IV. "On Peristaltic Induction of Electric Currents." By Professor WILLIAM THOMSON, F.R.S. Received May 10, 1856.

Recent observations on the propagation of electricity through wires in subaqueous and subterranean telegraphic cables have brought to light phenomena of induced electric currents, which, while they are essentially different from the phenomena of what has hitherto been called electro-dynamic induction, are exactly such as might have been anticipated from the well-established theory of electrical equilibrium, had experiment afforded the data of relation between electrostatical and electro-dynamic units wanted for determining what dimensions of wire would be required to render these phenomena sensible to ordinary observation. They present a very perfect analogy with the mutual influences of a number of elastic tubes bound together laterally throughout their lengths, and surrounded and filled with a liquid which is forced through one or more of them, while the others are left with their ends open (*uninsulated*), or stopped (*insulated*), or subjected to any other particular conditions. The hydrostatic pressure applied to force the liquid through any of the tubes will cause them to swell and to press against the others, which will thus, by peristaltic action, compel the liquid contained in them to move, in different parts of them, in one direction or the other. A long solid cylinder of an incompressible elastic solid*, bored out symmetrically in four, six, or more circular passages parallel to its length, will correspond to an ordinary telegraph cable containing the same number of copper wires separated from one another only by gutta-percha: and the hydraulic motion will follow rigorously the same laws as the electrical conduction, and will be expressed by identical language in mathematics, provided the lateral dimensions of the bores are so small in comparison with their lengths, or the viscosity of the liquid so great, that the motions are not sensibly affected by *inertia*, and are consequently dependent altogether on hydrostatic pressure and fluid friction. The electrical induction now alluded to depends on the electrostatic forces determined by Coulomb; but it would be in

* Such as india-rubber very approximately is in reality.

one respect a real, and in all respects an apparent, contradiction of terms, to speak of electrostatic induction of electric currents, and I therefore venture to introduce the term *peristaltic* to characterize that kind of induction by which currents are excited in elongated conductors through the variation of electrostatic potential in the surrounding matter. On the other hand, as any inductive excitation of electric motion might be called electro-dynamic induction, it will be convenient to distinguish the kind of electro-dynamic induction first discovered by Faraday, by a distinctive name; and as the term *electro-magnetic*, which has been so applied, appears correctly characteristic, I shall call *electro-magnetic induction* that kind of action by which electric currents are excited, or inequalities of electric potential sustained, in a conductor of electricity, by variations of magnetic or electro-magnetic potential, or by absolute or relative motion of the conductor itself across lines of magnetic or electro-magnetic force.

The most general problem of peristaltic induction is to determine the motion of electricity in any number of long conducting wires, insulated from one another within an uninsulated tube of conducting material, when subjected each to any prescribed electrical action at its extremities; without supposing any other condition regarding the sections and relative dispositions of the conductors than—(1), that their lateral dimensions and mutual distances are so small in proportion to their lengths, that the effects of peristaltic induction are paramount over those of electro-magnetic induction; and (2), that the section of the entire system of conductors, if not uniform in all parts, varies so gradually as to be sensibly uniform through every part of the length not a very large multiple of the largest lateral dimension. In the present communication I shall only give the general equations of motion by which the physical conditions to be satisfied are expressed for every case; and I shall confine the investigation of solutions to certain cases of uniform and symmetrical arrangement, such as are commonly used in the submarine telegraph cable.

At any time t, let q_1, q_2, q_3, &c. be the quantities of electricity with which the different wires are charged, per unit of length of each, at a distance x from one extremity, O, of the conducting system; and let v_1, v_2, v_3, &c. be the electrostatical potentials in the same parts of those conductors. Let $\varpi_1^{(1)}$, $\varpi_1^{(2)}$, $\varpi_1^{(3)}$, &c., $\varpi_2^{(1)}$, $\varpi_2^{(2)}$, $\varpi_2^{(3)}$, &c., $\varpi_3^{(1)}$, $\varpi_3^{(2)}$, $\varpi_3^{(3)}$, &c. be coefficients, such that the electrostatical potentials (v_1, v_2, &c.), due to stated charges (q_1, q_2, &c.) of the different wires, are expressed by the equations

$$\left.\begin{array}{l} v_1 = \varpi_1^{(1)} q_1 + \varpi_1^{(2)} q_2 + \varpi_1^{(3)} q_3 + \&c. \\ v_2 = \varpi_2^{(1)} q_1 + \varpi_2^{(2)} q_2 + \varpi_2^{(3)} q_3 + \&c. \\ v_3 = \varpi_3^{(1)} q_1 + \varpi_3^{(2)} q_2 + \varpi_3^{(3)} q_3 + \&c. \\ \quad \&c. \qquad\qquad \&c. \end{array}\right\} \quad . \quad . \quad . \quad (1).$$

If the sections of all the conductors are circular, these coefficients ($\varpi_1^{(1)}$, $\varpi_1^{(2)}$, &c.) may be easily determined numerically to any required degree of accuracy, in each particular case, by the *method of electro-statical images.* The electromotive force per unit of length at the position x will be, in the different wires,

$$\frac{dv_1}{dx}, \quad \frac{dv_2}{dx}, \quad \frac{dv_3}{dx},$$

respectively, and therefore if γ_1, γ_2, γ_3, &c. denote the strength of current at the same position, and k_1, k_2, k_3, &c. the resistances to conduction per unit of length in the different wires respectively, we have by the law of Ohm, applied to the action of peristaltic electromotive force,

$$k_1\gamma_1 = -\frac{dv_1}{dx}, \quad k_2\gamma_2 = -\frac{dv_2}{dx}, \quad k_3\gamma_3 = -\frac{dv_3}{dx} \quad . \quad . \quad (2).$$

Now unless the strength of current be uniform along any one of the wires, the charge of electricity will experience accumulation or diminution in any part of it by either more or less electricity flowing in on one side than out on the other; and the mathematical expression of these circumstances is clearly

$$\frac{dq_1}{dt} = -\frac{d\gamma_1}{dx}, \quad \frac{dq_2}{dt} = -\frac{d\gamma_2}{dx}, \quad \frac{dq_3}{dt} = -\frac{d\gamma_3}{dx} \quad . \quad . \quad . \quad (3).$$

Using in these equations the values of γ_1, γ_2, γ_3, &c. given by (2), and then substituting for v_1, v_2, v_3, &c. their expressions (1), we obtain

$$\left.\begin{array}{l} \frac{dq_1}{dt} = \frac{d}{dx}\left\{\frac{1}{k_1}\cdot\frac{d(\varpi_1^{(1)}q_1)}{dx} + \frac{1}{k_2}\frac{d(\varpi_1^{(2)}q_2)}{dx} + \frac{1}{k_3}\frac{d(\varpi_1^{(3)}q_3)}{dx} + \&c.\right\} \\ \frac{dq_2}{dt} = \frac{d}{dx}\left\{\frac{1}{k_1}\cdot\frac{d(\varpi_2^{(1)}q_1)}{dx} + \frac{1}{k_2}\cdot\frac{d(\varpi_2^{(2)}q_2)}{dx} + \frac{1}{k_3}\frac{d(\varpi_2^{(3)}q_3)}{dx} + \&c.\right\} \\ \frac{dq_3}{dt} = \frac{d}{dx}\left\{\frac{1}{k_1}\cdot\frac{d(\varpi_3^{(1)}q_1)}{dx} + \frac{1}{k_2}\cdot\frac{d(\varpi_3^{(2)}q_2)}{dx} + \frac{1}{k_3}\frac{d(\varpi_3^{(3)}q_3)}{dx} + \&c.\right\} \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \end{array}\right\} \quad (4),$$

which are the general equations of motion required.

It is to be observed that k_1, k_2, &c., $\varpi_1^{(1)}$, $\varpi_1^{(2)}$, $\varpi_2^{(1)}$, &c. will be functions of x if the section of the conducting system is heterogeneous in different positions along it; but in all cases in which each conductor is uniform, and uniformly situated with reference to the others along the whole length, these coefficients will be constant, and the equations become reduced to

$$\left.\begin{aligned}\frac{dq_1}{dt}&=\frac{\varpi_1^{(1)}}{k_1}\frac{d^2q_1}{dx^2}+\frac{\varpi_1^{(2)}}{k_2}\frac{d^2q_2}{dx^2}+\frac{\varpi_1^{(3)}}{k_3}\frac{d^2q_3}{dx^2}+\&c.\\ \frac{dq_2}{dt}&=\frac{\varpi_2^{(1)}}{k_1}\frac{d^2q_1}{dx^2}+\frac{\varpi_2^{(2)}}{k_2}\frac{d^2q_2}{dx^2}+\frac{\varpi_2^{(3)}}{k_3}\frac{d^2q_3}{dx^2}+\&c.\\ \frac{dq_3}{dt}&=\frac{\varpi_3^{(1)}}{k_1}\frac{d^2q_1}{dx^2}+\frac{\varpi_3^{(2)}}{k_2}\frac{d^2q_2}{dx^2}+\frac{\varpi_3^{(3)}}{k_3}\frac{d^2q_3}{dx^2}+\&c.\\ &\cdots\cdots\end{aligned}\right\}\quad . \quad . \quad (5).$$

The most obvious general method of treatment for integrating these equations, is to find elementary solutions by assuming

$$q_1=\mathrm{A}_1u,\quad q_2=\mathrm{A}_2u,\quad q_3=\mathrm{A}_3u,\ \ldots\ldots\ q_i=\mathrm{A}_iu,\quad .\quad .\quad (6),$$

where u satisfies the equation

$$\frac{du}{dt}=\varkappa\frac{d^2u}{dx^2}\quad .\ .\ .\ .\ .\ .\ .\ .\quad (7).$$

This will reduce the differential equations (5) to a set of linear equations among the coefficients A_1, A_2, ... A_i, giving by elimination an algebraic equation of the ith degree having i real roots, to determine $\varkappa$. The particular form of elementary solution of the equation (7) to be used may be chosen from among those given by Fourier, according to convenience, for satisfying the terminal conditions for the different wires.

In thinking on some applications of the preceding theory, I have been led to consider the following general question regarding the mutual influence of electrified conductors:—If, of a system of detached insulated conductors, one only be electrified with a given absolute charge of electricity, *will the potential excited in any one of the others be equal to that which the communication of an equal absolute charge to this other would excite in the first?* I now find that a general theorem communicated by myself to the Cambridge Mathematical Journal, and published in the Numbers for November 1842 and February 1843, but, as I afterwards (Jan. 1845) learned, first given by Green in his Essay on the Mathematical Theory of

Electricity and Magnetism (Nottingham, 1828), leads to an affirmative answer to this question.

The general theorem to which I refer is, that if, considering the forces due respectively to two different distributions of matter (whether real, or such as is imagined in theories of electricity and magnetism), we denote by N_1, N_2 their normal components at any point of a closed surface, or group of closed surfaces, S, containing all parts of each distribution of matter, and by V_1, V_2 the potentials at the same point due respectively to the two distributions, and if ds be an element of the surface S, the value of $\iint N_1 V_2 ds$ is the same as that of $\iint N_2 V_1 ds$ (each being equal to the integral $\iiint R_1 R_2 \sin\theta \, dx \, dy \, dz$ extended over the whole of space external to the surface S, at any point (x, y, z) of which external space the two resultants are denoted by R_1, R_2 respectively, and the angle between their directions by θ). To apply this with reference to the proposed question, let the first distribution of matter consist of a certain charge, q, communicated to one of a group of insulated conductors, and the inductive electrifications of the others, not one of which has any absolute charge; let the second distribution of matter consist of the electrifications of the same group of conductors when an equal quantity q is given to a second of them, and all the others are destitute of absolute charges; and let surface S be the group of the surfaces of the different conductors. Since the potential is constant through each separate conductor, the integral $\iint N_1 V_2 \, ds$ will be equal to the sum of a set of terms of the form $[V_2][\iint N_1 ds]$, where $[V_2]$ denotes the value in any of these conductors of the potential of the second distribution, and $[\iint N_1 ds]$ an integral including the whole surface of the same conductor, but no part of that of any of the others. Now by a well-known theorem, first given by Green, $[\iint N_1 ds]$ is equal to $4\pi q$ if q denote the absolute quantity of matter within the surface of the integral (as is the case for the first group of conductors), and vanishes if there be no distribution of matter, or (as is the case with each of the other conductors) if there be equal quantities of positive and negative matter within the surface over which the integral is extended. Hence if $[V_2]_1$ denote the potential in the first conductor due to the second distribution of matter, we have

$$\iint N_1 V_2 ds = 4\pi [V_2]_1 q.$$

Similarly, we have

$$\iint N_2 V_1 ds = 4\pi [V_1]_2 q.$$

Hence, by the general theorem, we conclude $[V_2]_1 = [V_1]_2$, and so demonstrate the affirmative answer to the question stated above.

I think it unnecessary to enter on details suited to the particular case of lateral electrostatic influence between neighbouring parts of a number of wires insulated from one another under a common conducting sheath, when uniform or varying electric currents are sent through by them; for which a particular demonstration in geometry of two dimensions, analogous to the demonstration of Green's theorem to which I have referred as involving the consideration of a triple integral for space of three dimensions, may be readily given; but, as a particular case of the general theorem I have now demonstrated, it is obviously true that the potential in one wire due to a certain quantity of electricity per unit of length in the neighbouring parts of another under the same sheath, is equal to the potential in this other, due to an equal electrification of the first.

Hence the following relations must necessarily subsist among the coefficients of mutual peristaltic induction in the general equations given above,

$$\varpi_1^{(2)} = \varpi_2^{(1)}; \quad \varpi_1^{(3)} = \varpi_3^{(1)}; \quad \varpi_2^{(3)} = \varpi_3^{(2)}; \text{ \&c.}$$

On the Solution of the Equations of Peristaltic Induction in symmetrical systems of Submarine Telegraph Wires.

The general method which has just been indicated for resolving the equations of electrical motion in any number of linear conductors subject to mutual peristaltic influence, fails when these conductors are symmetrically arranged within a symmetrical conducting sheath (and therefore actually in the case of any ordinary multiple wire telegraph cable), from the determinantal equation having sets of equal roots. Regular analytical methods are well known by which the solutions for such particular cases may be derived from the failing general solutions; but it is nevertheless interesting to investigate each particular case specially, so as to obtain its proper solution by a synthetical process, the simplest possible for the one case considered alone. In the present communication, the problem of peristaltic induction is thus treated for some of the most common cases of actual submarine telegraph cables, in which two or more wires of equal dimensions are insulated in symmetrical positions within a cylindrical conducting sheath of circular section.

CASE I.—*Two-wire Cable.*

In the general equations (according to the notation of the first part of this communication) we have $k_1=k_2$; $\varpi_1^{(1)}=\varpi_2^{(2)}$; and $\varpi_2^{(1)}=\varpi_1^{(2)}$: and it will be convenient now to denote the values of the members of these three equations by k, $\frac{1}{c}$, and $\frac{f}{c}$ respectively; that is, to express by k the galvanic resistance in each wire per unit of length, by c the electrostatical capacity of each per unit of length when the other is prevented from acquiring an absolute charge, and by f the proportion in which this exceeds the electrostatical capacity of each when the other has a charge equal to its own; or in other words, to assume c and f so that

$$\left.\begin{aligned} v_1 &= \frac{1}{c}q_1+\frac{f}{c}q_2 \\ v_2 &= \frac{f}{c}q_1+\frac{1}{c}q_2 \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (1),$$

if v_1 and v_2 be the potentials in the two wires in any part of the cable where they are charged with quantities of electricity respectively q_1 and q_2 per unit of length. The equations of electrical conduction along the two wires then become

$$\left.\begin{aligned} \frac{dv_1}{dt} &= \frac{1}{kc}\left(\frac{d^2v_1}{dx^2}+f\frac{d^2v_2}{dx^2}\right) \\ \frac{dv_2}{dt} &= \frac{1}{kc}\left(f\frac{d^2v_1}{dx^2}+\frac{d^2v_2}{dx^2}\right) \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (2).$$

From these we have, by addition and subtraction,

$$\frac{d\vartheta}{dt}=\frac{1+f}{kc}\frac{d^2\vartheta}{dx^2}, \text{ and } \frac{d\omega}{dt}=\frac{1-f}{kc}\frac{d^2\omega}{dx^2} \quad . \quad . \quad . \quad (3),$$

where ϑ and ω are such that

$$v_1=\vartheta+\omega, \quad v_2=\vartheta-\omega \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4).$$

If both wires reached to an infinite distance in each direction, the conditions to be satisfied in integrating the equations of motion would be simply that the initial distribution of electricity along each must be whatever is prescribed; that is, that

$$\left.\begin{aligned} & v_1=\phi_1(x), \text{ and } v_2=\phi_2(x) \\ \text{when} \quad & t=0 \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (5),$$

ϕ_1 and ϕ_2 denoting two arbitrary functions. Hence, according to Fourier, we have, for the integrals of the equations (3),

$$\left.\begin{aligned}\vartheta &= \sqrt{\frac{kc}{4(1+f)\pi}}\,.\,t^{-\frac{1}{2}}\int_{-\infty}^{\infty}\frac{1}{2}\{\phi_1(\xi)+\phi_2(\xi)\}\varepsilon^{-\frac{kc(\xi-x)^2}{4(1+f)t}}\,d\xi \\ \omega &= \sqrt{\frac{kc}{4(1-f)\pi}}\,t^{-\frac{1}{2}}\int_{-\infty}^{\infty}\frac{1}{2}\{\phi_1(\xi)-\phi_2(\xi)\}\varepsilon^{-\frac{kc(\xi-x)^2}{4(1-f)t}}\,d\xi\end{aligned}\right\} \quad (6),$$

and the solution of the problem is expressed in terms of these integrals by (4).

If now we suppose the cable to have one end at a finite distance from the part considered, for instance at the point O from which x is reckoned, and if at this end each wire is subjected to electric action so as to make its potential vary arbitrarily with the time, there will be the additional condition

$$\text{when}\quad \left.\begin{aligned} v_1=\psi_1(t),\ \text{and}\ v_2=\psi_2(t),\\ x=0\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (7),$$

to be fulfilled. In the other conditions, (5), only positive values of x have now to be considered, but they must be fulfilled in such a way as not to interfere with the prescribed values of the potentials at the ends of the wires; which may be done according to the principle of images, by still supposing the wires to extend indefinitely in both directions, and in the beginning to be symmetrically electrified with contrary electricities on the two sides of O. To express the new condition (7), a form of integral, investigated in a communication to the Royal Society ('Proceedings,' May 10, 1855, p. 385), may be used; and we thus have for the integrals of equations (3),

$$\left.\begin{aligned}\vartheta &= \sqrt{\frac{kc}{4(1+f)\pi}}\Big[t^{-\frac{1}{2}}\int_0^{\infty}\frac{1}{2}\{\phi_1(\xi)+\phi(\xi)\}\{\varepsilon^{-\frac{kc(\xi-x)^2}{4(1+f)t}}-\varepsilon^{-\frac{kc(\xi+x)^2}{4(1+f)t}}\}d\xi \\ &\qquad + x\int_0^t\frac{1}{2}\{\psi_1(\theta)+\psi_2(\theta)\}\varepsilon^{-\frac{kcx^2}{4(1+f)(t-\theta)}}\frac{d\theta}{(t-\theta)^{\frac{3}{2}}}\Big] \\ \omega &= \sqrt{\frac{kc}{4(1-f)\pi}}\Big[t^{-\frac{1}{2}}\int_0^{\infty}\frac{1}{2}\{\phi_1(\xi)\phi_2-(\xi)\}\{\varepsilon^{-\frac{kc(\xi-x)^2}{4(1-f)t}}-\varepsilon^{-\frac{kc(\xi+x)^2}{4(1-f)t}}\}d\xi \\ &\qquad + x\int_0^t\frac{1}{2}\{\psi_1(\theta)+\psi_2(\theta)\}\varepsilon^{-\frac{kcx^2}{4(1-f)(t-\theta)}}\frac{d\theta}{(t-\theta)^{\frac{3}{2}}}\Big]\end{aligned}\right\} \quad (8).$$

Lastly, instead of the cable extending indefinitely on one side of the end O, let it be actually limited at a point E. If the ends of the two wires at E be subjected to electric action, so as to make each vary arbitrarily with the time, the new conditions to be satisfied, in addition to the others, (5) and (7), will be

$$\left.\begin{array}{ll} & v_1=\chi_1(t) \text{ and } v_2=\chi_2(t) \\ \text{when} & x=a \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9),$$

if χ_1 and χ_2 denote two arbitrary functions, and a the length OE. Or, on the other hand, if they be connected together, so that a current may go from O to E along one and return along the other, the new conditions will be

$$\left.\begin{array}{ll} & v_1-v_2=0,\ \dfrac{d(v_1+v_2)}{dx}=0, \\ \text{when} & x=a \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)'.$$

Either of these requirements may be fulfilled in an obvious way by the *method of successive images*, and we so obtain the following respective solutions:—

$$\left.\begin{array}{l} \sqrt{\dfrac{kc}{4(1+f)\pi}}\Big[t^{-\frac{1}{2}}\displaystyle\int_0^a \frac{1}{2}\{\varphi_1(\xi)+\varphi_2(\xi)\}F_{(f)}(\xi,t)d\xi \\ \dfrac{d\theta}{(t-\theta)^{\frac{3}{2}}}\Big[\dfrac{1}{2}\{\psi_1(\theta)+\psi_2(\theta)\}\mathfrak{F}_{(f)}(x,t-\theta)+\dfrac{1}{2}\{\chi_1(\theta)+\chi_2(\theta)\}\mathfrak{F}(a-x,t-\theta)\Big]\Big] \\ \sqrt{\dfrac{kc}{4(1-f)\pi}}\Big[t^{-\frac{1}{2}}\displaystyle\int_0^a \frac{1}{2}\{\varphi_1(\xi)-\varphi_2(\xi)\}F_{(-f)}(\xi,t)d\xi \\ \dfrac{d\theta}{(t-\theta)^{\frac{3}{2}}}\Big[\dfrac{1}{2}\{\psi_1(\theta)-\psi_2(\theta)\}\mathfrak{F}_{(-f)}(x,t-\theta)+\dfrac{1}{2}\{\chi_1(\theta)-\chi_2(\theta)\}\mathfrak{F}_{(-f)}(a-x,t-\theta)\Big]\Big] \end{array}\right\}(10)$$

$$\left.\begin{array}{l} \vartheta=\sqrt{\dfrac{kc}{4(1+f)\pi}}\Big[t^{-\frac{1}{2}}\displaystyle\int_0^a \frac{1}{2}\{\varphi_1(\xi)+\varphi_2(\xi)\}E_{(f)}(\xi,t)d\xi \\ \qquad\qquad +\displaystyle\int_0^t \frac{1}{2}\{\psi_1(\theta)+\psi_2(\theta)\}\mathfrak{E}_{(f)}(x,t-\theta)\frac{d\theta}{(t-\theta)^{\frac{3}{2}}}\Big] \\ \omega=\sqrt{\dfrac{kc}{4(1-f)\pi}}\Big[t^{-\frac{1}{2}}\displaystyle\int_0^a \frac{1}{2}\{\varphi_1(\xi)-\varphi_2(\xi)\}F_{(-f)}(\xi,t)d\xi \\ \qquad\qquad +\displaystyle\int_0^t \frac{1}{2}\{\psi_1(\theta)-\psi_2(\theta)\}\mathfrak{F}_{(-f)}(x,t-\theta)\frac{d\theta}{(t-\theta)^{\frac{3}{2}}}\Big] \end{array}\right\}(10)',$$

where F, $\mathfrak{F}$, E, $\mathfrak{E}$ denote for brevity the following functions:—

$$\left.\begin{aligned}
F_{(f)}(\xi,t) &= \sum_{i=-\infty}^{i=\infty}\left\{\varepsilon^{-\frac{kc(x+2ia-\xi)^2}{4(1+f)t}}-\varepsilon^{-\frac{kc(x+2ia+\xi)^2}{4(1+f)t}}\right\}\\
\mathfrak{F}_{(f)}(x,t-\theta) &= \sum_{i=-\infty}^{i=\infty}(x+2ia)\varepsilon^{-\frac{kc(x+2ia)^2}{4(1+f)(t-\theta)}} = \frac{(1+f)(t-\theta)}{kc}\{F_{(f)}(\xi,t-\theta)\div\xi\}_{\xi=0}\\
E_{(f)}(\xi,t) &= \sum_{-\infty}^{\infty}(-1)^i\left\{\varepsilon^{-\frac{kc(x+2ia-\xi)^2}{4(1+f)t}}-\varepsilon^{-\frac{kc(x+2ia+\xi)^2}{4(1+f)t}}\right\}\\
\mathfrak{E}_{(f)}(x,t-\theta) &= \sum_{-\infty}^{\infty}(-1)^i(x+2ia)\varepsilon^{-\frac{kc(x+2ia)^2}{4(1+f)(t-\theta)}} = \frac{(1+f)(t-\theta)}{kc}\{E_{(f)}(\xi,t-\theta)\div\xi\}_{\xi=0}
\end{aligned}\right\}$$

Each of the functions F and E is clearly the difference between two periodical functions of $(\xi-x)$ and $(\xi+x)$; and each of the functions $\mathfrak{F}$ and $\mathfrak{E}$ is a periodical function of x simply. The expressions for these four functions, obtained by the ordinary formulæ for the expression of periodical functions in trigonometrical series, are as follows:—

$$\left.\begin{aligned}
F_{(f)}(\xi,t) &= \frac{2}{a}\sqrt{\frac{4(1+f)\pi t}{kc}}\sum_{i=1}^{i=\infty}\varepsilon^{-\frac{i^2\pi^2(1+f)t}{a^2kc}}\sin\frac{i\pi x}{a}\sin\frac{i\pi\xi}{a}\\
\mathfrak{F}_{(f)}(x,t-\theta) &= \frac{1}{2a^2}\left[\frac{4(1+f)\pi(t-\theta)}{kc}\right]^{\frac{3}{2}}\sum_1^{\infty}i\varepsilon^{-\frac{i^2\pi^2(1+f)(t-\theta)}{a^2kc}}\sin\frac{i\pi x}{a}\\
E_{(f)}(\xi,t) &= \frac{2}{a}\sqrt{\frac{4(1+f)\pi t}{kc}}\sum_1^{\infty}\varepsilon^{-\frac{(2i-1)^2\pi^2(1+f)t}{4a^2kc}}\sin\frac{(2i-1)\pi x}{2a}\sin\frac{(2i-1)\pi\xi}{2a}\\
\mathfrak{E}_{(f)}(x,t-\theta) &= \frac{1}{4a^2}\left[\frac{4(1+f)\pi(t-\theta)}{kc}\right]^{\frac{3}{2}}\sum_1^{\infty}(2i-1)\varepsilon^{-\frac{(2i-1)^2\pi^2(1+f)(t-\theta)}{4a^2kc}}\sin\frac{(2i-1)\pi x}{2a}
\end{aligned}\right\}(12).$$

Either (11) or (12) may be used to obtain explicit expressions for the solutions (10) and (10)′, in convergent series; but of the series so obtained, (11) converge very rapidly and (12) very slowly when t is small; and, on the contrary, (11) very slowly and (12) very rapidly when t is large. It is satisfactory, that, as t increases, the first set of series (11) do not cease to be, before the second set (12) become, convergent enough to be extremely convenient for practical computation.

The solutions obtained by using (12), in (10) and (10)′, are the same as would have been found by applying Fourier's ordinary process to derive from the elementary integral $\varepsilon^{-mt}\sin nx$ the effects of the initial arbitrary electrification of the wires, and employing a method given by Professor Stokes* to express the effects of the variations arbitrarily applied at the free ends of the wires.

Case II.—*Three-wire Cable.*

The equations of mutual influence between the wires may be clearly put under the forms

$$cv_1=q_1+f(q_2+q_3), \quad cv_2=q_2+f(q_3+q_1), \quad cv_3=q_3+f(q_1+q_2);$$

and the equations of electrical motion along them are then as follows:—

$$kc\frac{dq_1}{dt}=\frac{d^2q_1}{dx^2}+f\left(\frac{d^2q_2}{dx^2}+\frac{d^2q_3}{dx^2}\right), \quad kc\frac{dq_2}{dt}=\frac{d^2q_2}{dx^2}+f\left(\frac{d^2q_3}{dx^2}+\frac{d^2q_1}{dx^2}\right),$$
$$kc\frac{dq_3}{dt}=\frac{d^2q_3}{dx^2}+f\left(\frac{d^2q_1}{dx^2}+\frac{d^2q_2}{dx^2}\right).$$

If we assume

$$\sigma=q_1+q_2+q_3, \quad \omega_1=2q_1-q_2-q_3, \quad \omega_2=2q_2-q_3-q_1, \quad \omega_3=2q_3-q_1-q_2,$$

which give

$$q_1=\frac{1}{3}\sigma+\omega_1, \quad q_2=\frac{1}{3}\sigma+\omega_2, \quad q_3=\frac{1}{3}\sigma+\omega_3,$$

and require that $\omega_1+\omega_2+\omega_3=0$, we find by addition and subtraction, among the equations of conduction,

$$kc\frac{d\sigma}{dt}=(1+2f)\frac{d^2\sigma}{dx^2}$$

and

$$kc\frac{d\omega}{dt}=(1-f)\frac{d^2\omega}{dx^2},$$

where for ω may be substituted either ω_1, ω_2, or ω_3.

Case III.—*Four-wire Cable.*

The equations of mutual influence being

$$cv_1=q_1+f(q_2+q_4)+gq_3,$$

and other four symmetrical with this; and the equations of motion,

$$kc\frac{dq_1}{dt}=\frac{d^2q_1}{dx^2}+f\left(\frac{d^2q_2}{dx^2}+\frac{d^2q_4}{dx^2}\right)+g\frac{d^2q_3}{dx^2},$$

&c. &c. &c.,

* See Cambridge Phil. Trans. vol. viii. p. 533, "On the Critical Values of the sums of Periodic Series."

we may assume

$$q_1+q_2+q_3+q_4=\sigma, \quad q_1-q_3=\omega_1,$$
$$q_1-q_2+q_3-q_4=\vartheta, \quad q_2-q_4=\omega_2;$$

which give

$$q_1=\frac{1}{4}(\sigma+\vartheta+2\omega_1); \quad q_2=\frac{1}{4}(\sigma-\vartheta+2\omega_2);$$
$$q_3=\frac{1}{4}(\sigma+\vartheta-2\omega_1); \quad q_4=\frac{1}{4}(\sigma-\vartheta-2\omega_2);$$

and we find from the equations of conduction,

$$kc\frac{d\sigma}{dt}=(1+2f+g)\frac{d^2\sigma}{dx^2}; \; kc\frac{d\vartheta}{dt}=(1-2f+g)\frac{d^2\vartheta}{dx^2}; \; kc\frac{d\omega}{dt}=(1-g)\frac{d^2\omega}{dx^2}.$$

CASE IV.—*Cable of six wires symmetrically arranged.*

Equations of mutual influence,

$$cv_1=q_1+f(q_2+q_6)+g(q_3+q_5)+hq_4$$
&c. &c. &c.

Equations of conduction,

$$kc\frac{dq_1}{dt}=\frac{d^2q_1}{dx^2}+f\left(\frac{d^2q_2}{dx^2}+\frac{d^2q_6}{dx^2}\right)+g\left(\frac{d^2q_3}{dx^2}+\frac{d^2q_5}{dx^2}\right)+h\frac{d^2q_4}{dx^2}.$$

Then assuming

$$q_1+q_2+q_3+q_4+q_5+q_6=\sigma$$
$$q_1-q_4+q_3-q_6+q_5-q_2=\vartheta$$
$$3(q_1+q_4)-\sigma=\omega_1; \quad 3(q_3+q_6)-\sigma=\omega_2; \quad 3(q_5+q_2)-\sigma=\omega_3;$$
$$3(q_1-q_4)-\vartheta=\rho_1; \quad 3(q_3-q_6)-\vartheta=\rho_2; \quad 3(q_5-q_2)-\vartheta=\rho_3;$$

which require that

$$\omega_1+\omega_2+\omega_3=0, \text{ and } \rho_1+\rho_2+\rho_3=0;$$

we have

$$kc\frac{d\sigma}{dt}=[1+2(f+g)+h]\frac{d^2\sigma}{dx^2}; \quad kc\frac{d\vartheta}{dt}=[1-2(f-g)-h]\frac{d^2\vartheta}{dx^2};$$
$$kc\frac{d\omega}{dt}=[1-(f+g)+h]\frac{d^2\omega}{dx^2}; \quad kc\frac{d\rho}{dt}=[1+(f-g)-h]\frac{d^2\rho}{dx^2}.$$

These equations, integrated by the usual process to fulfil the prescribed conditions, determine σ, ϑ, ω_1, ω_2, ω_3, ρ_1, ρ_2, ρ_3; and we then have, for the solution of the problem,

$$q_1=\frac{1}{6}(\sigma+\vartheta+\omega_1+\rho_1); \; q_3=\frac{1}{6}(\sigma+\vartheta+\omega_2+\rho_2); \; q_5=\frac{1}{6}(\sigma+\vartheta+\omega_3+\rho_3);$$
$$q_4=\frac{1}{6}(\sigma-\vartheta+\omega_1-\rho_1); \; q_6=\frac{1}{6}(\sigma-\vartheta+\omega_2-\rho_2); \; q_2=\frac{1}{6}(\sigma-\vartheta+\omega_3-\rho_3).$$