such a cell is very long in making its appearance—so long, in fact, that we may be pardoned for regarding it as we should do the philosopher's stone or the elixir of life—things highly desirable, perhaps, but impossible of attainment.

Mr. Cooper's book is certain to appeal to a wide circle of readers, and we have no doubt whatever that it will at once take its place as the standard treatise on the subject.

Géométrographie ou Art des Constructions Géométriques. Par ÉMILE LEMOINE. C. Naud, 1902. Pp. 87. ("Scientia" Series, No. 18.)

MOST problems in geometrical construction admit of more than one solution, but among them there is generally one which involves the least number of operations, and is therefore the simplest. This simplest solution constitutes the *geometrographic* construction. The instruments employed consist of a straight-edge, dividers, and set-square. The various operations involved—adjusting the straight-edge so that it passes through one or two given points, drawing a straight line, setting the dividers to a given length, drawing a circle, &c.—are denoted by symbols. The complexity of the solution may then be ascertained from the symbolical expression for the operations involved, and the number of these latter is termed the *coefficient of simplicity* (as the author properly points out, the coefficient of complexity would be a more appropriate term). By a careful study of the problem, the author has in many cases succeeded in reducing considerably the coefficient of simplicity. One case is mentioned in which, by the joint efforts of a number of geometers, this coefficient was reduced from 78 (involving the tracing of 17 straight lines and 20 circles) to 35 (7 straight lines and 5 circles). The author gives the solutions of 69 problems, in some cases giving several solutions one of which (the simplest) is the geometrographic one. The construction is first explained, and is then followed by a symbolical formula, the coefficient of simplicity, and the number of straight lines and circles drawn in the course of the construction.

Théorie de la Lune. Par H. ANDOYER. Paris: C. Naud, 1902. Pp. 86. ("Scientia" Series, No. 17.)

IN this little book, the author develops, in the simplest possible form, the principal portions of the lunar theory, without, however, considering the numerical values of the various constants which appear in the equations. On account of the highly abstruse nature of the subject, the book is necessarily intended for specialists, and to them should prove very useful.

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XXXII. *On the Weights of Atoms.* *By* Lord KELVIN, *G.C.V.O.*

[Concluded from page 198.]

§ 52. A NEW method of finding an inferior limit to the number of molecules in a cubic centimetre of a gas, very different from anything previously thought of, and especially interesting to us in connexion with the wave-theory of light, was given by Lord Rayleigh *, in 1899, as a deduction from the dynamical theory of the blue sky which he had given 18 years earlier. Many previous writers, Newton included, had attributed the light from the sky, whether clear blue, or hazy, or cloudy, or rainy, to fine suspended particles which divert portions of the sunlight from its regular course; but no one before Rayleigh, so far as I know, had published any idea of how to explain the blueness of the cloudless sky. Stokes, in his celebrated paper on Fluorescence †, had given the true theory of what was known regarding the polarization of the blue sky in the following "significant remark" as Rayleigh calls it: "Now this result "appears to me to have no remote bearing on the question of "the directions of the vibrations in polarized light. So long "as the suspended particles are large compared with the waves "of light, reflexion takes place as it would from a portion of "the surface of a large solid immersed in the fluid, and no "conclusion can be drawn either way. But if the diameter

* Rayleigh, Collected Papers, vol. i. art. viii. p. 87.

† "On the Change of Refrangibility of Light," Phil. Trans. 1852, and Collected Papers, vol. iii.

"of the particles be small compared with the length of a wave "of light, it seems plain that the vibrations in a reflected ray "cannot be perpendicular to the vibrations in the incident "ray"; which implies that the light scattered in directions perpendicular to the exciting incident ray has everywhere its vibrations perpendicular to the plane of the incident ray and the scattered ray; provided the diameter of the molecule which causes the scattering is very small in comparison with the wave-length of the light. In conversation Stokes told me of this conclusion, and explained to me with perfect clearness and completeness its dynamical foundation; and applied it to explain the polarization of the light of a cloudless sky, viewed in a direction at right angles to the direction of the sun. But he did not tell me (though I have no doubt he knew it himself) why the light of the cloudless sky seen in any direction is blue, or I should certainly have remembered it.

§ 53. Rayleigh explained this thoroughly in his first paper (1871), and gave what is now known as Rayleigh's law of the blue sky; which is, that, provided the diameters of the suspended particles are small in comparison with the wave-lengths, the proportions of scattered light to incident light for different wave-lengths are inversely as the fourth powers of the wave-lengths. Thus, while the scattered light has the same colour as the incident light when homogeneous, the proportion of scattered light to incident light is seven times as great for the violet as for the red of the visible spectrum; which explains the intensely blue or violet colour of the clearest blue sky.

§ 54. The dynamical theory shows that the part of the light of the blue sky, looked at in a direction perpendicular to the direction of the sun, which is due to sunlight incident on a single particle of diameter very small in comparison with the wave-lengths of the illuminating light, consists of vibrations perpendicular to the plane of these two directions: that is to say, is completely polarized in the plane through the sun. In his 1871 paper *, Rayleigh pointed out that each particle is illuminated, not only by the direct light of the sun, but also by light scattered from other particles, and by earth-shine, and partly also by suspended particles of dimensions not small in comparison with the wave-lengths of the actual light; and he thus explained the observed fact that the polarization of even the clearest blue sky at 90° from the sun is not absolutely complete, though it is very nearly so.

* Collected Papers, vol. i. p. 94.

There is very little of polarization in the light from white clouds seen in any direction, or even from a cloudless sky close above the horizon seen at 90° from the sun. This is partly because the particles which give it are not small in comparison with the wave-lengths, and partly because they contribute much to illuminate one another in addition to the sunlight directly incident on them.

§ 55. For his dynamical foundation, Rayleigh definitely assumed the suspended particles to act as if the ether in their places were denser than undisturbed ether, but otherwise uninfluenced by the matter of the particles themselves. He tacitly assumed throughout that the distance from particle to particle is very great in comparison with the greatest diameter of each particle. He assumed these denser portions of ether to be of the same rigidity as undisturbed ether; but it is obvious that this last assumption could not largely influence the result, provided the greatest diameter of each particle is very small in comparison with its distance from next neighbour, and with the wave-lengths of the light: and, in fact, I have found from the investigation of §§ 41, 42 of Lecture XIV. for *rigid* spherical molecules embedded in ether, exactly the same result as Rayleigh's; which is as follows

$$k=\frac{8\pi^3 n}{3}\left(\frac{D'-D}{D}\frac{T}{\lambda^2}\right)^2=82\cdot67\,n\left(\frac{D'-D}{D}\frac{T}{\lambda^2}\right)^2 \quad . \quad . \quad (1);$$

where λ denotes the wave-length of the incident light supposed homogeneous; T the volume of each suspended particle; D the undisturbed density of the ether; D' the mean density of the ether within the particle; n the number of particles per cubic centimetre; and k the proportionate loss of homogeneous incident light, due to the scattering in all directions by the suspended particles per centimetre of air traversed. Thus

$$1-\epsilon^{-kx} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

is the loss of light in travelling a distance x (reckoned in centimetres) through ether as disturbed by the suspended particles.

It is remarkable that D' need not be uniform throughout the particle. It is also remarkable that the shape of the volume T may be anything, provided only its greatest diameter is very small in comparison with λ. The formula supposes $T(D'-D)$ the same for all the particles. We shall have to consider cases in which differences of T and D' for different particles are essential to the result; and to include these we shall have to use the formula

$$k=\frac{82{\cdot}67}{\lambda^4}\Sigma\left[\frac{(D'-D)\,T}{D}\right]^2 \quad . \quad . \quad . \quad . \quad (3),$$

where $\Sigma\left[\frac{(D'-D)\,T}{D}\right]^2$ denotes the sum of $\left[\frac{(D'-D)\,T}{D}\right]^2$ for all the particles in a cubic centimetre.

§ 56. Supposing now the number of suspended particles per cubic wave-length to be very great, and the greatest diameter of each to be small in comparison with its distance from next neighbour, we see that the virtual density of the ether vibrating among the particles is

$$D+\Sigma T(D'-D) \quad . \quad . \quad . \quad . \quad . \quad (4);$$

and therefore, if u and u' be the velocities of light in pure ether, and in ether as disturbed by the suspended particles, we have (Lecture VIII. p. 80)

$$u'^2=u^2\left[1+\Sigma\,\frac{T(D'-D)}{D}\right] \quad . \quad . \quad . \quad . \quad (5).$$

Hence, if μ denote the refractive index of the disturbed ether, that of pure ether being 1, we have

$$\mu=\left[1+\Sigma\frac{T(D'-D)}{D}\right]^{\frac{1}{2}} \quad . \quad . \quad . \quad . \quad (6);$$

and therefore, approximately,

$$\mu^2-1=\Sigma\,\frac{T(D'-D)}{D} \quad . \quad . \quad . \quad . \quad . \quad (7).$$

§ 57. In taking an example to illustrate the actual transparency of our atmosphere, Rayleigh says*; "Perhaps the "best data for a comparison are those afforded by the varying "brightness of stars at various altitudes. Bouguer and others "estimate about ·8 for the transmission of light through the "entire atmosphere from a star in the zenith. This corre- "sponds to 8·3 kilometres (the "height of the homogeneous "atmosphere" at 10° Cent.) of air at standard pressure." Hence for a medium of the transparency thus indicated we have $\epsilon^{-830000k}=\cdot 8$; which gives $1/k=3720000$ centimetres $=$ 37·2 kilometres.

* Phil. Mag. April 1899, p. 382.

§ 58. Suppose for a moment the want of perfect transparency thus defined to be *wholly* due to the fact that the *ultimate molecules* of air are not infinitely small and infinitely numerous, so that the "suspended particles" hitherto spoken of would be merely the molecules N_2, O_2; and suppose further $(D'-D)\,T$ to be the same for nitrogen and oxygen. The known refractivity of air ($\mu-1=\cdot 0003$), nearly enough the same for all visible light, gives by equation (7) above, with n instead of Σ,

$$\frac{n(D'-D)\,T}{D}=\cdot 0006.$$

Using this in (1) we find

$$k=\frac{29{\cdot}76}{n\lambda^4.10^6} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8),$$

for what the rate of loss on direct sunlight would be, per centimetre of air traversed, if the light were all of one wave-length, λ. But we have no such simplicity in Bouguer's datum regarding transparency for the actual mixture which constitutes sunlight: because the formula makes k^{-1} proportional to the fourth power of the wave-length; and every cloudless sunset and moonset and sunrise and moonrise over the sea, and every cloudless view of sun or moon below the horizon of the eye on a high mountain, proves the transparency to be in reality much greater for red light than for the average undimmed light of either luminary, though probably not so much greater as to be proportional to the fourth power of the wave-length. We may, however, feel fairly sure that Bouguer's estimate of the loss of light in passing vertically through the whole atmosphere is approximately true for the most luminous part of the spectrum corresponding to about the D line, wave-length $5{\cdot}89\,.\,10^{-5}$ cm., or (a convenient round number) $6\,.\,10^{-5}$ as Rayleigh has taken it. With this value for λ, and $3{\cdot}72\,.\,10^6$ centimetres for k^{-1}, (8) gives $n=8{\cdot}54.10^{18}$ for atmospheric air at 10° and at standard pressure. Now it is quite certain that a very large part of the loss of light estimated by Bouguer is due to suspended particles; and therefore it is certain that the number of molecules in a cubic centimetre of gas at standard temperature and pressure is considerably greater than $8{\cdot}54\,.\,10^{18}$.

§ 59. This conclusion drawn by Rayleigh from his dynamical theory of the absorption of light from direct rays through air, giving very decidedly an inferior limit to the number of molecules in a cubic centimetre of gas, is perhaps the most thoroughly well founded of all definite estimates hitherto made regarding sizes or numbers of atoms. We shall see (§§ 73 ... 79 below) that a much larger inferior limit is found on the same principles by careful consideration of the loss of light due to the ultimate molecules of pure air *and to suspended matter* undoubtedly existing in all parts of our atmosphere, even where absolutely cloudless, that is to say, warmer than the dew-point, and therefore having none of the liquid spherules of water which constitute cloud or mist.

§ 60. Go now to the opposite extreme from the tentative hypothesis of § 58 and, while assuming, as we know to be true, that the observed refractivity is wholly or almost wholly due to the ultimate molecules of air, suppose the opacity estimated by Bouguer to be wholly due to suspended particles which, for brevity, we shall call dust (whether dry or moist). These particles may be supposed to be generally of very unequal magnitudes: but, for simplicity, let us take a case in which they are all equal, and their number only 1/10000th of the $8{\cdot}54\,.\,10^{18}$, which in § 59 we found to give the true refractivity of air, with Bouguer's degree of opacity for $\lambda = 6\,.\,10^{-5}$. With the same opacity we now find the contribution to refractivity of the particles causing it, to be only 1/100th of the known refractivity of air. The number of particles of dust which we now have is $8{\cdot}54.10^{14}$ per cubic centimetre, or 1107 per cubic wave-length, which we may suppose to be almost large enough or quite large enough to allow the dynamics of § 56 for refractivity to be approximately true. But it seems to me almost certain that $8{\cdot}54.10^{14}$ is vastly greater than the greatest number of dust particles per cubic centimetre to which the well-known haziness of the clearest of cloudless air in the lower regions of our atmosphere is due; and that the true numbers, at different times and places, may probably be such as those counted by Aitken * at from 42500 (Hyères, 4 p.m. April 5, 1892) to 43 (Kingairloch, Argyllshire, 1 p.m. to 1.30 p.m. July 26, 1891).

§ 61. Let us, however, find how small the number of particles per cubic centimetre must be to produce Bouguer's degree of opacity, without the particles themselves being so

* Trans. R. S. E. 1894, vol. xxxvii. part iii. pp. 675, 672

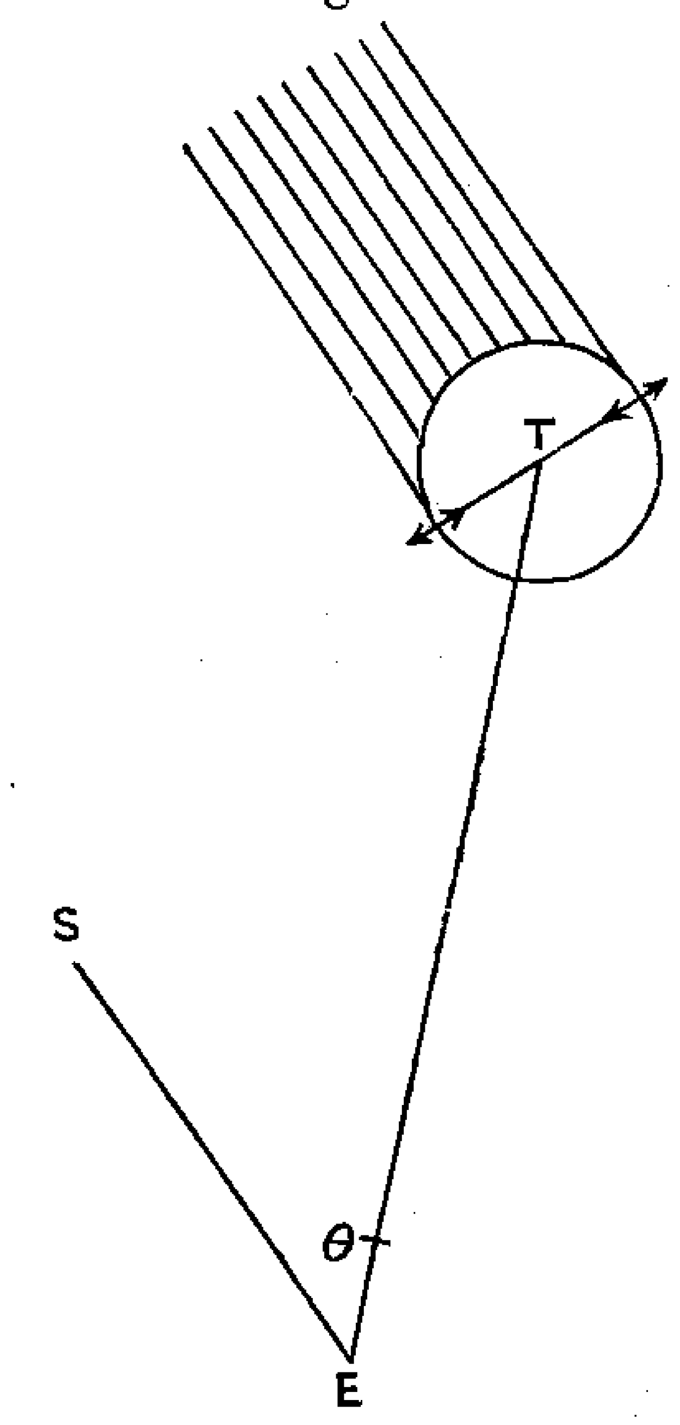
large in comparison with the wave-length as to exclude the application of Rayleigh's theory. Try for example $T = 10^{-3}.\lambda^3$ (that is to say, the volume of the molecule 1/1000th of the cubic wave-length, or roughly diameter of molecule 1/10th of the wave-length) which seems small enough for fairly approximate application of Rayleigh's theory; and suppose, merely to make an example, D' to be the optical density of water, D being that of ether; that is to say, $D'/D = (1{\cdot}3337)^2 = 1{\cdot}78$. Thus we have $(D'-D)\ T/D = {\cdot}0007\lambda^3$: and with $\lambda = 6\,.\,10^{-5}$, and with $k^{-1} = 3{\cdot}72\,.\,10^6$, (1) gives $n = 1{\cdot}48\,.\,10^6$, or about one and a half million particles per cubic centimetre. Though this is larger than the largest number for natural air counted by Aitken, it is interesting as showing that Bouguer's degree of opacity can be accounted for by suspended particles, few enough to give no appreciable contribution to refractivity, and yet not too large for Rayleigh's theory. But when we look through very clear air by day, and see how far from azure or deep blue is the colour of a few hundred metres, or a few kilometres of air with the mouth of a cave or the darkest shade of mountain or forest for background; and when in fine sunny weather we study the appearance of the *grayish* haze always, even on the clearest days, notably visible over the scenery among mountains or hills; and when by night at sea we see a lighthouse light at a distance of 45 or 50 kilometres, and perceive how little of redness it shows; and when we see the setting sun shorn of his brilliance sufficiently to allow us to look direct at his face, and yet only ruddy, rarely what could be called ruby red; it seems to me that we have strong evidence for believing that the want of perfect clearness of the lower regions of our atmosphere is in the main due to suspended particles, too large to allow approximate fulfilment of Rayleigh's law of fourth power of wave-length.

§ 62. But even if they were small enough for Rayleigh's theory the question would remain, Are they small enough and numerous enough to account for the refractivity of the atmosphere? To this we shall presently see we must answer undoubtedly "No"; and much less than Bouguer's degree of opacity, probably not as much as a quarter or a fifth of it, is due to the ultimate molecules of air. In a paper by Mr. Quirino Majorana in the Transactions of the R. Accademia dei Lincei (of which a translation is published in the Philosophical Magazine for May 1901), observations by himself in Sicily, at Catania and on Mount Etna, and by Mr. Gaudenzio Sella, on Monte Rosa in Switzerland, determining the ratio of the brightness of the sun's surface to the brightness of the sky seen in any direction, are described. This ratio they denote by r. One specially notable result of Mr. Majorana's is that "the value of r at the crater of Etna is about five times greater than at Catania." The barometric pressures were approximately 53·6 and 76 cms. of mercury. Thus the atmosphere above Catania was only 1·42 times the atmosphere above Etna, and yet it gave five times as much scattering of light by its particles, and by the particles suspended in it. This at once proves that a great part of the scattering must be due to suspended particles; and more of them than in proportion to the density in the air below the level of Etna than in the air above it. In Majorana's observations, it was found that "except for regions close to the horizon, the luminosity of the "sky had a sensibly constant value in all directions when "viewed from the summit of Etna." This uniformity was observed even for points in the neighbourhood of the sun, as near to it as he could make the observation without direct light from the sun getting into his instrument. I cannot but think that this apparent uniformity was only partial. It is quite certain that with sunlight shining down from above, and with equal light everywhere shining up from earth or sea or haze, illuminating the higher air, the intensities of the blue light seen in different directions above the crater would be largely different. This is proved by the following investigation; which is merely an application of Rayleigh's theory to the question before us. But from Majorana's narrative we may at all events assume that, as when observing from Catania, he also on Etna chose the least luminous part of the sky (Phil. Mag., May 1901, p. 561), for the recorded results (p. 562) of his observations.

§ 63. The diagram, fig. 1 below, is an ideal representation of a single molecule or particle, T, with sunlight falling on it indicated by parallel lines, and so giving rise to scattered light seen by an eye at E. We suppose the molecule or particle to be so massive relatively to its bulk of ether that it is practically unmoved by the ethereal vibration; and for simplicity at present we suppose the ether to move freely through the volume T, becoming denser without changing its velocity when it enters this fixed volume, and less dense when it leaves. In §§ 41, 42, of Lecture XV. above, and in Appendix A, a definite supposition, attributing to ether no other property than elasticity as of an utterly homogeneous perfectly elastic solid, and the exercise of mutual force between itself and ponderable matter occupying the same space, is explained: according to which the ether within the atom will

react upon moving ether outside just as it would if our present convenient temporary supposition of magically augmented density within the volume of an absolutely fixed molecule were realized in nature. For our present purpose, we may if we please, following Rayleigh, do away altogether with the ponderable molecule, and merely suppose T to be a denser

Fig. 1.



portion of the ether. And if its greatest diameter is small enough relatively to a wave-length, it will make no unnegligible difference whether we suppose the ether in T to have the same rigidity as the surrounding free ether, or suppose it perfectly rigid as in §§ 1—46 of Lecture XIV. dealing with a rigid globe embedded in ether.

§ 64. Resolving the incident light into two components having semi-ranges of vibration ϖ, ρ, in the plane of the paper and perpendicular to it; consider first the component in the plane having vibrations symbolically indicated by the arrowheads, and expressed by the following formula

$$\omega \sin \frac{2\pi u t}{\lambda},$$

where u is the velocity of light, and λ the wave-length. The greater density of the ether within T gives a reactive force on the surrounding ether outside, in the line of the primary vibration, and against the direction of its acceleration, of which the magnitude is

$$\frac{T(D'-D)\varpi}{D}\frac{2\pi u}{\lambda}\cos\frac{2\pi ut}{\lambda} \quad . \quad . \quad . \quad . \quad . \quad (9).$$

This alternating force produces a train of spherical waves spreading out from T in all directions, of which the displacement is, at greatest, very small in comparison with ϖ; and which at any point E at distance r from the centre of T, large in comparison with the greatest diameter of T, is given by the following expression *

$$\xi\cos\frac{2\pi}{\lambda}(ut-r),$$

with
$$\xi=\varpi\frac{\pi T(D'-D)}{r\lambda^2 D}\cos\theta \quad . \quad . \quad . \quad . \quad (10),$$

where θ is the angle between the direction of the sun and the line TE. This formula, properly modified to apply it to the other component of the primary vibration, that is, the component perpendicular to the plane of the paper, gives for the displacement at E due to this component

$$\eta\cos\frac{2\pi}{\lambda}(ut-r),$$

with
$$\eta=\rho\frac{\pi T(D'-D)}{r\lambda^2 D} \quad . \quad . \quad . \quad . \quad . \quad . \quad (11).$$

Hence for the quantity of light falling from T per unit of time, on unit area of a plane at E, perpendicular to ET, reckoned in convenient temporary units, we have

$$\xi^2+\eta^2=\left[\frac{\pi T(D'-D)}{r\lambda^2 D}\right]^2(\varpi^2\cos^2\theta+\rho^2) \quad . \quad (12).$$

§ 65. Consider now the scattered light emanating from a large horizontal plane stratum of air 1 cm. thick. Let T of fig. 1 be one of a vast number of particles in a portion of this

* This formula is readily found from §§ 41, 42 of Lecture XIV. The complexity of the formulas in §§ 8–40 is due to the inclusion in the investigation of forces and displacements at small distances from T, and to the condition imposed that T is a rigid spherical figure. The dynamics of §§ 33–36 with $c=0$, and the details of §§ 37–39 further simplified by taking $v=\infty$, lead readily to the formulas (10) and (11) in our present text.

stratum subtending a small solid angle Ω viewed at an angular distance β from the zenith by an eye at distance r. The volume of this portion of the stratum is $\Omega\sec\beta\, r^2$ cubic centimetres; and therefore, if Σ denotes summation for all the particles in a cubic centimetre, small enough for application of Rayleigh's theory, and q the quantity of light shed by them from the portion $\Omega\sec\beta\, r^2$ of the stratum, and incident on a square centimetre at E, perpendicular to ET, we have

$$q=\frac{\pi^2}{\lambda^4}\Sigma\left[\frac{T(D'-D)}{D}\right]^2\Omega\sec\beta(\varpi^2\cos^2\theta+\rho^2) \quad . \quad . \quad (13).$$

Summing this expression for the contributions by all the luminous elements of the sun and taking

$$\int q=Q$$

to denote this summation, we have instead of the factor

$$\varpi^2\cos^2\theta+\rho^2,$$

$$\cos^2\theta\int\varpi^2+\int\rho^2:$$

and we have
$$\int\omega^2=\int\rho^2=\tfrac{1}{2}S \quad . \quad . \quad . \quad . \quad . \quad . \quad (14),$$

where S denotes the total quantity of light from the sun falling perpendicularly on unit of area in the particular place of the atmosphere considered. Hence the summation of (13) for all the sunlight incident on the portion $\Omega\sec\beta\, r^2$ of the stratum, gives

$$Q=\frac{\pi^2}{\lambda^4}\Sigma\left[\frac{T(D'-D)}{D}\right]^2\Omega\sec\beta(\tfrac{1}{2}\cos^2\theta+\tfrac{1}{2})S \quad . \quad (15).$$

§ 66. To define the point of the sky of which the illumination is thus expressed, let ζ be the zenith distance of the sun, and ψ the azimuth, reckoned from the sun, of the place of the sky seen along the line ET. This place and the sun and the zenith are at the angles of a spherical triangle SZT, of which ST is equal to θ. Hence we have

$$\cos\theta=\cos\zeta\cos\beta+\sin\zeta\sin\beta\cos\psi \quad . \quad . \quad . \quad (16).$$

Let now, as an example, the sun be vertical: we have $\zeta=0$, $\theta=\beta$, and (15) becomes

$$Q=\frac{\pi^2}{\lambda^4}\Sigma\left[\frac{T(D'-D)}{D}\right]^2\Omega.\tfrac{1}{2}(\cos\beta+\sec\beta)S \quad . \quad . \quad (17).$$

This shows least luminosity of the sky around the sun at the zenith, increasing to ∞ at the horizon (easily interpreted). The law of increase is illustrated in the following table of values of $\frac{1}{2}$ (cos β + sec β) for every 10° of β from 0° to 90°.

β.	$\frac{1}{2}(\cos\beta+\sec\beta)$.	β.	$\frac{1}{2}(\cos\beta+\sec\beta)$.
0°	1·000	50°	1·099
10°	1·000	60°	1·250
20°	1·002	70°	1·633
30°	1·010	80°	2·966
40°	1·036	90°	∞

§ 67. Instead now of considering illumination on a plane perpendicular to the line of vision, consider the illumination by light from our one-centimetre-thick great* horizontal plane stratum of air, incident on a square centimetre of horizontal plane. The quantity of this light per unit of time coming from a portion of sky subtending a small solid angle Ω at zenith distance β is $Q\cos\beta$. Taking $\Omega=\sin\beta d\beta d\psi$ and integrating, we find for the light shed by the one-centimetre-thick horizontal stratum on a horizontal square centimetre of the ground,

$$\int_0^{2\pi} d\psi\int_0^{\frac{1}{2}\pi} d\beta\sin\beta . Q\frac{\cos\beta}{\Omega}=\frac{4\pi^3}{3\lambda^4}\Sigma\left[\frac{T(D'-D)}{D}\right]^2 S \quad . (18).$$

Now each molecule and particle of dust sheds as much light upwards as downwards. Hence (18) doubled expresses the quantity of light lost by direct rays from a vertical sun in crossing the one-centimetre-thick horizontal stratum. It agrees with the expression for k in (1) of § 55, as it ought to do.

§ 68. The expression (15) is independent of the distance of the stratum above the level of the observer's eye. Hence if H denote the height above this level, of the upper boundary of an ideal homogeneous atmosphere consisting of all the ultimate molecules and all the dust of the real atmosphere scattered uniformly through it, and if s denote the whole light on unit area of a plane at E perpendicular to ET, from all the molecules and dust in the solid angle Ω of the real atmosphere, due to the sun's direct light incident on them, we have

$$\frac{s}{S}=H\sec\beta\frac{\pi^2}{\lambda^4}\Sigma\left[\frac{T(D'-D)}{D}\right]^2\Omega.\tfrac{1}{2}(\cos^2\theta+1)\ . \quad . \quad . \quad (19);$$

provided we may, in the cases of application whatever they may be, neglect the diminution of the direct sunlight in its actual course through air, whether to the observer or to the portion of the air of which he observes the luminosity, and neglect the diminution of the scattered light from the air in its course through air to the observer. This proviso we shall see is practically fulfilled in Mr. Majorana's observations on the crater of Etna for zenith distances of the sun not exceeding 60°, and in Mr. Sella's observation on Monte Rosa in which the sun's zenith distance was 50°. But for Majorana's recorded observation on Etna at 5.50 a.m. when the sun's zenith distance was 81°·71, of which the secant is 6·927, there may have been an important diminution of the sun's light reaching the air vertically above the observer, and a considerably more important diminution of his light as seen direct by the observer. This would tend to make the sunlight reaching the observer less strong relatively to the skylight than according to (19); and might conceivably account for the first number in col. 3 being smaller than the first number in col. 4 of the Table of § 69 below; but it seems to me more probable that the smallness of the first two numbers in col. 3, showing considerably greater luminosity of sky than according to (19), may be partly or chiefly due to dust in the air overhead, optically swelled by moisture in the early morning. The largeness of the luminosity of the sky indicated by the smallness of the last number in col. 3 (376), in comparison with the last number of col. 4 (460), may conceivably be explained by earthshine from air and volcanic ash and rock and forest and vineyard and sea below the level of the crater adding considerably to the illumination which the sky experiences from above by direct sunlight. This addition would be much greater at 11 a.m., when the sun's zenith distance was 29°·9, than at 9 a.m., when it was 44°·6.

§ 69. The results of Majorana's observations from the crater of Etna are shown in the following Table, of which the first and third columns are quoted from the Philosophical Magazine for May 1901, and the second column has been kindly given to me in a letter by Mr. Majorana. The values of S/s shown in column 4 are calculated from § 68 (19), with the factor of

* We are neglecting the curvature of the earth, and supposing the density and composition of the air to be the same throughout the plane horizontal stratum to distances from the zenith very great in comparison with its height above the ground.

$\sec\beta(\cos^2\theta+1)$ taken to make it equal to Majorana's r for sun's zenith distance 44°·6, on the supposition that the region of sky observed was in each case (see § 62 above) in the position of minimum luminosity as given by (19). It is obvious that this position is in a vertical great circle through the sun, and

Col. 1.	Col. 2.	Col. 3.	Col. 4.	Col. 5.
Time.	Zenith distance of sun. ζ.	Ratio of luminosity of sun's disc to luminosity of sky. r.	$\frac{S}{s}$.	Zenith distance of least luminous part of sky. β.
5.50 A.M.	81·7	2570000	3280000	5°·5
7	68·0	3125000	3350000	14·4
8	56·1	3650000	3600000	21·7
9	44·6	3930000	3930000	27·8
11	29·9	3760000	4600000	33·6

on the opposite side of the zenith from the sun; and thus we have $\theta=\zeta+\beta$. Hence (19) becomes

$$\frac{s}{S}=H\frac{\pi^2}{\lambda^4}\Sigma\left[\frac{T(D'-D)}{D}\right]^2\Omega.\tfrac{1}{2}\sec\beta[\cos^2(\zeta+\beta)+1] \quad . \quad (20).$$

To make (20) a minimum we have

$$\tan\beta=\frac{2\sin 2(\beta+\zeta)}{3+\cos 2(\beta+\zeta)} \quad . \quad . \quad . \quad . \quad . \quad . \quad (21).$$

The value of β satisfying this equation for any given value of ζ is easily found by trial and error, guided by a short preliminary table of values of β for assumed values of $\beta+\zeta$. Col. 5 shows values of β thus found approximately enough to give the values of S/s shown in col. 4 for the several values of ζ.

§ 70. Confining our attention now to Majorana's observations at 9 A.M. when the sun's altitude was about 44°·6; let e be the proportion of the light illuminating the air over the crater of Etna which at that hour was due to air, earth, and water below; and therefore $1-e$ the proportion of the observed luminosity of the sky which was due to the direct rays of the sun, and expressed by § 68 (19). Thus, for $\beta=27°·8$, $\zeta=44°·6$, and $\theta=72°·4$, we have $S/s=3930000/(1-e)$, instead of the S/s of col. 4, § 69. With this, equation (20) gives

$$\Sigma\left[\frac{T(D'-D)}{D}\right]^2=\frac{\lambda^4(1-e)}{H\Omega}.4·18.10^{-8} \quad . \quad (22).$$

Here, in order that the comparison may be between the whole light of the sun and the light from an equal apparent area of the sky, we must take

$$\Omega=\pi/219·4^2*=1/15320,$$

being the apparent area of the sun's disc as seen from the earth. As to H, it is what is commonly called the "height of the homogeneous atmosphere" and, whether at the top of Etna or at sea-level, is

$$7·988.10^5\left(1+\frac{t}{273}\right)\text{ centimetres;}$$

where t denotes the temperature at the place above which H is reckoned. Taking this temperature as 15° C., we find

$$H=8·44.10^5\text{ centimetres.}$$

Thus (22) becomes

$$\Sigma\left[\frac{T(D'-D)}{D}\right]^2=\lambda^4(1-e).·759.10^{-9} \quad . \quad . \quad (23).$$

§ 71. Let us now denote by f and $1-f$ the proportions of (23) due respectively to the ultimate molecules of air and to dust. We have

$$n\left[\frac{T(D'-D)}{D}\right]^2=\lambda^4 f(1-e).·759.10^{-9} \quad . \quad (24);$$

where n denotes the number of the ultimate molecules in a cubic centimetre of the air at the top of Etna; and $T(D'-D)/D$ relates to any one of these molecules; any difference which there may be between oxygen and nitrogen being neglected. Now assuming that the refractivity of the atmosphere is practically due to the ultimate molecules, and that no appreciable part of it is due to the dust in the air, we have by § 56 (7),

$$·0002=n\frac{T(D'-D)}{2D} \quad . \quad . \quad . \quad . \quad . \quad (25),$$

the first number being approximately enough the refractivity

* The sun's distance from the earth is 219·4 times his radius.

of air at the crater of Etna (barometric pressure, 53·6 centimetres of mercury). Hence

$$\left[\frac{T(D'-D)}{D}\right]^2=\frac{1}{n^2}1{\cdot}6\,.\,10^{-7} \quad . \quad . \quad . \quad . \quad (26),$$

and using this in (24) we find

$$n=\frac{211}{\lambda^4 f(1-e)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (27).$$

Here, as in § 57 in connexion with Bouguer's estimate for loss of light in transmission through air, we have an essential uncertainty in respect to the effective wave-length; and, for the same reasons as in § 57, we shall take $\lambda=6\,.\,10^{-5}$ cm. as the proper mean for the circumstances under consideration. With this value of λ, (27) becomes

$$n=\frac{1}{f(1-e)}1{\cdot}63\,.\,10^{19} \quad . \quad . \quad . \quad . \quad (28).$$

§ 72. In Mr. Sella's observations on Monte Rosa the zenith distance of the sun was 50°, and the place of the sky observed was in the zenith. He found the brightness of the sun's disc to be about 5000000 times the brightness of the sky in the zenith. Dealing with this result as in §§ 70, 71, with $\beta=0$ in (20), and supposing the temperature of the air at the place of observation to have been 0° C., we find

$$n'=\frac{1}{f'(1-e')}2{\cdot}25\,.\,10^{19} \quad . \quad . \quad . \quad . \quad (29),$$

where e', f', and n' are the values of e, f, and n, at the place of observation on Monte Rosa. Denoting now by N the number of molecules in a cubic centimetre of air at 0° C. and pressure 75 centimetres of mercury, we have, by the laws of Boyle and Charles, on the supposition that the temperature of the air was 15° on the summit of Etna, and 0° on Monte Rosa

$$N=n\frac{75}{53{\cdot}6}\left(1+\frac{15}{273}\right)=n'\frac{75}{49},$$

or

$$N=1{\cdot}48n=1{\cdot}53n' \quad . \quad . \quad . \quad . \quad (30).$$

From these, with (28) and (29), we find

$$N=\frac{2{\cdot}41}{f(1-e)}\,.\,10^{19}=\frac{3{\cdot}44}{f'(1-e')}\,.\,10^{19} \quad . \quad . \quad (31).$$

§ 73. To estimate the values of e and e' as defined in §§ 70, 72, consider the albedos* of the earth as might be seen from a balloon in the blue sky observed by Majorana and G. Sella over Etna and over Monte Rosa respectively. These might be about ·2 and ·4, the latter much the greater because of the great amount of snow contributing to illuminate the sky over Monte Rosa. With so much of guess-work in our data we need not enter on the full theory of the contribution to sky-light by earthshine from below according to the principle of §§ 67, 68, interesting as it is; and we may take as very rough estimates ·2 and ·4 as the values of e and e'. Thus (31) becomes

$$N=\frac{3{\cdot}01}{f}\,.\,10^{19}=\frac{5{\cdot}73}{f'}\,.\,10^{19} \quad . \quad . \quad . \quad (32).$$

§ 74. Now it would only be if the whole light of the sky were due to the ultimate molecules on which the refractivity depends that f or f' could have so great a value as unity. If this were the case for the blue sky seen over Monte Rosa by G. Sella in 1900, we should have $f'=1$, and therefore $N=5{\cdot}73\,.\,10^{19}$. But it is most probable that even in the very clearest weather on the highest mountain, a considerable portion of the light of the sky is due to suspended particles much larger than the ultimate molecules N_2, O_2, of the atmosphere; and therefore the observations of the luminosity of the sky over Monte Rosa in the summer of 1900 render it probable that N is greater than $5{\cdot}73\,.\,10^{19}$. If now we take our estimate of § 50, for the number of molecules in a cubic cm. of air at 0°, and normal pressure, $N=10^{20}$, we have $1-f={\cdot}699$ and $1-f'={\cdot}427$; that is to say, according to the several assumptions we have made, ·699 of the whole light of the portion of sky observed over Etna by Majorana was due to dust, and only ·427 of that observed by Sella on

* Albedo is a word introduced by Lambert 150 years ago to signify the ratio of the total light emitted by a thoroughly unpolished solid or a mass of cloud to the total amount of the incident light. The albedo of an ideal perfectly white body is 1. My friend Professor Becker has kindly given me the following table of albedos from Müller's book *Die Photometrie der Gestirne* (Leipsic, 1897) as determined by observers and experimenters.

Mercury	0·14	Uranus	0·60
Venus	0·76	Neptune	0·52
Moon	0·34	Snow	0·78
Mars	0·22	White Paper	0·70
Jupiter	0·62	White Sandstone	0·24
Saturn	0·72	Damp Soil	0·08

Monte Rosa was due to dust. It is quite possible that this conclusion might be exactly true, and it is fairly probable that it is an approximation to the truth. But on the whole these observations indicate, so far so they can be trusted, the probability of at least as large a value as 10^{20} for N.

§ 75. All the observations referred to in §§ 57–74 are vitiated by essentially involving the physiological judgment of perception of difference of strengths of two lights of different colours. In looking at two very differently tinted shadows of a pencil side by side, one of them blue or violet cast by a comparatively near candle, the other reddish-yellow cast by a distant brilliantly white incandescent lamp or by a more distant electric arc-lamp, or by the moon, when practising Rumford's method of photometry, it is quite wonderful to find how unanimous half-a-dozen laboratory students, or even less skilled observers, are in declaring This is the stronger! or, That is the stronger! or, Neither is stronger than the other! When the two shadows are declared equally strong, the declaration is that the differently tinted lights from the two shadowed places side by side on the white paper are, according to the physiological perception by the eye, equally strong. But this has no meaning in respect to any definite component parts of the two lights; and the unanimity, or the greatness of the majority, of the observers declaring it only proves a physiological agreement in the perceptivity of healthy average eyes (from which colour-blind eyes would no doubt differ wildly). Two circular areas of white paper in Sella's observations on Monte Rosa, a circle and a surrounding area of ground glass in Majorana's observations with his own beautiful sky-photometer on Etna, are seen illuminated respectively by diminished sunlight of unchanged tint and by light from the blue sky. The sun-lit areas seem reddish-yellow by contrast with the sky-lit areas which are azure blue. What is meant when the two areas differing so splendidly are declared to be equally luminous? The nearest approach to an answer to this question is given at the end of § 71 above, and is eminently unsatisfactory. The same may be truly said of the dealing with Bouguer's datum in § 57, though the observers on whom Bouguer founded do not seem to have been disturbed by knowledge that there was anything indefinite in what they were trying to define or to find by observation.

§ 76. To obtain results not vitiated by the imperfection of the physiological judgment described in § 75, Newton's prismatic analysis of the light observed, or something equivalent to it, is necessary. This was done by Rayleigh himself for the blue light of the sky actually before he had worked out his dynamical theory. He compared the prismatic spectrum of light from the zenith with that of sunlight diffused through white paper; and by aid of a curve drawn from about thirty comparisons ranging over the spectrum from C to beyond F, found the following results for four different wave-lengths.

	C.	D.	b^3.	F.
Wave-length .	656·2	589·2	517·3	486·2
Calculated . .	1	1·54	2·52	3·34
Observed . .	1	1·64	2·84	3·60

On these he makes the following remarks:—"It should "be noticed that the sky compared with diffused light was "even bluer than theory makes it, on the supposition that the "diffused light through the paper may be taken as similar to "that whose scattering illuminates the sky. It is possible "that the paper was slightly yellow; or the cause may lie in "the yellowness of sunlight as it reaches us compared with "the colour it possesses in the upper regions of the atmosphere. "It would be a mistake to lay any great stress on the obser-"vations in their present incomplete form; but at any rate "they show that a colour more or less like that of the sky "would result from taking the elements of white light in "quantities proportional to λ^{-4}. I do not know how it may "strike others; but individually I was not prepared for so "great a difference as the observations show, the ratio for "F being more than three times as great as for C." For myself I thoroughly agree with this last sentence of Rayleigh's. There can be no doubt of the trustworthiness of his observational results; but it seems to me most probable, or almost certain, that the yellowness or orange-colour of the sunlight seen through the paper, caused by larger absorption of green, blue, and violet rays, may explain the extreme relative richness in green, blue, and violet rays which the results show for the zenith blue sky observed.

§ 77. An elaborate series of researches on the blue of the sky on twenty-two days from July 1900 to February 1901 is described in a very interesting paper, "Ricerche sul Bleu del Cielo," a dissertation presented to the Royal University of Rome by Dr. Giuseppe Zettwuch, as a thesis for his degree of Doctor in Physics. In these researches prismatically analysed light from the sky was compared with prismatically analysed direct sunlight reduced by passage through a narrow slit; and the results were therefore not vitiated by unequal absorptions of direct sunlight in the apparatus. A translation of the author's own account of his conclusions is published in the Philosophical Magazine for August 1902; by which it will be seen that the blueness of the sky, even when of most serene azure, was always much less deep than the true Rayleigh blue defined by the λ^{-4} law. Hence, according to Rayleigh's theory (see § 53 above) much of the light must always have come from particles not exceedingly small in proportion to the wave-length. Thus in Zettwuch's researches we have a large confirmation of the views expressed in §§ 54, 58, 61, 74 above, and §§ 78, 79 below.

§ 78. Through the kindness of Professor Becker, I am now able to supplement Bouguer's 170-year old information with the results of an admirable extension of his investigation by Professor Müller of the Potsdam Observatory, in which the proportion (denoted by p in the formula below) transmitted down to sea-level of homogeneous light entering our atmosphere vertically is found for all wave-lengths from $4{\cdot}4 \,.\, 10^{-5}$ to $6{\cdot}8 \,.\, 10^{-5}$, by comparison of the solar spectrum with the spectrum of a petroleum flame for different zenith distances of the sun. It is to be presumed, although I do not find it so stated, that only the clearest atmosphere available at Potsdam was used in these observations. For the sake of comparison with Rayleigh's theory, Professor Becker has arithmetically resolved Müller's results into two parts; one constant, and the other varying inversely as the fourth power of the wave-length, expressed in the following formula* modified to facilitate comparison with §§ 57–59 above:

$$p = \epsilon^{-(\cdot 0887 + \cdot 0772 z^{-4})} = \cdot 9152 \epsilon^{-\cdot 0772 z^{-4}} \quad . \quad . \quad . \quad (33),$$

where $z = \lambda \div \cdot 6 \,.\, 10^{-5}$. In respect to the two factors here shown, we may say roughly that the first factor is due to suspended particles too large, and the second to particles not too large, for the application of Rayleigh's law. For the case of $\lambda = 6 \,.\, 10^{-5}$ ($z = 1$) this gives

$$p = \epsilon^{-(\cdot 0887 + \cdot 0772)} = \cdot 9152 \,.\, \cdot 9258 = \cdot 847 \quad . \quad . \quad (34).$$

§ 79. Taking now the last term in the index and the last factor shown in (34) and dealing with it according to §§ 57–59 above, and still, as in § 55, using k to denote the proportionate loss of light per centimetre due to particles small enough for Rayleigh's theory, whether "suspended particles" or ultimate molecules of air or both, we have $\epsilon^{-830000k} = \cdot 9258$ which gives $k^{-1} = 10{\cdot}75 \,.\, 10^6$ cms. Hence if, as in § 58, we suppose for a moment the want of perfect transparency thus defined to be wholly due to the ultimate molecules of air, we should have by the dynamics of refractivity $n \dfrac{T(D' - D)}{D} = \cdot 0006$; and thence by (1) of § 55 with $\lambda = 6 \,.\, 10^{-5}$ we should find for the number of molecules per cubic centimetre $n = 2{\cdot}469 \,.\, 19^{19}$. But it is quite certain that a part, and most probably a large part, of the want of transparency produced by particles small enough for Rayleigh's theory is due to "suspended particles" larger than the ultimate molecules: and we infer that the number of ultimate molecules per cubic centimetre is greater than, and probably very much greater than, $2{\cdot}469 \,.\, 10^{19}$. Thus from the surer and more complete data of Müller regarding extinction of light of different wave-lengths traversing the air, we find an inferior limit for the number of molecules per cubic centimetre nearly three times as great as that which Rayleigh showed to be proved from Bouguer's datum.

* Müller, *Die Photometrie der Gestirne*, p. 140.

§ 80. Taking, somewhat arbitrarily, as the result of §§ 23–77 that the number of molecules in a cubic centimetre of a perfect gas at standard temperature and pressure is 10^{20}, we have the following interesting table of conclusions regarding the weights of atoms and the molecular dimensions of liquefied gases, of water, of ice, and of solid metals.

Substance.	Mass of atom or of H_2O in grammes.	Density.	Number of atoms in cub. cm.	Distance from centre to centre if ranged in cubic order with actual density.
H	$0{\cdot}45 \,.\, 10^{-24}$	Liquid at 17° absolute... ·090	$200 \,.\, 10^{21}$	$1{\cdot}71 \,.\, 10^{-8}$
O	7·15 ,,	,, ,, freezing-point 1·27	178 ,,	1·78 ,,
H_2O	8·05 ,,	Water .... 1·00	124 ,,	2·00 ,,
H_2O	8·05 ,,	Ice .... ·917	114 ,,	2·06 ,,
H_2O	8·05 ,,	Vapour at 0° C...... $\cdot 487 \,.\, 10^{-5}$	$605 \,.\, 10^{15}$	118·2 ,,
N	6·29 ,,	Liquid .... 1·047	$166 \,.\, 10^{21}$	1·82 ,,
Argon	17·81 ,,	,, ... 1·212	68·1 ,,	2·45 ,,
Gold	88·52 ,,	Solid........................19·32	218 ,,	1·66 ,,
Silver	48·47 ,,	,, ...10·53	217 ,,	1·66 ,,
Copper	28·43 ,,	,, ... 8·95	311 ,,	1·475 ,,
Iron	25·15 ,,	,, ... 7·86	313 ,,	1·47 ,,
Zinc	29·30 ,,	,, ... 7·15	244 ,,	1·60 ,,