpha = \Sigma myz$, $\beta = \Sigma mzx$, $\gamma = \Sigma mxy$. From its derivation we see that this quantity is *essentially positive*. Hence when, by a proper linear transformation, it is deprived of the terms containing the products of λ, μ, ν, it will be brought to the form

$$Mk^2 = A\lambda^2 + B\mu^2 + C\nu^2 = Q,$$

where A, B, C are essentially positive. They are evidently the moments of inertia about the new rectangular axes of co-ordinates,

and λ, μ, ν the corresponding direction-cosines of the axis round which the moment of inertia is to be found. Moment of inertia about any axis.

Let $A > B > C$, if they are unequal. Then

$$A\lambda^2 + B\mu^2 + C\nu^2 = Q(\lambda^2 + \mu^2 + \nu^2)$$

shows that Q cannot be greater than A, nor less than C. Also, if A, B, C be equal, Q is equal to each.

If a, b, c be the radii of gyration about the new axes of x, y, z,

$$A = Ma^2,\ B = Mb^2,\ C = Mc^2,$$

and the above equation gives

$$k^2 = a^2\lambda^2 + b^2\mu^2 + c^2\nu^2.$$

But if x, y, z be any point in the line whose direction-cosines are λ, μ, ν, and r its distance from the origin, we have

$$\frac{x}{\lambda} = \frac{y}{\mu} = \frac{z}{\nu} = r, \text{ and therefore}$$

$$k^2r^2 = a^2x^2 + b^2y^2 + c^2z^2.$$

If, therefore, we consider the ellipsoid whose equation is

$$a^2x^2 + b^2y^2 + c^2z^2 = \epsilon^4,$$

we see that it intercepts on the line whose direction-cosines are λ, μ, ν—and about which the radius of gyration is k, a length r which is given by the equation

$$k^2r^2 = \epsilon^4;$$

or the rectangle under any radius-vector of this ellipsoid and the radius of gyration about it is constant. Its semi-axes are evidently $\frac{\epsilon^2}{a}$, $\frac{\epsilon^2}{b}$, $\frac{\epsilon^2}{c}$ where ϵ may have any value we may assign.

Thus it is evident that

282. For every rigid body there may be described about any point as centre, an ellipsoid (called *Poinsot's Momental Ellipsoid**) which is such that the length of any radius-vector is Momental ellipsoid.

* The definition is not Poinsot's, but ours. The momental ellipsoid as we define it is fairly called Poinsot's, because of the splendid use he has made of it in his well-known kinematic representation of the solution of the problem —to find the motion of a rigid body with one point held fixed but otherwise influenced by no forces—which, with Sylvester's beautiful theorem completing it so as to give a purely kinematical mechanism to show the time which the body takes to attain any particular position, we reluctantly keep back for our Second Volume.

Momental ellipsoid.

inversely proportional to the radius of gyration of the body about that radius-vector as axis.

Principal axes.

The axes of this ellipsoid are, and might be defined as, the *Principal Axes* of inertia of the body for the point in question: but the best definition of principal axes of inertia is given below. First take two preliminary lemmas:—

Equilibration of Centrifugal Forces.

(1) If a rigid body rotate round any axis, the centrifugal forces are reducible to a single force perpendicular to the axis of rotation, and to a couple (§ 234 above) having its axis parallel to the line of this force.

(2) But in particular cases the couple may vanish, or both couple and force may vanish and the centrifugal forces be in equilibrium. The force vanishes if, and only if, the axis of rotation passes through the body's centre of inertia.

Definition of Principal Axes of Inertia.

DEF. (1). Any axis is called a principal axis of a body's inertia, or simply a principal axis of the body, if when the body rotates round it the centrifugal forces either balance or are reducible to a single force.

DEF. (2). A principal axis not through the centre of inertia is called a principal axis of inertia for the point of itself through which the resultant of centrifugal forces passes.

DEF. (3). A principal axis which passes through the centre of inertia is a principal axis for every point of itself.

The proofs of the lemmas may be safely left to the student as exercises on § 559 below; and from the proof the identification of the principal axes as now defined with the principal axes of Poinsot's momental ellipsoid is seen immediately by aid of the analysis of § 281.

283. The proposition of § 280 shows that the moment of inertia of a rigid body about any axis is equal to that which the mass, if collected at the centre of inertia, would have about this axis, together with that of the body about a parallel axis through its centre of inertia. It leads us naturally to investigate the relation between principal axes for any point and principal axes for the centre of inertia. The following investigation proves the remarkable theorem of § 284, which was first given in 1811 by Binet in the *Journal de l'École Polytechnique.*

Principal axes.

Let the origin, O, be the centre of inertia, and the axes the principal axes at that point. Then, by §§ 280, 281, we have for the moment of inertia about a line through the point P (ξ, η, ζ), whose direction-cosines are λ, μ, ν;

$$Q = A\lambda^2 + B\mu^2 + C\nu^2 + M\{(\mu\zeta - \nu\eta)^2 + (\nu\xi - \lambda\zeta)^2 + (\lambda\eta - \mu\xi)^2\}$$
$$= \{A + M(\eta^2 + \zeta^2)\}\lambda^2 + \{B + M(\zeta^2 + \xi^2)\}\mu^2 + \{C + M(\xi^2 + \eta^2)\}\nu^2$$
$$- 2M(\mu\nu\eta\zeta + \nu\lambda\zeta\xi + \lambda\mu\xi\eta).$$

Substituting for Q, A, B, C their values, and dividing by M, we have

$$k^2 = (a^2 + \eta^2 + \zeta^2)\lambda^2 + (b^2 + \zeta^2 + \xi^2)\mu^2 + (c^2 + \xi^2 + \eta^2)\nu^2$$
$$- 2(\eta\zeta\mu\nu + \zeta\xi\nu\lambda + \xi\eta\lambda\mu).$$

Let it be required to find λ, μ, ν so that the direction specified by them may be a principal axis. Let $s = \lambda\xi + \mu\eta + \nu\zeta$, *i.e.* let s represent the projection of OP on the axis sought.

The axes of the ellipsoid

$$(a^2 + \eta^2 + \zeta^2)x^2 + \ldots\ldots - 2(\eta\zeta yz + \ldots\ldots) = H \;\ldots\ldots(a),$$

are found by means of the equations

$$\left.\begin{aligned} (a^2 + \eta^2 + \zeta^2 - p)\lambda - \xi\eta\mu - \zeta\xi\nu = 0 \\ -\xi\eta\lambda + (b^2 + \zeta^2 + \xi^2 - p)\mu - \eta\zeta\nu = 0 \\ -\zeta\xi\lambda - \eta\zeta\mu + (c^2 + \xi^2 + \eta^2 - p)\nu = 0 \end{aligned}\right\}\ldots\ldots\ldots\ldots\ldots(b).$$

If, now, we take f to denote OP, or $(\xi^2 + \eta^2 + \zeta^2)^{\frac{1}{2}}$, these equations, where p is clearly the square of the radius of gyration about the axis to be found, may be written

$$(a^2 + f^2 - p)\lambda - \xi(\xi\lambda + \eta\mu + \zeta\nu) = 0,$$
$$\text{etc.} = \text{etc.},$$

or

$$(a^2 + f^2 - p)\lambda - \xi s = 0,$$
$$\text{etc.} = \text{etc.},$$

or

$$\left.\begin{aligned} (a^2 - K)\lambda - \xi s = 0 \\ (b^2 - K)\mu - \eta s = 0 \\ (c^2 - K)\nu - \zeta s = 0 \end{aligned}\right\}\ldots\ldots\ldots\ldots\ldots\ldots(c)$$

where $K = p - f^2$. Hence

$$\lambda = \frac{\xi s}{a^2 - K}, \text{ etc.}$$

Multiply, in order, by ξ, η, ζ, add, and divide by s, and we get

$$\frac{\xi^2}{a^2 - K} + \frac{\eta^2}{b^2 - K} + \frac{\zeta^2}{c^2 - K} = 1 \;\ldots\ldots\ldots\ldots\ldots\ldots(d).$$

Principal axes.

By (c) we see that (λ, μ, ν) is the direction of the normal through the point P, (ξ, η, ζ) of the surface represented by the equation

$$\frac{x^2}{a^2-K}+\frac{y^2}{b^2-K}+\frac{z^2}{c^2-K}=1 \quad\ldots\ldots\ldots\ldots(e),$$

which is obviously a surface of the second degree confocal with the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1 \quad\ldots\ldots\ldots\ldots(f),$$

and passing through P in virtue of (d), which determines K accordingly. The three roots of this cubic are clearly all real; one of them is less than the least of a^2, b^2, c^2, and positive or negative according as P is within or without the ellipsoid (f). And if $a>b>c$, the two others are between c^2 and b^2, and between b^2 and a^2, respectively. The addition of f^2 to each gives the square of the radius of gyration round the corresponding principal axis. Hence

Binet's Theorem.

284. The principal axes for any point of a rigid body are normals to the three surfaces of the second order through that point, confocal with the ellipsoid, which has its centre at the centre of inertia, and its three principal diameters co-incident with the three principal axes for that point, and equal respectively to the doubles of the radii of gyration round them. This ellipsoid is called the *Central Ellipsoid*.

Central ellipsoid.

Kinetic symmetry round a point:

285. A rigid body is said to be kinetically symmetrical about its centre of inertia when its moments of inertia about three principal axes through that point are equal; and therefore necessarily the moments of inertia about *all* axes through that point equal, § 281, and all these axes principal axes. About it uniform spheres, cubes, and in general any complete crystalline solid of the first system (see chapter on Properties of Matter), are kinetically symmetrical.

round an axis.

A rigid body is kinetically symmetrical about an *axis* when this axis is one of the principal axes through the centre of inertia, and the moments of inertia about the other two, and therefore about any line in their plane, are equal. A spheroid, a square or equilateral triangular prism or plate, a circular ring, disc, or cylinder, or any complete crystal of the second or fourth system, is kinetically symmetrical about its axis.

Energy in abstract dynamics.

286. The only actions and reactions between the parts of a system, not belonging palpably to the conservative class, which we shall consider in abstract dynamics, are those of friction between solids sliding on solids, except in a few instances in which we shall consider the general character and ultimate results of effects produced by viscosity of fluids, imperfect elasticity of solids, imperfect electric conduction, or imperfect magnetic retentiveness. We shall also, in abstract dynamics, consider forces as applied to parts of a limited system arbitrarily from without. These we shall call, for brevity, the applied forces.

287. The law of energy may then, in abstract dynamics, be expressed as follows:—

The whole work done in any time, on any limited material system, by applied forces, is equal to the whole effect in the forms of potential and kinetic energy produced in the system, together with the work lost in friction.

288. This principle may be regarded as comprehending the whole of abstract dynamics, because, as we now proceed to show, the conditions of equilibrium and of motion, in every possible case, may be immediately derived from it.

Equilibrium.

289. A material system, whose relative motions are unresisted by friction, is in equilibrium in any particular configuration if, and is not in equilibrium unless, the work done by the applied forces is equal to the potential energy gained, in any possible infinitely small displacement from that configuration. This is the celebrated principle of "virtual velocities" which Lagrange made the basis of his *Mécanique Analytique*. The ill-chosen name "virtual velocities" is now falling into disuse.

Principle of virtual velocities.

290. To prove it, we have first to remark that the system cannot possibly move away from any particular configuration except by work being done upon it by the forces to which it is subject: it is therefore in equilibrium if the stated condition is fulfilled. To ascertain that nothing less than this condition can secure its equilibrium, let us first consider a system having only one degree of freedom to move. Whatever forces act on the whole system, we may always hold it in equilibrium by a single force applied to any one point of the system in its line

Principle of virtual velocities. of motion, opposite to the direction in which it tends to move, and of such magnitude that, in any infinitely small motion in either direction. it shall resist, or shall do, as much work as the other forces, whether applied or internal, altogether do or resist. Now, by the principle of superposition of forces in equilibrium, we might, without altering their effect, apply to any one point of the system such a force as we have just seen would hold the system in equilibrium, and another force equal and opposite to it. All the other forces being balanced by one of these two, they and it might again, by the principle of superposition of forces in equilibrium, be removed; and therefore the whole set of given forces would produce the same effect, whether for equilibrium or for motion, as the single force which is left acting alone. This single force, since it is in a line in which the point of its application is free to move, must move the system. Hence the given forces, to which this single force has been proved equivalent, cannot possibly be in equilibrium unless their whole work for an infinitely small motion is nothing, in which case the single equivalent force is reduced to nothing. But whatever amount of freedom to move the whole system may have, we may always, by the application of frictionless constraint, limit it to one degree of freedom only; —and this may be freedom to execute any particular motion whatever, possible under the given conditions of the system. If, therefore, in any such infinitely small motion, there is variation of potential energy uncompensated by work of the applied forces, constraint limiting the freedom of the system to only this motion will bring us to the case in which we have just demonstrated there cannot be equilibrium. But the application of constraints limiting motion cannot possibly disturb equilibrium, and therefore the given system under the actual conditions cannot be in equilibrium in any particular configuration if there is more work done than resisted in any possible infinitely small motion from that configuration by all the forces to which it is subject*.

Neutral equilibrium. **291.** If a material system, under the influence of internal and applied forces, varying according to some definite law, is

* [This attempt to deduce the principle of virtual velocities from the equation of energy alone can hardly be regarded as satisfactory. H. L.]

balanced by them in any position in which it may be placed, its equilibrium is said to be neutral. This is the case with any spherical body of uniform material resting on a horizontal plane. A right cylinder or cone, bounded by plane ends perpendicular to the axis, is also in neutral equilibrium on a horizontal plane. Practically, any mass of moderate dimensions is in neutral equilibrium when its centre of inertia only is fixed, since, when its longest dimension is small in comparison with the earth's radius, gravity is, as we shall see, approximately equivalent to a single force through this point.

Stable equilibrium. But if, when displaced infinitely little in any direction from a particular position of equilibrium, and left to itself, it commences and continues vibrating, without ever experiencing more than infinitely small deviation in any of its parts, from the position of equilibrium, the equilibrium in this position is said to be stable. A weight suspended by a string, a uniform sphere in a hollow bowl, a loaded sphere resting on a horizontal plane with the loaded side lowest, an oblate body resting with one end of its shortest diameter on a horizontal plane, a plank, whose thickness is small compared with its length and breadth, floating on water, etc. etc., are all cases of stable equilibrium; if we neglect the motions of rotation about a vertical axis in the second, third, and fourth cases, and horizontal motion in general, in the fifth, for all of which the equilibrium is neutral.

Unstable equilibrium. If, on the other hand, the system can be displaced in any way from a position of equilibrium, so that when left to itself it will not vibrate within infinitely small limits about the position of equilibrium, but will move farther and farther away from it, the equilibrium in this position is said to be unstable. Thus a loaded sphere resting on a horizontal plane with its load as high as possible, an egg-shaped body standing on one end, a board floating edgeways in water, etc. etc., would present, if they could be realised in practice, cases of unstable equilibrium.

When, as in many cases, the nature of the equilibrium varies with the direction of displacement, if unstable for any possible displacement it is practically unstable on the whole. Thus a coin standing on its edge, though in neutral equilibrium for displacements in its plane, yet being in unstable equilibrium

Unstable equilibrium.

for those perpendicular to its plane, is practically unstable. A sphere resting in equilibrium on a saddle presents a case in which there is stable, neutral, or unstable equilibrium, according to the direction in which it may be displaced by rolling, but, practically, it would be unstable.

Test of the nature of equilibrium.

292. The theory of energy shows a very clear and simple test for discriminating these characters, or determining whether the equilibrium is neutral, stable, or unstable, in any case. If there is just as much work resisted as performed by the applied and internal forces in any possible displacement the equilibrium is neutral, but not unless. If in every possible infinitely small displacement from a position of equilibrium they do less work among them than they resist, the equilibrium is thoroughly stable, and not unless. If in any or in every infinitely small displacement from a position of equilibrium they do more work than they resist, the equilibrium is unstable. It follows that if the system is influenced only by internal forces, or if the applied forces follow the law of doing always the same amount of work upon the system passing from one configuration to another by all possible paths, the whole potential energy must be constant, in all positions, for neutral equilibrium; must be a minimum for positions of thoroughly stable equilibrium; must be either an absolute maximum, or a maximum for some displacements and a minimum for others when there is unstable equilibrium*.

Deduction of the equations of motion of any system.

293. We have seen that, according to D'Alembert's principle, as explained above (§ 264), forces acting on the different points of a material system, and their reactions against the accelerations which they actually experience in any case of motion, are in equilibrium with one another. Hence in any actual case of motion, not only is the actual work done by the forces equal to the kinetic energy produced in any infinitely small time, in virtue of the actual accelerations; but so also is the work which would be done by the forces, in any infinitely small time, if the velocities of the points constituting the system, were at any instant changed to any possible infinitely small velocities, and the accelerations unchanged. This statement, when put in

* [It will be observed that these criteria are stated rather than proved. See § 337 *post*. H. L.]

Deduction of the equations of motion of any system.

the concise language of mathematical analysis, constitutes Lagrange's application of the "principle of virtual velocities" to express the conditions of D'Alembert's equilibrium between the forces acting, and the resistances of the masses to acceleration. It comprehends, as we have seen, every possible condition of every case of motion. The "equations of motion" in any particular case are, as Lagrange has shown, deduced from it with great ease.

Let m be the mass of any one of the material points of the system; x, y, z its rectangular co-ordinates at time t, relatively to axes fixed in direction (§ 249) through a point reckoned as fixed (§ 245); and X, Y, Z the components, parallel to the same axes, of the whole force acting on it. Thus $-m\dfrac{d^2x}{dt^2}$, $-m\dfrac{d^2y}{dt^2}$, $-m\dfrac{d^2z}{dt^2}$ are the components of the reaction against acceleration. And these, with X, Y, Z, for the whole system, must fulfil the conditions of equilibrium. Hence if δx, δy, δz denote any arbitrary variations of x, y, z consistent with the conditions of the system, we have

Indeterminate equation of motion of any system.

$$\Sigma\left\{\left(X - m\frac{d^2x}{dt^2}\right)\delta x + \left(Y - m\frac{d^2y}{dt^2}\right)\delta y + \left(Z - m\frac{d^2z}{dt^2}\right)\delta z\right\} = 0 \;..(1),$$

where Σ denotes summation to include all the particles of the system. This may be called the indeterminate, or the variational, equation of motion. Lagrange used it as the foundation of his whole kinetic system, deriving from it all the common equations of motion, and his own remarkable equations in generalized co-ordinates (presently to be given). We may write it otherwise as follows:

$$\Sigma m\,(\ddot{x}\delta x + \ddot{y}\delta y + \ddot{z}\delta z) = \Sigma\,(X\delta x + Y\delta y + Z\delta z) \quad ......(2),$$

where the first member denotes the work done by forces equal to those required to produce the real accelerations, acting through the spaces of the arbitrary displacements; and the second member the work done by the actual forces through these imagined spaces.

If the moving bodies constitute a conservative system, and if V denote its potential energy in the configuration specified by (x, y, z, etc.), we have of course (§§ 241, 273)

$$\delta V = -\Sigma\,(X\delta x + Y\delta y + Z\delta z) \;.................(3),$$

and therefore the indeterminate equation of motion becomes

$$\Sigma m(\ddot{x}\delta x + \ddot{y}\delta y + \ddot{z}\delta z) = -\delta V \quad\text{.................(4)},$$

Of conservative system.

where δV denotes the excess of the potential energy in the configuration $(x + \delta x,\ y + \delta y,\ z + \delta z$, etc.) above that in the configuration $(x, y, z$, etc.).

One immediate particular result must of course be the common equation of energy, which must be obtained by supposing δx, δy, δz, etc., to be the actual variations of the co-ordinates in an infinitely small time δt. Thus if we take $\delta x = \dot{x}\delta t$, etc., and divide both members by δt, we have

$$\Sigma(X\dot{x} + Y\dot{y} + Z\dot{z}) = \Sigma m(\ddot{x}\dot{x} + \ddot{y}\dot{y} + \ddot{z}\dot{z}) \quad\text{...........(5)}.$$

Equation of energy.

Here the first member is composed of Newton's *Actiones Agentium*; with his *Reactiones Resistentium* so far as friction, gravity, and molecular forces are concerned, subtracted: and the second consists of the portion of the *Reactiones* due to acceleration. As we have seen above (§ 214), the second member is the rate of increase of $\Sigma\frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$ per unit of time. Hence, denoting by v the velocity of one of the particles, and by W the integral of the first member multiplied by dt, that is to say, the integral work done by the working and resisting forces in any time, we have

$$\Sigma\tfrac{1}{2}mv^2 = W + E_0 \quad\text{.......................(6)},$$

E_0 being the initial kinetic energy. This is the integral equation of energy. In the particular case of a conservative system, W is a function of the co-ordinates, irrespectively of the time, or of the paths which have been followed. According to the previous notation, with besides V_0 to denote the potential energy of the system in its initial configuration, we have $W = V_0 - V$, and the integral equation of energy becomes

$$\Sigma\tfrac{1}{2}mv^2 = V_0 - V + E_0,$$

or, if E denote the sum of the potential and kinetic energies, a constant,

$$\Sigma\tfrac{1}{2}mv^2 = E - V \quad\text{.......................(7)}.$$

The general indeterminate equation gives immediately, for the motion of a system of free particles,

$$m_1\ddot{x}_1 = X_1,\ \ m_1\ddot{y}_1 = Y_1,\ \ m_1\ddot{z}_1 = Z_1,\ \ m_2\ddot{x}_2 = X_2,\ \text{ etc.}$$

Of these equations the three for each particle may of course be treated separately if there is no mutual influence between the particles: but when they exert force on one another, X_1, Y_1, etc., will each in general be a function of all the co-ordinates.

Constraint introduced into the indeterminate equation.

From the indeterminate equation (1) Lagrange, by his method of multipliers, deduces the requisite number of equations for determining the motion of a rigid body, or of any system of connected particles or rigid bodies, thus:—Let the number of the particles be i, and let the connexions between them be expressed by n equations,

$$\left.\begin{matrix} F(x_1, y_1, z_1, x_2, \ldots) = 0 \\ F_{,}(x_1, y_1, z_1, x_2, \ldots) = 0 \\ \text{etc.} \qquad \text{etc.} \end{matrix}\right\} \quad\text{.................(8)}$$

being the *kinematical equations* of the system. By taking the variations of these we find that every possible infinitely small displacement δx_1, δy_1, δz_1, δx_2, ... must satisfy the n linear equations

$$\frac{dF}{dx_1}\delta x_1 + \frac{dF}{dy_1}\delta y_1 + \text{etc.} = 0,\ \ \frac{dF_{,}}{dx_1}\delta x_1 + \frac{dF_{,}}{dy_1}\delta y_1 + \text{etc.} = 0,\ \ \text{etc.} \ldots(9).$$

Multiplying the first of these by λ, the second by $\lambda_{,}$, etc., adding to the indeterminate equation, and then equating the coefficients of δx_1, δy_1, etc., each to zero, we have

$$\left.\begin{matrix} \lambda\dfrac{dF}{dx_1} + \lambda_{,}\dfrac{dF_{,}}{dx_1} + \ldots + X_1 - m_1\dfrac{d^2x_1}{dt^2} = 0 \\ \lambda\dfrac{dF}{dy_1} + \lambda_{,}\dfrac{dF_{,}}{dy_1} + \ldots + Y_1 - m_1\dfrac{d^2y_1}{dt^2} = 0 \\ \text{etc.} \qquad\qquad \text{etc.} \end{matrix}\right\} \quad\text{..........(10)}.$$

Determinate equations of motion deduced.

These are in all $3i$ equations to determine the n unknown quantities λ, $\lambda_{,}$, ..., and the $3i - n$ independent variables to which x_1, y_1, ... are reduced by the kinematical equations (8). The same equations may be found synthetically in the following manner, by which also we are helped to understand the precise meaning of the terms containing the multipliers λ, $\lambda_{,}$, etc.

First let the particles be free from constraint, but acted on both by the given forces X_1, Y_1, etc., and by forces depending on mutual distances between the particles and upon their positions relatively to fixed objects subject to the law of conservation, and having for their potential energy

$$-\tfrac{1}{2}(kF^2 + k_{,}F_{,}^2 + \text{etc.}),$$

so that components of the forces actually experienced by the different particles shall be

Determinate equations of motion deduced.

$$X_1 + kF\frac{dF}{dx} + k_{\prime}F_{\prime}\frac{dF_{\prime}}{dx_1} + \text{etc.} + \tfrac{1}{2}\left(F^2\frac{dk}{dx_1} + F_{\prime}^2\frac{dk_{\prime}}{dx_1} + \text{etc.}\right)$$

etc., etc.

Hence the equations of motion are

$$\left.\begin{array}{l} m_1\dfrac{d^2x_1}{dt^2} = X_1 + kF\dfrac{dF}{dx_1} + k_{\prime}F_{\prime}\dfrac{dF_{\prime}}{dx_1} + \text{etc.} + \tfrac{1}{2}\left(F^2\dfrac{dk}{dx_1} + F_{\prime}^2\dfrac{dk_{\prime}}{dx_1} + \text{etc.}\right) \\ m_1\dfrac{d^2y_1}{dt^2} = \text{etc.} \\ \quad \text{etc.,} \qquad \text{etc.} \end{array}\right\}\text{(11)}.$$

Now suppose k, $k_{\prime}$, etc. to be infinitely great:—in order that the *forces on the particles* may not be infinitely great, we must have

$$F = 0,\quad F_{\prime} = 0,\ \text{etc.},$$

that is to say, the equations of condition (8) must be fulfilled; and the last groups of terms in the second members of (11) now disappear because they contain the squares of the infinitely small quantities F, $F_{\prime}$, etc. Put now $kF = \lambda$, $k_{\prime}F_{\prime} = \lambda_{\prime}$, etc., and we have equations (10). This second mode of proving Lagrange's equations of motion of a constrained system corresponds precisely to the imperfect approach to the ideal case which can be made by real mechanism. The levers and bars and guide-surfaces cannot be infinitely rigid. Suppose then k, $k_{\prime}$, etc. to be finite but very great quantities, and to be some functions of the co-ordinates depending on the elastic qualities of the materials of which the guiding mechanism is composed:—equations (11) will express the motion, and by supposing k, $k_{\prime}$, etc. to be greater and greater we approach more and more nearly to the ideal case of absolutely rigid mechanism constraining the precise fulfilment of equations (8).

The problem of finding the motion of a system subject to any *unvarying* kinematical conditions whatever, under the action of any given forces, is thus reduced to a question of pure analysis. In the still more general problem of determining the motion when certain parts of the system are constrained to move in a specified manner, the equations of condition (8) involve not only the co-ordinates, but also t, the time. It is easily seen however that the equations (10) still hold, and with (8) fully determine the motion. For:—consider the equations of equilibrium of the particles acted on by any forces X_1', Y_1', etc., and constrained by

proper mechanism to fulfil the equations of condition (8) with the actual values of the parameters for any particular value of t. The equations of equilibrium will be uninfluenced by the fact that some of the parameters of the conditions (8) have different values at different times. Hence, with $X_1 - m_1\dfrac{d^2x_1}{dt^2}$, $Y_1 - m_1\dfrac{d^2y_1}{dt^2}$, instead of X_1', Y_1', etc., according to D'Alembert's principle, the equations of motion will still be (8), (9), and (10) quite independently of whether the parameters of (8) are all constant, or have values varying in any arbitrary manner with the time.

Determinate equations of motion deduced.

Equation of energy.

To find the equation of energy multiply the first of equations (10) by $\dot{x}_1$, the second by $\dot{y}_1$, etc., and add. Then remarking that in virtue of (8) we have

$$\frac{dF}{dx_1}\dot{x}_1 + \frac{dF}{dy_1}\dot{y}_1 + \text{etc.} + \left(\frac{dF}{dt}\right) = 0,$$

$$\frac{dF_{\prime}}{dx_1}\dot{x}_1 + \frac{dF_{\prime}}{dy_1}\dot{y}_1 + \text{etc.} + \left(\frac{dF_{\prime}}{dt}\right) = 0,$$

partial differential coefficients of F, $F_{\prime}$, etc. with reference to t being denoted by $\left(\dfrac{dF}{dt}\right)$, $\left(\dfrac{dF_{\prime}}{dt}\right)$, etc.; and denoting by T the kinetic energy or $\tfrac{1}{2}\Sigma m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$, we find

$$\frac{dT}{dt} = \Sigma(X\dot{x} + Y\dot{y} + Z\dot{z}) - \lambda\left(\frac{dF}{dt}\right) - \lambda_{\prime}\left(\frac{dF_{\prime}}{dt}\right) - \text{etc.} = 0 \ldots.(12).$$

When the kinematic conditions are "*unvarying*," that is to say, when the equations of condition are equations among the co-ordinates with constant parameters, we have

$$\left(\frac{dF}{dt}\right) = 0,\quad \left(\frac{dF_{\prime}}{dt}\right) = 0,\ \text{etc.},$$

and the equation of energy becomes

$$\frac{dT}{dt} = \Sigma(X\dot{x} + Y\dot{y} + Z\dot{z}) \ldots\ldots\ldots\ldots\ldots(13),$$

showing that in this case the fulfilment of the equations of condition involves neither gain nor loss of energy. On the other hand, equation (12) shows how to find the work performed or consumed in the fulfilment of the kinematical conditions when they are not unvarying.

Equation of energy.

As a simple example of varying constraint, which will be very easily worked out by equations (8) and (10), perfectly illustrating the general principle, the student may take the case of a particle acted on by any given forces and free to move anywhere in a plane which is kept moving with any given uniform or varying angular velocity round a fixed axis.

Gauss's principle of least constraint.

When there are connexions between any parts of a system, the motion is in general not the same as if all were free. If we consider any particle during any infinitely small time of the motion, and call the product of its mass into the square of the distance between its positions at the end of this time, on the two suppositions, the *constraint*: the sum of the constraints is a minimum. This follows easily from (1).

Impact.

294. When two bodies, in relative motion, come into contact, pressure begins to act between them to prevent any parts of them from jointly occupying the same space. This force commences from nothing at the first point of collision, and gradually increases per unit of area on a gradually increasing surface of contact. If, as is always the case in nature, each body possesses some degree of elasticity, and if they are not kept together after the impact by cohesion, or by some artificial appliance, the mutual pressure between them will reach a maximum, will begin to diminish, and in the end will come to nothing, by gradually diminishing in amount per unit of area on a gradually diminishing surface of contact. The whole process would occupy not greatly more or less than an hour if the bodies were of such dimensions as the earth, and such degrees of rigidity as copper, steel, or glass. It is finished, probably, within a thousandth of a second if they are globes of any of these substances not exceeding a yard in diameter.

295. The whole amount, and the direction, of the *"Impact"* experienced by either body in any such case, are reckoned according to the "change of momentum" which it experiences. The amount of the impact is measured by the amount, and its direction by the direction, of the change of momentum which is produced. The component of an impact in a direction parallel to any fixed line is similarly reckoned according to the component change of momentum in that direction.

296. If we imagine the whole time of an impact divided into a very great number of equal intervals, each so short that the force does not vary sensibly during it, the component change of momentum in any direction during any one of these intervals will (§ 220) be equal to the force multiplied by the measure of the interval. Hence the component of the impact is equal to the sum of the forces in all the intervals, multiplied by the length of each interval. Impact.

Let P be the component force in any direction at any instant, τ, of the interval, and let I be the amount of the corresponding component of the whole impact. Then

$$I = \int P d\tau.$$

297. Any force in a constant direction acting in any circumstances, for any time great or small, may be reckoned on the same principle; so that what we may call its whole amount during any time, or its *"time-integral,"* will measure, or be measured by, the whole momentum which it generates in the time in question. But this reckoning is not often convenient or useful except when the whole operation considered is over before the position of the body, or configuration of the system of bodies, involved, has altered to such a degree as to bring any other forces into play, or alter forces previously acting, to such an extent as to produce any sensible effect on the momentum measured. Thus if a person presses gently with his hand, during a few seconds, upon a mass suspended by a cord or chain, he produces an effect which, if we know the degree of the force at each instant, may be thoroughly calculated on elementary principles. No approximation to a full determination of the motion, or to answering such a partial question as "how great will be the whole deflection produced?" can be founded on a knowledge of the *"time-integral"* alone. If, for instance, the force be at first very great and gradually diminish, the effect will be very different from what it would be if the force were to increase very gradually and to cease suddenly, even although the time-integral were the same in the two cases. But if the same body is "struck a blow," in a horizontal direction, either by the hand, or by a mallet or other somewhat

Time-integral.

Time-integral.

hard mass, the action of the force is finished before the suspending cord has experienced any sensible deflection from the vertical. Neither gravity nor any other force sensibly alters the effect of the blow. And therefore the whole momentum at the end of the blow is sensibly equal to the "amount of the impact," which is, in this case, simply the time-integral.

Ballistic pendulum.

298. Such is the case of Robins' *Ballistic Pendulum*, a massive cylindrical block of wood cased in a cylindrical sheath of iron closed at one end and moveable about a horizontal axis at a considerable distance above it—employed to measure the velocity of a cannon or musket-shot. The shot is fired into the block in a horizontal direction along the axis of the block and perpendicular to the axis of suspension. The impulsive penetration is so nearly instantaneous, and the inertia of the block so large compared with the momentum of the shot, that the ball and pendulum are moving on as one mass before the pendulum has been sensibly deflected from the vertical. This is essential to the regular use of the apparatus. The iron sheath with its flat end must be strong enough to guard against splinters of wood flying sidewise, and to keep in the bullet.

299. Other illustrations of the cases in which the time-integral gives us the complete solution of the problem may be given without limit. They include all cases in which the direction of the force is always coincident with the direction of motion of the moving body, and those special cases in which the time of action of the force is so short that the body's motion does not, during its lapse, sensibly alter its relation to the direction of the force, or the action of any other forces to which it may be subject. Thus, in the vertical fall of a body, the time-integral gives us at once the change of momentum; and the same rule applies in most cases of forces of brief duration, as in a "drive" in cricket or golf.

Direct impact of spheres

300. The simplest case which we can consider, and the one usually treated as an introduction to the subject, is that of the collision of two smooth spherical bodies whose centres before collision were moving in the same straight line. The force between them at each instant must be in this line, because of

Direct impact of spheres.

the symmetry of circumstances round it; and by the third law it must be equal in amount on the two bodies. Hence (LEX II.) they must experience changes of motion at equal rates in contrary directions; and at any instant of the impact the integral amounts of these changes of motion must be equal. Let us suppose, to fix the ideas, the two bodies to be moving both before and after impact in the same direction in one line: one of them gaining on the other before impact, and either following it at a less speed, or moving along with it, as the case may be, after the impact is completed. Cases in which the former is driven backwards by the force of the collision, or in which the two moving in opposite directions meet in collision, are easily reduced to dependence on the same formula by the ordinary algebraic convention with regard to positive and negative signs.

In the standard case, then, the quantity of motion lost, up to any instant of the impact, by one of the bodies, is equal to that gained by the other. Hence at the instant when their velocities are equalized they move as one mass with a momentum equal to the sum of the momenta of the two before impact. That is to say, if v denote the common velocity at this instant, we have

$$(M+M')\,v = MV + M'V',$$

or

$$v = \frac{MV + M'V'}{M+M'},$$

if M, M' denote the masses of the two bodies, and V, V' their velocities before impact.

During this first period of the impact the bodies have been, on the whole, coming into closer contact with one another, through a compression or deformation experienced by each, and resulting, as remarked above, in a fitting together of the two surfaces over a finite area. No body in nature is perfectly inelastic; and hence, at the instant of closest approximation, the mutual force called into action between the two bodies continues, and tends to separate them. Unless prevented by natural surface cohesion or welding (such as is always found, as we shall see later in our chapter on Properties of Matter, however hard and well polished the surfaces may

Direct impact of spheres.

be), or by artificial appliances (such as a coating of wax, applied in one of the common illustrative experiments; or the coupling applied between two railway carriages when run together so as to push in the springs, according to the usual practice at railway stations), the two bodies are actually separated by this force, and move away from one another. Newton found that, *provided the impact is not so violent as to make any sensible permanent indentation in either body,* the relative velocity of separation after the impact bears a proportion to their previous relative velocity of approach, which is constant for the same two bodies. This proportion, always less than unity, approaches more and more nearly to it the harder the bodies are. Thus with balls of compressed wool he found it $\frac{5}{9}$, iron nearly the same, glass $\frac{15}{16}$. The results of more recent experiments on the same subject have confirmed Newton's law. These will be described later. In any case of the collision of two balls, let e denote this proportion, to which we give the name *Coefficient of Restitution;** and, with previous notation, let in addition U, U' denote the velocities of the two bodies after the conclusion of the impact; in the standard case each being positive, but $U' > U$. Then we have

Effect of elasticity.

Newton's experiments.

$$U' - U = e\,(V - V')$$

and, as before, since one has lost as much momentum as the other has gained,

$$MU + M'U' = MV + M'V'.$$

From these equations we find

$$(M + M')\,U = MV + M'V' - eM'\,(V - V'),$$

with a similar expression for U'.

Also we have, as above,

$$(M + M')\,v = MV + M'V'.$$

Hence, by subtraction,

$$(M + M')\,(v - U) = eM'\,(V - V') = e\,\{M'V - (M + M')\,v + MV\}$$

* In most modern treatises this is called a "coefficient of elasticity," which is clearly a mistake; suggested, it may be, by Newton's words, but inconsistent with his facts, and utterly at variance with modern language and modern knowledge regarding elasticity.

Direct impact of spheres.

and therefore $\qquad v - U = e\,(V - v).$

Of course we have also $\quad U' - v = e\,(v - V').$

These results may be put in words thus:—The *relative* velocity of either of the bodies with regard to the centre of inertia of the two is, after the completion of the impact, reversed in direction, and diminished in the ratio $e : 1$.

301. Hence the loss of kinetic energy, being, according to §§ 267, 280, due only to change of kinetic energy relative to the centre of inertia, is to this part of the whole as $1 - e^2 : 1$.

Thus

$$\begin{aligned}
\text{Initial kinetic energy} &= \tfrac{1}{2}\,(M + M')\,v^2 + \tfrac{1}{2}M\,(V - v)^2 + \tfrac{1}{2}M'\,(v - V')^2.\\
\text{Final}\quad ,, \quad ,, &= \tfrac{1}{2}\,(M + M')\,v^2 + \tfrac{1}{2}M\,(v - U)^2 + \tfrac{1}{2}M'\,(U' - v)^2.\\
\text{Loss} &= \tfrac{1}{2}\,(1 - e^2)\,\{M\,(V - v)^2 + M'\,(v - V')^2\}.
\end{aligned}$$

302. When two elastic bodies, the two balls supposed above for instance, impinge, some portion of their previous kinetic energy will always remain in them as vibrations. A *portion* of the loss of energy (miscalled the effect of imperfect elasticity) is necessarily due to this cause in every real case.

Distribution of energy after impact.

Later, in our chapter on Properties of Matter, it will be shown as a result of experiment, that forces of elasticity are, to a very close degree of accuracy, simply proportional to the strains (§ 154), within the limits of elasticity, in elastic solids which, like metals, glass, etc., bear but small deformations without permanent change. Hence when two such bodies come into collision, sometimes with greater and sometimes with less mutual velocity, but with all other circumstances similar, the velocities of all particles of either body, at corresponding times of the impacts, will be always in the same proportion. Hence the velocity of separation of the centres of inertia after impact will bear a constant proportion to the previous velocity of approach; which agrees with the Newtonian Law*. It is therefore probable that a very sensible portion, if not the whole, of the loss of energy in the visible motions of two elastic bodies, after impact, experimented on by Newton, may have been due

Newton's experimental law consistent with *perfect* elasticity.

* [On the theory of impact propounded by Hertz the extent of the area of contact will vary with the relative velocity before impact, and the argument from dynamical similarity does not apply. H.L.]

Distribution of energy after impact.

to vibrations; but unless some other cause also was largely operative, it is difficult to see how the loss was so much greater with iron balls than with glass.

303. In certain definite extreme cases, imaginable although not realizable, no energy will be spent in vibrations, and the two bodies will separate, each moving simply as a rigid body, and having in this simple motion the whole energy of work done on it by elastic force during the collision. For instance, let the two bodies be cylinders, or prismatic bars with flat ends, of the same kind of substance, and of equal and similar transverse sections; and let this substance have the property of compressibility with perfect elasticity, in the direction of the length of the bar, and of absolute resistance to change in every transverse dimension. Before impact, let the two bodies be placed with their lengths in one line, and their transverse sections (if not circular) similarly situated, and let one or both be set in motion in this line. The result, as regards the motions of the two bodies after the collision, will be sensibly the same if they are of any real ordinary elastic solid material, provided the greatest transverse diameter of each is very small in comparison with its length. Then, if the lengths of the two be equal, they will separate after impact with the same relative velocity as that with which they approached, and neither will retain any vibratory motion after the end of the collision.

304. If the two bars are of unequal length, the shorter will, after the impact, be exactly in the same state as if it had struck another of its own length, and it therefore will move as a rigid body after the collision. But the other will, along with a motion of its centre of gravity, calculable from the principle that its whole momentum must (§ 267) be changed by an amount equal exactly to the momentum gained or lost by the first, have also a vibratory motion, of which the whole kinetic and potential energy will make up the deficiency of energy which we shall presently calculate in the motions of the centres of inertia. For simplicity, let the longer body be supposed to be at rest before the collision. Then the shorter on striking it will be left at rest; this boing clearly the result in the case of

Distribution of energy after impact.

$e=1$ in the preceding formulæ (§ 300) applied to the impact of one body striking another of equal mass previously at rest. The longer bar will move away with the same momentum, and therefore with less velocity of its centre of inertia, and less kinetic energy of this motion, than the other body had before impact, in the ratio of the smaller to the greater mass. It will also have a very remarkable vibratory motion, which, when its length is more than double of that of the other, will consist of a wave running backwards and forwards through its length, and causing the motion of its ends, and, in fact, of every particle of it, to take place by "fits and starts," not continuously. The full analysis of these circumstances, though very simple, must be reserved until we are especially occupied with waves, and the kinetics of elastic solids. It is sufficient at present to remark, that the motions of the centres of inertia of the two bodies after impact, whatever they may have been previously, are given by the preceding formulæ with for e the value $\frac{M'}{M}$, where M' and M are the smaller and the larger mass respectively.

305. The mathematical theory of the vibrations of solid elastic spheres has not yet been worked out; and its application to the case of the vibrations produced by impact presents considerable difficulty. Experiment, however, renders it certain, that but a small part of the whole kinetic energy of the previous motions can remain in the form of vibrations after the impact of two equal spheres of glass or of ivory. This is proved, for instance, by the common observation, that one of them remains nearly motionless after striking the other previously at rest; since, the velocity of the common centre of inertia of the two being necessarily unchanged by the impact, we infer that the second ball acquires a velocity nearly equal to that which the first had before striking it. But it is to be expected that unequal balls of the same substance coming into collision will, by impact, convert a very sen ible proportion of the kinetic energy of their previous motions into energy of vibrations; and generally, that the same will be the case when equal or unequal masses of different substances come into colli-

Distribution of energy after impact.

sion; although for one particular proportion of their diameters, depending on their densities and elastic qualities, this effect will be a minimum, and possibly not much more sensible than it is when the substances are the same and the diameters equal.

306. It need scarcely be said that in such cases of impact as that of the tongue of a bell, or of a clock-hammer striking its bell (or spiral spring as in the American clocks), or of piano-forte hammers striking the strings, or of a drum struck with the proper implement, a large part of the kinetic energy of the blow is spent in generating vibrations.

Moment of an impact about an axis.

307. The *Moment of an impact* about any axis is derived from the line and amount of the impact in the same way as the moment of a velocity or force is determined from the line and amount of the velocity or force, §§ 235, 236. If a body is struck, the change of its moment of momentum about any axis is equal to the moment of the impact round that axis. But, without considering the measure of the impact, we see (§ 267) that the moment of momentum round any axis, lost by one body in striking another, is, as in every case of mutual action, equal to that gained by the other.

Ballistic pendulum.

Thus, to recur to the ballistic pendulum—the line of motion of the bullet at impact may be in any direction whatever, but the only part which is effective is the component in a plane perpendicular to the axis. We may therefore, for simplicity, consider the motion to be in a line perpendicular to the axis, though not necessarily horizontal. Let m be the mass of the bullet, v its velocity, and p the distance of its line of motion from the axis. Let M be the mass of the pendulum with the bullet lodged in it, and k its radius of gyration. Then if ω be the angular velocity of the pendulum when the impact is complete,

$$mvp = Mk^2\omega,$$

from which the solution of the question is easily determined.

For the kinetic energy after impact is changed (§ 241) into its equivalent in potential energy when the pendulum reaches its position of greatest deflection. Let this be given by the angle θ: then the height to which the centre of inertia is raised is $h(1-\cos\theta)$ if h be its distance from the axis. Thus

Ballistic pendulum.

$$Mgh(1-\cos\theta) = \tfrac{1}{2}Mk^2\omega^2 = \tfrac{1}{2}\frac{m^2v^2p^2}{Mk^2},$$

or

$$2\sin\frac{\theta}{2} = \frac{mvp}{Mk\sqrt{gh}},$$

an expression for the chord of the angle of deflection. In practice the chord of the angle θ is measured by means of a light tape or cord attached to a point of the pendulum, and slipping with small friction through a clip fixed close to the position occupied by that point when the pendulum hangs at rest.

Work done by impact.

308. *Work done by an impact* is, in general, the product of the impact into half the sum of the initial and final velocities of the point at which it is applied, resolved in the direction of the impact. In the case of direct impact, such as that treated in § 300, the initial kinetic energy of the body is $\tfrac{1}{2}MV^2$, the final $\tfrac{1}{2}MU^2$, and therefore the gain, by the impact, is

$$\tfrac{1}{2}M(U^2 - V^2),$$

or, which is the same,

$$M(U-V)\,.\,\tfrac{1}{2}(U+V).$$

But $M(U-V)$ is (§ 295) equal to the amount of the impact. Hence the proposition: the extension of which to the most general circumstances is easily seen.

Let ι be the amount of the impulse up to time τ, and I the whole amount, up to the end, T. Thus,—

$$\iota = \int_0^\tau P d\tau,\quad I = \int_0^T P d\tau;\ \text{also}\ P = \frac{d\iota}{d\tau}.$$

Whatever may be the conditions to which the body struck is subjected, the change of velocity in the point struck is proportional to the amount of the impulse up to any part of its whole time, so that, if $\mathfrak{M}$ be a constant depending on the masses and conditions of constraint involved, and if U, v, V denote the component velocities of the point struck, in the direction of the impulse, at the beginning, at the time τ, and at the end, respectively, we have

$$v = U + \frac{\iota}{\mathfrak{M}},\quad V = U + \frac{I}{\mathfrak{M}}.$$

Hence, for the rate of the doing of work by the force P, at the instant t, we have

$$Pv = PU + \frac{\iota P}{\mathfrak{M}}.$$

Work done by impact.

Hence for the whole work (W) done by it,

$$W = \int_0^T \left(PU + \frac{\iota P}{\mathfrak{M}} \right) d\tau$$
$$= UI + \frac{1}{\mathfrak{M}} \int_0^I \iota d\iota = UI + \tfrac{1}{2} \frac{I^2}{\mathfrak{M}}$$
$$= UI + \tfrac{1}{2} I (V - U) = I \cdot \tfrac{1}{2} (U + V).$$

309. It is worthy of remark, that if any number of impacts be applied to a body, their whole effect will be the same whether they be applied together or successively (provided that the whole time occupied by them be infinitely short), although the work done by each particular impact is in general different according to the order in which the several impacts are applied. The whole amount of work is the sum of the products obtained by multiplying each impact by half the sum of the components of the initial and final velocities of the point to which it is applied.

Equations of impulsive motion.

310. The effect of any stated impulses, applied to a rigid body, or to a system of material points or rigid bodies connected in any way, is to be found most readily by the aid of D'Alembert's principle; according to which the given impulses, and the impulsive reaction against the generation of motion, measured in amount by the momenta generated, are in equilibrium; and are therefore to be dealt with mathematically by applying to them the equations of equilibrium of the system.

Let P_1, Q_1, R_1 be the component impulses on the first particle, m_1, and let $\dot{x}_1$, $\dot{y}_1$, $\dot{z}_1$ be the components of the velocity instantaneously acquired by this particle. Component forces equal to $(P_1 - m_1\dot{x}_1)$, $(Q_1 - m_1\dot{y}_1)$, ... must equilibrate the system, and therefore we have (§ 290)

$$\Sigma \{(P - m\dot{x})\, \delta x + (Q - m\dot{y})\, \delta y + (R - m\dot{z})\, \delta z\} = 0 \ldots\ldots\ldots(a)$$

where δx_1, δy_1, ... denote the components of any infinitely small displacements of the particles possible under the conditions of the system. Or, which amounts to the same thing, since any possible infinitely small displacements are simply proportional to any possible velocities in the same directions,

$$\Sigma \{(P - m\dot{x})\, u + (Q - m\dot{y})\, v + (Q - m\dot{z})\, w\} = 0 \ldots\ldots\ldots\ldots(b)$$

Equations of impulsive motion.

where u_1, v_1, w_1 denote any possible component velocities of the first particle, etc.

One particular case of this equation is of course had by supposing u_1, v_1, ... to be equal to the velocities $\dot{x}_1$, $\dot{y}_1$, ... actually acquired; and, by halving, etc., we find

$$\Sigma (P \cdot \tfrac{1}{2}\dot{x} + Q \cdot \tfrac{1}{2}\dot{y} + R \cdot \tfrac{1}{2}\dot{z}) = \tfrac{1}{2}\Sigma m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) \ldots\ldots(c).$$

This agrees with § 308 above.

Theorem of Euler, extended by Lagrange.

311. Euler discovered that the kinetic energy acquired from rest by a rigid body in virtue of an impulse fulfils a maximum-minimum condition. Lagrange* extended this proposition to a system of bodies connected by any invariable kinematic relations, and struck with any impulses. Delaunay found that it is really always a maximum *when the impulses are given, and when different motions possible under the conditions of the system, and fulfilling the law of energy* [§ 310 (c)], *are considered.* Farther, Bertrand shows that the energy actually acquired is not merely a "maximum," but exceeds the energy of any other motion fulfilling these conditions; and that the amount of the excess is equal to the energy of the motion which must be compounded with either to produce the other.

Equation of impulsive motion.

Let $\dot{x}_1'$, $\dot{y}_1'$ ... be the component velocities of any motion whatever fulfilling the equation (c), which becomes

$$\tfrac{1}{2}\Sigma (P\dot{x}' + Q\dot{y}' + R\dot{z}') = \tfrac{1}{2}\Sigma m (\dot{x}'^2 + \dot{y}'^2 + \dot{z}'^2) = T' \ldots\ldots(d).$$

If, then, we take $\dot{x}_1' - \dot{x}_1 = u_1$, $\dot{y}_1' - \dot{y}_1 = v_1$, etc., we have

$$T' - T = \tfrac{1}{2}\Sigma m \{(2\dot{x} + u)\, u + (2\dot{y} + v) v + (2\dot{z} + w) w\}$$
$$= \Sigma m (\dot{x}u + \dot{y}v + \dot{z}w) + \tfrac{1}{2}\Sigma m (u^2 + v^2 + w^2) \ldots\ldots(e).$$

But, by (b),

$$\Sigma m (\dot{x}u + \dot{y}v + \dot{z}w) = \Sigma (Pu + Qv + Rw) \ldots\ \ldots..(f);$$

and, by (c) and (d),

$$\Sigma (Pu + Qv + Rw) = 2T' - 2T \ldots\ldots\ldots\ldots\ldots(g).$$

Hence (e) becomes

$$T' - T = 2(T' - T) + \tfrac{1}{2}\Sigma m (u^2 + v^2 + w^2),$$

whence $$T - T' = \tfrac{1}{2}\Sigma m (u^2 + v^2 + w^2) \ldots\ldots\ldots\ldots\ldots(h),$$

which is Bertrand's result.

* *Mécanique Analytique*, 2nde partie, 3me section, § 37.

Liquid set in motion impulsively.

312. The energy of the motion generated suddenly in a mass of incompressible liquid given at rest completely filling a vessel of any shape, when the vessel is suddenly set in motion, or when it is suddenly bent out of shape in any way whatever, subject to the condition of not changing its volume, *is less than the energy of any other motion it can have with the same motion of its bounding surface*. The consideration of this theorem, which, so far as we know, was first published in the *Cambridge and Dublin Mathematical Journal* [Feb. 1849], has led us to a general *minimum* property regarding motion acquired by any system when *any prescribed velocities* are generated suddenly in any of its parts; announced in the *Proceedings of the Royal Society of Edinburgh* for April, 1863. It is, that provided impulsive forces are applied to the system only at places where the velocities to be produced are prescribed, the kinetic energy is *less* in the actual motion than in any other motion which the system can take, and which has the same values for the prescribed velocities. The excess of the energy of any possible motion above that of the actual motion is (as in Bertrand's theorem) equal to the energy of the motion which must be compounded with either to produce the other. The proof is easy:—here it is:—

Equations (d), (e), and (f) hold as in § (311). But now each velocity component, u_1, v_1, w_1, u_2, etc. vanishes for which the component impulse P_1, Q_1, R_1, P_2, etc. does not vanish (because $\dot{x}_1+u_1$, $\dot{y}_1+v_1$, etc. fulfil the prescribed velocity conditions). Hence every product P_1u_1, Q_1v_1, etc. vanishes. Hence now instead of (g) and (h) we have

$$\Sigma(\dot{x}u+\dot{y}v+\dot{z}w)=0 \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\; (g'),$$

and

$$T'-T=\tfrac{1}{2}\Sigma m(u^2+v^2+w^2) \;\ldots\ldots\ldots\ldots\ldots\; (h').$$

We return to the subject in §§ 316, 317 as an illustration of the use of Lagrange's generalized co-ordinates; to the introduction of which into Dynamics we now proceed.

Impulsive motion referred to generalized co-ordinates.

313. The method of generalized co-ordinates explained above (§ 204) is extremely useful in its application to the dynamics of a system; whether for expressing and working out the details of any particular case in which there is any

finite number of degrees of freedom, or for proving general principles applicable even to cases, such as that of a liquid, as described in the preceding section, in which there may be an infinite number of degrees of freedom. It leads us to generalize the measure of inertia, and the resolution and composition of forces, impulses, and momenta, on dynamical principles corresponding with the kinematical principles explained in § 204, which gave us generalized component velocities: and, as we shall see later, the generalized equations of continuous motion are not only very convenient for the solution of problems, but most *instructive* as to the nature of relations, however complicated, between the motions of different parts of a system. In the meantime we shall consider the generalized expressions for the impulsive generation of motion. We have seen above (§ 308) that the kinetic energy acquired by a system given at rest and struck with any given impulses, is equal to half the sum of the products of the component forces multiplied each into the corresponding component of the velocity acquired by its point of application, when the ordinary system of rectangular co-ordinates is used. Precisely the same statement holds on the generalized system, and if stated as the convention agreed upon, it suffices to define the generalized components of impulse, those of velocity having been fixed on kinematical principles (§ 204). Generalized components of momentum of any specified motion are, of course, equal to the generalized components of the impulse by which it could be generated from rest.

Generalized components of impulse or momentum.

(*a*) Let ψ, ϕ, θ, ... be the generalized co-ordinates of a material system at any time; and let $\dot{\psi}$, $\dot{\phi}$, $\dot{\theta}$, ... be the corresponding generalized velocity-components, that is to say, the rates at which ψ, ϕ, θ, ... increase per unit of time, at any instant, in the actual motion. If x_1, y_1, z_1 denote the common rectangular co-ordinates of one particle of the system, and $\dot{x}_1$, $\dot{y}_1$, $\dot{z}_1$ its component velocities, we have

$$\left.\begin{aligned} \dot{x}_1 &= \frac{dx_1}{d\psi}\dot{\psi}+\frac{dx_1}{d\phi}\dot{\phi}+\text{etc.} \\ \dot{y}_1 &= \frac{dy_1}{d\psi}\dot{\psi}+\frac{dy_1}{d\phi}\dot{\phi}+\text{etc.} \\ &\quad\text{etc.} \qquad\quad \text{etc.} \end{aligned}\right\} \;\ldots\ldots\ldots\ldots\ldots\; (1).$$

Hence the kinetic energy, which is $\Sigma \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$, in terms of rectangular co-ordinates, becomes a quadratic function of $\dot{\psi}$, $\dot{\phi}$, etc., when expressed in terms of generalized co-ordinates, so that if we denote it by T we have

Generalized expression for kinetic energy.

$$T = \tfrac{1}{2}\{(\psi, \psi)\,\dot{\psi}^2 + (\phi, \phi)\,\dot{\phi}^2 + \ldots + 2\,(\psi, \phi)\,\dot{\psi}\dot{\phi} + \ldots\} \ldots\ldots (2),$$

where (ψ, ψ), (ϕ, ϕ), (ψ, ϕ), etc., denote various functions of the co-ordinates, determinable according to the conditions of the system. The only condition essentially fulfilled by these coefficients is, that they must give a finite positive value to T for all values of the variables.

(*b*) Again let (X_1, Y_1, Z_1), (X_2, Y_2, Z_2), etc., denote component forces on the particles (x_1, y_1, z_1), (x_2, y_2, z_2), etc., respectively; and let $(\delta x_1, \delta y_1, \delta z_1)$, etc., denote the components of any infinitely small motions possible without breaking the conditions of the system. The work done by those forces, upon the system when so displaced, will be

$$\Sigma\,(X\delta x + Y\delta y + Z\delta z) \ldots\ldots\ldots\ldots\ldots\ldots\ldots (3).$$

To transform this into an expression in terms of generalized co-ordinates, we have

$$\left.\begin{array}{l} \delta x_1 = \dfrac{dx_1}{d\psi}\,\delta\psi + \dfrac{dx_1}{d\phi}\,\delta\phi + \text{etc.} \\ \delta y_1 = \dfrac{dy_1}{d\psi}\,\delta\psi + \dfrac{dy_1}{d\phi}\,\delta\phi + \text{etc.} \\ \qquad\ \text{etc.} \qquad\qquad \text{etc.} \end{array}\right\} \ldots\ldots\ldots\ldots\ldots (4),$$

and it becomes

$$\Psi\delta\psi + \Phi\delta\phi + \text{etc.} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5),$$

Generalized components of force,

where

$$\left.\begin{array}{l} \Psi = \Sigma\left(X\dfrac{dx}{d\psi} + Y\dfrac{dy}{d\psi} + Z\dfrac{dz}{d\psi}\right) \\ \Phi = \Sigma\left(X\dfrac{dx}{d\phi} + Y\dfrac{dy}{d\phi} + Z\dfrac{dz}{d\phi}\right) \\ \qquad\quad \text{etc.} \qquad\quad \text{etc.} \end{array}\right\} \ldots\ldots\ldots\ldots\ldots (6).$$

These quantities, Ψ, Φ, etc., are clearly *the generalized components of the force on the system.*

Let $\underset{\cdot}{\Psi}$, $\underset{\cdot}{\Phi}$, etc. denote component impulses, generalized on the same principle; that is to say, let

of impulse.

$$\underset{\cdot}{\Psi} = \int_0^\tau \Psi dt, \quad \underset{\cdot}{\Phi} = \int_0^\tau \Phi dt, \text{ etc.,}$$

where Ψ, Φ, ... denote generalized components of the continuous force acting at any instant of the infinitely short time τ, within which the impulse is completed.

Impulsive generation of motion referred to generalized co-ordinates.

If this impulse is applied to the system, previously in motion in the manner specified above, and if $\delta\dot{\psi}$, $\delta\dot{\phi}$, ... denote the resulting augmentations of the components of velocity, the means of the component velocities before and after the impulse will be

$$\dot{\psi} + \tfrac{1}{2}\delta\dot{\psi}, \quad \dot{\phi} + \tfrac{1}{2}\delta\dot{\phi}, \ \ldots\ldots$$

Hence, according to the general principle explained above for calculating the work done by an impulse, the whole work done in this case is

$$\underset{\cdot}{\Psi}\,(\dot{\psi} + \tfrac{1}{2}\delta\dot{\psi}) + \underset{\cdot}{\Phi}\,(\dot{\phi} + \tfrac{1}{2}\delta\dot{\phi}) + \text{etc.}$$

To avoid unnecessary complications, let us suppose $\delta\dot{\psi}$, $\delta\dot{\phi}$, etc., to be each infinitely small. The preceding expression for the work done becomes

$$\underset{\cdot}{\Psi}\dot{\psi} + \underset{\cdot}{\Phi}\dot{\phi} + \text{etc.};$$

and, as the effect produced by this work is augmentation of kinetic energy from T to $T + \delta T$, we must have

$$\delta T = \underset{\cdot}{\Psi}\dot{\psi} + \underset{\cdot}{\Phi}\dot{\phi} + \text{etc.}$$

Now let the impulses be such as to augment $\dot{\psi}$ to $\dot{\psi} + \delta\dot{\psi}$, and to leave the other component velocities unchanged. We shall have

$$\underset{\cdot}{\Psi}\dot{\psi} + \underset{\cdot}{\Phi}\dot{\phi} + \text{etc.} = \frac{dT}{d\dot{\psi}}\,\delta\dot{\psi}.$$

Dividing both members by $\delta\dot{\psi}$, and observing that $\dfrac{dT}{d\dot{\psi}}$ is a linear function of $\dot{\psi}$, $\dot{\phi}$, etc., we see that $\dfrac{\underset{\cdot}{\Psi}}{\delta\dot{\psi}}$, $\dfrac{\underset{\cdot}{\Phi}}{\delta\dot{\psi}}$, etc., must be equal to the coefficients of $\dot{\psi}$, $\dot{\phi}$, ... respectively in $\dfrac{dT}{d\dot{\psi}}$.

(*c*) From this we see, further, that the impulse required to produce the component velocity $\dot{\psi}$ from rest, or to generate it in the system moving with any other possible velocity, has for its components

$$(\psi, \psi)\,\dot{\psi}, \ (\psi, \phi)\,\dot{\psi}, \ (\psi, \theta)\,\dot{\psi}, \text{ etc.}$$

Hence we conclude that to generate the whole resultant velocity $(\dot{\psi}, \dot{\phi}, \ldots)$ from rest, requires an impulse, of which the components, if denoted by ξ, η, ζ, ..., are expressed as follows :—

Momenta in terms of velocities.

$$\left.\begin{aligned}\xi &= (\psi,\psi)\dot\psi + (\phi,\psi)\dot\phi + (\theta,\psi)\dot\theta + \ldots\\ \eta &= (\psi,\phi)\dot\psi + (\phi,\phi)\dot\phi + (\theta,\phi)\dot\theta + \ldots\\ \zeta &= (\psi,\theta)\dot\psi + (\phi,\theta)\dot\phi + (\theta,\theta)\dot\theta + \ldots\\ &\text{etc.}\end{aligned}\right\}\ldots\ldots\ldots\ldots(7),$$

where it must be remembered that, as seen in the original expression for T, from which they are derived, (ϕ, ψ) means the same thing as (ψ, ϕ), and so on. The preceding expressions are the differential coefficients of T with reference to the velocities; that is to say,

$$\xi = \frac{dT}{d\dot\psi},\quad \eta = \frac{dT}{d\dot\phi},\quad \zeta = \frac{dT}{d\dot\theta}\ldots\ldots\ldots\ldots\ldots\ldots(8).$$

(*d*) The second members of these equations being linear functions of $\dot\psi$, $\dot\phi$, ..., we may, by ordinary elimination, find $\dot\psi$, $\dot\phi$, etc., in terms of ξ, η, etc., and the expressions so obtained are of course linear functions of the last-named elements. And, since T is a quadratic function of $\dot\psi$, $\dot\phi$, etc., we have

$$2T = \xi\dot\psi + \eta\dot\phi + \zeta\dot\theta + \text{etc.}\ldots\ldots\ldots\ldots\ldots\ldots(9).$$

Kinetic energy in terms of momentums and velocities.

From this, on the supposition that T, $\dot\psi$, $\dot\phi$, ... are expressed in terms of ξ, η, ..., we have by differentiation

$$2\frac{dT}{d\xi} = \dot\psi + \xi\frac{d\dot\psi}{d\xi} + \eta\frac{d\dot\phi}{d\xi} + \zeta\frac{d\dot\theta}{d\xi} + \text{etc.}$$

Now the algebraic process by which $\dot\psi$, $\dot\phi$, etc., are obtained in terms of ξ, η, etc., shows that, inasmuch as the coefficient of $\dot\phi$ in the expression, (7), for ξ, is equal to the coefficient of $\dot\psi$, in the expression for η, and so on; the coefficient of η in the expression for $\dot\psi$ must be equal to the coefficient of ξ in the expression for $\dot\phi$, and so on; that is to say,

$$\frac{d\dot\psi}{d\eta} = \frac{d\dot\phi}{d\xi},\quad \frac{d\dot\psi}{d\zeta} = \frac{d\dot\theta}{d\xi},\ \text{etc.}$$

Hence the preceding expression becomes

$$2\frac{dT}{d\xi} = \dot\psi + \xi\frac{d\dot\psi}{d\xi} + \eta\frac{d\dot\psi}{d\eta} + \zeta\frac{d\dot\psi}{d\zeta} + \ldots = 2\dot\psi,$$

and therefore

Velocities in terms of momentums.

$$\left.\begin{aligned}\dot\psi &= \frac{dT}{d\xi}\,.\\ \text{Similarly}\qquad \dot\phi &= \frac{dT}{d\eta},\ \text{etc.}\end{aligned}\right\}\ldots\ldots\ldots\ldots\ldots\ldots(10).$$

Velocities in terms of momentums.

These expressions solve the direct problem,—to find the velocity produced by a given impulse (ξ, η, ...), when we have the kinetic energy, T, expressed as a quadratic function of the components of the impulse.

(*e*) If we consider the motion simply, without reference to the impulse required to generate it from rest, or to stop it, the quantities ξ, η, ... are clearly to be regarded as the components of the momentum of the motion, according to the system of generalized co-ordinates.

Reciprocal relation between momentums and velocities in two motions.

(*f*) The following algebraic relation will be useful:—

$$\xi_{,}\dot\psi + \eta_{,}\dot\phi + \zeta_{,}\dot\theta + \text{etc.} = \xi\dot\psi_{,} + \eta\dot\phi_{,} + \zeta\dot\theta_{,} + \text{etc}\ldots\ldots\ldots\ldots(11),$$

where, ξ, η, ψ, ϕ, etc., having the same signification as before, $\xi_{,}$, $\eta_{,}$, $\zeta_{,}$, etc., denote the impulse-components corresponding to any other values, $\dot\psi_{,}$, $\dot\phi_{,}$, $\dot\theta_{,}$, etc., of the velocity-components. It is proved by observing that each member of the equation becomes a symmetrical function of $\dot\psi$, $\dot\psi_{,}$; $\dot\phi$, $\dot\phi_{,}$; etc.; when for $\xi_{,}$, $\eta_{,}$, etc., their values in terms of $\dot\psi_{,}$, $\dot\phi_{,}$, etc., and for ξ, η, etc., their values in terms of $\dot\psi$, $\dot\phi$, etc., are substituted.

Application of generalized co-ordinates to theorems of § 311.

314. A material system of any kind, given at rest, and subjected to an impulse in any specified direction, and of any given magnitude, moves off so as to take the greatest amount of kinetic energy which the specified impulse can give it, subject to § 308 or § 309 (*c*).

Let ξ, η, ... be the components of the given impulse, and $\dot\psi$, $\dot\phi$, ... the components of the actual motion produced by it, which are determined by the equations (10) above. Now let us suppose the system be guided, by means of merely directive constraint, to take, from rest, under the influence of the given impulse, some motion ($\dot\psi_{,}$, $\dot\phi_{,}$, ...) different from the actual motion; and let $\xi_{,}$, $\eta_{,}$, ... be the impulse which, with this constraint removed, would produce the motion ($\dot\psi_{,}$, $\dot\phi_{,}$, ...). We shall have, for this case, as above,

$$T_{,} = \tfrac{1}{2}(\xi_{,}\dot\psi_{,} + \eta_{,}\dot\phi_{,} + \ldots).$$

But $\xi_{,} - \xi$, $\eta_{,} - \eta$ ... are the components of the impulse experienced in virtue of the constraint we have supposed introduced. They neither perform nor consume work on the system when moving as directed by this constraint; that is to say,

$$(\xi_{,} - \xi)\dot\psi_{,} + (\eta_{,} - \eta)\dot\phi_{,} + (\zeta_{,} - \zeta)\dot\theta_{,} + \text{etc.} = 0\ldots\ldots\ldots\ldots(12);$$

Application of generalized co-ordinates to theorems of § 311.

and therefore

$$2T_{,} = \xi\dot\psi_{,} + \eta\dot\phi_{,} + \zeta\dot\theta_{,} + \text{etc.} \quad \ldots\ldots\ldots\ldots(13).$$

Hence we have

$$2(T - T_{,}) = \xi(\dot\psi - \dot\psi_{,}) + \eta(\dot\phi - \dot\phi_{,}) + \text{etc.}$$
$$= (\xi - \xi_{,})(\dot\psi - \dot\psi_{,}) + (\eta - \eta_{,})(\dot\phi - \dot\phi_{,}) + \text{etc.}$$
$$+ \xi_{,}(\dot\psi - \dot\psi_{,}) + \eta_{,}(\dot\phi - \dot\phi_{,}) + \text{etc.}$$

But, by (11) and (12) above, we have

$$\xi_{,}(\dot\psi - \dot\psi_{,}) + \eta_{,}(\dot\phi - \dot\phi_{,}) + \text{etc.} = (\xi - \xi_{,})\dot\psi_{,} + (\eta - \eta_{,})\dot\phi_{,} + \text{etc.} = 0,$$

and therefore we have finally

$$2(T - T_{,}) = (\xi - \xi_{,})(\dot\psi - \dot\psi_{,}) + (\eta - \eta_{,})(\dot\phi - \dot\phi_{,}) + \text{etc.} \quad \ldots(14).$$

Theorems of § 311 in terms of generalized co-ordinates.

that is to say, T exceeds $T_{,}$ by the amount of the kinetic energy that would be generated by an impulse $(\xi - \xi_{,}, \eta - \eta_{,}, \zeta - \zeta_{,}, \text{etc.})$ applied simply to the system, which is essentially positive. In other words,

315. If the system is guided to take, under the action of a given impulse, any motion $(\dot\psi_{,}, \dot\phi_{,}, \ldots)$ different from the natural motion $(\dot\psi, \dot\phi, \ldots)$, it will have less kinetic energy than that of the natural motion, by a difference equal to the kinetic energy of the motion $(\dot\psi - \dot\psi_{,}, \dot\phi - \dot\phi_{,}, \ldots)$.

COR. If a set of material points are struck independently by impulses each given in amount, more kinetic energy is generated if the points are perfectly free to move each independently of all the others, than if they are connected in any way. And the deficiency of energy in the latter case is equal to the amount of the kinetic energy of the motion which geometrically compounded with the motion of either case would give that of the other.

Problems whose data involve impulses and velocities.

(*a*) Hitherto we have either supposed the motion to be fully given, and the impulses required to produce them, to be to be found; or the impulses to be given and the motions produced by them to be to be found. A not less important class of problems is presented by supposing as many linear equations of condition between the impulses and components of motion to be given as there are degrees of freedom of the system to move (or independent co-ordinates). These equations, and as many more supplied by (8) or their equivalents (10), suffice for the complete solution of the problem, to determine the impulses and the motion.

(*b*) A very important case of this class is presented by prescribing, among the velocities alone, a number of linear equations with constant terms, and supposing the impulses to be so directed and related as to do no work on any velocities satisfying another prescribed set of linear equations with no constant terms; the whole number of equations of course being equal to the number of independent co-ordinates of the system. The equations for solving this problem need not be written down, as they are obvious; but the following reduction is useful, as affording the easiest proof of the *minimum* property stated below. Problems whose data involve impulses and velocities.

(*c*) The given equations among the velocities may be reduced to a set, each homogeneous, except one equation with a constant term. Those homogeneous equations diminish the number of degrees of freedom; and we may transform the co-ordinates so as to have the number of independent co-ordinates diminished accordingly. Farther, we may choose the new co-ordinates, so that the linear function of the velocities in the single equation with a constant term may be one of the new velocity-components; and the linear functions of the velocities appearing in the equation connected with the prescribed conditions as to the impulses may be the remaining velocity-components. Thus the impulse will fulfil the condition of doing no work on any other component velocity than the one which is given, and the general problem—

316. Given any material system at rest: let any parts of it be set in motion suddenly with any specified velocities, possible according to the conditions of the system; and let its other parts be influenced only by its connexions with the parts set in motion; required the motion: General problem (compare § 312).

takes the following very simple form:—An impulse of the character specified as a particular component, according to the generalized method of co-ordinates, acts on a material system; its amount being such as to produce a given velocity-component of the corresponding type. It is required to find the motion.

The solution of course is to be found from the equations

$$\dot\psi = A, \qquad \eta = 0, \qquad \zeta = 0 \ldots\ldots\ldots\ldots\ldots\ldots(15)$$

(which are the special equations of condition of the problem) and the general kinetic equations (7), or (10). Choosing the latter, and denoting by $[\xi, \xi]$, $[\xi, \eta]$, etc., the coefficients of $\frac{1}{2}\xi^2$, $\xi\eta$, etc., General problem (compare § 312).

in T, we have

$$\xi = \frac{A}{[\xi, \xi]}, \quad \dot{\phi} = \frac{[\xi, \eta]}{[\xi, \xi]} A, \quad \dot{\theta} = \frac{[\xi, \zeta]}{[\xi, \xi]} A, \text{ etc.} \qquad\text{............(16)}$$

for the result.

This result possesses the remarkable property, that the kinetic energy of the motion expressed by it is less than that of any other motion which fulfils the prescribed condition as to velocity. For, if $\xi_{,}$, $\eta_{,}$, $\zeta_{,}$, etc., denote the impulses required to produce any other motion, $\dot{\psi}_{,}$, $\dot{\phi}_{,}$, $\dot{\theta}_{,}$, etc., and $T_{,}$ the corresponding kinetic energy, we have, by (9),

$$2T_{,} = \xi_{,}\dot{\psi}_{,} + \eta_{,}\dot{\phi}_{,} + \zeta_{,}\dot{\theta}_{,} + \text{etc.}$$

But by (11),

$$\xi_{,}\dot{\psi} + \eta_{,}\dot{\phi} + \zeta_{,}\dot{\theta} + \text{etc.} = \xi\dot{\psi}_{,},$$

since, by (15), we have $\eta = 0$, $\xi = 0$, etc. Hence

$$2T_{,} = \xi\dot{\psi}_{,} + \xi_{,}(\dot{\psi}_{,} - \dot{\psi}) + \eta_{,}(\dot{\phi}_{,} - \dot{\phi}) + \zeta_{,}(\dot{\theta}_{,} - \dot{\theta}) + \ldots$$

Now let also this second case ($\dot{\psi}_{,}$, $\dot{\phi}_{,}, \ldots$) of motion fulfil the prescribed velocity-condition $\dot{\psi}_{,} = A$. We shall have

$$\xi_{,}(\dot{\psi}_{,} - \dot{\psi}) + \eta_{,}(\dot{\phi}_{,} - \dot{\phi}) + \zeta_{,}(\dot{\theta}_{,} - \dot{\theta}) + \ldots$$
$$= (\xi_{,} - \xi)(\dot{\psi}_{,} - \dot{\psi}) + (\eta_{,} - \eta)(\dot{\phi}_{,} - \dot{\phi}) + (\zeta_{,} - \zeta)(\dot{\theta}_{,} - \dot{\theta}) + \ldots.$$

since $\dot{\psi}_{,} - \dot{\psi} = 0$, $\eta = 0$, $\zeta = 0, \ldots$. Hence if $\mathfrak{T}$ denote the kinetic energy of the differential motion ($\dot{\psi}_{,} - \dot{\psi}$, $\dot{\phi}_{,} - \dot{\phi}, \ldots$) we have

$$2T_{,} = 2T + 2\mathfrak{T} \qquad\text{.............................(17)};$$

but $\mathfrak{T}$ is essentially positive and therefore $T_{,}$, the kinetic energy of any motion fulfilling the prescribed velocity-condition, but differing from the actual motion, is greater than T the kinetic energy of the actual motion; and the amount, $\mathfrak{T}$, of the difference is given by the equation

$$2\mathfrak{T} = \eta_{,}(\dot{\phi}_{,} - \dot{\phi}) + \zeta_{,}(\dot{\theta}_{,} - \dot{\theta}) + \text{etc.} \qquad\text{...........(18)},$$

or in words,

Kinetic energy a minimum in this case.

317. The solution of the problem is this:—The motion actually taken by the system is the motion which has less kinetic energy than any other fulfilling the prescribed velocity-conditions. And the excess of the energy of any other such motion, above that of the actual motion, is equal to the energy of the motion which must be compounded with either to produce the other.

Kinetic energy a minimum in this case.

In dealing with cases it may often happen that the use of the co-ordinate system required for the application of the solution (16) is not convenient; but in all cases, even in such as in examples (2) and (3) below, which involve an infinite number of degrees of freedom, the minimum property now proved affords an easy solution.

Impact of a smooth rigid plane of infinite mass on a free rigid body at rest.

Example (1). Let a smooth plane, constrained to keep moving with a given normal velocity, q, come in contact with a free inelastic rigid body at rest: to find the motion produced. The velocity-condition here is, that the motion shall consist of any motion whatever giving to the point of the body which is struck a stated velocity, q, perpendicular to the impinging plane, compounded with any motion whatever giving to the same point any velocity parallel to this plane. To express this condition, let u, v, w be rectangular component linear velocities of the centre of gravity, and let ϖ, ρ, σ be component angular velocities round axes through the centre of gravity parallel to the line of reference. Thus, if x, y, z denote the co-ordinates of the point struck relatively to these axes through the centre of gravity, and if l, m, n be the direction cosines of the normal to the impinging plane, the prescribed velocity-condition becomes

$$(u + \rho z - \sigma y)\, l + (v + \sigma x - \varpi z)\, m + (w + \varpi y - \rho x)\, n = -q \qquad\text{.........(a)},$$

the negative sign being placed before q on the understanding that the motion of the impinging plane is obliquely, if not directly, *towards* the centre of gravity, when l, m, n are each positive If, now, we suppose the rectangular axes through the centre of gravity to be principal axes of the body, and denote by Mf^2, Mg^2, Mh^2 the moments of inertia round them, we have

$$T = \tfrac{1}{2} M (u^2 + v^2 + w^2 + f^2\varpi^2 + g^2\rho^2 + h^2\sigma^2) \qquad\text{...............(b)}.$$

This must be made a minimum subject to the equation of condition (a). Hence, by the ordinary method of indeterminate multipliers,

$$\left.\begin{array}{c} Mu + \lambda l = 0,\ Mv + \lambda m = 0,\ Mw + \lambda n = 0 \\ Mf^2\varpi + \lambda(ny - mz) = 0,\ Mg^2\rho + \lambda(lz - nx) = 0,\ Mh^2\sigma + \lambda(mx - ly) = 0 \end{array}\right\} (c).$$

These six equations give each of them explicitly the value of one of the six unknown quantities u, v, w, ϖ, ρ, σ, in terms of λ and data. Using the values thus found in (a), we have an equation to determine λ; and thus the solution is completed. The first three of equations (c) show that λ, which has entered as an indeterminate multiplier, is to be interpreted as the measure of the amount of the impulse.

Generation of motion by impulse in an inextensible cord or chain.

Example (2). A stated velocity in a stated direction is communicated impulsively to each end of a flexible inextensible cord forming any curvilinear arc: it is required to find the initial motion of the whole cord.

Let x, y, z be the co-ordinates of any point P in it, and $\dot{x}$, $\dot{y}$, $\dot{z}$ the components of the required initial velocity. Let also s be the length from one end to the point P.

If the cord were extensible, the rate per unit of time of the stretching per unit of length which it would experience at P, in virtue of the motion $\dot{x}$, $\dot{y}$, $\dot{z}$, would be

$$\frac{dx}{ds}\frac{d\dot{x}}{ds}+\frac{dy}{ds}\frac{d\dot{y}}{ds}+\frac{dz}{ds}\frac{d\dot{z}}{ds}.$$

Hence, as the cord is inextensible, by hypothesis,

$$\frac{dx}{ds}\frac{d\dot{x}}{ds}+\frac{dy}{ds}\frac{d\dot{y}}{ds}+\frac{dz}{ds}\frac{d\dot{z}}{ds}=0\ldots\ldots\ldots\ldots\ldots\ldots\ldots (a).$$

Subject to this, the kinematical condition of the system, and

$$\left.\begin{array}{l}\dot{x}=u\\ \dot{y}=v\\ \dot{z}=w\end{array}\right\}\text{ when } s=0,\qquad \left.\begin{array}{l}\dot{x}=u'\\ \dot{y}=v'\\ \dot{z}=w'\end{array}\right\}\text{ when } s=l,$$

l denoting the length of the cord, and (u, v, w), (u', v', w'), the components of the given velocities at its two ends: it is required to find $\dot{x}$, $\dot{y}$, $\dot{z}$ at every point, so as to make

$$\int_0^l \tfrac{1}{2}\mu\,(\dot{x}^2+\dot{y}^2+\dot{z}^2)\,ds\ldots\ldots\ldots\ldots\ldots\ldots\ldots (b)$$

a minimum, μ denoting the mass of the string per unit of length, at the point P, which need not be uniform from point to point; and of course

$$ds=(dx^2+dy^2+dz^2)^{\frac{1}{2}}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (c).$$

Multiplying (a) by λ, an indeterminate multiplier, and proceeding as usual according to the method of variations, we have

$$\int_0^l\left\{\mu\,(\dot{x}\delta\dot{x}+\dot{y}\delta\dot{y}+\dot{z}\delta\dot{z})+\lambda\left(\frac{dx}{ds}\frac{d\delta\dot{x}}{ds}+\frac{dy}{ds}\frac{d\delta\dot{y}}{ds}+\frac{dz}{ds}\frac{d\delta\dot{z}}{ds}\right)\right\}ds=0,$$

in which we may regard x, y, z as known functions of s, and this it is convenient we should make independent variable. Integrating "by parts" the portion of the first member which contains λ, and attending to the terminal conditions, we find, according to the regular process, for the equations containing the solution

Generation of motion by impulse in an inextensible cord or chain.

$$\mu\dot{x}=\frac{d}{ds}\left(\lambda\frac{dx}{ds}\right),\quad \mu\dot{y}=\frac{d}{ds}\left(\lambda\frac{dy}{ds}\right),\quad \mu\dot{z}=\frac{d}{ds}\left(\lambda\frac{dz}{ds}\right)\ldots\ldots\ldots\ldots (d).$$

These three equations with (a) suffice to determine the four unknown quantities, $\dot{x}$, $\dot{y}$, $\dot{z}$, and λ. Using (d) to eliminate $\dot{x}$, $\dot{y}$, $\dot{z}$ from (a), we have

$$0=\frac{d\frac{1}{\mu}}{ds}\left\{\frac{dx}{ds}\frac{d}{ds}\left(\lambda\frac{dx}{ds}\right)+\ldots\right\}+\frac{1}{\mu}\left\{\frac{dx}{ds}\frac{d^2}{ds^2}\left(\lambda\frac{dx}{ds}\right)+\ldots\right\}.$$

Taking now s for independent variable, and performing the differentiation here indicated, with attention to the following relations:—

$$\frac{dx^2}{ds^2}+\ldots=1,\quad \frac{dx}{ds}\frac{d^2x}{ds^2}+\ldots=0,$$

$$\frac{dx}{ds}\frac{d^3x}{ds^3}+\ldots+\left(\frac{d^2x}{ds^2}\right)^2+\ldots=0,$$

and the expression (§ 9) for ρ, the radius of curvature, we find

$$\frac{1}{\mu}\frac{d^2\lambda}{ds^2}+\frac{d\left(\frac{1}{\mu}\right)}{ds}\frac{d\lambda}{ds}-\frac{\lambda}{\mu\rho^2}=0\ldots\ldots\ldots\ldots\ldots\ldots (e).$$

a linear differential equation of the second order to determine λ, when μ and ρ are given functions of s.

The interpretation of (d) is very obvious. It shows that λ is the impulsive tension at the point P of the string; and that the velocity which this point acquires instantaneously is the resultant of $\frac{1}{\mu}\frac{d\lambda}{ds}$ tangential, and $\frac{\lambda}{\rho\mu}$ towards the centre of curvature. The differential equation (e) therefore shows the law of transmission of the instantaneous tension along the string, and proves that it depends solely on the mass of the cord per unit of length in each part, and the curvature from point to point, but not at all on the plane of curvature, of the initial form. Thus, for instance, it will be the same along a helix as along a circle of the same curvature.

Generation of motion by impulse in an inextensible cord or chain.

With reference to the fulfilling of the six terminal equations, a difficulty occurs inasmuch as $\dot{x}$, $\dot{y}$, $\dot{z}$ are expressed by (d) immediately, without the introduction of fresh arbitrary constants, in terms of λ, which, as the solution of a differential equation of the second degree, involves only two arbitrary constants. The explanation is, that at any point of the cord, at any instant, any velocity in any direction perpendicular to the tangent may be generated without at all altering the condition of the cord even at points infinitely near it. This, which seems clear enough without proof, may be demonstrated analytically by transforming the kinematical equation (a) thus. Let f be the component tangential velocity, q the component velocity towards the centre of curvature, and p the component velocity perpendicular to the osculating plane. Using the elementary formulas for the direction cosines of these lines (§ 9), and remembering that s is now independent variable, we have

$$\dot{x}=f\frac{dx}{ds}+q\frac{\rho d^2x}{ds^2}+p\frac{\rho\,(dzd^2y-dyd^2z)}{ds^3},\quad \dot{y}=\text{etc.}$$

Substituting these in (a) and reducing, we find

$$\frac{df}{ds}=\frac{q}{\rho}\quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (f),$$

a form of the kinematical equation of a flexible line which will be of much use to us later.

We see, therefore, that if the tangential components of the impressed terminal velocities have any prescribed values, we may give besides, to the ends, any velocities whatever perpendicular to the tangents, without altering the motion acquired by any part of the cord. From this it is clear also, that the directions of the terminal impulses are necessarily tangential; or, in other words, that an impulse inclined to the tangent at either end, would generate an infinite transverse velocity.

To express, then, the terminal conditions, let F and F' be the tangential velocities produced at the ends, which we suppose known. We have, for any point, P, as seen above from (d),

$$f=\frac{1}{\mu}\frac{d\lambda}{ds}\quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (g),$$

and hence when

$$\left.\begin{array}{ll} & s=0,\ \dfrac{1}{\mu}\dfrac{d\lambda}{ds}=F \\ \text{and when} & s=l,\ \dfrac{1}{\mu}\dfrac{d\lambda}{ds}=F' \end{array}\right\}\quad\ldots\ldots\ldots\ldots\ldots\ldots (h),$$

which suffice to determine the constants of integration of (d). Or if the data are the tangential impulses, I, I', required at the ends to produce the motion, we have

$$\left.\begin{array}{ll} \text{when} & s=0,\ \lambda=I, \\ \text{and when} & s=l,\ \lambda=I' \end{array}\right\}\quad\ldots\ldots\ldots\ldots\ldots\ldots (i).$$

Or if either end be free, we have $\lambda=0$ at it, and any prescribed condition as to impulse applied, or velocity generated, at the other end.

The solution of this problem is very interesting, as showing how rapidly the propagation of the impulse falls off with "change of direction" along the cord. The reader will have no difficulty in illustrating this by working it out in detail for the case of a cord either uniform or such that $\mu\dfrac{d\frac{1}{\mu}}{ds}$ is constant, and given in the form of a circle or helix. When μ and ρ are constant, for instance, the impulsive tension decreases in the proportion of 1 to ϵ per space along the curve equal to ρ. The results have curious, and dynamically most interesting, bearings on the motions of a whip lash, and of the rope in harpooning a whale.

Impulsive motion of incompressible liquid.

Example (3). Let a mass of incompressible liquid be given at rest completely filling a closed vessel of any shape; and let, by suddenly commencing to change the shape of this vessel, any arbitrarily prescribed normal velocities be suddenly produced in the liquid at all points of its bounding surface, subject to the condition of not altering the volume: It is required to find the instantaneous velocity of any interior point of the fluid.

Let x, y, z be the co-ordinates of any point P of the space occupied by the fluid, and let u, v, w be the components of the required velocity of the fluid at this point. Then ρ being the density of the fluid, and $\iiint$ denoting integration throughout the space occupied by the fluid, we have

$$T=\iiint\tfrac{1}{2}\rho\,(u^2+v^2+w^2)\,dxdydz\quad\ldots\ldots\ldots\ldots (a),$$

Impulsive motion of incompressible liquid.

which, subject to the kinematical condition (§ 193),

$$\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}=0 \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\; (b),$$

must be the least possible, with the given surface values of the normal component velocity. By the method of variation we have

$$\iiint\left\{\rho(u\delta u+v\delta v+w\delta w)+\lambda\left(\frac{d\delta u}{dx}+\frac{d\delta v}{dy}+\frac{d\delta w}{dz}\right)\right\}dxdydz=0 \ldots.\; (c).$$

But integrating by parts we have

$$\iiint\lambda\left(\frac{d\delta u}{dx}+\frac{d\delta v}{dy}+\frac{d\delta w}{dz}\right)dxdydz=\iint\lambda\,(\delta udydz+\delta vdzdx+\delta wdxdy)$$
$$-\iiint\left(\delta u\frac{d\lambda}{dx}+\delta v\frac{d\lambda}{dy}+\delta w\frac{d\lambda}{dz}\right)dxdydz \;\ldots..(d),$$

and if l, m, n denote the direction cosines of the normal at any point of the surface, dS an element of the surface, and $\iint$ integration over the whole surface, we have

$$\iint\lambda\,(\delta udydz+\delta vdzdx+\delta wdxdy)=\iint\lambda\,(l\delta u+m\delta v+n\delta w)\,dS=0,$$

since the normal component of the velocity is given, which requires that $l\delta u+m\delta v+n\delta w=0$. Using this in going back with the result to (c), (d), and equating to zero the coefficients of δu, δv, δw, we find

$$\rho u=\frac{d\lambda}{dx},\quad \rho v=\frac{d\lambda}{dy},\quad \rho w=\frac{d\lambda}{dz}\;\ldots\ldots\ldots\ldots\ldots\ldots\ldots(e).$$

These, used to eliminate u, v, w from (b), give

$$\frac{d}{dx}\left(\frac{1}{\rho}\frac{d\lambda}{dx}\right)+\frac{d}{dy}\left(\frac{1}{\rho}\frac{d\lambda}{dy}\right)+\frac{d}{dz}\left(\frac{1}{\rho}\frac{d\lambda}{dz}\right)=0\;\ldots\ldots\ldots.(f),$$

an equation for the determination of λ, whence by (e) the solution is completed.

The condition to be fulfilled, besides the kinematical equation (b), amounts to this merely,—that $\rho\,(udx+vdy+wdz)$ must be a complete differential. If the fluid is homogeneous, ρ is constant, and $udx+vdy+wdz$ must be a complete differential; in other words, the motion suddenly generated must be of the "non-rotational" character [§ 190, (i)] throughout the fluid mass. The equation to determine λ becomes, in this case,

$$\frac{d^2\lambda}{dx^2}+\frac{d^2\lambda}{dy^2}+\frac{d^2\lambda}{dz^2}=0\;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(g).$$

Impulsive motion of incompressible liquid.

From the hydrodynamical principles explained later it will appear that λ, the function of which $\rho\,(udx+vdy+wdz)$ is the differential, is the impulsive pressure at the point (x, y, z) of the fluid. Hence we may infer that the equation (f), with the condition that λ shall have a given value at every point of a certain closed surface, has a possible and a determinate solution for every point within that surface. This is precisely the same problem as the determination of the permanent temperature at any point within a heterogeneous solid of which the surface is kept permanently with any non-uniform distribution of temperature over it, (f) being Fourier's equation for the uniform conduction of heat through a solid of which the conducting power at the point (x, y, z) is $\frac{1}{\rho}$. The possibility and the determinateness of this problem (with an exception regarding multiply continuous spaces, to be fully considered in Vol. II.) were both proved above [Chap. I. App. A, (e)] by a demonstration, the comparison of which with the present is instructive. The other case of superficial condition—that with which we have commenced here—shows that the equation (f), with $l\frac{d\lambda}{dx}+m\frac{d\lambda}{dy}+n\frac{d\lambda}{dz}$ given arbitrarily for every point of the surface, has also (with like qualification respecting multiply continuous spaces) a possible and single solution for the whole interior space. This, as we shall see in examining the mathematical theory of magnetic induction, may also be inferred from the general theorem (e) of App. A above, by supposing α to be zero for all points without the given surface, and to have the value $\frac{1}{\rho}$ for any internal point (x, y, z).

Lagrange's equations of motion in terms of generalized co-ordinates

318. The equations of continued motion of a set of free particles acted on by any forces, or of a system connected in any manner and acted on by any forces, are readily obtained in terms of Lagrange's Generalized Co-ordinates by the regular and direct process of analytical transformation, from the ordinary forms of the equations of motion in terms of Cartesian (or rectilineal rectangular) co-ordinates. It is convenient first to effect the transformation for a set of free particles acted on by any forces. The case of any system with invariable connexions, or with connexions varied in a given manner, is then to be dealt with by supposing one or more of the generalized co-ordinates to be constant: or to be given functions of the time. Thus the generalized equations of motion are merely those for the reduced number of the co-ordinates remaining un-given; and their integration determines these co-ordinates.

deduced direct by transformation from the equations of motion in terms of Cartesian co-ordinates.

Let m_1, m_2, etc. be the masses, x_1, y_1, z_1, x_2, etc. be the co-ordinates of the particles; and X_1, Y_1, Z_1, X_2, etc. the components of the forces acting upon them. Let ψ, ϕ, etc. be other variables equal in number to the Cartesian co-ordinates, and let there be the same number of relations given between the two sets ot variables; so that we may either regard ψ, ϕ, etc. as known functions of x_1, y_1, etc., or x_1, y_1, etc. as known functions of ψ, ϕ, etc. Proceeding on the latter supposition we have the equations (*a*), (1), of § 313; and we have equations (*b*), (6), of the same section for the generalized components Ψ, Φ, etc. of the force on the system.

For the Cartesian equations of motion we have

$$X_1 = m_1 \frac{d^2x_1}{dt^2}, \quad Y_1 = m_1 \frac{d^2y_1}{dt^2}, \quad Z_1 = m_1 \frac{d^2z_1}{dt^2}, \quad X_2 = m_2 \frac{d^2x_2}{dt^2} \text{ etc.} \ldots (19).$$

Multiplying the first by $\frac{dx_1}{d\psi}$, the second by $\frac{dy_1}{d\psi}$, and so on, and adding all the products, we find by 313 (6)

$$\Psi = m_1 \left(\frac{d^2x_1}{dt^2}\frac{dx_1}{d\psi} + \frac{d^2y_1}{dt^2}\frac{dy_1}{d\psi} + \frac{d^2z_1}{dt^2}\frac{dz_1}{d\psi} \right) + m_2 (\text{etc.}) + \text{etc.} \ldots (20).$$

Now

$$\frac{d^2x_1}{dt^2}\frac{dx_1}{d\psi} = \frac{d}{dt}\left(\dot{x}_1 \frac{dx_1}{d\psi}\right) - \dot{x}_1 \frac{d}{dt}\frac{dx_1}{d\psi} = \frac{d}{dt}\left(\dot{x}_1 \frac{d\dot{x}_1}{d\dot{\psi}}\right) - \dot{x}_1 \frac{d\dot{x}_1}{d\psi}$$

$$= \frac{d}{dt}\left\{ \tfrac{1}{2} \frac{d(\dot{x}_1^2)}{d\dot{\psi}} \right\} - \tfrac{1}{2} \frac{d(\dot{x}_1^2)}{d\psi} \ldots\ldots (21).$$

Using this and similar expressions with reference to the other co-ordinates in (20), and remarking that

$$\tfrac{1}{2} m_1 (\dot{x}_1^2 + \dot{y}_1^2 + \dot{z}_1^2) + \tfrac{1}{2} m_2 (\text{etc.}) + \text{etc.} = T \ldots\ldots (22),$$

if, as before, we put T for the kinetic energy of the system; we find

$$\Psi = \frac{d}{dt}\frac{dT}{d\dot{\psi}} - \frac{dT}{d\psi} \ldots\ldots\ldots (23).$$

Lagrange's equations of motion in terms of generalized co-ordinates deduced direct by transformation from the equations of motion in terms of Cartesian co-ordinates.

The substitutions of $\frac{d\dot{x}_1}{d\dot{\psi}}$ for $\frac{dx_1}{d\psi}$ and of $\frac{d\dot{x}_1}{d\psi}$ for $\frac{d}{dt}\frac{dx_1}{d\psi}$ used above, suppose $\dot{x}_1$ to be a function of the co-ordinates, and of the generalized velocity-components, as shown in equations (1) of § 313. It is on this supposition [which makes T a quadratic function of the generalized velocity-components with functions of the co-ordinates as coefficients as shown in § 313 (2)] that the differentiations $\frac{d}{d\dot{\psi}}$ and $\frac{d}{d\psi}$ in (23) are performed. Proceeding similarly with reference to ϕ, etc., we find expressions similar to (23) for Φ, etc., and thus we have for the equations of motion in terms of the generalized co-ordinates

$$\left.\begin{aligned} \frac{d}{dt}\frac{dT}{d\dot{\psi}} - \frac{dT}{d\psi} &= \Psi, \\ \frac{d}{dt}\frac{dT}{d\dot{\phi}} - \frac{dT}{d\phi} &= \Phi, \\ \text{etc.} & \end{aligned}\right\} \ldots\ldots\ldots (24).$$

It is to be remarked that there is nothing in the preceding transformation which would be altered by supposing t to appear in the relations between the Cartesian and the generalized co-ordinates: thus if we suppose these relations to be

$$\left.\begin{aligned} F(x_1, y_1, z_1, x_2, \ldots\ldots \psi, \phi, \theta, \ldots\ldots t) &= 0 \\ F_1(x_1, y_1, z_1, x_2, \ldots\ldots \psi, \phi, \theta, \ldots\ldots t) &= 0 \\ \text{etc.} & \end{aligned}\right\} \ldots\ldots\ldots (25),$$

we now, instead of § 313 (1), have

$$\left.\begin{aligned} \dot{x}_1 &= \left(\frac{dx_1}{dt}\right) + \frac{dx_1}{d\psi}\dot{\psi} + \frac{dx_1}{d\phi}\dot{\phi} + \text{etc.} \\ \dot{y}_1 &= \left(\frac{dy_1}{dt}\right) + \frac{dy_1}{d\psi}\dot{\psi} + \frac{dy_1}{d\phi}\dot{\phi} + \text{etc.} \\ \text{etc.} & \end{aligned}\right\} \ldots\ldots\ldots (26),$$

where $\left(\frac{dx_1}{dt}\right)$ denotes what the velocity-component $\dot{x}_1$ would be if ψ, ϕ, etc. were constant; being analytically the partial differential coefficient with reference to t of the formula derived from (26) to express x_1 as a function of t, ψ, ϕ, θ, etc.

Using (26) in (22) we now find instead of a homogeneous quadratic function of $\dot{\psi}$, $\dot{\phi}$, etc., as in (2) of § 313, a mixed

Lagrange's equations of motion in terms of generalized co-ordinates deduced direct by transformation from the equations of motion in terms of Cartesian co-ordinates.

function of zero degree and first and second degrees, for the kinetic energy, as follows :—

$$T = K + (\psi)\dot{\psi} + (\phi)\dot{\phi} + \ldots + \tfrac{1}{2}\{(\psi,\psi)\dot{\psi}^2 + (\phi,\phi)\dot{\phi}^2 + \ldots 2(\psi,\phi)\dot{\psi}\dot{\phi}\ldots\} \ldots (27),$$

where

$$\left.\begin{aligned}
K &= \tfrac{1}{2}\Sigma m\left\{\left(\left(\frac{dx}{dt}\right)\right)^2 + \left(\left(\frac{dy}{dt}\right)\right)^2 + \left(\left(\frac{dz}{dt}\right)\right)^2\right\} \\
(\psi) &= \Sigma m\left\{\left(\frac{dx}{dt}\right)\frac{dx}{d\psi} + \left(\frac{dy}{dt}\right)\frac{dy}{d\psi} + \left(\frac{dz}{dt}\right)\frac{dz}{d\psi}\right\}, \text{ etc.} \\
(\psi,\psi) &= \Sigma m\left\{\left(\frac{dx}{d\psi}\right)^2 + \left(\frac{dy}{d\psi}\right)^2 + \left(\frac{dz}{d\psi}\right)^2\right\}, \text{ etc.} \\
(\psi,\phi) &= \Sigma m\left(\frac{dx}{d\psi}\frac{dx}{d\phi} + \frac{dy}{d\psi}\frac{dy}{d\phi} + \frac{dz}{d\psi}\frac{dz}{d\phi}\right), \text{ etc.} \\
&\text{etc.}
\end{aligned}\right\} \ldots (28);$$

K, (ψ), (ϕ), (ψ, ψ), (ψ, ϕ), etc. being thus in general each a known function of t, ψ, ϕ, etc.

Equations (24) above are Lagrange's celebrated equations of motion in terms of generalized co-ordinates. It was first pointed out by Vieille* that they are applicable not only when ψ, ϕ, etc. are related to x_1, y_1, z_1, x_2, etc. by invariable relations as supposed in Lagrange's original demonstration, but also when the relations involve t in the manner shown in equations (25). Lagrange's original demonstration, to be found in the Fourth Section of the Second Part of his *Mécanique Analytique*, consisted of a transformation from Cartesian to generalized co-ordinates of the indeterminate equation of motion; and it is the same demonstration with unessential variations that has been hitherto given, so far as we know, by all subsequent writers including ourselves in our first edition (§ 329). It seems however an unnecessary complication to introduce the indeterminate variations δx, δy, etc.; and we find it much simpler to deduce Lagrange's generalized equations by direct transformation from the equations of motion (19) of a free particle†.

* Sur les équations différentielles de la dynamique, *Liouville's Journal*, 1849, p. 201.

† [The proof by direct transformation was given by Sir W. R. Hamilton, *Phil. Trans.*, 1835, p. 96. H. L.]

Lagrange's generalized form of the equations of motion expanded.

When the kinematic relations are invariable, that is to say when t does not appear in the equations of condition (25), we find from (27) and (28),

$$T = \tfrac{1}{2}\{(\psi,\psi)\dot{\psi}^2 + 2(\psi,\phi)\dot{\psi}\dot{\phi} + (\phi,\phi)\dot{\phi}^2 + \ldots\} \ldots\ldots (29),$$

$$\left.\begin{aligned}
\frac{d}{dt}\frac{dT}{d\dot{\psi}} = {} & (\psi,\psi)\ddot{\psi} + (\psi,\phi)\ddot{\phi} + \ldots \\
& + \left\{\frac{d(\psi,\psi)}{d\psi}\dot{\psi} + \frac{d(\psi,\psi)}{d\phi}\dot{\phi} + \ldots\right\}\dot{\psi} \\
& + \left\{\frac{d(\psi,\phi)}{d\psi}\dot{\psi} + \frac{d(\psi,\phi)}{d\phi}\dot{\phi} + \ldots\right\}\dot{\phi} \\
& + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}\right\} \ldots\ldots\ldots (29'),$$

and

$$\frac{dT}{d\psi} = \tfrac{1}{2}\left\{\frac{d(\psi,\psi)}{d\psi}\dot{\psi}^2 + 2\frac{d(\psi,\phi)}{d\psi}\dot{\psi}\dot{\phi} + \frac{d(\phi,\phi)}{d\psi}\dot{\phi}^2 + \ldots\right\} \; (29'').$$

Hence the ψ-equation of motion expanded in this, the most important class of cases, is as follows:

$$\left.\begin{aligned}
&(\psi,\psi)\ddot{\psi} + (\psi,\phi)\ddot{\phi} + \ldots + Q_\psi(T) = \Psi, \\
&\text{where} \\
&Q_\psi(T) = \tfrac{1}{2}\left\{\frac{d(\psi,\psi)}{d\psi}\dot{\psi}^2 + 2\frac{d(\psi,\psi)}{d\phi}\dot{\psi}\dot{\phi} + \left[2\frac{d(\psi,\phi)}{d\phi} - \frac{d(\phi,\phi)}{d\psi}\right]\dot{\phi}^2 + \ldots\right\} \\
&\qquad\ldots\ldots\ldots\ldots\ldots\ldots (29''').
\end{aligned}\right\}$$

Remark that $Q_\psi(T)$ is a quadratic function of the velocity-components derived from that which expresses the kinetic energy (T) by the process indicated in the second of these equations, in which ψ appears singularly, and the other co-ordinates symmetrically with one another.

Equation of energy.

Multiply the ψ-equation by $\dot{\psi}$, the ϕ-equation by $\dot{\phi}$, and so on; and add. In what comes from $Q_\psi(T)$ we find terms

$$+2\frac{d(\psi,\psi)}{d\phi}\dot{\psi}\dot{\phi}\,.\,\dot{\psi}, \text{ and } -\frac{d(\psi,\psi)}{d\phi}\dot{\psi}^2\,.\,\dot{\phi};$$

which together yield $\quad + \dfrac{d(\psi,\psi)}{d\phi}\dot{\psi}^2\,.\,\dot{\phi}$.

With this, and the rest simply as shown in (29'''), we find

$$\begin{aligned}
&[(\psi,\psi)\ddot{\psi} + (\psi,\phi)\ddot{\phi} + \ldots]\dot{\psi} \\
&+ [(\psi,\phi)\ddot{\psi} + (\phi,\phi)\ddot{\phi} + \ldots]\dot{\phi} \\
&+ \ldots\ldots\ \ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}$$

Equation of energy.

$$+\frac{dT}{d\psi}\dot\psi+\frac{dT}{d\phi}\dot\phi+\ldots \qquad =\Psi\dot\psi+\Phi\dot\phi+\ldots\ldots\ldots\ldots(29^{\text{iv}}),$$

or

$$\frac{dT}{dt}=\Psi\dot\psi+\Phi\dot\phi+\ldots\ldots\ldots\ldots\ldots\ldots\ldots(29^{\text{v}}).$$

Hamilton's form.

When the kinematical relations are invariable, that is to say, when t does not appear in the equations of condition (25), the equations of motion may be put under a slightly different form first given by Hamilton, which is often convenient; thus:—Let T, $\dot\psi$, $\dot\phi$,..., be expressed in terms of ξ, η,..., the impulses required to produce the motion from rest at any instant [§ 313 (d)]; so that T will now be a homogeneous quadratic function, and $\dot\psi$, $\dot\phi$, ... each a linear function, of these elements, with coefficients—functions of ψ, ϕ, etc., depending on the kinematical conditions of the system, but not on the particular motion. Thus, denoting, as in § 322 (29), by ∂, partial differentiation with reference to ξ, η, ..., ψ, ϕ,..., considered as independent variables, we have [§ 313 (10)]

$$\dot\psi=\frac{\partial T}{d\xi},\qquad \dot\phi=\frac{\partial T}{d\eta},\ \ldots\ldots\ldots\ldots\ldots\ldots(30),$$

and, allowing d to denote, as in what precedes, the partial differentiations with reference to the system $\dot\psi$, $\dot\phi$, ..., ψ, ϕ, ..., we have [§ 313 (8)]

$$\xi=\frac{dT}{d\dot\psi},\qquad \eta=\frac{dT}{d\dot\phi},\ \ldots\ldots\ldots\ldots\ldots\ldots(31).$$

The two expressions for T being, as above, § 313,

$$T=\tfrac{1}{2}\{(\psi,\psi)\dot\psi^2+\ldots+2(\psi,\phi)\dot\psi\dot\phi+\ldots\}=\tfrac{1}{2}\{[\psi,\psi]\xi^2+\ldots+2[\psi,\phi]\xi\eta+\ldots\}(32),$$

the second of these is to be obtained from the first by substituting for $\dot\psi$, $\dot\phi$...., their expressions in terms of ξ, η, ... Hence

$$\frac{\partial T}{d\psi}=\frac{dT}{d\psi}+\frac{dT}{d\dot\psi}\frac{\partial\dot\psi}{d\psi}+\frac{dT}{d\dot\phi}\frac{\partial\dot\phi}{d\psi}+\ldots=\frac{dT}{d\psi}+\xi\frac{\partial}{d\psi}\frac{\partial T}{d\xi}+\eta\frac{\partial}{d\psi}\frac{\partial T}{d\eta}+\ldots$$
$$=\frac{dT}{d\psi}+\frac{\partial}{d\psi}\left(\xi\frac{\partial T}{d\xi}+\eta\frac{\partial T}{d\eta}+\ldots\right)=\frac{dT}{d\psi}+2\frac{\partial T}{d\psi}.$$

From this we conclude

$$\frac{\partial T}{d\psi}=-\frac{dT}{d\psi};\ \text{and, similarly,}\ \frac{\partial T}{d\phi}=-\frac{dT}{d\phi},\ \text{etc.}\ \ldots\ldots(33).$$

Hence Lagrange's equations become

$$\frac{d\xi}{dt}+\frac{\partial T}{d\psi}=\Psi,\ \text{etc.}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(34).$$

Hamilton's form.

In § 327 below a purely analytical proof will be given of Lagrange's generalized equations of motion, establishing them directly as a deduction from the principle of "Least Action," independently of any expression either of this principle or of the equations of motion in terms of Cartesian co-ordinates. In their Hamiltonian form they are also deduced in § 330 (33) from the principle of Least Action ultimately, but through the beautiful "Characteristic Equation" of Hamilton.

319. Hamilton's form of Lagrange's equations of motion in terms of generalized co-ordinates expresses that what is required to prevent any one of the components of momentum from varying is a corresponding component force equal in amount to the rate of change of the kinetic energy per unit increase of the corresponding co-ordinate, with all components of momentum constant: and that whatever is the amount of the component force, its excess above this value measures the rate of increase of the component momentum.

In the case of a conservative system, the same statement takes the following form:—The rate at which any component momentum increases per unit of time is equal to the rate, per unit increase of the corresponding co-ordinate, at which the sum of the potential energy, and the kinetic energy for constant momentums, diminishes. This is the celebrated "canonical form" of the equations of motion of a system, though why it has been so called it would be hard to say.

"Canonical form" of Hamilton's general equations of motion of a conservative system.

Let V denote the potential energy, so that [§ 293 (3)]

$$\Psi\delta\psi+\Phi\delta\phi+\ \ldots=-\delta V,$$

and therefore

$$\Psi=-\frac{dV}{d\psi},\qquad \Phi=-\frac{dV}{d\phi},\ \ldots$$

Let now U denote the algebraic expression for the sum of the potential energy, V, in terms of the co-ordinates, ψ, ϕ..., and the kinetic energy, T, in terms of the co-ordinates and the components of momentum, ξ, η,.... Then

also

$$\left.\begin{aligned}\frac{d\xi}{dt}&=-\frac{\partial U}{d\psi},\ \text{etc.}\\ \frac{d\psi}{dt}&=\frac{\partial U}{d\xi},\ \text{etc.}\end{aligned}\right\}\ldots\ldots\ldots\ldots\ldots\ldots\ldots(35),$$

the latter being equivalent to (30), since the potential energy does not contain ξ, η, etc.

In the following examples we shall adhere to Lagrange's form (24), as the most convenient for such applications.

Examples of the use of Lagrange's generalized equations of motion;—polar co-ordinates.

Example (A).—Motion of a single point (m) referred to polar co-ordinates (r, θ, ϕ). From the well-known geometry of this case we see that δr, $r\delta\theta$, and $r\sin\theta\delta\phi$ are the amounts of linear displacement corresponding to infinitely small increments, δr, $\delta\theta$, $\delta\phi$, of the co-ordinates: also that these displacements are respectively in the direction of r, of the arc $r\delta\theta$ (of a great circle) in the plane of r and the pole, and of the arc $r\sin\theta\delta\phi$ (of a small circle in a plane perpendicular to the axis); and that they are therefore at right angles to one another. Hence if F, G, H denote the components of the force experienced by the point, in these three rectangular directions, we have

$$F=R,\quad Gr=\Theta,\ \text{ and }\ Hr\sin\theta=\Phi\,;$$

R, Θ, Φ being what the generalized components of force (§ 313) become for this particular system of co-ordinates. We also see that $\dot r$, $r\dot\theta$, and $r\sin\theta\dot\phi$ are three components of the velocity, along the same rectangular directions. Hence

$$T=\tfrac{1}{2}m(\dot r^2+r^2\dot\theta^2+r^2\sin^2\theta\dot\phi^2).$$

From this we have

$$\frac{dT}{d\dot r}=m\dot r,\ \frac{dT}{d\dot\theta}=mr^2\dot\theta,\ \frac{dT}{d\dot\phi}=mr^2\sin^2\theta\dot\phi\,;$$

$$\frac{dT}{dr}=mr(\dot\theta^2+\sin^2\theta\dot\phi^2),\ \frac{dT}{d\theta}=mr^2\sin\theta\cos\theta\dot\phi^2,\ \frac{dT}{d\phi}=0.$$

Hence the equations of motion become

$$m\left\{\frac{d\dot r}{dt}-r(\dot\theta^2+\sin^2\theta\dot\phi^2)\right\}=F,$$

$$m\left\{\frac{d\,(r^2\dot\theta)}{dt}-r^2\sin\theta\cos\theta\dot\phi^2\right\}=Gr,$$

$$m\frac{d\,(r^2\sin^2\theta\dot\phi)}{dt}=Hr\sin\theta\,;$$

or, according to the ordinary notation of the differential calculus,

$$m\left\{\frac{d^2r}{dt^2}-r\left(\frac{d\theta^2}{dt^2}+\sin^2\theta\frac{d\phi^2}{dt^2}\right)\right\}=F,$$

$$m\left\{\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right)-r^2\sin\theta\cos\theta\frac{d\phi^2}{dt^2}\right\}=Gr,$$

$$m\frac{d}{dt}\left(r^2\sin^2\theta\frac{d\phi}{dt}\right)=Hr\sin\theta.$$

Examples of the use of Lagrange's generalized equations of motion;—polar co-ordinates.

If the motion is confined to one plane, that of r, θ, we have $\frac{d\phi}{dt}=0$, and therefore $H=0$, and the two equations of motion which remain are

$$m\left(\frac{d^2r}{dt^2}-r\frac{d\theta^2}{dt^2}\right)=F,\quad m\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right)=Gr.$$

These equations might have been written down at once in terms of the second law of motion from the kinematical investigation of § 32, in which it was shown that $\frac{d^2r}{dt^2}-r\frac{d\theta^2}{dt^2}$, and $\frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right)$ are the components of acceleration along and perpendicular to the radius-vector, when the motion of a point in a plane is expressed according to polar co-ordinates, r, θ.

The same equations, with ϕ instead of θ, are obtained from the polar equations in three dimensions by putting $\theta=\frac{1}{2}\pi$, which implies that $G=0$, and confines the motion to the plane (r, ϕ).

Dynamical problem.

Example (B).—Two particles are connected by a string; one of them, m, moves in any way on a smooth horizontal plane, and the string, passing through a smooth infinitely small aperture in this plane, bears the other particle m', hanging vertically downwards, and only moving in this vertical line: (the string remaining always stretched in any practical illustration, but, in the problem, being of course supposed capable of transmitting negative tension with its two parts straight.) Let l be the whole length of the string, r that of the part of it from m to the aperture in the plane, and let θ be the angle between the direction of r and a fixed line in the plane. We have

$$T=\tfrac{1}{2}\{m(\dot r^2+r^2\dot\theta^2)+m'\dot r^2\},$$

$$\frac{dT}{d\dot r}=(m+m')\dot r,\qquad \frac{dT}{d\dot\theta}=mr^2\dot\theta,$$

$$\frac{dT}{dr}=mr\theta^2,\qquad \frac{dT}{d\theta}=0.$$

Also, there being no other external force than gm', the weight of the second particle,

$$R=-gm',\quad \Theta=0.$$

Examples of the use of Lagrange's generalized equations of motion; dynamical problem.

Hence the equations of motion are

$$(m+m')\ddot{r} - mr\dot{\theta}^2 = -m'g, \qquad m\frac{d(r^2\dot{\theta})}{dt} = 0.$$

The motion of m' is of course that of a particle influenced only by a force towards a fixed centre; but the law of this force, P (the tension of the string), is remarkable. To find it we have (§ 32), $P = m(-\ddot{r} + r\dot{\theta}^2)$. But, by the equations of the motion,

$$\ddot{r} - r\dot{\theta}^2 = -\frac{m'}{m+m'}(g + r\dot{\theta}^2), \text{ and } \dot{\theta} = \frac{h}{mr^2},$$

where h (according to the usual notation) denotes the moment of momentum of the motion, being an arbitrary constant of integration. Hence

$$P = \frac{mm'}{m+m'}\left(g + \frac{h^2}{m^2}r^{-3}\right).$$

Case of stable equilibrium due to motion.

The particular case of projection which gives m a circular motion and leaves m' at rest is interesting, inasmuch as (§ 350, below) the motion of m is stable, and therefore m' is in stable equilibrium.

Examples continued; C (a), folding door.

Example (C).—A rigid body m is supported on a fixed axis, and another rigid body n is supported on the first, by another axis; the motion round each axis being perfectly free.

Case (a).—*The second axis parallel to the first.* At any time, t, let ϕ and ψ be the inclinations of a fixed plane through the first axis to the plane of it and the second axis, and to a plane through the second axis and the centre of inertia of the second body. These two co-ordinates, ϕ, ψ, it is clear, completely specify the configuration of the system. Now let a be the distance of the second axis from the first, and b that of the centre of inertia of the second body from the second axis. The velocity of the second axis will be $a\dot{\phi}$; and the velocity of the centre of inertia of the second body will be the resultant of two velocities

$$a\dot{\phi}, \text{ and } b\dot{\psi},$$

in lines inclined to one another at an angle equal to $\psi - \phi$, and its square will therefore be equal to

$$a^2\dot{\phi}^2 + 2ab\dot{\phi}\dot{\psi}\cos(\psi - \phi) + b^2\dot{\psi}^2.$$

Hence, if m and n denote the masses, j the radius of gyration of the first body about the fixed axis, and k that of the second body about a parallel axis through its centre of inertia; we have, according to §§ 280, 281,

Examples continued; C (a), folding door.

$$T = \tfrac{1}{2}\{mj^2\dot{\phi}^2 + n[a^2\dot{\phi}^2 + 2ab\dot{\phi}\dot{\psi}\cos(\psi-\phi) + b^2\dot{\psi}^2 + k^2\dot{\psi}^2]\}.$$

Hence we have,

$$\frac{dT}{d\dot{\phi}} = mj^2\dot{\phi} + na^2\dot{\phi} + nab\cos(\psi-\phi)\dot{\psi}; \quad \frac{dT}{d\dot{\psi}} = nab\cos(\psi-\phi)\dot{\phi} + n(b^2+k^2)\dot{\psi};$$

$$\frac{dT}{d\phi} = -\frac{dT}{d\psi} = nab\sin(\psi-\phi)\dot{\phi}\dot{\psi}.$$

The most general supposition we can make as to the applied forces, is equivalent to assuming a couple, Φ, to act on the first body, and a couple, Ψ, on the second, each in a plane perpendicular to the axes; and these are obviously what the generalized components of stress become in this particular co-ordinate system, ϕ, ψ. Hence the equations of motion are

$$(mj^2 + na^2)\ddot{\phi} + nab\frac{d[\dot{\psi}\cos(\psi-\phi)]}{dt} - nab\sin(\psi-\phi)\dot{\phi}\dot{\psi} = \Phi,$$

$$nab\frac{d[\dot{\phi}\cos(\psi-\phi)]}{dt} + n(b^2+k^2)\ddot{\psi} + nab\sin(\psi-\phi)\dot{\phi}\dot{\psi} = \Psi.$$

If there is no other applied force than gravity, and if, as we may suppose without losing generality, the two axes are horizontal, the potential energy of the system will be

$$gmh(1-\cos\phi) + gn\{a[1-\cos(\phi+A)] + b[1-\cos(\psi+A)]\},$$

the distance of the centre of inertia of the first body from the fixed axis being denoted by h, the inclination of the plane through the fixed axis and the centre of inertia of the first body, to the plane of the two axes, being denoted by A, and the fixed plane being so taken that $\phi = 0$ when the former plane is vertical. By differentiating this, with reference to ϕ and ψ, we therefore have

$$-\Phi = gmh\sin\phi + gna\sin(\phi+A), \quad -\Psi = gnb\sin(\psi+A).$$

We shall examine this case in some detail later, in connexion with the interference of vibrations, a subject of much importance in physical science.

When there are no applied or intrinsic working forces, we have $\Phi = 0$ and $\Psi = 0$: or, if there are mutual forces between the two bodies, but no forces applied from without, $\Phi + \Psi = 0$. In

Examples the use of Lagrange's generalized equations of motion.

either of these cases we have the following first integral :—

$$(mj^2 + na^2)\,\dot\phi + m'ab\cos(\psi-\phi)\,(\dot\phi+\dot\psi) + n\,(b^2+k^2)\,\dot\psi = C\,;$$

obtained by adding the two equations of motion and integrating. This, which clearly expresses the constancy of the whole moment of momentum, gives $\dot\phi$ and $\dot\psi$ in terms of $(\dot\psi-\dot\phi)$ and $(\psi-\phi)$. Using these in the integral equation of energy, provided the mutual forces are functions of $\psi-\phi$, we have a single equation between $\dfrac{d(\psi-\phi)}{dt}$, $(\psi-\phi)$, and constants, and thus the full solution of the problem is reduced to quadratures. [It is worked out fully below, as Sub-example G_1.]

C (b). Motion of governing masses in Watt's centrifugal governor: also of gimballed compass-bowl.

Case (*b*).—*The second axis perpendicular to the first.* For simplicity suppose the pivoted axis of the second body, n, to be a principal axis relatively [§ 282 Def. (2)] to the point, N, in which it is cut by a plane perpendicular to it through the fixed axis of the first body, m. Let NE and NF be n's two other principal axes. Denote now by

h the distance from N to m's fixed axis;

k, e, f the radii of gyration of n round its three principal axes through N;

j the radius of gyration of m round its fixed axis;

θ the inclination of NE to m's fixed axis;

ψ the inclination of the plane parallel to n's pivoted axis through m's fixed axis, to a fixed plane through the latter.

Remarking that the component angular velocities of n round NE and NF are $\dot\psi\cos\theta$ and $\dot\psi\sin\theta$, we find immediately

$$T = \tfrac{1}{2}\{[mj^2 + n\,(h^2 + e^2\cos^2\theta + f^2\sin^2\theta)]\,\dot\psi^2 + nk^2\dot\theta^2\},$$

or, if we put

$$mj^2 + n\,(h^2+f^2) = G,\;\; n\,(e^2-f^2) = D\,;$$
$$T = \tfrac{1}{2}\{(G + D\cos^2\theta)\,\dot\psi^2 + nk^2\dot\theta^2\}.$$

The farther working out of this case we leave as a simple but most interesting exercise for the student. We may return to it later, as its application to the theory of centrifugal chronometric regulators is very important.

Motion of a rigid body pivoted on one of its principal axes mounted on a gimballed bowl.

Example (C′). Take the case C (*b*) and mount a third body M upon an axis OC fixed relatively to n in any position parallel to NE. Suppose for simplicity O to be the centre of inertia of M and OC one of its principal axes; and let OA, OB be its two other principal axes relative to O. The notation being in other respects the same as in Example C (*b*), denote now farther by A, B, C the moments of inertia of M round OA, OB, OC; ϕ the angle between the plane AOC and the plane through the fixed axis of m perpendicular to the pivoted axis of n; ϖ, ρ, σ the component angular velocities of M round OA, OB, OC.

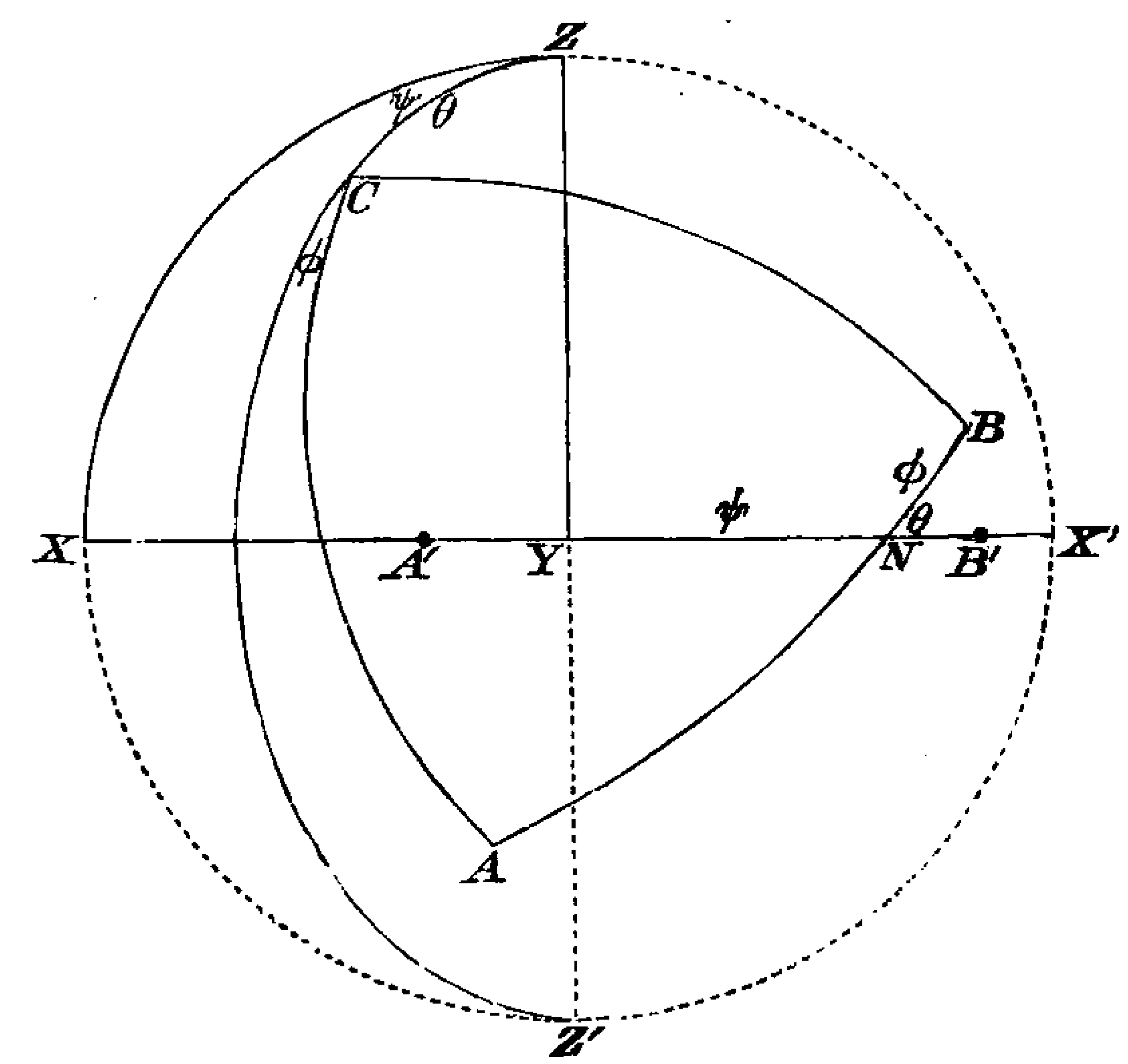
In the annexed diagram, taken from § 101 above, ZCZ' is a

Letter O at centre of sphere concealed by Y.

$\widehat{XA'} = \psi + \phi$,

$\widehat{YN} = \psi$,

$\widehat{NB'} = \phi$.



circle of unit radius having its centre at O and its plane parallel to the fixed axis of m and perpendicular to the pivoted axis of n.

The component velocities of C in the direction of the arc ZC and perpendicular to it are $\dot\theta$ and $\dot\psi\sin\theta$; and the component angular velocity of the plane ZCZ' round OC is $\dot\psi\cos\theta$. Hence

$$\varpi = \dot\theta\sin\phi - \dot\psi\sin\theta\cos\phi,$$
$$\rho = \dot\theta\cos\phi + \dot\psi\sin\theta\sin\phi,$$
and
$$\sigma = \dot\psi\cos\theta + \dot\phi.$$

[Compare § 101.]

Motion of a rigid body pivoted on one of its principal axes mounted on a gimballed bowl.

The kinetic energy of the motion of M relatively to O, its centre of inertia, is (§ 281)

$$\tfrac{1}{2}(A\varpi^2 + B\rho^2 + C\sigma^2);$$

and (§ 280) its whole kinetic energy is obtained by adding the kinetic energy of a material point equal to its mass moving with the velocity of its centre of inertia. This latter part of the kinetic energy of M is most simply taken into account by supposing n to include a material point equal to M placed at O; and using the previous notation k, e, f for radii of gyration of n on the understanding that n now includes this addition. Hence for the present example, with the preceding notation G, D, we have

$$T = \tfrac{1}{2}\{(G + D\cos^2\theta)\dot{\psi}^2 + nk^2\dot{\theta}^2\} \\ + A(\dot{\theta}\sin\phi - \dot{\psi}\sin\theta\cos\phi)^2 + B(\dot{\theta}\cos\phi + \dot{\psi}\sin\theta\sin\phi)^2 \\ + C(\dot{\psi}\cos\theta + \dot{\phi})^2\}.$$

Rigid body rotating freely; referred to the ψ, ϕ, θ co-ordinates (§ 101).

From this the three equations of motion are easily written down.

By putting $G = 0$, $D = 0$, and $k = 0$, we have the case of the motion of a free rigid body relatively to its centre of inertia.

Gyroscopes and gyrostats.

By putting $B = A$ we fall on a case which includes gyroscopes and gyrostats of every variety; and have the following much simplified formula:

$$T = \tfrac{1}{2}\{[G + A + (D - A)\cos^2\theta]\dot{\psi}^2 + (nk^2 + A)\dot{\theta}^2 + C(\dot{\psi}\cos\theta + \phi)^2\},$$

or

$$T = \tfrac{1}{2}\{(E + F\cos^2\theta)\dot{\psi}^2 + (nk^2 + A)\dot{\theta}^2 + C(\dot{\psi}\cos\theta + \dot{\phi})^2\},$$

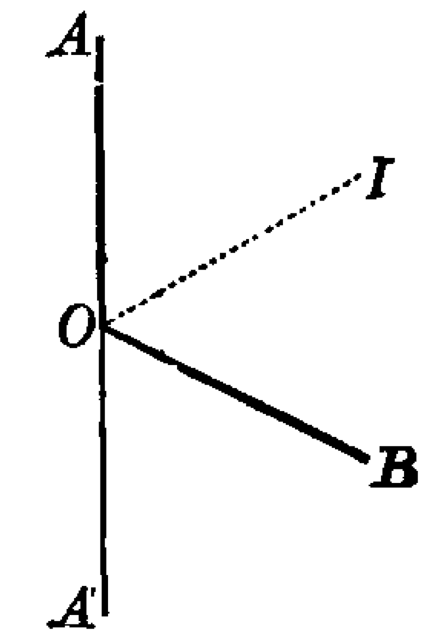
if we put $E = G + A$, and $F = D - A$.

Gyroscopic pendulum.

Example (D).—*Gyroscopic pendulum.*—A rigid body, P, is attached to one axis of a universal flexure joint (§ 109), of which the other is held fixed, and a second body, Q, is supported on P by a fixed axis, in line with, or parallel to, the first-mentioned arm of the joint. For simplicity, we shall suppose Q to be kinetically symmetrical about its bearing axis, and OB to be a principal axis of an ideal rigid body, PQ, composed of P and a mass so distributed along the bearing axis of the actual body Q as to have the same centre of inertia and the same moments of inertia round axes perpendicular to it. Let AO be the fixed arm, O the joint, OB the movable arm bearing the body P, and coinciding with, or parallel to, the axis of Q. Let $BOA' = \theta$; let ϕ be the

Gyroscopic pendulum.

angle which the plane AOB makes with a fixed plane of reference, through OA, chosen so as to contain a second principal axis of the imagined rigid body, PQ, when OB is placed in line with AO; and let ψ be the angle between a plane of reference in Q through its axis of symmetry and the plane of the two principal axes of PQ already mentioned. These three co-ordinates (θ, ϕ, ψ) clearly specify the configuration of the system at any time, t. Let the moments of inertia of the imagined rigid body PQ, round its principal axis OB, the other principal axis referred to above, and the remaining one, be denoted by $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ respectively; and let $\mathfrak{A}'$ be the moment of inertia of Q round its bearing axis.



We have seen (§ 109) that, with the kind of joint we have supposed at O, every possible motion of a body rigidly connected with OB, is resolvable into a rotation round OI, the line bisecting the angle AOB, and a rotation round the line through O perpendicular to the plane AOB. The angular velocity of the latter is θ, according to our present notation. The former would give to any point in OB the same absolute velocity by rotation round OI, that it has by rotation with angular velocity $\dot{\phi}$ round AA'; and is therefore equal to

$$\frac{\sin A'OB}{\sin IOB}\dot{\phi} = \frac{\sin\theta}{\cos\frac{1}{2}\theta}\dot{\phi} = 2\dot{\phi}\sin\tfrac{1}{2}\theta.$$

This may be resolved into $2\dot{\phi}\sin^2\frac{1}{2}\theta = \dot{\phi}(1 - \cos\theta)$ round OB, and $2\dot{\phi}\sin\frac{1}{2}\theta\cos\frac{1}{2}\theta = \dot{\phi}\sin\theta$ round the perpendicular to OB, in plane AOB. Again, in virtue of the symmetrical character of the joint with reference to the line OI, the angle ϕ, as defined above, will be equal to the angle between the plane of the two first-mentioned principal axes of body P, and the plane AOB. Hence the axis of the angular velocity $\dot{\phi}\sin\theta$, is inclined to the principal axis of moment $\mathfrak{B}$ at an angle equal to ϕ. Resolving therefore this angular velocity, and $\dot{\theta}$, into components round the axes of $\mathfrak{B}$ and $\mathfrak{C}$, we find, for the whole component angular velocities of the imagined rigid body PQ, round these axes, $\dot{\phi}\sin\theta\cos\phi + \dot{\theta}\sin\phi$, and $-\dot{\phi}\sin\theta\sin\phi + \dot{\theta}\cos\phi$, respectively. The whole kinetic energy, T, is composed of that of the imagined rigid body PQ, and that of Q about axes through its centre of

Gyroscopic pendulum.

inertia: we therefore have

$$2T=\mathfrak{A}(1-\cos\theta)^2\dot\phi^2+\mathfrak{B}(\dot\phi\sin\theta\cos\phi+\dot\theta\sin\phi)^2+\mathfrak{C}(\dot\phi\sin\theta\sin\phi-\dot\theta\cos\phi)^2 + \mathfrak{A}'\{\dot\psi-\dot\phi(1-\cos\theta)\}^2.$$

Hence $\dfrac{dT}{d\dot\psi}=\mathfrak{A}'\{\dot\psi-\dot\phi(1-\cos\theta)\}$, $\dfrac{dT}{d\psi}=0$,

$$\frac{dT}{d\dot\phi}=\mathfrak{A}(1-\cos\theta)^2\dot\phi+\mathfrak{B}(\dot\phi\sin\theta\cos\phi+\dot\theta\sin\phi)\sin\theta\cos\phi$$
$$+\mathfrak{C}(\dot\phi\sin\theta\sin\phi-\dot\theta\cos\phi)\sin\theta\sin\phi-\mathfrak{A}'\{\dot\psi-\dot\phi(1-\cos\theta)\}(1-\cos\theta),$$

$$\frac{dT}{d\phi}=-\mathfrak{B}(\dot\phi\sin\theta\cos\phi+\dot\theta\sin\phi)(\dot\phi\sin\theta\sin\phi-\dot\theta\cos\phi)$$
$$+\mathfrak{C}(\dot\phi\sin\theta\sin\phi-\dot\theta\cos\phi)(\dot\phi\sin\theta\cos\phi+\dot\theta\sin\phi),$$

$$\frac{dT}{d\dot\theta}=\mathfrak{B}(\dot\phi\sin\theta\cos\phi+\dot\theta\sin\phi)\sin\phi-\mathfrak{C}(\dot\phi\sin\theta\sin\phi-\dot\theta\cos\phi)\cos\phi$$

and $$\frac{dT}{d\theta}=\mathfrak{A}(1-\cos\theta)\sin\theta\dot\phi^2+\mathfrak{B}\cos\theta\cos\phi\dot\phi(\dot\phi\sin\theta\cos\phi+\dot\theta\sin\phi)$$
$$+\mathfrak{C}\cos\theta\sin\phi\dot\phi(\dot\phi\sin\theta\sin\phi-\dot\theta\cos\phi)-\mathfrak{A}'\sin\theta\dot\phi\{\dot\psi-(1-\cos\theta)\dot\phi\}.$$

Now let a couple, G, act on the body Q, in a plane perpendicular to its axis, and let L, M, N act on P, in the plane perpendicular to OB, in the plane $A'OB$, and in the plane through OB perpendicular to the diagram. If ψ is kept constant, and ϕ varied, the couple G will do or resist work in simple addition with L. Hence, resolving $L+G$ and N into components round OI, and perpendicular to it, rejecting the latter, and remembering that $2\sin\frac{1}{2}\theta\dot\phi$ is the angular velocity round OI, we have

$$\Phi=2\sin\tfrac{1}{2}\theta\{-(L+G)\sin\tfrac{1}{2}\theta+N\cos\tfrac{1}{2}\theta\}=\{-(L+G)(1-\cos\theta)+N\sin\theta\}.$$

Also, obviously

$$\Psi=G,\quad \Theta=M.$$

Using these several expressions in Lagrange's general equations (24), we have the equations of motion of the system. They will be of great use to us later, when we shall consider several particular cases of remarkable interest and of very great importance.

Example of varying relation without constraint (rotating axes).

Example (E).—*Motion of a free particle referred to rotating axes.* Let x, y, z be the co-ordinates of a moving particle referred to axes rotating with a constant or varying angular velocity round the axis OZ. Let x_1, y_1, z, be its co-ordinates referred to the same axis, OZ, and two axes OX_1, OY_1, fixed in the plane perpendicular to it. We have

Example of varying relation without constraint (rotating axes).

$$x_1=x\cos\alpha-y\sin\alpha,\ \ y_1=x\sin\alpha+y\cos\alpha;$$
$$\dot x_1=\dot x\cos\alpha-\dot y\sin\alpha-(x\sin\alpha+y\cos\alpha)\dot\alpha,\ \ \dot y_1=\text{etc.}$$

where α, the angle X_1OX, must be considered as a given function of t. Hence

$$T=\tfrac{1}{2}m\{\dot x^2+\dot y^2+\dot z^2+2(x\dot y-y\dot x)\dot\alpha+(x^2+y^2)\dot\alpha^2\},$$
$$\frac{dT}{d\dot x}=m(\dot x-y\dot\alpha),\ \ \frac{dT}{d\dot y}=m(\dot y+x\dot\alpha),\ \ \frac{dT}{d\dot z}=m\dot z,$$
$$\frac{dT}{dx}=m(\dot y\dot\alpha+x\dot\alpha^2),\ \ \frac{dT}{dy}=m(-\dot x\dot\alpha+y\dot\alpha^2),\ \ \frac{dT}{dz}=0.$$

Also,

$$\frac{d}{dt}\frac{dT}{d\dot x}=m(\ddot x-\dot y\dot\alpha-y\ddot\alpha),\ \ \frac{d}{dt}\frac{dT}{d\dot y}=m(\ddot y+\dot x\dot\alpha+x\ddot\alpha),$$

and hence the equations of motion are

$$m(\ddot x-2\dot y\dot\alpha-x\dot\alpha^2-y\ddot\alpha)=X,\ \ m(\ddot y+2\dot x\dot\alpha-y\dot\alpha^2+x\ddot\alpha)=Y,\ \ m\ddot z=Z,$$

X, Y, Z denoting simply the components of the force on the particle, parallel to the moving axes at any instant. In this example t enters into the relation between fixed rectangular axes and the co-ordinate system to which the motion is referred; but there is no constraint. The next is given as an example of varying, or kinetic, constraint.

Example of varying relation due to kinetic constraint.

Example (F).—*A particle, influenced by any forces, and attached to one end of a string of which the other is moved with any constant or varying velocity in a straight line.* Let θ be the inclination of the string at time t, to the given straight line, and ϕ the angle between two planes through this line, one containing the string at any instant, and the other fixed. These two co-ordinates (θ, ϕ) specify the position, P, of the particle at any instant, the length of the string being a given constant, a, and the distance OE, of its other end E, from a fixed point, O, of the line in which it is moved, being a given function of t, which we shall denote by u. Let x, y, z be the co-ordinates of the particle referred to three fixed rectangular axes. Choosing OX as the given straight line, and YOX the fixed plane from which ϕ is measured, we have

$$x=u+a\cos\theta,\ \ y=a\sin\theta\cos\phi,\ \ z=a\sin\theta\sin\phi,$$
$$\dot x=\dot u-a\sin\theta\dot\theta;$$

Example of varying relation due to kinetic constraint.

and for $\dot{y}$, $\dot{z}$ we have the same expressions as in Example (A). Hence

$$T = \mathfrak{T} + \tfrac{1}{2} m\,(\dot{u}^2 - 2\dot{u}\dot{\theta} a \sin\theta)$$

where $\mathfrak{T}$ denotes the same as the T of Example (A), with $\dot{r} = 0$, and $r = a$. Hence, denoting as there, by G and H the two components of the force on the particle, perpendicular to EP, respectively in the plane of θ and perpendicular to it, we find, for the two required equations of motion,

$$m\,\{a\,(\ddot{\theta} - \sin\theta\cos\theta\dot{\phi}^2) - \sin\theta\,\ddot{u}\} = G, \text{ and } ma\,\frac{d\,(\sin^2\theta\dot{\phi})}{dt} = H.$$

These show that the motion is the same as if E were fixed, and a force equal to $-m\ddot{u}$ were applied to the particle in a direction parallel to EX; a result that might have been arrived at at once by superimposing on the whole system an acceleration equal and opposite to that of E, to effect which on P the force $-m\ddot{u}$ is required.

Example (F′). Any case of varying relations such that in 318 (27) the coefficients (ψ, ψ), (ψ, ϕ) ... are independent of t. Let $\mathfrak{T}$ denote the quadratic part, L the linear part, and K [as in § 318 (27)] the constant part of T in respect to the velocity components, so that

$$\left.\begin{aligned} \mathfrak{T} &= \tfrac{1}{2}\,\{(\psi,\psi)\,\dot{\psi}^2 + 2\,(\psi,\phi)\,\dot{\psi}\dot{\phi} + (\phi,\phi)\,\dot{\phi}^2 + \ldots\} \\ L &= (\psi)\,\dot{\psi} + (\phi)\,\dot{\phi} + \ldots \\ K &= (\psi,\ \phi,\ \theta,\ \ldots) \end{aligned}\right\} \quad\ldots\ldots.(a),$$

where (ψ,ψ), (ψ,ϕ), (ϕ,ϕ) ... denote functions of the co-ordinates without t, and (ψ), (ϕ), ..., $(\psi,\ \phi,\ \theta,\ \ldots)$ functions of the co-ordinates and, may be also, of t; and

$$T = \mathfrak{T} + L + K \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.(b).$$

We have $$\frac{dK}{d\dot{\psi}} = 0.$$

Hence the contribution from K to the first member of the ψ-equation of motion is simply $-\dfrac{dK}{d\psi}$. Again we have

$$\frac{dL}{d\dot{\psi}} = (\psi);$$

hence $$\frac{d}{dt}\frac{dL}{d\dot{\psi}} = \frac{d\,(\psi)}{d\psi}\,\dot{\psi} + \frac{d\,(\psi)}{d\phi}\,\dot{\phi} + \text{etc.} + \left(\frac{d\,(\psi)}{dt}\right).$$

Example of varying relation due to kinetic constraint.

Farther we have

$$\frac{dL}{d\psi} = \frac{d\,(\psi)}{d\psi}\,\dot{\psi} + \frac{d\,(\phi)}{d\psi}\,\dot{\phi} + \ldots.$$

Hence the whole contribution from L to the ψ-equation of motion is

$$\left(\frac{d\,(\psi)}{d\phi} - \frac{d\,(\phi)}{d\psi}\right)\dot{\phi} + \left(\frac{d\,(\psi)}{d\theta} - \frac{d\,(\theta)}{d\psi}\right)\dot{\theta} + \ldots + \left(\frac{d\,(\psi)}{dt}\right)\ldots..(c).$$

Lastly, the contribution from $\mathfrak{T}$ is the same as the whole from T in § 318 (29‴); so that we have

$$\frac{d}{dt}\frac{d\mathfrak{T}}{d\dot{\psi}} - \frac{d\mathfrak{T}}{d\psi} = (\psi,\psi)\,\ddot{\psi} + (\psi,\phi)\,\ddot{\phi} + \ldots$$
$$+\tfrac{1}{2}\left\{\frac{d\,(\psi,\psi)}{d\psi}\,\dot{\psi}^2 + 2\,\frac{d\,(\psi,\psi)}{d\phi}\,\dot{\psi}\dot{\phi} + \left[2\,\frac{d\,(\psi,\phi)}{d\phi} - \frac{d\,(\phi,\phi)}{d\psi}\right]\dot{\phi}^2 + \ldots\right\} \quad (d),$$

and the completed ψ-equation of motion is

$$\frac{d}{dt}\frac{d\mathfrak{T}}{d\dot{\psi}} - \frac{d\mathfrak{T}}{d\psi} + \left(\frac{d\,(\psi)}{d\phi} - \frac{d\,(\phi)}{d\psi}\right)\dot{\phi} + \left(\frac{d\,(\psi)}{d\theta} - \frac{d\,(\theta)}{d\psi}\right)\dot{\theta} + \ldots$$
$$+ \left(\frac{d\,(\psi)}{dt}\right) - \frac{dK}{d\psi} = \Psi\ldots\ldots\ldots(e).$$

It is important to remark that the coefficient of $\dot{\phi}$ in this ψ-equation is equal but of opposite sign to the coefficient of $\dot{\psi}$ in the ϕ-equation. [Compare Example G (19) below.]

Equation of energy.

Proceeding as in § 318 (29^{iv}) (29^{v}), we have in respect to $\mathfrak{T}$ precisely the same formulas as there in respect to T. The terms involving first powers of the velocities simply, balance in the sum: and we find finally

$$\frac{d\mathfrak{T}}{dt} + \left(\frac{dL}{dt}\right) - \frac{d_{(\psi,\phi,\ldots)}K}{dt} = \Psi\dot{\psi} + \Phi\dot{\phi} + \ldots\ldots\ldots\ldots\ldots(f),$$

where $d_{(\psi,\phi,\ldots)}$ denotes differentiation on the supposition of $\psi, \phi, \ldots$ variable; and t constant, where it appears explicitly.

Now with this notation we have

$$\frac{dL}{dt} = \left(\frac{dL}{dt}\right) + \frac{d_{(\psi,\phi,\ldots)}L}{dt} + (\psi)\,\ddot{\psi} + (\phi)\,\ddot{\phi} + \ldots,$$

and $$\frac{dK}{dt} = \left(\frac{dK}{dt}\right) + \frac{d_{(\psi,\phi,\ldots)}K}{dt}.$$

Hence from (f) we have

$$\frac{dT}{dt} = \frac{d\,(\mathfrak{T} + L + K)}{dt} = \Psi\dot{\psi} + \Phi\dot{\phi} + \ldots + \frac{d_{(\psi,\phi,\ldots)}L}{dt} + (\psi)\,\ddot{\psi} + (\phi)\,\ddot{\phi} + \ldots$$
$$+ 2\,\frac{d_{(\psi,\phi,\ldots)}K}{dt} + \left(\frac{dK}{dt}\right)\ldots\ldots\ldots\ldots(g).$$

Exercise for student.

Take, for illustration, Examples (E) and (F) from **above**; **in** which we have

$$[\text{Example (E)}] \qquad \mathfrak{T} = \tfrac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2),$$

$$L = m\dot{a}\,(x\dot{y} - y\dot{x}),$$

$$K = \tfrac{1}{2} m\dot{a}^2 (x^2 + y^2),$$

$$\text{and } [\text{Example (F)}] \qquad \mathfrak{T} = \tfrac{1}{2} ma^2 (\sin^2\theta\dot{\phi}^2 + \dot{\theta}^2),$$

$$L = -m\dot{u}a \sin\theta\dot{\theta},$$

$$K = \tfrac{1}{2} m\dot{u}^2.$$

Write out explicitly in each case equations (f) and (g), and verify them by direct work from the equations of motion forming the conclusions of the examples as treated above (remembering that $\dot{a}$ and $\dot{u}$ are to be regarded as given explicit functions of t).

Ignoration of co-ordinates.

Example (G).—*Preliminary to Gyrostatic connexions and to Fluid Motion.* Let there be one or more co-ordinates χ, χ', etc. which do not appear in the coefficients of velocities in the expression for T; that is to say let $\frac{dT}{d\chi} = 0$, $\frac{dT}{d\chi'} = 0$, etc. The equations corresponding to these co-ordinates become

$$\frac{d}{dt}\frac{dT}{d\dot{\chi}} = \text{X}, \quad \frac{d}{dt}\frac{dT}{d\dot{\chi}'} = \text{X}', \text{ etc.} \quad \text{..................(1).}$$

Farther let us suppose that the force-components X, X′, etc. corresponding to the co-ordinates χ, χ', etc. are each zero: we shall have

$$\frac{dT}{d\dot{\chi}} = C, \quad \frac{dT}{d\dot{\rho}} = C', \text{ etc.} \quad \text{................(2);}$$

or, expanded according to previous notation [318 (29)],

$$\left.\begin{array}{l} (\psi,\chi)\,\dot{\psi} + (\phi,\chi)\,\dot{\phi} + \ldots + (\chi,\chi)\,\dot{\chi} + (\chi,\chi')\,\dot{\chi}' + \ldots \;\; = C \\ (\psi,\chi')\,\dot{\psi} + (\phi,\chi')\,\dot{\phi} + \ldots + (\chi',\chi)\,\dot{\chi} + (\chi',\chi')\,\dot{\chi}' + \ldots = C' \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{array}\right\} \ldots(3).$$

Hence, if we put

$$\left.\begin{array}{l} (\psi,\chi)\,\dot{\psi} + (\phi,\chi)\,\dot{\phi} + \ldots \;\; = P \\ (\psi,\chi')\,\dot{\psi} + (\phi,\chi')\,\dot{\phi} + \ldots = P' \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{array}\right\} \text{................}\mathbf{(4),}$$

Ignoration of co-ordinates.

we have

$$\left.\begin{array}{l} (\chi,\chi)\,\dot{\chi} + (\chi,\chi')\,\dot{\chi}' + \ldots \;\; = C - P \\ (\chi',\chi)\,\dot{\chi} + (\chi',\chi')\,\dot{\chi}' + \ldots = C' - P' \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{array}\right\} \text{............(5).}$$

Resolving these for $\dot{\chi}$, $\dot{\chi}'$, ... we find

$$\dot{\chi} = \frac{\begin{vmatrix} (\chi',\chi'), (\chi',\chi''), \ldots \\ (\chi'',\chi'), (\chi'',\chi''), \ldots \\ \ldots\ldots\ldots\ldots \end{vmatrix} (C-P) + \begin{vmatrix} (\chi'',\chi'), (\chi'',\chi''), \ldots \\ (\chi''',\chi'), (\chi''',\chi''), \ldots \\ \ldots\ldots\ldots\ldots \end{vmatrix} (C'-P') + \ldots}{\begin{vmatrix} (\chi,\chi), (\chi,\chi'), (\chi,\chi''), \ldots \\ (\chi',\chi), (\chi',\chi'), (\chi',\chi''), \ldots \\ (\chi'',\chi), (\chi'',\chi'), (\chi'',\chi''), \ldots \\ \ldots\ldots\ldots\ldots\ldots \end{vmatrix}} \quad (6),$$

and symmetrical expressions for $\dot{\chi}'$, $\dot{\chi}''$, ..., or, as we may write them short,

$$\left.\begin{array}{l} \dot{\chi} = (C, C)\,(C - P) + (C, C')\,(C' - P') \;\; + \ldots \\ \dot{\chi}' = (C', C)\,(C - P) + (C', C')\,(C' - P') + \ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{array}\right\} \text{.........(7),}$$

where (C, C), (C, C'), (C', C'), ... denote functions of the retained co-ordinates ψ, ϕ, θ, ... It is to be remembered that, because $(\chi,\chi') = (\chi',\chi)$, $(\chi,\chi'') = (\chi'',\chi)$, we see from (6) that

$$(C, C') = (C', C),\ (C, C'') = (C'', C),\ (C', C'') = (C'', C'), \text{ and so on...(8).}$$

The following formulas for $\dot{\chi}$, $\dot{\chi}'$, ..., condensed in respect to C, C', C'' by aid of the notation (14) below, and expanded in respect to ψ, ϕ, ..., by (4), will also be useful.

$$\left.\begin{array}{l} \dot{\chi} = \dfrac{dK}{dC} - (M\dot{\psi} + N\dot{\phi} + \ldots) \\[2ex] \dot{\chi}' = \dfrac{dK}{dC'} - (M'\dot{\psi} + N'\dot{\phi} + \ldots) \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{array}\right\} \text{............(9),}$$

where

$$\left.\begin{array}{l} M = (C, C)\,.\,(\psi,\chi) + (C, C')\,.\,(\psi,\chi') \;\; + \ldots \\ N = (C, C)\,.\,(\phi,\chi) + (C, C')\,.\,(\phi,\chi') + \ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ M' = (C', C)\,.\,(\psi,\chi) + (C', C')\,.\,(\psi,\chi') + \ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{array}\right\} \text{.......(10).}$$

The elimination of $\dot{\chi}$, $\dot{\chi}'$, ... from T by these expressions for

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them is facilitated by remarking that, as it is a quadratic function of $\dot\psi, \dot\phi, \ldots \dot\chi, \dot\chi', \ldots$, we have

$$T = \tfrac{1}{2}\left\{\dot\psi \frac{dT}{d\dot\psi} + \dot\phi\frac{dT}{d\dot\phi} + \ldots + \dot\chi\frac{dT}{d\dot\chi} + \dot\chi'\frac{dT}{d\dot\chi'} + \ldots\right\}.$$

Hence by (3),

$$T = \tfrac{1}{2}\left\{\dot\psi \frac{dT}{d\dot\psi} + \dot\phi\frac{dT}{d\dot\phi} + \ldots + \dot\chi C + \dot\chi' C' + \ldots\right\},$$

so that we have now only first powers of $\dot\chi, \dot\chi', \ldots$ to eliminate. Gleaning out $\dot\chi, \dot\chi', \ldots$ from the first group of terms, and denoting by T_0 the part of T not containing $\dot\chi, \dot\chi', \ldots$, we find

$$\begin{aligned} T = T_0 + \tfrac{1}{2}\{&[(\psi,\chi)\dot\psi + (\phi,\chi)\dot\phi + \ldots + C]\dot\chi \\ &+ [(\psi,\chi')\dot\psi + (\phi,\chi')\dot\phi + \ldots + C']\dot\chi' \\ &+ \ldots\ldots\ldots\ldots\ldots\ldots\},\end{aligned}$$

or, according to the notation of (4),

$$T = T_0 + \tfrac{1}{2}\{(C+P)\dot\chi + (C'+P')\dot\chi' + \ldots\}.$$

Eliminating now $\dot\chi, \dot\chi', \ldots$ by (7) we find

$$T = T_0 + \tfrac{1}{2}\{(C,C)(C^2-P^2) + 2(C,C')(CC'-PP') + (C',C')(C'^2-P'^2) + \ldots\}\ldots\ldots(11).$$

It is remarkable that only second powers, and products, *not first powers*, of the velocity-components $\dot\psi, \dot\phi, \ldots$ appear in this expression. We may write it thus:—

$$T = \mathfrak{T} + K \ldots\ldots\ldots\ldots\ldots\ldots(12),$$

where $\mathfrak{T}$ denotes a quadratic function of $\dot\psi, \dot\phi, \ldots$, as follows:—

$$\mathfrak{T} = T_0 - \tfrac{1}{2}\{(C,C)P^2 + 2(C,C')PP' + (C',C')P'^2 + \ldots\}\ldots(13),$$

and K a quantity independent of $\dot\psi, \dot\phi, \ldots$, as follows:—

$$K = \tfrac{1}{2}\{(C,C)C^2 + 2(C,C')CC' + (C',C')C'^2 + \ldots\}\ldots\ldots(14).$$

Next, to eliminate $\dot\chi, \dot\chi', \ldots$ from the Lagrange's equations, we have, in virtue of (12) and of the constitutions of T, $\mathfrak{T}$, and K,

$$\frac{dT}{d\dot\psi} + \frac{dT}{d\dot\chi}\frac{d\dot\chi}{d\dot\psi} + \frac{dT}{d\dot\chi'}\frac{d\dot\chi'}{d\dot\psi} + \text{etc.} = \frac{d\mathfrak{T}}{d\dot\psi}\ldots\ldots\ldots(15),$$

where $\dfrac{d\dot\chi}{d\dot\psi}$, $\dfrac{d\dot\chi'}{d\dot\psi}$, etc. are to be found by (7) or (9), and therefore are simply the coefficients of $\dot\psi$ in (9); so that we have

$$\frac{d\dot\chi}{d\dot\psi} = -M,\quad \frac{d\dot\chi'}{d\dot\psi} = -M'\ldots\ldots\ldots\ldots(16),$$

where M, M' are functions of $\psi, \phi, \ldots$ explicitly expressed by (10). Using (16) in (15) we find

$$\frac{dT}{d\dot\psi} = \frac{d\mathfrak{T}}{d\dot\psi} + CM + C'M' + \text{etc.}\ldots\ldots\ldots(17).$$

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Again remarking that $\mathfrak{T} + K$ contains ψ, both as it appeared originally in T, and as farther introduced in the expressions (7) for $\dot\chi, \dot\chi', \ldots$, we see that

$$\begin{aligned}\frac{d}{d\psi}(\mathfrak{T}+K) &= \frac{dT}{d\psi} + \frac{dT}{d\dot\chi}\frac{d\dot\chi}{d\psi} + \frac{dT}{d\dot\chi'}\frac{d\dot\chi'}{d\psi} + \ldots \\ &= \frac{dT}{d\psi} + C\frac{d\dot\chi}{d\psi} + C'\frac{d\dot\chi'}{d\psi} + \ldots\end{aligned}$$

And by (9) we have

$$\frac{d\dot\chi}{d\psi} = -\left(\dot\psi\frac{dM}{d\psi} + \dot\phi\frac{dN}{d\psi} + \ldots\right) + \frac{d}{d\psi}\frac{dK}{dC};$$

which, used in the preceding, gives

$$\frac{d}{d\psi}(\mathfrak{T}+K) = \frac{dT}{d\psi} - C\left(\dot\psi\frac{dM}{d\psi} + \dot\phi\frac{dN}{d\psi} + \ldots\right) - C'\left(\dot\psi\frac{dM'}{d\psi} + \dot\phi\frac{dN'}{d\psi} + \ldots\right) - \text{etc.} + 2\frac{dK}{d\psi}.$$

Hence

$$\frac{dT}{d\psi} = \frac{d\mathfrak{T}}{d\psi} - \frac{dK}{d\psi} + \Sigma C\left(\dot\psi\frac{dM}{d\psi} + \dot\phi\frac{dN}{d\psi} + \ldots\right)\ldots\ldots\ldots(18),$$

where Σ denotes summation with regard to the constants C, C', etc.

Using this and (17) in the Lagrange's ψ-equation, we find finally for the ψ-equation of motion in terms of the non-ignored co-ordinates alone, and conclude the symmetrical equations for ϕ, etc., as follows,

$$\left.\begin{aligned}
&\frac{d}{dt}\left(\frac{d\mathfrak{T}}{d\dot\psi}\right) - \frac{d\mathfrak{T}}{d\psi} + \Sigma C\left\{\left(\frac{dM}{d\phi} - \frac{dN}{d\psi}\right)\dot\phi + \left(\frac{dM}{d\theta} - \frac{dO}{d\psi}\right)\dot\theta + \ldots\right\} + \frac{dK}{d\psi} = \Psi \\
&\frac{d}{dt}\left(\frac{d\mathfrak{T}}{d\dot\phi}\right) - \frac{d\mathfrak{T}}{d\phi} + \Sigma C\left\{\left(\frac{dN}{d\psi} - \frac{dM}{d\phi}\right)\dot\psi + \left(\frac{dN}{d\theta} - \frac{dO}{d\phi}\right)\dot\theta + \ldots\right\} + \frac{dK}{d\phi} = \Phi \\
&\frac{d}{dt}\left(\frac{d\mathfrak{T}}{d\dot\theta}\right) - \frac{d\mathfrak{T}}{d\theta} + \Sigma C\left\{\left(\frac{dO}{d\psi} - \frac{dM}{d\theta}\right)\dot\psi + \left(\frac{dO}{d\phi} - \frac{dN}{d\theta}\right)\dot\phi + \ldots\right\} + \frac{dK}{d\theta} = \Theta \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}\right\}(19).$$

[Compare Example F′ (e) above. It is important to remark that in each equation of motion the first power of the related velocity-component disappears; and the coefficient of each of the other velocity-components in this equation is equal but of opposite sign to the coefficient of the velocity-component corresponding to this equation, in the equation corresponding to that other velocity-component.]

Equation of Energy.

The equation of energy, found as above [§ 318 (29^{iv}) and (29^{v})], is

$$\frac{d(\mathfrak{T}+K)}{dt} = \Psi\dot{\psi} + \Phi\dot{\phi} + \text{etc.} \quad\ldots\ldots\ldots\ldots\ldots(20).$$

The interpretation, considering (12), is obvious. The contrast with Example F′ (*g*) is most instructive.

Sub-Example (G$_1$).—Take, from above, Example C, case (*a*): and put $\phi = \psi + \theta$; also, for brevity, $mj^2 + na^2 = B$, $n(b^2+k^2) = A$, and $nab = c$. We have*

$$T = \tfrac{1}{2}\{A\dot{\psi}^2 + 2c\dot{\psi}(\dot{\psi}+\dot{\theta})\cos\theta + B(\dot{\psi}+\dot{\theta})^2\};$$

and from this find

$$\frac{dT}{d\psi} = 0, \quad \frac{dT}{d\dot{\psi}} = A\dot{\psi} + c(2\dot{\psi}+\dot{\theta})\cos\theta + B(\dot{\psi}+\dot{\theta});$$

$$\frac{dT}{d\theta} = -c\dot{\psi}(\dot{\psi}+\dot{\theta})\sin\theta, \quad \frac{dT}{d\dot{\theta}} = c\dot{\psi}\cos\theta + B(\dot{\psi}+\dot{\theta}).$$

Here the co-ordinate θ alone, and not the co-ordinate ψ, appears in the coefficients. Suppose now $\Psi = 0$ [which is the case considered at the end of C (*a*) above]. We have $\frac{dT}{d\dot{\psi}} = C$, and deduce

$$\dot{\psi} = \frac{C - (c\cos\theta + B)\dot{\theta}}{A + B + 2c\cos\theta},$$

$$T = \tfrac{1}{2}\left(\dot{\psi}\frac{dT}{d\dot{\psi}} + \dot{\theta}\frac{dT}{d\dot{\theta}}\right) = \tfrac{1}{2}\{\dot{\psi}C + \dot{\theta}[(c\cos\theta + B)\dot{\psi} + B\dot{\theta}]\}$$

$$= \tfrac{1}{2}\{\dot{\psi}[C + (c\cos\theta + B)\dot{\theta}] + B\dot{\theta}\}$$

$$= \tfrac{1}{2}\left\{\frac{C^2 - (c\cos\theta + B)^2\dot{\theta}^2}{A + B + 2c\cos\theta} + B\dot{\theta}^2\right\} = \tfrac{1}{2}\frac{C^2 + (AB - c^2\cos^2\theta)\dot{\theta}^2}{A + B + 2c\cos\theta}.$$

Hence $$\mathfrak{T} = \tfrac{1}{2}\frac{AB - c^2\cos^2\theta}{A + B + 2c\cos\theta}\dot{\theta}^2,$$

and $$K = \tfrac{1}{2}\frac{C^2}{A + B + 2c\cos\theta}:$$

* Remark that, according to the alteration from $\psi, \dot{\psi}, \phi, \dot{\phi}$, to $\psi, \dot{\psi}, \theta, \dot{\theta}$, as independent variables,

$$\frac{dT}{d\psi} = \left(\frac{dT}{d\psi}\right) + \left(\frac{dT}{d\phi}\right), \quad \frac{dT}{d\theta} = \left(\frac{dT}{d\phi}\right);$$

and $$\frac{dT}{d\dot{\psi}} = \left(\frac{dT}{d\dot{\psi}}\right) + \left(\frac{dT}{d\dot{\phi}}\right), \quad \frac{dT}{d\dot{\theta}} = \left(\frac{dT}{d\dot{\phi}}\right);$$

where ( ) indicates the original notation of C (*a*).

and the one equation of the motion becomes Equation Energy.

$$\frac{d}{dt}\left(\frac{AB - c^2\cos^2\theta}{A + B + 2c\cos\theta}\dot{\theta}\right) - \tfrac{1}{2}\dot{\theta}^2\frac{d}{d\theta}\left(\frac{AB - c^2\cos^2\theta}{A + B + 2c\cos\theta}\right) = \Theta - \frac{dK}{d\theta};$$

which is to be fully integrated first by multiplying by $d\theta$ and integrating once; and then solving for dt and integrating again with respect to θ. The first integral, being simply the equation of energy integrated, is [Example G (20)]

$$\mathfrak{T} = \int\Theta\, d\theta - K;$$

and the final integral is

$$t = \int d\theta\sqrt{\frac{AB - \cos^2\theta}{2(A + B + 2c\cos\theta)(\int\Theta d\theta - K)}}.$$

In the particular case in which the motion commences from rest, or is such that it can be brought to rest by proper applications of force-components, Ψ, Φ, etc. without any of the force-components X, X′, etc., we have $C = 0$, $C' = 0$, etc.; and the elimination of $\dot{\chi}$, $\dot{\chi}'$, etc. by (3) renders T a homogeneous quadratic function of $\dot{\psi}$, $\dot{\phi}$, etc. without C, C', etc.; and the equations of motion become Ignoration of co-ordinates.

$$\left.\begin{array}{l} \dfrac{d}{dt}\dfrac{dT}{d\dot{\psi}} - \dfrac{dT}{d\psi} = \Psi \\ \dfrac{d}{dt}\dfrac{dT}{d\dot{\phi}} - \dfrac{dT}{d\phi} = \Phi \\ \dfrac{d}{dt}\dfrac{dT}{d\dot{\theta}} - \dfrac{dT}{d\theta} = \Theta \\ \quad\text{etc.} \qquad \text{etc.} \end{array}\right\} \ldots\ldots\ldots\ldots\ldots\ldots\ldots(21).$$

We conclude that on the suppositions made, the elimination of the velocity-components corresponding to the non-appearing co-ordinates gives an expression for the kinetic energy in terms of the remaining velocity-components and corresponding co-ordinates which may be used in the generalised equations just as if these were the sole co-ordinates. The reduced number of equations of motion thus found suffices for the determination of the co-ordinates which they involve without the necessity for knowing or finding the other co-ordinates. If the farther question be put,—to determine the ignored co-ordinates, it is to be answered by a simple integration of equations (7) with $C = 0$, $C' = 0$, etc.

One obvious case of application for this example is a system in which any number of fly wheels, that is to say, bodies which are

Ignoration of co-ordinates.

kinetically symmetrical round an axis (§ 285), are pivoted frictionlessly on any moveable part of the system. In this case with the particular supposition $C=0$, $C'=0$, etc., the result is simply that the motion is the same as if each fly wheel were deprived of moment of inertia round its bearing axis, that is to say reduced to a line of matter fixed in the position of this axis and having unchanged moment of inertia round any axis perpendicular to it. But if C, C', etc. be not each zero we have a case embracing a very interesting class of dynamical problems in which the motion of a system having what we may call gyrostatic links or connexions is the subject. Example (D) above is an example, in which there is just one fly wheel and one moveable body on which it is pivoted. The ignored co-ordinate is ψ; and supposing now Ψ to be zero, we have

$$\dot\psi - \dot\phi(1-\cos\theta) = C \qquad\qquad (a).$$

If we suppose $C=0$ all the terms having $\mathfrak{A}'$ for a factor vanish and the motion is the same as if the fly wheel were deprived of inertia round its bearing axis, and we had simply the motion of the "ideal rigid body PQ" to consider. But when C does not vanish we eliminate $\dot\psi$ from the equations by means of (a). It is important to remark that in every case of Example (G) in which $C=0$, $C'=0$, etc. the motion at each instant possesses the property (§ 312 above) of having less kinetic energy than any other motion for which the velocity-components of the non-ignored co-ordinates have the same values.

Take for another example the final form of Example C′ above, putting B for C, and A for nk^2+A. We have

$$T=\tfrac12\{(E+F\cos^2\theta)\dot\psi^2+B(\dot\psi\cos\theta+\dot\phi)^2+A\dot\theta^2\} \quad\ldots(22).$$

Here neither ψ nor ϕ appears in the coefficients. Let us suppose $\Phi=0$, and eliminate $\dot\phi$, to let us ignore ϕ. We have

$$\frac{dT}{d\dot\phi}=B(\dot\psi\cos\theta+\dot\phi)=C.$$

Hence
$$\dot\phi=\frac{C}{B}-\dot\psi\cos\theta \quad\ldots\ldots(23),$$

$$\mathfrak{T}=\tfrac12\{(E+F\cos^2\theta)\dot\psi^2+A\dot\theta^2\}\quad\ldots\ldots(24),$$

and
$$K=\tfrac12\frac{C^2}{B}\quad\ldots\ldots(25).$$

The place of $\dot\chi$ in (9) above is now taken by $\dot\phi$, and comparing with (23) we find

$$M=\cos\theta,\ N=0,\ O=0.$$

Ignoration of co-ordinates.

Hence, and as K is constant, the equations of motion (19) become

$$\left.\begin{aligned}\frac{d}{dt}\frac{d\mathfrak{T}}{d\dot\psi}-\frac{d\mathfrak{T}}{d\psi}-C\sin\theta\dot\theta&=\Psi\\ \text{and}\quad \frac{d}{dt}\frac{d\mathfrak{T}}{d\dot\theta}-\frac{d\mathfrak{T}}{d\theta}+C\sin\theta\dot\psi&=\Theta\end{aligned}\right\}\quad\ldots\ldots(26);$$

and, using (24) and expanding,

$$\left.\begin{aligned}\frac{d\{(E+F\cos^2\theta)\dot\psi\}}{dt}-C\sin\theta\dot\theta&=\Psi\\ A\ddot\theta+F\sin\theta\cos\theta\dot\psi^2+C\sin\theta\dot\psi&=\Theta\end{aligned}\right\}\quad\ldots\ldots(27).$$

A most important case for the "ignoration of co-ordinates" is presented by a large class of problems regarding the motion of frictionless incompressible fluid in which we can ignore the infinite number of co-ordinates of individual portions of the fluid and take into account only the co-ordinates which suffice to specify the whole boundary of the fluid, including the bounding surfaces of any rigid or flexible solids immersed in the fluid*. The analytical working out of Example (G) shows in fact that when the motion is such as could be produced from rest by merely moving the boundary of the fluid without applying force to its individual particles otherwise than by the transmitted fluid pressure we have exactly the case of $C=0$, $C'=0$, etc.: and Lagrange's generalized equations with the kinetic energy expressed in terms of velocity-components completely specifying the motion of the boundary are available. Thus,

Kinetics of a perfect liquid.

320. Problems in fluid motion of remarkable interest and importance, not hitherto attacked, are very readily solved by the aid of Lagrange's generalized equations of motion. For brevity we shall designate a mass which is absolutely incompressible, and absolutely devoid of resistance to change of shape, by the simple appellation of a *liquid*. We need scarcely say that matter perfectly satisfying this definition does not exist in nature: but we shall see (under properties of matter) how nearly it is approached by water and other common real liquids. And we shall find that much practical and interesting information regarding their true motions is obtained by deduc-

* [This bold transition to the case of a system including a continuous medium, with an infinity of ignored coordinates, has been felt by some writers to require verification. The necessary steps in the hydrodynamical application have been supplied by Kirchhoff and Boltzmann. H. L.]

Kinetics of a perfect liquid.

tions from the principles of abstract dynamics applied to the ideal perfect liquid of our definition. It follows from Example (G) above (and several other proofs, some of them more synthetical in character, will be given in our Second Volume,) that the motion of a homogeneous liquid, whether of infinite extent, or contained in a finite closed vessel of any form, with any rigid or flexible bodies moving through it, if it has ever been at rest, is the same at each instant as that determinate motion (fulfilling, § 312, the condition of having the least possible kinetic energy) which would be impulsively produced from rest by giving instantaneously to every part of the bounding surface, and of the surface of each of the solids within it, its actual velocity at that instant. So that, for example, however long it may have been moving, if all these surfaces were suddenly or gradually brought to rest, the whole fluid mass would come to rest at the same time. Hence, if none of the surfaces is flexible, but we have one or more rigid bodies moving in any way through the liquid, under the influence of any forces, the kinetic energy of the whole motion at any instant will depend solely on the finite number of co-ordinates and component velocities, specifying the position and motion of those bodies, whatever may be the positions reached by particles of the fluid (expressible only by an infinite number of co-ordinates). And an expression for the whole kinetic energy in terms of such elements, finite in number, is precisely what is wanted, as we have seen, as the foundation of Lagrange's equations in any particular case.

It will clearly, in the hydrodynamical, as in all other cases, be a homogeneous quadratic function of the components of velocity, if referred to an invariable co-ordinate system; and the coefficients of the several terms will in general be functions of the co-ordinates, the determination of which follows immediately from the solution of the minimum problem of Example (3) § 317, in each particular case.

Example (1).—*A ball set in motion through a mass of incompressible fluid extending infinitely in all directions on one side of an infinite plane, and originally at rest.* Let x, y, z be the co-ordinates of the centre of the ball at time t, with reference to rectangular axes through a fixed point O of the bounding plane, with OX perpendicular to this plane. If at any instant either

Kinetics of a perfect liquid.

component $\dot{y}$ or $\dot{z}$ of the velocity be reversed, the kinetic energy will clearly be unchanged, and hence no terms $\dot{y}\dot{z}$, $\dot{z}\dot{x}$, or $\dot{x}\dot{y}$ can appear in the expression for the kinetic energy: which, on this account, and because of the symmetry of circumstances with reference to y and z, is

$$T=\tfrac{1}{2}\{P\dot{x}^2+Q(\dot{y}^2+\dot{z}^2)\}.$$

Also, we see that P and Q are functions of x simply, since the circumstances are similar for all values of y and z. Hence, by differentiation,

$$\frac{dT}{d\dot{x}}=P\dot{x},\quad \frac{dT}{d\dot{y}}=Q\dot{y},\quad \frac{dT}{d\dot{z}}=Q\dot{z},$$

$$\frac{d}{dt}\left(\frac{dT}{d\dot{x}}\right)=P\ddot{x}+\frac{dP}{dx}\dot{x}^2,\quad \frac{d}{dt}\left(\frac{dT}{d\dot{y}}\right)=Q\ddot{y}+\frac{dQ}{dx}\dot{y}\dot{x},\text{ etc.},$$

$$\frac{dT}{dx}=\tfrac{1}{2}\left\{\frac{dP}{dx}\dot{x}^2+\frac{dQ}{dx}(\dot{y}^2+\dot{z}^2)\right\},\qquad \frac{dT}{dy}=0,\text{ etc.},$$

and the equations of motion are

$$P\ddot{x}+\tfrac{1}{2}\left\{\frac{dP}{dx}\dot{x}^2-\frac{dQ}{dx}(\dot{y}^2+\dot{z}^2)\right\}=X,$$

$$Q\ddot{y}+\frac{dQ}{dx}\dot{y}\dot{x}=Y,\quad Q\ddot{z}+\frac{dQ}{dx}\dot{z}\dot{x}=Z.$$

Principles sufficient for a practical solution of the problem of determining P and Q will be given later. In the meantime, it is obvious that each decreases as x increases. Hence the equations of motion show that

Effect of a rigid plane on the motion of a ball through a liquid.

321. A ball projected through a liquid perpendicularly *from* an infinite plane boundary, and influenced by no other forces than those of fluid pressure, experiences a gradual acceleration, quickly approximating to a limiting velocity which it sensibly reaches when its distance from the plane is many times its diameter. But if projected *parallel* to the plane, it experiences, as the resultant of fluid pressure, a resultant attraction towards the plane. The former of these results is easily proved by first considering projection *towards* the plane (in which case the motion of the ball will obviously be retarded), and by taking into account the general principle of reversibility (§ 272) which has perfect application in the ideal case of a perfect liquid. The second result is less easily foreseen without