Note on a Passage in Fourier's Heat

by William Thomson

In finding the motion of heat in a sphere, Fourier expands a function Fx, arbitrary between the limits $x = 0$ and $x = X$, in a series of the form

$$a_1 \sin n_1 x + a_2 \sin n_2 x + \&c.$$

where n_1, n_2, &c. are the successive roots of the equation

$$\frac{\tan nX}{nX} = 1 - hX$$

Now Fourier gives no demonstration of the possibility of this expansion, but he merely determines what the coefficients a_1, a_1, &c would be, if the function were represented by a series of this form. Poisson arrives, by another method, at the same conclusion as Fourier, and then states this objection to Fourier's solution; but, as is remarked by Mr Kelland (Theory of Heat, p. 81, note), he "does not appear, as far as I can see, to get over the difficulty." The writer of the following article hopes that the demonstration in it will be considered as satisfactory, and consequently as removing the difficulty.

Let $n_i X = \varepsilon_i$, $\frac{\pi x}{X} = x'$, and $Fx = fx'$

Then the preceding series will take the form

$$a_1 \sin \frac{\varepsilon_1 x}{\pi} + a_2 \sin \frac{\varepsilon_2 x}{\pi} + \&c.,$$

the accents being omitted above x.

Now it is shewn by Fourier, that

$$\varepsilon_i = \left(\frac{2i-1}{2} - c_i\right)\pi,$$