shorter period merely sufficed to *diminish* the depression. Probably closely connected with this action is the gradual decrease Quincke * has observed that time produces in the form of a bubble of gas in a liquid, and of a drop of mercury. To this class of facts are also nearly related the decrease in the intensity of Quincke's † electrical diaphragm-currents, and that which ‡ I have shown to take place in the electromotive force produced when water is forced through capillary tubes. Elster § has recently extended the observation to a similar variation in the electromotive force set up by liquids flowing over the surfaces of solids. Dorn ‖ has investigated at some length the cause of this action in the case of tubes, and has shown that it is capable of modification in various ways, some of which appear capable of exercising a corresponding control over the above-described depression of a liquid in a capillary tube at a temperature near the critical.

It is proposed to continue the still incomplete portions of this inquiry in a paper to the Society next session.

In conclusion I beg leave to express my thanks to Professor McLeod, not only for having advised me to extend my observations to higher temperatures than those at first employed, but also for the willingness he has always shown to aid me with valuable suggestions.

Summary of Contents.

1. When a tube enclosing a capillary tube dipping into alcohol, ether, or sulphurous anhydride is heated, the liquid sinks in the capillary, and rises by expansion in the outer tube. Between 2° and 3° C. below the critical temperatures of these liquids both surfaces become level; and on continuing to heat, the concave meniscus in the capillary tube is seen below that in the external tube. The extent of this depression depends on the diameter &c. of the capillary tube, and on the nature of its internal surface. When the end of a capillary tube dips very slightly below the surface of the liquid, it is level in the capillary and external tubes at the disappearance of the liquid.

2. In some capillary tubes the liquid is not depressed, but disappears at the level of the liquid in which they are immersed on first heating. Once heated, long contact between liquid and tube is necessary to prevent the formation of the depression on again heating. For two tubes which were examined, this time was in each case about 20 hours; a shorter period merely sufficed to *diminish* the depresion. The depression is the result of an action between the liquid and the inner glass surface of the capillary tube.

* Pogg. *Ann.* Bd. clx. S. 570.

† *Ibid.* Bd. cx. S. 56.

‡ *Ibid.* 1877, S. 345.

§ *Inaugural-Dissertation über die in freien Wasserstrahlen auftretenden electromotorischen Kräfte.* Leipzig, 1879.

‖ Wiedemann's *Ann.* Bd. ix. 1880, S. 523. Compare also Helmholtz, Wied. *Ann.* vii. p. 337 (1879).

3. Indications that surfaces exercise a slight action in determining the position at which the liquid condenses in the external tube have been observed.

4. By reflecting a bright line of light from the apparently convex and well-defined surface of ether in a tube of 20 millims. diameter at a temperature near the critical, it may be inferred to remain concave until it loses the power of reflecting when it is plane. The apparent convexity is the result of refraction, or, perhaps, of an action resembling mirage.

5. The black ill-defined band which immediately succeeds the disappearance of the liquid surface is the result of too rapid heating, and possibly due to the mixing of liquid and vapour when they are of nearly equal density. When very slowly heated, as described, the defined concave surface is gradually obliterated, and is last seen as a fine and often waving line. Under this condition also the volume of the liquid at its disappearance is greater than when it is rapidly heated. When the liquid is vaporized by rapid heating, it has a higher temperature and larger volume at the time of disappearance than it has when first condensed by cooling; slowly heated and cooled, these volumes and temperatures are more nearly the same.

Royal Indian Engineering College,
June 1880.

XXIV. *Vibrations of a Columnar Vortex.* *By* Sir William Thomson*.

THIS is a case of fluid-motion, in which the stream-lines are approximately circles, with their centres in one line (the axis of the vortex) and the velocities approximately constant, and approximately equal at equal distances from the axis. As a preliminary to treating it, it is convenient to express the equations of motion of a homogeneous incompressible inviscid fluid (the description of fluid to which the present investigation is confined) in terms of "columnar coordinates," r, θ, z—that is, coordinates such that $r\cos\theta=x$, $r\sin\theta=y$.

If we call the density unity, and if we denote by $\dot{x}, \dot{y}, \dot{z}$ the velocity-components of the fluid particle which at time t is

* From the Proceedings of the Royal Society of Edinburgh, March 1, 1880.

passing through the point (x, y, z), and by $\frac{d}{dt}, \frac{d}{dx}, \frac{d}{dy}, \frac{d}{dz}$ differentiations respectively on the supposition of x, y, z constant, t, y, z constant, t, x, z constant, and t, x, y constant, the ordinary equations of motion are

$$\left.\begin{aligned}-\frac{dp}{dx}&=\frac{d\dot{x}}{dt}+\dot{x}\frac{d\dot{x}}{dx}+\dot{y}\frac{d\dot{x}}{dy}+\dot{z}\frac{d\dot{x}}{dz},\\-\frac{dp}{dy}&=\frac{d\dot{y}}{dt}+\dot{x}\frac{d\dot{y}}{dx}+\dot{y}\frac{d\dot{y}}{dy}+\dot{z}\frac{d\dot{y}}{dz},\\-\frac{dp}{dz}&=\frac{d\dot{z}}{dt}+\dot{x}\frac{d\dot{z}}{dx}+\dot{y}\frac{d\dot{z}}{dy}+\dot{z}\frac{d\dot{z}}{dz},\end{aligned}\right\}\quad . \quad . \quad (1)$$

and

$$\frac{d\dot{x}}{dx}+\frac{d\dot{y}}{dy}+\frac{d\dot{z}}{dz}=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

To transform to the columnar coordinates, we have

$$\left.\begin{aligned}x&=r\cos\theta, \quad y=r\sin\theta,\\\dot{x}&=\dot{r}\cos\theta-r\dot{\theta}\sin\theta,\\\dot{y}&=\dot{r}\sin\theta+r\dot{\theta}\cos\theta,\\\frac{d}{dx}&=\cos\theta\frac{d}{dr}-\sin\theta\frac{d}{rd\theta},\\\frac{d}{dy}&=\sin\theta\frac{d}{dr}+\cos\theta\frac{d}{rd\theta}.\end{aligned}\right\}\quad . \quad . \quad . \quad . \quad . \quad (3)$$

The transformed equations are

$$\left.\begin{aligned}-\frac{dp}{dr}&=\frac{d\dot{r}}{dt}+\dot{r}\frac{d\dot{r}}{dr}-\frac{(r\dot{\theta})^2}{r}+\dot{\theta}\frac{d\dot{r}}{d\theta}+\dot{z}\frac{d\dot{r}}{dz},\\-\frac{dp}{rd\theta}&=r\frac{d\dot{\theta}}{dt}+\dot{r}\frac{d(r\dot{\theta})}{dr}+\dot{r}\dot{\theta}+\dot{\theta}\frac{d(r\dot{\theta})}{d\theta}+\dot{z}\frac{d(r\dot{\theta})}{dz},\\-\frac{dp}{dz}&=\frac{d\dot{z}}{dt}+\dot{r}\frac{d\dot{z}}{dr}+\dot{\theta}\frac{d\dot{z}}{d\theta}+\dot{z}\frac{d\dot{z}}{dz},\end{aligned}\right\}\quad (4)$$

and

$$\frac{d\dot{r}}{dr}+\frac{\dot{r}}{r}+\frac{d(r\dot{\theta})}{rdr}+\frac{d\dot{z}}{dz}=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Now let the motion be approximately in circles round Oz, with velocity everywhere approximately equal to T, a function of r; and to fulfil these conditions, assume

$$\left.\begin{aligned}&\dot{r}=\rho\cos mz\sin(nt-i\theta), \quad r\dot{\theta}=\mathrm{T}+\tau\cos mz\cos(nt-i\theta),\\&\dot{z}=w\sin mz\sin(nt-i\theta), \quad p=\mathrm{P}+\varpi\cos mz\cos(nt-i\theta),\\&\text{with} \qquad \mathrm{P}=\int\frac{\mathrm{T}^2dr}{r},\end{aligned}\right\}(6)$$

where ρ, τ, w, and ϖ are functions of r, each infinitely small in comparison with T. Substituting in (4) and (5) and neglecting squares and products of the infinitely small quantities, we find

$$\left.\begin{aligned}-\frac{d\varpi}{dr}&=\left(n-i\frac{\mathrm{T}}{r}\right)\rho-2\frac{\mathrm{T}}{r}\tau,\\-\frac{i\varpi}{r}&=-\left(n-i\frac{\mathrm{T}}{r}\right)\tau+\left(\frac{\mathrm{T}}{r}+\frac{d\mathrm{T}}{dr}\right)\rho,\\+m\varpi&=\left(n-i\frac{\mathrm{T}}{r}\right)w,\end{aligned}\right\}\quad . \quad . \quad (7)$$

$$\frac{d\rho}{dr}+\frac{\rho}{r}+\frac{i\tau}{r}+mw=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

Taking (7), eliminating ϖ, and resolving for ρ, τ, we find

$$\left.\begin{aligned}\rho&=\frac{1}{m\mathrm{D}}\left(n-i\frac{\mathrm{T}}{r}\right)\left\{\left(n-i\frac{\mathrm{T}}{r}\right)\frac{dw}{dr}-\frac{i}{r}\left(\frac{\mathrm{T}}{r}+\frac{d\mathrm{T}}{dr}\right)w\right\},\\\tau&=\frac{1}{m\mathrm{D}}\left\{\left(\frac{\mathrm{T}}{r}+\frac{d\mathrm{T}}{dr}\right)\left(n-i\frac{\mathrm{T}}{r}\right)\frac{dw}{dr}+\frac{i}{r}\left[\frac{\mathrm{T}^2}{r^2}-\frac{d\mathrm{T}^2}{dr^2}-\left(n-i\frac{\mathrm{T}}{r}\right)^2\right]w\right\},\\\text{where} \quad \mathrm{D}&=\frac{2\mathrm{T}}{r}\left(\frac{\mathrm{T}}{r}+\frac{d\mathrm{T}}{dr}\right)-\left(n-i\frac{\mathrm{T}}{r}\right)^2.\end{aligned}\right\}\ . \ (9)$$

For the particular case of $m=0$, or motion in two dimensions (r, θ), it is convenient to put

$$\frac{-w}{m}=\phi. \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

In this case the motion which superimposed on $\dot{r}=0$ and $r\dot{\theta}=\mathrm{T}$ gives the disturbed motion is irrotational, and $\phi\sin(nt-i\theta)$ is its velocity-potential. It is also to be remarked that, when m does not vanish, the superimposed motion is irrotational where, if at all, and only where $\mathrm{T}=\text{const.}/r$; and that whenever it is irrotational, ϕ, as given by (10), is its velocity-potential.

Eliminating ρ and τ from (8) by (9), we have a linear differential equation of the second order for w. The integration of this, and substitution of the result in (9), give w, ρ, and τ in terms of r, and the two arbitrary constants of integration which, with m, n, and i, are to be determined to fulfil whatever surface-conditions, or initial conditions, or conditions of maintenance are prescribed for any particular problem.

Crowds of exceedingly interesting cases present themselves. Taking one of the simplest to begin:—

CASE I.

Let $\mathrm{T}=\omega r$ (ω const.), ... (11)

$$\left.\begin{array}{ll}\dot{r}=c\cos mz\sin(nt-i\theta) & \text{when approximately } r=a,\\ \dot{r}=\mathfrak{c}\cos mz\sin(nt-i\theta) & \qquad\text{"}\qquad\text{"}\qquad r=\mathfrak{a},\\ \multicolumn{2}{l}{c,\ \mathfrak{c},\ m,\ n,\ a,\ a' \text{ being any given quantities and } i}\\ \multicolumn{2}{l}{\quad\text{any given integer.}}\end{array}\right\} \quad . \quad (12)$$

The condition $T=\omega r$ simplifies (9) to

$$\left.\begin{aligned}\rho&=\frac{(n-i\omega)\left\{\left(n-i\omega\right)\dfrac{dw}{dr}-\dfrac{2i\omega}{r}w\right\}}{m\{4\omega^2-(n-i\omega)^2\}},\\ \tau&=\frac{(n-i\omega)\left\{2\omega\dfrac{dw}{dr}-\dfrac{i(n-i\omega)}{r}\omega\right\}}{m\{4\omega^2-(n-i\omega)^2\}};\end{aligned}\right\} \quad . \quad (13)$$

and the elimination of ρ and τ by these from (8) gives

$$+\frac{d^2w}{dr^2}+\frac{1}{r}\frac{dw}{dr}-\frac{i^2w}{r^2}+m^2\frac{4\omega^2-(n-i\omega)^2}{(n-i\omega)^2}w=0; \quad (14)$$

or

$$\left.\begin{aligned}&\frac{d^2w}{dr^2}+\frac{1}{r}\frac{dw}{dr}-\frac{i^2w}{r^2}+\nu^2w=0,\\ \text{where}\quad &\nu=m\sqrt{\frac{4\omega^2-(n-i\omega)^2}{(n-i\omega)^2}};\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (15)$$

or

$$\left.\begin{aligned}&\frac{d^2w}{dr^2}+\frac{1}{r}\frac{dw}{dr}-\frac{i^2w}{r^2}-\sigma^2w=0,\\ \text{where}\quad &\sigma=m\sqrt{\frac{(n-i\omega)^2-4\omega^2}{(n-i\omega)^2}}.\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (16)$$

Hence if J_i, $\mathfrak{J}_i$ denote Bessel's functions of order i, and of the first and second kinds* (that is to say, J_i finite or zero for infinitely small values of r, and $\mathfrak{J}_i$ finite or zero for infinitely great values of r), and if I_i and $\mathfrak{I}_i$ denote the corresponding real functions with ν imaginary, we have

$$w=CJ_i(\nu r)+\mathfrak{C}\mathfrak{J}_i(\nu r), \quad . \quad . \quad . \quad . \quad (17)$$

or

$$w=CI_i(\sigma r)+\mathfrak{C}\mathfrak{I}_i(\sigma r), \quad . \quad . \quad . \quad . \quad (18)$$

where C and $\mathfrak{C}$ denote arbitrary constants, to be determined in the present case by the equations of condition (12). These are equivalent to $\rho=c$ when $r=a$, and $\rho=\mathfrak{c}$ when $r=\mathfrak{a}$, and, when (16) is used for w in (13), give two simple equations to determine C and $\mathfrak{C}$.

* Compare 'Proceedings,' March 17, 1879, "Gravitational Oscillations of Rotating Water." Solution II. (Case of Circular Basins). Phil. Mag. August 1880, p. 114.

The problem thus solved is the finding of the periodic disturbance in the motion of rotating liquid in a space between two boundaries which are concentric circular cylindric when undisturbed, produced by infinitely small simple harmonic normal motion of these boundaries, distributed over them according to the simple harmonic law in respect to the coordinates z, θ. The most interesting Subcase is had by supposing the inner boundary evanescent ($\mathfrak{a}=0$), and the liquid continuous and undisturbed throughout the space contained by the outer cylindric boundary of radius a. This, as is easily seen, makes $w=0$ when $r=0$, except for the case $i=1$, and essentially, without exception, requires that $\mathfrak{c}$ be zero. Thus the solution for w becomes

$$w=CJ_i(\nu r), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

or

$$w=CI_i(\sigma r); \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

and the condition $\rho=c$ when $r=a$ gives, by (13),

$$C=\frac{\nu^2 m}{\nu J'_i(\nu a)-\dfrac{2i\omega}{(n-i\omega)a}J_i(\nu a)}, \quad . \quad . \quad . \quad (21)$$

or the corresponding I formula.

By summation after the manner of Fourier, we find the solution for any arbitrary distribution of the generative disturbance over the cylindric surface (or over each of the two if we do not confine ourselves to the Subcase), and for any arbitrary periodic function of the time. It is to be remarked that (6) represents an undulation travelling round the cylinder with linear velocity na/i at the surface, or angular velocity n/i throughout. To find the interior effect of a *standing* vibration produced at the surface, we must add to the solution (6), or any sum of solutions of the same type, a solution, or a sum of solutions, in all respects the same, except with $-n$ in place of n.

It is also to be remarked that great enough values of i make ν^2 negative, and therefore ν imaginary; and for such the solutions in terms of σ and the I_i, $\mathfrak{I}_i$ functions must be used.

Case II.—*Hollow Irrotational Vortex in a fixed Cylindric Tube.*

Conditions :—

$$\left.\begin{aligned}&T=\frac{c}{r};\ \dot{r}=0 \text{ when } r=a;\\ \text{and}\quad &P+p=0 \text{ for the disturbed orbit, } r=\mathfrak{a}+\textstyle\int\dot{r}_{\mathfrak{a}}dt,\end{aligned}\right\}(22)$$

a and a being the radii of the hollow cylindric interior, or free boundary, and of the external fixed boundary, and $\dot{r}_a$ the value of $\dot{r}$ when r is approximately equal to a. The condition $T=c/r$ simplifies (9) and (14) to

$$\rho=-\frac{1}{m}\frac{dw}{dr}, \text{ and } \tau=\frac{iw}{mr}, \quad . \quad . \quad . \quad . \quad (23)$$

$$\frac{d^2w}{dr^2}+\frac{1}{r}\frac{dw}{dr}-\frac{i^2w}{r^2}-m^2w^4; \quad . \quad . \quad . \quad . \quad (24)$$

and by (7) we have

$$\varpi=\frac{1}{m}\left(n-\frac{ic}{r^2}\right)w. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

Hence

$$w=\mathrm{C}\mathrm{I}_i(mr)+\mathfrak{C}\mathfrak{I}_i(mr); \quad . \quad . \quad . \quad . \quad . \quad (26)$$

and the equation of condition for the fixed boundary (radial velocity zero there) gives

$$\mathrm{C}\mathrm{I}'_i(ma)+\mathfrak{C}\mathfrak{I}'_i(ma)=0. \quad . \quad . \quad . \quad . \quad (27)$$

To find the other equation of condition, we must first find an expression for the disturbance from circular figure of the free inner boundary. Let for a moment r, θ be the coordinates of one and the same particle of fluid. We shall have

$$\theta=\int\dot{\theta}dt; \text{ and } r=\int\dot{r}dt+r_0,$$

where r_0 denotes the radius of the "mean circle" of the particle's path.

Hence, to a first approximation,

$$\theta=\frac{ct}{r^2}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (28)$$

and therefore, by (6),

$$\dot{r}=\rho\cos mz\sin\left(n-\frac{ic}{r^2}\right)t;$$

whence

$$r=r_0-\frac{\rho}{n-\frac{ic}{r^2}}\cos mz\cos(nt-i\theta). \quad . \quad (29)$$

Hence the equation of the free boundary is

$$r=a-\frac{\rho_{(r=a)}}{n-i\omega}\cos(nt-i\theta), \quad . \quad . \quad . \quad . \quad (30)$$

where

$$\omega=\frac{c}{a^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (31)$$

Hence at (r, θ, z) of this surface we have, from $P=\int\frac{T^2dr}{r}$, of (6) above,

$$P=\frac{T^2}{r}(r-a)$$

$$=-\frac{c^2}{a^3}\frac{\rho(r=a)}{n-i\omega}\cos mz\cos(nt-i\theta). \quad . \quad (32)$$

Hence, and by (6), and (26), and (25), and (23), the condition $P+p=0$ at the free boundary gives

$$\frac{c^2}{a^3}\left[\mathrm{C}\mathrm{I}'_i(ma)+\mathfrak{C}\mathfrak{I}'_i(ma)\right]+\frac{(n-i\omega)^2}{m}\left[\mathrm{C}\mathrm{I}_i(ma)+\mathfrak{C}\mathfrak{I}_i(ma)\right]=0. \quad . \quad . \quad . \quad . \quad (33)$$

Eliminating $\mathrm{C}/\mathfrak{C}$ from this by (27), we get an equation to determine n, by which we find

$$n=\omega(i\pm\sqrt{N}), \quad . \quad . \quad . \quad . \quad . \quad . \quad (34)$$

where N is an essentially positive numeric.

II.—Subcase.

A very interesting Subcase is that of $a=\infty$, which, by (27), makes $C=0$, and therefore, by (33), gives

$$N=ma\frac{-\mathfrak{I}'(ma)}{\mathfrak{I}(ma)}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (35)$$

Whether in Case II. or Subcase II., we see that the disturbance consists of an undulation travelling round the cylinder with angular velocity

$$\omega\left(1+\frac{\sqrt{N}}{i}\right) \text{ or } \omega\left(1-\frac{\sqrt{N}}{i}\right),$$

or of two such undulations superimposed on one another, travelling round the cylinder with angular velocities greater than and (algebraically) less than the angular velocity of the mass of the liquid at its free surfaces by equal differences. The propagation of the wave of greater velocity is in the same direction as that in which the liquid revolves; the propagation of the other is in the contrary direction when $N>i^2$ (as it certainly is in some cases).

If the free surface be started in motion with one or other of the two principal angular velocities (34), or linear velocities $a\omega\left(1\pm\frac{\sqrt{N}}{i}\right)$, and the liquid be then left to itself, it will perform the simple harmonic undulatory movement represented by (6), (26), (23). But if the free surface be displaced to the corrugated form (30), and then left free either at rest or with any other distribution of normal velocity than either of those, the corrugation will, as it were, split into two sets of waves travelling with the two different velocities $a\omega\left(1\pm\frac{\sqrt{N}}{i}\right)$.

The case $i=0$ is clearly exceptional, and can present no undulations travelling round the cylinder. It will be considered later.

The case $i=1$ is particularly important and interesting. To evaluate N for it, remark that

$$\left.\begin{array}{l} I_1(mr)=I'_0(mr) \\ \not{I}_1(mr)=\not{I}'_0(mr). \end{array}\right\} \quad \text{and} \quad . \quad . \quad . \quad . \quad . \quad (36)$$

Now the general solution of (24) is

$$\left.\begin{array}{l} w=\left(E+D\log\frac{1}{mr}\right)\left(1+\frac{m^2r^2}{2^2}+\frac{m^4r^4}{2^2.4^2}+\&c.\right) \\ \qquad +D\left(\frac{m^2r^2}{2^2}S_1+\frac{m^2r^2}{2^2.4^2}S_2+\&c.\right), \end{array}\right\} . \ (36^*)$$

where E and D are constants. Hence, according to our notation,

$$I_0(mr)=1+\frac{m^2r^2}{2^2}+\frac{m^4r^4}{2^2.4^2}+\&c., \quad . \quad . \quad . \quad (37)$$

the constant factor being taken so as to make $I_0(0)=1$.

Stokes* investigated the relation between E and D to make $w=0$ when $r=\infty$, and found it to be

$$\left.\begin{array}{l} E/D=\log 8+\pi^{-\frac{1}{2}}\Gamma'\tfrac{1}{2}=+2\cdot079442-1\cdot963510=\cdot11593; \\ \text{or, to 20 places,} \\ E/D=\cdot11593\ 15156\ 58412\ 44881. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \end{array}\right\} (38)$$

Hence, and by convenient assumption for constant factor,

$$\left.\begin{array}{l} \not{I}_0(mr)=\log\frac{1}{mr}\left(1+\frac{m^2r^2}{2^2}+\frac{m^4r^4}{2^2.4^2}+\&c.\right) \\ \qquad +\frac{m^2r^2}{2^2}(S_1+\cdot11593)+\frac{m^4r^4}{2^2.4^2}(S_2+\cdot11593)+\&c. \end{array}\right\} (39)$$

It is to be remarked that the series in (36) and (39) are convergent, however great be mr; though for values of mr

* "On the Effect of Internal Friction on the Motion of Pendulums," equations (93) and (106). (Camb. Phil. Trans. Dec. 1850.)

P.S.—I am informed by Mr. J. W. L. Glaisher that Gauss, in section 32 of his "Disquisitiones Generales circa seriem infinitam $1+\frac{\alpha.\beta}{1.\gamma}x+\&c.$," (*Opera*, vol. iii. p. 155), gives the value of $-\pi^{-\frac{1}{2}}\Gamma'\tfrac{1}{2}$, or $-\psi(-\tfrac{1}{2})$, in his

exceeding 6 or 7 the semiconvergent expressions* will give the values of the functions nearly enough for most practical purposes, with much less arithmetical labour.

From (37) and (39) we find, by differentiation,

$$\left.\begin{array}{l} I_1(mr)=\frac{mr}{2}+\frac{m^3r^3}{2^2.4}+\frac{m^5r^5}{2^2.4^2.6}+\&c., \\ I'_1(mr)=\frac{1}{2}+\frac{3m^2r^2}{2^2.4}+\frac{5m^4r^4}{2^2.4^2.6}+\&c. \end{array}\right\} \quad . \quad . \quad (40)$$

$$\left.\begin{array}{l} \not{I}_1(mr)=\frac{1}{mr}-\frac{mr}{2^2}[-1+2(S_1+\cdot1159315)] \\ \qquad +\frac{m^3r^3}{2^2.4^2}[-1+2(S_2+\cdot1159315)]+\&c. \\ \qquad -\log\frac{1}{mr}\left(\frac{mr}{2}+\frac{m^3r^3}{2^2.4}+\frac{m^5r^5}{2^2.4^2 6}+\&c\right), \\ \not{I}'_1(mr)=\frac{-1}{m^2r^2}-\frac{1}{2^2}[-3+2(S_1+\cdot1159315)] \\ \qquad +\frac{m^2r^2}{2^2.4^2}[7-6(S_2-\cdot1159315)]+\&c. \\ \qquad -\log\frac{1}{mz}\left(\frac{1}{2}+\frac{3m^2r^2}{2^2.4}+\frac{5m^4r^4}{2^2.4^2.6}+\&c.\right). \end{array}\right\} \quad (41)$$

For an illustration of Case II. with $i=1$, suppose ma to be very small. Remarking that $S_1=1$, we have

$$N=\frac{-ma\not{I}'_1(ma)}{\not{I}_1(ma)}=\frac{1+\frac{m^2a^2}{2}\left[\log\frac{1}{m}-\frac{1}{2}+\cdot1159\right]}{1-\frac{m^2a^2}{2}\left[\log\frac{1}{ma}+\frac{1}{2}+\cdot1159\right]}$$

$$=1+m^2a^2\left(\log\frac{1}{ma}+\cdot1159\right). \quad .. \quad (42)$$

Hence in this case, at all events, $N>i^2$; and the angular velocity of the slow wave, in the reverse direction to that of the

notation, to 23 places as follows:—

1·96351 00260 21423 47944 099.

Thus it appears that the last figure in Stokes's result (106) ought, as in the text, to be 0 instead of 2. In Callet's Tables we find

log. 8=2·07944 15416 79835 92825;

and subtracting the former number from this, we have the value of E to 20 places given the text.

* Stokes, ibid.

liquid's revolution, is

$$-n=\tfrac{1}{2}\,\omega m^2 a^2\left(\log\frac{1}{ma}+\cdot1159\right). \quad . \quad . \quad . \quad (43)$$

This is very small in comparison with

$$2\omega+\tfrac{1}{2}\,\omega m^2 a^2\left(\log\frac{1}{ma}+\cdot1159\right), \quad . \quad . \quad . \quad (44)$$

the angular velocity of the direct wave; and therefore clearly, if the initial normal velocity of the surface when left free after being displaced from its cylindrical figure of equilibrium be zero or any thing small, the amplitude of the quicker direct wave will be very small in proportion to that of the reverse slow one.

Case III.

A slightly disturbed vortex column in liquid extending through all space between two parallel planes; the undisturbed column consisting of a core of uniform vorticity (that is to say, rotating like a solid), surrounded by irrotationally revolving liquid with no slip at the cylindric interface. Denoting by a the radius of this cylinder, we have

$$\left.\begin{aligned} T&=\omega r \quad \text{when} \quad r<a,\\ \text{and} \qquad T&=\omega\frac{a^2}{r} \quad\;\; ,, \qquad r>a.\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (45)$$

Hence (13), (14) hold for $r<a$, and (23), (24) for $r>a$.

Going back to the form of assumption (6), we see that it suits the condition of rigid boundary planes if Oz be perpendicular to them, O in one of them, and the distance between them π/m.

The conditions to be fulfilled at the interface between core and surrounding liquid are that ρ and w must have the same values on the two sides of it: it is easily proved that this implies also equal values of τ on the two sides. The equality of ρ on the two sides of the interface gives, by (13) and (23),

$$\left\{\frac{(i\omega-u)\left[(i\omega-n)\dfrac{dw}{dr}+\dfrac{2i\omega}{r}w\right]}{4\omega^2-(i\omega-n)^2}\right\}^{\text{internal}}_{r=a}=-\left(\frac{dw}{dr}\right)^{\text{external}}_{r=a}; \quad . \quad (46)$$

and from this and the equality of w on the two sides we have

$$\frac{(i\omega-n)\left[(i\omega-n)\left(\dfrac{dw}{wdr}\right)^{\text{internal}}_{r=a}+\dfrac{2i\omega}{a}\right]}{4\omega^2-(i\omega-n)^2}=-\left(\frac{dw}{wdr}\right)^{\text{external}}_{r=a}. \quad (47)$$

The condition that the liquid extends to infinity all round makes $w=0$ when $r=\infty$. Hence the proper integral of (24) is of the form $\mathfrak{K}_i$: and the condition of undisturbed continuity through the axis shows that the proper integral of (13) is of the form J_i. Hence

$$\left.\begin{aligned} w&=CJ_i(\nu r) \text{ for } r<a,\\ \text{and} \qquad w&=\mathfrak{C}\mathfrak{K}_i(mr) \;\; ,, \;\; r>a,\end{aligned}\right\}, \quad . \quad . \quad . \quad . \quad (48)$$

by which (47) becomes

$$\frac{(i\omega-n)\left[(i\omega-n)\dfrac{\nu J'_i(\nu a)}{J_i(\nu a)}+\dfrac{2i\omega}{a}\right]}{4\omega^2-(i\omega-n)^2}=\frac{-m\mathfrak{K}'_i(ma)}{\mathfrak{K}_i(ma)}; \quad (49)$$

or by (15),

$$\frac{J'_i(q)}{qJ_i(q)}+\frac{i}{q^2\lambda}=\frac{-\mathfrak{K}'_i(ma)}{ma\mathfrak{K}_i(ma)}; \quad . \quad . \quad . \quad (50)$$

where

$$\lambda=\frac{i\omega-n}{2\omega}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (51)$$

and

$$q^2=m^2a^2\frac{1-\lambda^2}{\lambda^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (52)$$

Remarking that $J_i(q)$ is the same for positive and negative values of q, and that it passes from positive through zero to a finite negative maximum, thence through zero to a finite positive maximum, and so on an infinite number of times, while q is increased from 0 to ∞, we see that while λ is increased from -1 to 0, the first member of (50) passes an infinite number of times continuously through all real values from $-\infty$ to $+\infty$, and that it does the same when λ is diminished from $+1$ to 0. Hence (50), regarded as a transcendental equation in λ, has an infinite number of roots between -1 and 0 and an infinite number between 0 and $+1$. And it has no roots except between -1 and $+1$, because its second member is clearly positive, whatever be ma; and its first member is essentially real and negative for all real values of λ except between -1 and $+1$, as we see by remarking that when $\lambda^2>1$ $-q^2$ is real and positive, and $-J'_i(q)/qJ_i(q)$ is real and $>i/(-q^2)$; while $i/q^2\lambda$, whether positive or negative, is of less absolute value than $i/(-q^2)$.

Each of the infinite number of values of λ yielded by (50) gives, by (51) and (13), a solution of the problem of finding simple harmonic vibrations of a columnar vortex, with m of any assumed value. All possible simple harmonic vibrations are thus found: and summation, after the manner of Fourier, for different values of m, with different amplitudes and different epochs, gives every possible motion, deviating infinitely little from the undisturbed motion in circular orbits.

The simplest Subcase, that of $i=0$, is curiously interesting. For it (50), (51), (52) give

$$\frac{J'_0(q)}{qJ_0(q)}=\frac{-\mathfrak{J}'_0(ma)}{ma\mathfrak{J}_0(ma)}, \quad . \quad . \quad . \quad . \quad . \quad (53)$$

and

$$n=\frac{2\omega ma}{\sqrt{(m^2a^2+q^2)}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (54)$$

The successive roots of (53), regarded as a transcendental equation in q, lie between the 1st, 3rd, 5th... roots of $J_0(q)=0$, in order of ascending values of q, and the next greater roots of $J'_0(q)=0$, coming nearer and nearer down to the roots of J_0 the greater they are. They are easily calculated by aid of Hansen's Tables of Bessel's functions J_0 and J_1 (which is equal to J'_0) from $q=0$ to $q=20$*. When ma is a small fraction of unity, the second member of (53) is a large number; and even the smallest root exceeds by but a small fraction the first root of $J_0(q)=0$, which, according to Hansen's Table, is 2·4049, or, approximately enough for the present, 2·4. In every case in which q is very large in comparison with ma, whether ma is small or not, (54) gives

$$n=\frac{2\omega ma}{q} \text{ approximately.}$$

Now, going back to (6), we see that the summation of two solutions to constitute waves propagated along the length of the column gives:—

$$\left.\begin{array}{ll}\dot{r}=-\rho\sin(nt-mz); & r\dot{\theta}=T+\tau\cos(nt-mz);\\ \dot{z}=w\cos(nt-mz); & p=+\varpi\cos(nt-mz).\end{array}\right\} \quad (55)$$

The velocity of propagation of these waves is n/m. Hence, when q is large in comparison with ma, the velocity of longitudinal waves is $2\omega a/q$, or $2/q$ of the translational velocity of the surface of the core in its circular orbit. This is 1/1·2, or $\frac{5}{6}$ of the translational velocity, in the case of ma small, and the *mode* corresponding to the smallest root of (53). A full examination of the internal motion of the core, as expressed by (55), (13), (48), (15) is most interesting and instructive. It must form a more developed communication to the Royal Society.

* Republished in Lömmel's *Besselsche Functionen*, Leipzig, 1868.

The Subcase of $i=1$, and ma very small, is particularly interesting and important. In it we have, by (42), for the second member of (50), approximately,

$$\frac{-\mathfrak{J}'_1(ma)}{ma\mathfrak{J}_1(ma)}=\frac{1}{m^2a^2}\left[1+m^2a^2\left(\log\frac{1}{ma}+\cdot1159\right)\right]. \quad . \quad (56)$$

In this case the smallest root, q, is comparable with ma, and all the others are large in comparison with ma. To find the smallest, remark that when q is very small we have, to a second approximation,

$$\frac{J'_1(q)}{qJ_1(q)}=\frac{1}{q^2}-\frac{1}{4}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (57)$$

Hence (50), with $i=1$, becomes, to a first approximation,

$$\frac{1}{q^2}\left(1+\frac{1}{\lambda}\right)=\frac{1}{m^2a^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (58)$$

This and (52), used to find the two unknowns λ and q^2, give

$$\lambda=\tfrac{1}{2}, \text{ and } q^2=3m^2a^2,$$

for a first approximation. Now, with $i=1$, (51) becomes

$$\lambda=\tfrac{1}{2}\left(1-\frac{n}{\omega}\right),$$

and therefore n/ω is infinitely small. Hence (52) gives for a second approximation,

$$q^2=3m^2a^2\left(1+\frac{8n}{3\omega}\right), \quad . \quad . \quad . \quad . \quad . \quad (59)$$

and we have

$$\frac{1}{q^2\lambda}=\frac{2}{3}\,\frac{1}{m^2a^2}\left(1-\frac{5n}{3\omega}\right). \quad . \quad . \quad . \quad . \quad (60)$$

Using now (57), (59), (60), and (56) in (50), we find, to a second approximation,

$$\frac{1}{3ma^2}\left(1-\frac{8n}{3\omega}\right)-\frac{1}{4}+\frac{2}{3ma^2}\left(1-\frac{5n}{3\omega}\right)$$
$$=\frac{1}{m^2a^2}\left[1+m^2a^2\left(\log\frac{1}{ma}+\cdot1159\right)\right],$$

whence

$$\frac{-n}{\omega}=\frac{1}{2}m^2a^2\left(\log\frac{1}{ma}+\frac{1}{4}+\cdot1159\right). \quad . \quad . \quad (61)$$

Compare this result with (43) above. The fact that, as in (43), $-n$ is positive in (61), shows that in this case also the direction in which the disturbance travels round the cylinder is *retrograde* (or opposite to that of the translation of fluid in the undisturbed vortex); and, as was to be expected, the values of $-n$ are approximately equal in the two cases when ma is small enough; but it is smaller by a relatively small difference in (60) than in (43), as was also to be expected.

The case of ma small and $i>1$ has a particularly simple approximate solution for the smallest q-root of the transcendental (50). With any value of i instead of unity we still have (58), as a first approximation for q small. Eliminating q^2/m^2a^2 between this and (52), we still find $\lambda=\frac{1}{2}$; but instead of $n=0$ by (51), we now have $n=(i-1)\omega$. Thus is proved the solution for waves of deformation of sectional figure travelling round a cylindrical vortex, announced thirteen years ago without proof in my first article respecting Vortex Motion*.

* "Vortex Atoms," Proc. Roy. Soc. Edinb. Feb. 18, 1867.

XXV. *On the Diagrammatic and Mechanical Representation of Propositions and Reasonings.*

To the Editors of the Philosophical Magazine and Journal.

GENTLEMEN,

MR. VENN has kindly sent me a copy of his very interesting paper in the Philosophical Magazine for July, in which he explains a method which he has invented for solving logical problems by means of diagrams. The method is certainly ingenious, and for verifying analytical solutions of easy and elementary problems it would, I think, be useful in the hands of a teacher; but I cannot agree with its inventor's estimate of its practical utility in other respects, much less with his opinion as to its superiority over rival methods. Speaking of his diagram for five-letter problems, Mr. Venn says:—

"It must be admitted that such a diagram is not quite so simple to draw as one might wish it to be; but then we must remember what are the alternatives before any one who wishes to grapple effectively with five terms and all the thirty-two possibilities which they yield. *He must either write down, or in some way or other have set before him, all those thirty-two compounds of which* X Y Z W V *is a sample; that is, he must contemplate the array produced by* 160 *letters.*"

Fom the words in italics it is evident that Mr. Venn does not yet appreciate the advantages of my own method, which assuredly lays one under no such onerous obligation as he mentions. It grapples effectively, not merely with problems of five terms, but with problems of six, seven, eight, or even more terms; and it does so because it does *not* oblige one to take into separate consideration all those perplexing possibilities with which Mr. Venn's and similar methods are hampered. That the readers of this Magazine may be able to judge fairly as to the respective capabilities of Mr. Venn's method and mine, I will first solve one of his four-letter problems, and then a six-letter problem of my own, which though exceedingly easy by my method, would, if I am not greatly mistaken, subject his diagrammatic method to a severe strain.

"Every X is either Y or Z; every Y is either Z or W; every Z is either W or X; and every W is either X or Y: what further condition, if any, is needed to ensure that every XY shall be W?"

This is a special case of the following more general problem:—

Given a series of implications, A : a, B : b, C : c, &c.; what is the weakest implication that need be added to these data to justify the inference $m : n$?

The answer is $mn' : \mathrm{A}a' + \mathrm{B}b' + \mathrm{C}c' + \ldots$

When A, a, B, b, &c. are complex expressions involving m or n or both, great simplification may be effected by substituting in these expressions 1 for m and n', and therefore 0 for m' and n. In Mr. Venn's problem the data are (when expressed in my notation)

$$x : y+z, \quad y : z+w, \quad z : w+x, \quad w : x+y,$$

and the weakest addition to the premises to justify the inference $xy : w$ is therefore

$$xyw' : xy'z' + yz'w' + zw'x' + wx'y'.$$

Substituting 1 for every x, y, and w' (and therefore 0 for every x', y', and w) in the consequent of this implication, the implication becomes $xyw' : z'$, which is equivalent to $xyw'z : 0$, or $xyz : w$, the result required. In actual practical working these substitutions of unity and zero would be made mentally while writing down the consequent of the required implication, so that the result may fairly be said to follow directly from mere inspection of the data.

This and the other problems given by Mr. Venn are much too easy: the following problem, involving six letters, would be a fairer test of the power of his method; and I should much like to see his solution of it.

Taking $ax+by : cd'$ as the symbolical expression of the statement "whenever the event A happens with X, or B with Y, then C happens without D," and so on for similar state-