25. The vortex-density at any point of a vortex is the circulation of an infinitesimal filament through this point, divided by the volume of the complete filament. The vortex-density remains always unchanged for the same portion of fluid. By definition it is the same all along any one vortex-filament.

26. Divide a vortex into infinitesimal filaments inversely as their densities, so that their circulations are equal; and let the circulation of each be $\frac{1}{n}$ of unity. Take the projection of all the filaments on one plane. $\frac{1}{n}$ of the sum of the areas of these projections is (V. M. §§ 6, 62) equal to the component impulse of the vortex perpendicular to that plane. Take the projections of the filaments on three planes at right angles to one another, and find the centre of gravity of the areas of these three sets of projections. Find, according to Poinsot's method, the resultant axis, force, and couple of the three forces equal respectively to $\frac{1}{n}$ of the sums of the areas, and acting in lines through the three centres of gravity perpendicular to the three planes. This will be the resultant axis; the force resultant of the impulse, and the couple resultant of the vortex.

The last of these (that is to say, the couple) is also called the rotational moment of the vortex (V. M. § 6).

27. *Definition* IV.—The moment of a plane area round any axis is the product of the area multiplied into the distance from that axis of the perpendicular to its plane through its centre of gravity.

Definition V.—The area of the projection of a closed curve on the plane for which the area of projection is a maximum will be called the area of projection of the curve, or simply the area of the curve. The area of the projection on any plane perpendicular to the plane of the resultant area is of course zero.

Definition VI.—The resultant axis of a closed curve is a line through the centre of gravity, and perpendicular to the plane of its resultant area. The resultant areal moment of a closed curve is the moment round the resultant axis of the areas of its projections on two planes at right angles to one another, and parallel to this axis. It is understood, of course, that the areas of the projections on these two planes are not evanescent generally, except for the case of a plane curve, and that their zero-values are generally the sums of equal positive and negative portions. Thus their moments are not in general zero.

Thus, according to these definitions, the resultant impulse of a vortex-filament of infinitely small cross section and of unit circulation is equal to the resultant area of its curve. The resultant axis of a vortex is the same as the resultant axis of the curve; and the rotational moment is equal to the resultant areal moment of the curve.

28. Consider for a moment a vortex-filament in an infinite liquid with no disturbing influence of other vortices, or of solids immersed in the liquid. We now see, from the constancy of the impulse (proved generally in V. M. § 19), that the resultant area, and the resultant areal moment of the curve formed by the filament, remain constant however its curve may become contorted; and its resultant axis remains the same line in space. Hence, whatever motions and contortions the vortex-filament may experience, if it has any motion of translation through space this motion must be on the average along the resultant axis.

29. Consider now the actual vortex made up of an infinite number of infinitely small vortex-filaments. If these be of volumes inversely proportional to their vortex-densities (§ 25), so that their circulations are equal, we now see from the constancy of the impulse that the sum of the resultant areas of all the vortex-filaments remains constant; and so does the sum of their rotational moments: and the resultant areal axis of them all regarded as one system is a fixed line in space. Hence, as in the case of a vortex-filament, the translation, if any, through space is on the average along its resultant axis. All this, of course, is on the supposition that there is no other vortex, and no solid immersed in the liquid, and no bounding surface of the liquid near enough to produce any sensible influence on the given vortex.

XVI. *On Gravitational Oscillations of Rotating Water.* *By* Sir William Thomson.*

THIS is really Laplace's subject in his Dynamical Theory of the Tides; where it is dealt with in its utmost generality except one important restriction—the motion of each particle to be infinitely nearly horizontal, and the velocity to be always equal for all particles in the same vertical. This implies that the greatest depth must be small in comparison with the distance that has to be travelled to find the deviation from levelness of the water-surface altered by a sensible fraction of its maximum amount. In the present short communication I

* From the Proceedings of the Royal Society of Edinburgh, March 17, 1879. Communicated by the Author.

adopt this restriction; and, further, instead of supposing the water to cover the whole or a large part of the surface of a solid spheroid as does Laplace, I take the simpler problem of an area of water so small that the equilibrium-figure of its surface is not sensibly curved. Imagine a basin of water of any shape, and of depth not necessarily uniform, but, at greatest, small in comparison with the least diameter. Let this basin and the water in it rotate round a vertical axis with angular velocity ω so small that the greatest equilibrium-slope due to it may be a small fraction of the radian: in other words, the angular velocity must be small in comparison with $\sqrt{\frac{g}{\frac{1}{2}A}}$, where g denotes gravity, and A the greatest diameter of the basin. The equations of motion are

$$\left.\begin{aligned}\frac{du}{dt}-2\omega v&=-\frac{1}{\rho}\frac{dp}{dx},\\ \frac{dv}{dt}+2\omega u&=-\frac{1}{\rho}\frac{dp}{dy};\end{aligned}\right\}\quad\ldots\ldots\quad(1)$$

where u and v are the component velocities of any point of the fluid in the vertical column through the point (xy), relatively to horizontal axes Ox, Oy revolving with the basin; p the pressure at any point x, y, z of this column; and ρ the uniform density of the liquid. The terms $\omega^2 x$, $\omega^2 y$, which appear in ordinary dynamical equations referred to rotating axes, represent components of centrifugal force, and therefore do not appear in these equations. Let now D be the mean depth and D$+h$ the actual depth at any time t in the position (xy). The "equation of continuity" is

$$\frac{d(\mathrm{D}u)}{dx}+\frac{d(\mathrm{D}v)}{dy}=-\frac{dh}{dt}.\quad\ldots\ldots\quad(2)$$

Lastly, by the condition that the pressure at the free surface is constant, and that the difference of pressures at any two points in the fluid is equal to $g\times$ difference of levels, we have

$$\left.\begin{aligned}\frac{dp}{dx}&=g\rho\frac{dh}{dx},\\ \frac{dp}{dy}&=g\rho\frac{dh}{dy}.\end{aligned}\right\}\quad\ldots\ldots\quad(3)$$

Hence for the case of gravitational oscillations (1) becomes

$$\left.\begin{aligned}\frac{du}{dt}-2\omega v&=-g\frac{dh}{dx},\\ \frac{dv}{dt}+2\omega u&=-g\frac{dh}{dy}.\end{aligned}\right\}\quad\ldots\ldots\quad(4)$$

From (1) or (4) we find, by differentiation &c.,

$$\frac{d}{dt}\left(\frac{dv}{dx}-\frac{du}{dy}\right)+2\omega\left(\frac{du}{dx}+\frac{dv}{dy}\right)=0,\quad\ldots\quad(5)$$

which is the equation of vortex motion in the circumstances.

These equations reduced to polar coordinates, with the following notation,

$$x=r\cos\theta,\quad y=r\sin\theta,$$

$$u=\zeta\cos\theta-\tau\sin\theta,\quad v=\zeta\sin\theta+\tau\cos\theta,$$

become

$$\frac{\mathrm{D}\zeta}{r}+\frac{d(\mathrm{D}\zeta)}{dr}+\frac{d(\mathrm{D}\tau)}{rd\theta}=-\frac{dh}{dt},\quad\ldots\ldots\quad(2')$$

$$\left.\begin{aligned}\frac{d\zeta}{dt}-2\omega\tau&=-g\frac{dh}{dr},\\ \frac{d\tau}{dt}+2\omega\zeta&=-g\frac{dh}{rd\theta},\end{aligned}\right\}\quad\ldots\ldots\quad(4')$$

$$\frac{d}{dt}\left(\frac{\tau}{r}+\frac{d\tau}{dr}-\frac{d\zeta}{rd\theta}\right)+2\omega\left(\frac{\zeta}{r}+\frac{d\zeta}{dr}+\frac{d\tau}{rd\theta}\right)=0.\quad(5')$$

In these equations D may be any function of the coordinates. Cases of special interest in connexion with Laplace's tidal equations are had by supposing D to be a function of r alone. For the present, however, we shall suppose D to be constant. Then (2) used in (5) or (2′) in (5′) gives, after integration with respect to t,

$$\frac{dv}{dx}-\frac{du}{dy}=2\omega\frac{h}{\mathrm{D}},\quad\ldots\ldots\quad(6)$$

or, in polar coordinates,

$$\frac{\tau}{r}+\frac{d\tau}{dr}-\frac{d\zeta}{rd\theta}=2\omega\frac{h}{\mathrm{D}}.\quad\ldots\ldots\quad(6')$$

These equations (6), (6′) are instructive and convenient, though they contain nothing more than is contained in (2) or (2′), and (4) or (4′).

Separating u and v in (4), or ζ and τ in (4′), we find

$$\left.\begin{aligned}\frac{d^2u}{dt^2}+4\omega^2u&=-g\left(\frac{d}{dt}\frac{dh}{dx}+2\omega\frac{dh}{dy}\right),\\ \text{and}\qquad\frac{d^2v}{dt^2}+4\omega^2v&=g\left(2\omega\frac{dh}{dx}-\frac{d}{dt}\frac{dh}{dy}\right),\end{aligned}\right\}\quad\ldots\quad(7)$$

or, in polar coordinates,

$$\left.\begin{aligned}\frac{d^2\zeta}{dt^2}+4\omega^2\zeta&=-g\left(\frac{d}{dt}\frac{dh}{dr}+2\omega\frac{dh}{rd\theta}\right),\\ \frac{d^2\tau}{dt^2}+4\omega^2\tau&=\quad g\left(2\omega\frac{dh}{dr}-\frac{d}{dt}\frac{dh}{rd\theta}\right).\end{aligned}\right\}\quad . \quad . \quad . \quad (7')$$

Using in (7) (7′), in (2) (2′), with D constant, or in (6) (6′), we find

$$g\mathrm{D}\left(\frac{d^2h}{dx^2}+\frac{d^2h}{dy^2}\right)=\frac{d^2h}{dt^2}+4\omega^2h,\quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

and

$$g\mathrm{D}\left(\frac{d^2h}{dr^2}+\frac{1}{r}\frac{dh}{dr}+\frac{d^2h}{rd\theta^2}\right)=\frac{d^2h}{dt^2}+4\omega^2h.\quad . \quad . \quad (8')$$

It is to be remarked that (8) and (8′) are satisfied with u or v substituted for h.

I. Solutions for Rectangular Coordinates.

The general type solution of (8) is $h=\epsilon^{\alpha x}\epsilon^{\beta y}\epsilon^{\gamma t}$, where α, β, γ are connected by the equation

$$\alpha^2+\beta^2=\frac{\gamma^2+4\omega^2}{g\mathrm{D}}.\quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

For waves or oscillations we must have $\gamma=\sigma\sqrt{-1}$, where σ is real.

I *a. Nodal Tesseral Oscillations.*

For nodal oscillations of the tesseral type we must have $\alpha=m\sqrt{-1}$, $\beta=n\sqrt{-1}$, where m and n are real; and by putting together properly the imaginary constituents we find

$$h=\mathrm{C}\,{\sin\atop\cos}\,\sigma t\,{\sin\atop\cos}\,mx\,{\sin\atop\cos}\,ny,\quad . \quad . \quad . \quad . \quad (10)$$

where m, n, σ are connected by the equation

$$m^2+n^2=\frac{\sigma^2-4\omega^2}{g\mathrm{D}}.\quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

Finding the corresponding values of u and v, we see what the boundary-conditions must be to allow these tesseral oscillations to exist in a sea of any shape. No bounding-line can be drawn at every part of which the horizontal component velocity perpendicular to it is zero. Therefore to produce or permit oscillations of the simple harmonic type in respect to form, water must be forced in and drawn out alternately all round the boundary, or those parts of it (if not all) for which the horizontal component perpendicular to it is not zero. Hence the oscillations of water in a rotating rectangular trough are not of the simple harmonic type in respect to form, and the problem of finding them remains unsolved.

If $\omega=0$, we fall on the well-known solution for waves in a non-rotating trough, which are of the simple harmonic type.

I *b. Waves or Oscillations in an endless Canal with straight parallel sides.*

For waves in a canal parallel to x, the solution is

$$h=\mathrm{H}\epsilon^{-ly}\cos(mx-\sigma t);\quad . \quad . \quad . \quad . \quad (12)$$

where l, m, σ satisfy the equation

$$m^2-l^2=\frac{\sigma^2-4\omega^2}{g\mathrm{D}},\quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

in virtue of (9) or (11).

Using these in (7), we find that v vanishes throughout if we make

$$l=\frac{2\omega m}{\sigma};\quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

and with this value for l in (12) we find, by (7),

$$u=\mathrm{H}\frac{gm}{\sigma}\epsilon^{-ly}\cos(mx-\sigma t);\quad . \quad . \quad . \quad (15)$$

and using (14) and (13) we find

$$m^2=\frac{\sigma^2}{g\mathrm{D}},\quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

from which we infer that the velocity of propagation of waves is the same for the same period as in a fixed canal. Thus the influence of rotation is confined to the effect of the factor $\epsilon^{-2\omega m/\sigma.y}$. Many interesting results follow from the interpretation of this factor with different particular suppositions as to the relation between the period of the oscillation $\left(\frac{2\pi}{\sigma}\right)$, the period of the rotation $\left(\frac{2\pi}{\omega}\right)$, and the time required to travel at the velocity $\frac{\sigma}{m}$ across the canal. The more approximately nodal character of the tides on the north coast of the English Channel than on the south or French coast, and of the tides on the west or Irish side of the Irish Channel than on the east or English side, is probably to be accounted for on the principle represented by this factor, taken into account along with frictional resistance, in virtue of which the tides of the English Channel may be roughly represented by more powerful waves travelling from west to east, combined with less powerful waves travelling from east to west, and those of the southern part of the Irish Channel by more powerful waves travelling from south to north combined with less powerful waves travelling from north to south. The problem of standing oscillations in an endless rotating canal is solved by the following equations:—

$$\left.\begin{aligned} h&=\mathrm{H}\{\epsilon^{-ly}\cos(mx-\sigma t)-\epsilon^{ly}(\cos mx+\sigma t)\};\\ u&=\mathrm{H}\frac{gm}{\sigma}\{\epsilon^{-ly}\cos(mx-\sigma t)+\epsilon^{ly}\cos(mx-\sigma t)\};\\ v&=0.\end{aligned}\right\} \quad . \ (17)$$

If we give ends to the canal, we fall upon the unsolved problem referred to above of tesseral oscillations. If instead of being rigorously straight we suppose the canal to be circular and endless, provided the breadth of the canal be small in comparison with the radius of the circle, the solution (17) still holds. In this case, if c denote the circumference of the canal, we must have $m=\frac{2i\pi}{c}$, where i is an integer.

II. Oscillations and Waves in Circular Basin (Polar Coordinates).

Let

$$h=\mathrm{P}\cos(i\theta-\sigma t) \quad . \ . \ . \ . \ . \ . \ . \ (18)$$

be the solution for height, where P is a function of r. By (8′) P must satisfy the equation

$$\frac{d^2\mathrm{P}}{dr^2}+\frac{1}{r}\frac{d\mathrm{P}}{dr}-\frac{i^2\mathrm{P}}{r^2}+\frac{\sigma^2-4\omega^2}{g\mathrm{D}}\mathrm{P}=0; \quad . \ . \ . \ (19)$$

and by (7′) we find

$$\left.\begin{aligned} \zeta&=\frac{g}{\sigma^2-4\omega^2}\sin(i\theta-\sigma t)\left(\sigma\frac{d\mathrm{P}}{dr}-2\omega i\frac{\mathrm{P}}{r}\right),\\ \tau&=\frac{-g}{\sigma^2-4\omega^2}\cos(i\theta-\sigma t)\left(2\omega\frac{d\mathrm{P}}{dr}-\sigma i\frac{\mathrm{P}}{r}\right).\end{aligned}\right\} \quad . \ (20)$$

This is the solution for water in a circular basin, with or without a central circular island. Let a be the radius of the basin; and if there be a central island let a' be its radius. The boundary conditions to be fulfilled are $\zeta=0$ when $r=a$ and when $r=a'$. The ratio of one to the other of the two constants of integration of (19), and the speed σ of the oscillation, are the two unknown quantities to be found by these two equations. The ratio of the constants is immediately eliminated; and the result is a transcendental equation for σ. There is no difficulty, only a little labour, in thus finding as many as

we please of the fundamental modes, and working out the whole motion of the system for each. The roots of this equation, which are found to be all real by the Fourier-Sturm-Liouville theory, are the speeds* of the successive fundamental modes, corresponding to the different circular nodal subdivisions of the i diametral divisions implied by the assumed value of i. Thus, by giving to i the successive values 0, 1, 2, 3, &c., and solving the transcendental equation so found for each, we find all the fundamental modes of vibration of the mass of matter in the supposed circumstances.

If there is no central island, the solution of (19) which must be taken is that for which P and its differential coefficients are all finite when $r=0$. Hence P is what is called a Bessel's function of the first kind and of order i, and, according to the established notation†, we have

$$\mathrm{P}=\mathrm{J}_i\left(r\sqrt{\frac{\sigma^2-4\omega^2}{g\mathrm{D}}}\right). \quad . \ . \ . \ . \ (21)$$

The solution found above for an endless circular canal is fallen upon by giving a very great value to i. Thus, if we put $\frac{2\pi r}{i}=\lambda$ so that λ may denote wave-length, we have $\frac{i}{r}=\frac{2\pi}{\lambda}$, which will now be the m of former notation. We must now neglect the term $\frac{1}{r}\frac{dh}{dr}$ in (19); and thus the differential equation becomes

$$\frac{d^2h}{dr^2}+\left(\frac{\sigma^2-4\omega^2}{g\mathrm{D}}-m^2\right)h=0,$$

or

$$\frac{d^2h}{dr^2}-l^2h=0, \quad . \ . \ . \ . \ . \ . \ . \ . \ (22)$$

where l^2 denotes $m^2-\frac{\sigma^2-4\omega^2}{g\mathrm{D}}$. A solution of this equation is $h=\epsilon^{-ly}$, where $y=a-r$; and using this in (20) above, we find

* In the last two or three tidal reports of the British Association the word "speed," in reference to a simple harmonic function, has been used to designate the angular velocity of a body moving in a circle in the same period. Thus, if T be the period, $\frac{2\pi}{\mathrm{T}}$ is the speed; *vice versa*, if σ be the speed, $\frac{2\pi}{\sigma}$ is the period.

† Neumann, *Theorie der Bessel'schen Functionen* (Leipzig, 1867), § 5; and Lommel, *Studien über die Bessel'schen Functionen* (Leipzig, 1868), § 29.

$$\zeta=\frac{-g}{\sigma^2-4\omega^2}\,\mathrm{C}\sin(mx-\sigma t)(\sigma l-2\omega m)\epsilon^{-ly},\quad \text{where}\quad mx=i\theta.$$

Hence, to make $\zeta=0$ at each boundary, we have $\sigma l=2\omega m$, which makes $\zeta=0$, not only at the boundaries, but throughout the space for which the approximate equation (22) is sufficiently nearly true. And, putting for l^2 its value above, we have

$$4\omega^2m^2=\sigma^2\left(m^2-\frac{\sigma^2-4\omega^2}{g\mathrm{D}}\right),$$

whence

$$m^2=\frac{\sigma^2}{g\mathrm{D}},$$

which agrees with (16) above.

I hope in a future communication to the Royal Society to go in detail into particular cases, and to give details of the solutions at present indicated, some of which present great interest in relation to tidal theory, and also in relation to the abstract theory of vortex motion. The characteristic differences between cases in which σ is greater than 2ω or less than 2ω are remarkably interesting, and of great importance in respect to the theory of diurnal tides in the Mediterranean, or other more or less nearly closed seas in middle latitudes, and of the lunar fortnightly tide of the whole ocean. It is to be remarked that the preceding theory is applicable to waves or vibrations in any narrow lake or portion of the sea covering not more than a few degrees of the earth's surface, if for ω we take the component of the earth's angular velocity round a vertical through the locality—that is to say, $\omega=\gamma\sin l$, where γ denotes the earth's angular velocity, and l the latitude.

XVII. *On the Resolving-power of Telescopes.* By LORD RAYLEIGH, *F.R.S., Professor of Experimental Physics in the University of Cambridge**.

ALTHOUGH I have recently treated of this subject in the Philosophical Magazine†, its importance induces me to return to it in order to explain how easily it may be investigated in the laboratory. There can be no reason why the experiment about to be described should not be included in every course on physical optics.

The only work on this subject with which I am acquainted is that of Foucault‡, who investigated the resolving-power of a telescope of 10 centimetres aperture on a distant scale illuminated by direct sunshine. In this form the experiment is troublesome and requires expensive apparatus—difficulties which are entirely obviated by the plan which I have followed of using a much smaller aperture.

The object, on which the resolving-power of the telescope is tested is a grating of fine wires, constructed on the plan employed by Fraunhofer for diffraction-gratings. A stout brass wire or rod is bent into a horseshoe, and its ends are screwed. On these screws fine wire is wound of diameter equal to about half the pitch, and secured with solder. The wires on one side being now cut away, we obtain a grating of considerable accuracy. A wire grating thus formed is preferable to a scale ruled on paper, and placed in front of a lamp presents a very suitable subject for examination. The one that I employed has 50 wires to the inch, and for security is mounted in a frame between two plates of glass. For rough purposes a piece of common gauze with 30 or 40 meshes to the inch may be substituted with good effect.

For the sake of definiteness of wave-length the grating was backed by a soda-flame, though fair results are obtainable with a common paraffine-lamp. The telescope is a small instrument mounted on a stand, and provided with a cap by means of which various diaphragms can be conveniently fitted in front of the object-glass. The apertures in these diaphragms may be either circular or rectangular. In the latter case the length of the slit is placed parallel to the wires of the grating, and we have the advantage of greater illumination than with a circle of equal width. The observation consists in ascertaining the greatest distance at which the wires can be seen resolved. For this purpose the telescope, focused all the while, is gradually drawn back until in the judgment of the observer the periodic structure is no longer seen; and the distance between the grating and the diaphragm is then measured with a steel tape. The distance thus determined is more definite than might be expected, the differences in the case of various observers not usually amounting to more than 2 or 3 per cent.

Two slits were tried, half an inch long, and of widths ·107, ·196 inch respectively. These widths were measured by inserting a graduated wedge. It was found, however, that the graduations could not be trusted; so that the wedge was in fact used merely to convey the length to be measured to a pair of callipers reading to one thousandth of an inch. The distances at which resolution just ceased were estimated respectively as 91·5 and 168·5 inches, corresponding to angular

* Communicated by the Author.

† Oct., Nov., and Dec. 1879, Jan. 1880.

‡ "Mémoires sur la construction des télescopes," *Annales de l'Observatoire*, t. v.; also Verdet's *Leçons d'optique physique*, t. i. p. 309.