The Zeno's paradox in quantum theory

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We seek a quantum-theoretic expression for the probability that an unstable particle prepared initially in a well defined state ρ will be found to decay *sometime during a given interval*. It is argued that probabilities like this which pertain to continuous monitoring possess operational meaning. A simple natural approach to this problem leads to the conclusion that an unstable particle which is continuously observed to see whether it decays will never be found to decay! Since recording the track of an unstable particle (which can be distinguished from its decay products) *approximately* realizes such continuous observations, the above conclusion seems to pose a paradox which we call Zeno's paradox in quantum theory. The relation of this result to that of some previous works and its implications and possible resolutions are briefly discussed. The mathematical transcription of the above-mentioned conclusion is a structure theorem concerning semigroups. Although special cases of this theorem are known, the general formulation and the proof given here are believed to be new. We also note that the known "no-go" theorem concerning the semigroup law for the reduced evolution of any physical system (including decaying systems) is subsumed under our theorem as a direct corollary.

1. INTRODUCTION

The object of this paper is to discuss a seemingly paradoxical result in quantum theory concerning temporal evolution of a dynamical system under continuous observation during a period of time. For reasons that will become clear shortly we call this complex of deductions *Zeno's paradox in quantum theory.*

Let us consider schematically the theory of an unstable quantum system. Naturally the states corresponding to the decay products also should be included in the space of all states which we take to be a Hilbert space $\mathcal{H}$. Let us denote the (orthogonal) projection onto the subspace spanned by the *undecayed* (unstable) states of the system by E. This projection E thus represents the observable that corresponds to the "yes-no experiment" for determining whether the system is in an *undecayed* state or in a *decayed* state. The evolution in time of the states of the total system will be described by a unitary group $\mathrm{U}(t) \equiv \exp(-iHt)$ labeled by the real time parameter t. In this setting the quantity

$$q(t) = \mathrm{Tr}[\rho \mathrm{U}^*(t) E \mathrm{U}(t)] \tag{1}$$

is interpreted as the probability, *at the instant t*, for finding the system undecayed when at time 0 it was prepared in the state ρ. Correspondingly, the probability that, *at the instant t*, the system will be found to have decayed is the complementary quantity

$$p(t) = 1 - q(t) = \mathrm{Tr}[\rho \mathrm{U}^*(t) E^\perp \mathrm{U}(t)], \quad E^\perp = I - E. \tag{2}$$

All these are, of course, standard.

Quantum theory, in fact, provides an unambiguous algorithm for computing the probability distributions of *time*, given the knowledge of the initial state of the of) *time*, given the knowledge of the initial state of the same system and its law of time evolution. Expressions (1) and (2) are only particular instances of this well-known algorithm.

In contrast to the above-mentioned probabilities which refer to a specified instant of time we may consider the following probabilities for which quantum theory has no ready expressions:

(1) The probability that the system prepared in the undecayed state ρ at time 0 is found to decay *sometime during the interval* $\Delta = [0, t]$. We denote this by $P(0, t; \rho)$.

(2) The probability $Q(0, t; \rho)$ that no decay is found *throughout the interval* Δ when the initial state of the system was known to be ρ.

(3) The probability that the system prepared initially in the state ρ will be found to be undecayed *throughout* $[0, t_1] \equiv \Delta$, but found to decay *sometime* during the subsequent period $[t_1, t] \equiv \Delta_2$, $0 < t_1 < t$. We denote this by $R(0, t_1, t; \rho)$.

It is important to distinguish the probabilities $Q(0, t; \rho)$ from $q(t)$ since there is the temptation to identify them [and hence also $P(0, t; \rho)$ with $p(t)$].[1] The probability $q(t)$, however, refers to outcomes of measurement of E at the time t, *the system being left unobserved* after the initial state preparation until t.

The operational meaning (if any!) of the probabilities P, Q, R on the other hand is to be found in terms of the outcomes of continuously ongoing measurement of E during the entire interval of time Δ. The notion of such continuously ongoing observations (or, equivalently, measurements) is obviously an idealization.

We may consider the process of continuing observation as the limiting case of successions of (practically) instantaneous measurements (of E) as the intervals between successive measurements approach zero. Since there does not seem to be any principle, internal to quantum theory, that forbids the duration of a single measurement or the dead time between successive measurements from being arbitrarily small, the process of continuous observation seems to be an admissible process in quantum theory.

It may be argued, however, that what are measurable or not are governed not only by the fundamental principles of quantum theory but also by the actual constituents of the real world and their interactions.

The concept of continuous observation would indeed be bereft of any physical meaning *if* it could be established that the fundamental constituents of the real world and the interaction between them are such as to exclude the possibility of arbitrarily frequent observations. But, on the one hand, we cannot claim as final our present knowledge of the constituents and interactions of the real world. On the other hand, to agree that there is a limitation on the frequency of observation amounts to claiming the existence of an elementary and indivisible unit of time. Though the existence of an elementary interval of time is an exciting possibility, it is not part of the currently accepted and tested physical theories.

We, thus, feel that the notion of continuous observation should be accepted, at least for the present, as physically meaningful and quantum theory should be pressed to yield an answer to questions relating the probabilities pertaining to such observations.

Continuous observation processes seem to be realized in practice also, at least approximately, by the tracks of unstable charged particles in bubble chambers and other detecting media. The observation of the track amounts practically to a more or less continuous monitoring of the existence of the unstable particle and thus a measurement of E during the period of the particle's flight through the detection chamber. We are therefore led to accept as operationally meaningful the $P(0,t;\rho)$, $Q(0,t;\rho)$, and $R(0,t_1,t;\rho)$. To be a *complete* theory, quantum theory must provide an algorithm for computing these probabilities.

In the next section we describe what appears to be the natural approach to determining quantum-theoretic expressions for these probabilities. Our investigation leads to the paradoxical result mentioned at the beginning of this section: An unstable particle observed continuously whether it has decayed or not will *never* be found to decay! Since this evokes the famous paradox of Zeno denying the possibility of motion to a flying arrow, we call this result the Zeno's paradox in quantum theory

In fact, if E is taken to be the projection to the set of localized states of a particle (or, a quantum arrow) in a given region D of space, then one concludes that the particle will never be found to arrive in a disjoint region D' provided it is *continuously observed* whether it has entered D' or not: The "arrow" cannot move to where it is not!

This result acquires an even more picturesque and paradoxical formulation when it is applied to the "hellish contraption" considered in the Schrödinger's cat paradox.[2] It may be recalled that the contraption consists of an unstable (quantum) particle placed in a box equipped with an efficient counter and a cat inside a steel chamber. If the particle decays, the counter triggers and, in its turn, activates a tiny hammer which breaks a container of cyanide in the steel chamber. Monitoring the vital functions of the cat amounts to observing if the particle has decayed or not. In view of the Zeno's paradox formulated above, should we conclude that the particle will never decay? Will the cat escape the cruel death awaiting it, against which it has no defense, provided its vital signs are constantly watched with loving care?

The mathematical transcription of Zeno's paradox is a structure theorem concerning a class of strongly continuous semigroups. This theorem is formulated and proved in Sec. 3 of this paper and may possess some intrinsic interest apart from its application in the present context of a theory of continuous observation. As a byproduct of this investigation we find that the known result[3] concerning the incompatibility of the semiboundedness of the Hamiltonian H with the requirement that $E\exp(iHt)E$ form a strictly contractive semigroup can be subsumed under the above-mentioned theorem as one of its direct corollaries.

Some of the implications and possible resolutions of the quantum Zeno's paradox will be briefly discussed in the concluding section of the paper.

Finally it may be mentioned that the conclusion called here the Zeno's paradox in quantum theory has been noted in some previous works,[4–6] but the present analysis of the problem is carried out in a more general and mathematically rigorous setting than the previous works. This, we feel, is not merely a dispensable luxury, but is necessary to locate the precise assumptions on which the "Zeno's paradox" rests.

2. QUANTUM THEORETICAL EXPRESSIONS FOR $P(0,t;\rho)$ AND RELATED PROBABILITIES

The three probability functions P, Q, R introduced in the previous section relate to the results of continuous observation throughout an interval of time. By their very definitions they must obey the relations

$$P(0,t;\rho)+Q(0,t;\rho)=1$$

and

$$R(0,t_1,t;\rho)=Q(0,t_1;\rho)\,P(0,t-t_1;\rho_1),$$

where ρ_1 is the state in which the system (prepared initially in the state ρ) finds itself at t_1 after being continuously observed *and* found to be undecayed throughout $[0,t_1]$. We may therefore concentrate our attention on calculating Q and ρ_1.

We start with the system in the state ρ and make a series of $n+1$ measurements, which are idealized to be instantaneous, at times 0, t/n, $2t/n,\ldots,(n-1)t/n$, and t. We seek the probability $Q(\Delta,n;\rho)$ that it be found undecayed in *each* of these measurements. It is natural to assume that $Q(\Delta;\rho)\equiv Q(0,t;\rho)$ can be evaluated as the limit of $Q(\Delta,n;\rho)$ when $n\to\infty$, provided the limit exists.

Let us denote by $\rho(n,t)$ the state in which the system finds itself after the $(n+1)$ measurements at 0, t/n, $2t/n,\ldots,t$ *and* being found to be undecayed in each of these measurements. Now, according to the orthodox theory of measurement, if a measurement of E on the system is carried out yielding the result "yes" (that is, "undecayed"), then the state of the system collapses to a new (unnormalized!) state ρ' of the form

$$\rho'=\sum_j A_j^*\rho A_j \tag{3}$$

with

$$\sum_j A_j^* A_j=E. \tag{4}$$

The collapsed state ρ' given by (3) is, in general, not uniquely determined by the measured observable E and the observed outcome but depends also on the details of the measuring apparatus. This circumstance is reflected in the nonuniqueness of the operators A_j satisfying (4).

The mapping (3) of the density matrices is very closely related to the "completely positive maps" defined by

$$\rho \to \sum_\alpha V_\alpha \rho V_\alpha^* = \mathcal{A}(V_\alpha)\rho,$$

$$\sum_\alpha V_\alpha^* V_\alpha = 1.$$

The "state collapse" caused by "nonselective" measurements of E is described by such maps. They will be considered in a future publication in the context of repeated and continuous nonselective measurements.

Quantum theory envisages also the possibility of *ideal* measurements under which the collapse of the state proceeds according to the simple law

$$\rho \to \rho' = E\rho E \tag{5}$$

when the measurement of E on the state ρ yields the result "undecayed."

The considerations of this paper will be restricted to such ideal measurements only, since in such cases we can exploit the positive definiteness of the Hamiltonian in a direct manner. If we were to consider the more general collapses (3) we would have to proceed more indirectly using the von Neumann–Liouville generator which is however not positive definite. The study of the probabilities $Q(\Delta, \rho)$, etc. would then involve new technical problems obscuring the essentials of Zeno's paradox. We plan to present the study of the more general situation in a subsequent paper.[7]

Accordingly, to determine $\rho(n, t)$ we allow the system to collapse at each measurement according to (5) but at the intervening time intervals it undergoes the usual Schrödinger time development. The (unnormalized) state $\rho(n, t)$ is then easily seen to be

$$\rho(n, t) = T_n(t)\,\rho\, T_n^*(t), \tag{6}$$

where

$$T_n(t) = [E\mathrm{U}(t/n)\,E]^n \equiv [E\exp(-iHt/n)\,E]^n. \tag{7}$$

Moreover, it is also easy to show that the standard interpretation of the quantum theoretical formalism entails the formula

$$Q(n, \Delta; \rho) = \mathrm{Tr}[T_n(t)\,\rho\, T_n^*(t)]. \tag{8}$$

In fact, (8) is a special case of a more general formula for the probability connections between several successive observations.[8] It is important, however, to bear in mind that the general formula discussed in[8] [and, *a fortiori*, formula (8)] holds only under the assumption that the successive measurements under consideration are *ideal* in the sense described above. For nonideal successive measurements these formulas do not yield correct probability connections.

Returning to ideal measurements we have to proceed to the limit for $n \to \infty$. We define

$$\rho(t) = \operatorname*{s-lim}_{n\to\infty} \rho(n, t), \tag{9}$$

$$Q(\Delta; \rho) = \lim_{n\to\infty} Q(\Delta, n; \rho), \tag{10}$$

provided the limits on the right-hand side exist. Hence, if the limit

$$\operatorname*{s-lim}_{n\to\infty} T_n(t) \equiv \operatorname*{s-lim}_{n\to\infty} [E\mathrm{U}(t/n)\,E]^n = T(t)$$

exists for $t \geq 0$, then we may make the identification

$$\rho(t) = \{\mathrm{Tr}[T(t)\,\rho T^*(t)]\}^{-1} \cdot T(t)\,\rho T^*(t) \tag{11}$$

for the resultant (normalized) state obtained as a result of *continuous observation* and verification that the system remained *undecayed throughout the interval*. The probability $Q(\Delta; \rho)$ for this outcome is given by

$$Q(\Delta; \rho) = \lim_{n\to\infty} \mathrm{Tr}[T_n(t)\,\rho T_n^*(t)] = \mathrm{Tr}[\rho T^*(t)\, T(t)]. \tag{12}$$

Once $Q(\Delta; \rho)$ is obtained in this manner we may calculate $P(\Delta; \rho)$ to be

$$P(\Delta; \rho) = \mathrm{Tr}[\rho(I - T^*(t))\, T(t)]. \tag{13}$$

For a given group $\mathrm{U}(t)$ of time-evolution the existence of the operator $T(t)$ for $t \geq 0$ imposes a nontrivial restriction on the projection E. This restriction may be viewed as a *necessary* condition in order that the observable represented by E admits a *continuous ideal* measurement.

It is known[9] that the operators $T(t)$ (if they exist) form a strongly continuous semigroup for $t > 0$. The continuity of $T(t)$ at $t = 0$ does not generally follow from the existence of $T(t)$, but on physical grounds, we shall *assume it*;

$$\operatorname*{s-lim}_{t\to 0_+} T(t) = E. \tag{14}$$

This condition expresses the essentially desirable requirement that the probability $Q(\Delta; \rho)$ given by (12) approaches the probability $\mathrm{Tr}(\rho E)$ as $t \to 0_+$ that the system is undecayed initially.

To prove the existence of $T(t)$ and its continuity at the origin, (14), in specific examples of physical interest poses nontrivial mathematical problems. We hope to consider these in a subsequent paper.

3. ZENO'S PARADOX IN QUANTUM THEORY

In the preceding section we arrived at formula (13) for the probability $P(\Delta; \rho)$ that the system prepared initially in the undecayed state ρ will be observed to decay *sometime* during the interval $\Delta = [0, t]$. Despite the natural derivation of (13) we now show that the probability $P(\Delta; \rho)$ vanishes for all finite intervals Δ provided that the initial state was undecayed,

$$\mathrm{Tr}(\rho E) = 1. \tag{15}$$

We are thus led to the paradoxical conclusion that an unstable particle will not decay as long as it is kept under continuous observation as to whether it decays or not. The mathematical transcription of this statement is the following theorem.

Theorem 1: Let $\mathrm{U}(t) \equiv \exp(-iHt)$, t real, designate a strongly continuous one-parameter group of unitary operators in the (separable) Hilbert space $\mathcal{H}$. Let E denote an orthogonal projection in $\mathcal{H}$. Assume that:

(i) The self-adjoint generator H of the group $\mathrm{U}(t)$ is *semibounded*.

(ii) There exists an (antiunitary) operator θ such that

$$\theta E \theta^{-1} = E,$$

$$\theta \mathrm{U}(t)\, \theta^{-1} = \mathrm{U}(-t) \quad \text{for all } t.$$

(iii) $\text{s-lim}_{n\to\infty} [E\mathrm{U}(t/n)\, E]^n \equiv T(t)$ exists for all $t \geq 0$.

(iv) $\text{s-lim}_{n\to 0_+} T(t) = E$.

Then $\text{s-lim}_{n\to\infty}[E\mathrm{U}(t/n)\, E]^n \equiv T(t)$ exists for *all* real t and possesses the following properties:

(a) The function $t \to T(t)$ is strongly continuous and for *all* real t and s satisfies the semigroup law:

$$T(t)\, T(s) = T(t+s),$$

(b) and

$$T^*(t) = T(-t).$$

Remarks: (1) The conclusions of the theorem imply the relation:

$$T^*(t)\, T(t) = E \quad \text{for } \textit{all real } t \tag{16}$$

so that $P(\Delta;\rho) = \mathrm{Tr}[\rho(I-E)] = 0$ for all ρ satisfying (15).

(2) With θ interpreted as the time-reversal (or CPT) operation, the assumption (ii) of the theorem turns out to be only a weak version of T or CPT invariance of the theory. Moreover it should be noted that assumption (iii) is used only once in the proof for concluding the existence of the strong limit (iii) for $t < 0$ as well.

(3) It is easy to give a relatively elementary proof of the theorem under the additional assumption that E is a one-dimensional projection onto a vector in the domain of H. The theorem is also known to hold in the special case that $H = -\nabla^2$ and E is given by

$$(E\psi)(x) = \chi(x)\, \psi(x), \quad \psi \in L^2(R^3),$$

where χ is the characteristic function of a (suitably smooth) region of R^3.[10] Theorem 1 generalizes this result to arbitrary semibounded H and arbitrary projection E.

(4) The semiboundedness of H is necessary: Consider the following counterexample. Let $V(t)$ be the operator family

$$(V(t)\,\psi) = \psi(x-t), \quad \psi \in L^2(R).$$

Let E be defined by

$$(E\psi)(x) = \begin{cases} \psi(x) & x \leq 0, \\ 0 & x > 0. \end{cases}$$

It is then easy to verify that

$$E\,V(t)\,E\,V(s)\,E = E\,V(t+s)\,E$$

for all $t, s \geq 0$, so that

$$\text{s-}\lim_{n\to\infty} [E\,V(t/n)\,E]^n = E\,V(t)\,E = T(t)$$

for all $t \geq 0$, but

$$\begin{aligned} T^*(t)\, T(t) &= E\,V(t^*)\,E\,V(t)\,E \\ &= V^*(t)\,E\,V(t) \neq E \quad \text{for all } t > 0. \end{aligned}$$

Thus the conclusion of the theorem is violated, though the assumptions in its formulation except the semiboundedness of the self-adjoint generator $V(t)$ are met. [Strictly speaking, assumption (ii) about the existence of θ is also not satisfied, but it was necessary only to prove the existence of $T(-t)$ and $T(-t)$ is trivially verified to exist in the present example.]

We now turn to the

Proof of Theorem 1: The existence of

$$T(t) \equiv \text{s-}\lim_{n\to\infty} [E\mathrm{U}(t/n)\, E]^n \tag{17}$$

for *all* real t follows immediately from the assumed existence of $T(t)$ for *positive* t and assumption (ii). In fact for $t \geq 0$

$$\begin{aligned} T(-t) &\equiv \text{s-}\lim_{n\to\infty} [E\mathrm{U}(-t/n)\, E]^n \\ &= \text{s-}\lim_{n\to\infty} \theta[E\mathrm{U}(t/n)\, E]^n\, \theta^{-1} \\ &= \theta\, T(t)\, \theta^{-1}. \end{aligned} \tag{18}$$

To prove assertion (b) we observe that

$$\begin{aligned} [E\mathrm{U}(-t/n)\, E]^n &= \{[E\mathrm{U}(t/n)\, E]^n\}^* \\ &\to T^*(t) \quad \text{weakly as } n\to\infty. \end{aligned}$$

On the other hand,

$$[E\mathrm{U}(-t/n)\, E]^n \to T(-t) \quad \text{strongly as } n\to\infty.$$

The assertion (b) follows immediately.

It remains to prove (a), especially assertion (16). Let us make a slight notational change and write

$$W(t) \equiv T(-t) = \text{s-}\lim_{n\to\infty} [E\exp(iHt/n)\, E]^n. \tag{19}$$

The statement (a) can be transcribed into a corresponding statement on $W(t)$. We shall also assume, without loss of generality, that H is self-adjoint and positive.

For convenience in exposition we shall break up the proof of statement (a) into the following three lemmas:

Lemma 1: Let H be a positive self-adjoint operator of the Hilbert space $\mathcal{H}$ and let $F_n(z)$ be the operator-valued function defined by

$$F_n(z) \equiv [E\exp(iHz/n)\, E]^n. \tag{20}$$

Then

(1) $F_n(z)$ is defined and continuous (in the strong operator topology) for all complex z with $\mathrm{Im}z \geq 0$ and it is *holomorphic* in the open upper half-plane $\mathrm{Im}z > 0$.

(2) The function $F_n(z)$ has the integral representation

$$F_n(z) = \frac{(z+i)^2}{2\pi i} \int_{-\infty}^{\infty} \frac{F_n(t)}{(t+i)^2(t-z)}\, dt, \quad \mathrm{Im}z > 0. \tag{21}$$

(3)

$$\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{F_n(t)\,dt}{(t+i)^2(t-z)}=0,\quad \operatorname{Im}z<0. \tag{22}$$

Proof of Lemma 1: The assertion (1) follows from the positive self-adjointness of H and its proof is standard. To prove assertion (2) we start with Cauchy's integral formula for the function $F_n(z)/(z+i)^2$ which is holomorphic in the open upper half-plane,

$$\frac{F_n(z)}{(z+i)^2}=\frac{1}{2\pi i}\oint_C\frac{F_n(z')}{(z'+i)^2(z'-z)}\,dz',\quad \operatorname{Im}z>0,$$

where C is *any* simple *closed* rectifiable contour enclosing the point z and contained entirely in the open upper half-plane. A similar integral representation holds of course for the holomorphic function $F_n(z)$ itself. But with the choice we have made the integrand vanishes faster than $|z'|^{-1}$ as $|z'|\to\infty$. Hence if we choose the closed contour C to be the axis running from $-\infty+i\epsilon$ to $+\infty+i\epsilon$ and an infinite semicircle we could rewrite the contour integral as an open line integral

$$F_n(z)=\frac{(z+i)^2}{2\pi i}\int_{-\infty}^{\infty}\frac{F_n(t+i\epsilon)}{(t+i+i\epsilon)^2(t-z+i\epsilon)},\tag{21$'$}$$

$\operatorname{Im}z>\epsilon>0.$

The (operator) norm of this integrand is dominated by the integrable function

$$(1+t^2)^{-1}\cdot(\operatorname{Im}z-\epsilon_0)^{-1}$$

for all ϵ with $0\leqslant\epsilon<\epsilon_0<\operatorname{Im}z$. Moreover,

$$\operatorname*{s-lim}_{\epsilon\to0_+}F_n(t+i\epsilon)=F_n(t).$$

Hence the conditions for the application of Lebesgue's dominated convergence theorem for operator-valued integrals[11] are met and (21′) goes over to the desired representation (21) in the limit $\epsilon\to0_+$. The relation (22) is similarly obtained from the vanishing of the contour integral

$$\frac{1}{2\pi i}\oint_C\frac{F_n(z')}{(z'+i)^2(z'-z)}\,dz'\quad\text{for}\quad \operatorname{Im}z<0.$$

Lemma 2: With the same notation as in Lemma 1 let us assume that

$$W(t)\equiv\operatorname*{s-lim}_{n\to\infty}F_n(t)\equiv\operatorname*{s-lim}_{n\to\infty}[E\exp(iHt/n)E]^n$$

exists for all real t. Then:

(1) $W(z)\equiv\text{s-lim}_{n\to\infty}F_n(z)$ exists for all z with $\operatorname{Im}z>0$.

(2) The function $W(z)$ is holomorphic in the *open* upper half-plane and satisfies the semigroup composition law

$$W(z_1)\,W(z_2)=W(z_1+z_2).\tag{23}$$

(3) There exists a nonnegative and self-adjoint operator B and a projection G such that

$$GB=BG=B,$$

and

$$W(z)=G\exp(iBz)\,G,\ \operatorname{Im}z>0.\tag{24}$$

Proof of Lemma 2: To prove (1) we start with the representation (21) for $F_n(z)$. By assumption, $W(t)\equiv\text{s-lim}_{n\to\infty}F_n(t)$ for all real t and

$$\left\|\frac{F_n(t)}{(t+i)^2(t-z)}\right\|\leqslant\frac{(1+t^2)^{-1}}{|\operatorname{Im}z|}\quad\text{for all } n.$$

We can therefore apply again the Lebesgue theorem on dominated convergence and conclude that

$$W(z)=\operatorname*{s-lim}_{n\to\infty}F_n(z)$$

exists and has the representation

$$W(z)=\frac{(z+i)^2}{2\pi i}\int_{-\infty}^{\infty}\frac{W(t)}{(t+i)^2(t-z)}\,dt,\quad \operatorname{Im}z>0.\tag{25}$$

From the well known Vitali's theorem[11] we can conclude that $W(z)$ is holomorphic in the open upper half-plane.

To prove the semigroup property of $W(z)$ we show first that this law holds for pure imaginary values,

$$W(is)\,W(it)=W(i(t+s))$$

for all positive t and s.

To this end, first consider the case where t and s are rationally related so that there exist positive integers p,q for which

$$\frac{s+t}{r(p+q)}=\frac{s}{rp}=\frac{t}{rq}$$

for all integers r. For such s,t we can deduce

$$\left[E\exp\left(-H\frac{t+s}{r(p+q)}\right)E\right]^{r(p+q)}=\left[E\exp\left(-H\frac{s}{rp}\right)E\right]^{rp}\left[E\exp\left(-H\frac{t}{rq}\right)E\right]^{rq}$$

which, in the limit $r\to\infty$ yields

$$W(i(t+s))=W(is)\,W(it).$$

Once this is established for rationally related positive s and t by continuity it can be extended to all positive s and t. Since $W(is)$ is holomorphic it is, *a fortiori*, continuous for $s>0$.

To prove assertion (3) we observe that the operators

$$W(is)\equiv\operatorname*{s-lim}_{n\to\infty}[E\exp(-Hs/n)E]^n$$

form a semigroup of self-adjoint operators for $s>0$. According to a well known structure theorem for such semigroups[12] there exists a *self-adjoint nonnegative* operator B and a projection G such that

$$BG=GB=B$$

and

$$W(is)=G\exp(-Bs)\,G,\quad s>0.$$

The function $z\to G\exp(iBz)\,G$ is holomorphic in the open upper half-plane and assumes the same values $W(is)$ as the holomorphic function $W(z)$ for $z=is$ $(s>0)$. The uniqueness of holomorphic functions, then, immediately establishes the representation (24). The semigroup property (23) for z_1,z_2 in the open upper half-plane follows from (24) and the commutativity of G and B.

Lemma 3: Under the assumptions of Lemma 2, we have the weak limit for operators along the real axis

$$\operatorname*{w-lim}_{\eta\to 0_+} W(s+i\eta) \equiv W(s) \quad \text{for almost all real } s. \tag{26}$$

Proof of Lemma 3: To obtain this weak limit let us start from the integral representation (25) rewritten in the form

$$W(s+i\eta)=\frac{(s+i+i\eta)^2}{2\pi i}\int_{-\infty}^{\infty}\frac{W(t)}{(t+i)^2(t-s-i\eta)}\,dt,\quad \eta>0.$$

On the other hand, from (22) and the Lesbegue dominated convergence theorem

$$0=\frac{(s+i+i\eta)^2}{2\pi i}\int_{-\infty}^{\infty}\frac{W(t)}{(t+i)^2(t-s+i\eta)}\,dt,\quad \eta>0.$$

Therefore,

$$W(s+i\eta)=\frac{(s+i+i\eta)^2}{\pi}\int_{-\infty}^{\infty}\frac{W(t)}{(t+i)^2}\,\frac{\eta}{(t-s)^2+\eta^2}\,dt.$$

For any two vectors ψ, ϕ in $\mathcal{H}$ we may write

$$(\psi, W(s+i\eta)\,\phi) = \frac{(s+i+i\eta)^2}{\pi}\int_{-\infty}^{\infty}\frac{(\psi, W(t)\,\phi)}{(t+i)^2}\,\frac{\eta}{(t-s)^2+\eta^2}\,dt.$$

Since the quantity $(\psi, W(t)\,\phi)/(t+i)^2$ considered as a function of t is integrable, it follows that

$$\lim_{\eta\to 0_+}(\psi, W(s+i\eta)\,\phi)=(\psi, W(s)\,\phi) \tag{27}$$

for almost all s.

To complete the assertion of Lemma 3 a technical difficulty is to be resolved. For a given pair ψ, ϕ of vectors, the assertion (29) has been shown to hold for almost all s. The exceptional set (of measure zero) where this result may not hold may appear to depend on the pair ψ, ϕ chosen. To show that there is at most a single *null* set outside which (27) holds for *all* pairs ψ, ϕ we proceed as follows: Let $\mathcal{D}$ be a *countable* dense subset of the separable Hilbert space $\mathcal{H}$ and let $\mathcal{N}$ be the union of the *countable* family of exceptional null sets corresponding to all pairs ψ, ϕ with $\psi\in\mathcal{D}$, $\phi\in\mathcal{D}$. This set $\mathcal{N}$ is a set of measure zero and the weak limit (26) holds everywhere outside this set for ψ, ϕ in $\mathcal{D}$, but then (27) will hold in the complement of $\mathcal{N}$ for all pairs ψ, ϕ not necessarily in $\mathcal{D}$. In fact, writing

$$A(s,\eta)=W(s+i\eta)-W(s)$$

we may obtain

$$(\psi, A(s,\eta)\,\phi)=(\psi-\psi_n, A(s,\eta)\,\phi)+(\psi_n, A(s,n)(\phi-\phi_n)) + (\psi_n, A(s,\eta)\,\phi_n).$$

We see that for s outside the exceptional set $\mathcal{N}$ the first two terms on the right-hand side tend to zero as $n\to\infty$, since we may choose

$$\operatorname*{s-lim}_{n\to\infty}\psi_n=\psi,\quad \operatorname*{s-lim}_{n\to\infty}\phi_n=\phi.$$

The third term, by hypothesis, goes to zero as $\eta\to 0_+$ since ψ_n, ϕ_n are chosen to lie in $\mathcal{D}$. [The proof of this lemma incorporates a suggestion due to K. Sinha.]

The proof of assertion (a) of the theorem may now be easily completed by combining the conclusions of the preceding lemmas,

$$W(s)=\operatorname*{w-lim}_{\eta\to 0_+}W(s+i\eta)=\operatorname*{w-lim}_{\eta\to 0_+}G\exp[iB(s+i\eta)]\,G$$
$$=G\exp(iBs)\,G \quad \text{for almost all real } s. \tag{24'}$$

Thus $W(s)\,W^*(s)=G$ for almost all real s. According to assumption (iv) in the statement of Theorem 1,

$$\operatorname*{w-lim}_{s\to 0_+}W(s)\,W^*(s)=\operatorname*{w-lim}_{s\to 0_+}T(-s)\,T^*(-s)$$
$$=\operatorname*{w-lim}_{s\to 0_+}T^*(s)\,T(s)=E. \tag{28}$$

Thus $G=E$ and we may rewrite (24) in the form

$$W(s)=E\exp(iBs)\,E \text{ for almost all } s,$$
$$EB=BE=B. \tag{29}$$

But we can now strengthen this relation for $W(s)$ to read

$$W(s)=E\exp(iBs)\,E \quad \text{for } all\ s \tag{30}$$

in view of the strong continuity of $W(s)$. Combining (29) and (30) we immediately deduce the validity of assertion (a).

Although not of primary interest for the discussion in this paper, we recall the known result[3] that if H is semibounded, the operators $E\mathrm{U}(t)E$ cannot form a semigroup for $t\geqslant 0$ except in the event of E commuting with $\mathrm{U}(t)$ for all real t. We may subsume this result as a corollary to Theorem 1.

Corollary: Let the self-adjoint operator H be semibounded, let E be an orthogonal projection and let $\mathrm{U}(t)$ stand for $\exp(-iHt)$. If $\{E\mathrm{U}(t)E \mid t\geqslant 0\}$ form a semigroup, then

$$E\mathrm{U}(t)=\mathrm{U}(t)E \quad \text{for all real } t. \tag{31}$$

Proof: The semigroup property for $E\mathrm{U}(t)E$, i.e., the relation

$$E\mathrm{U}(t)E\mathrm{U}(s)E=E\mathrm{U}(t+s)E \quad \text{for } all\ real\ s,t \tag{32}$$

will imply

$$E\mathrm{U}(t)E\mathrm{U}^*(t)E=E \quad \text{for } all\ real\ t \tag{32'}$$

and hence

$$E\leqslant \mathrm{U}(t)E\mathrm{U}^*(t) \quad \text{for } all\ real\ t.$$

Multiplying this equation from the left by $\mathrm{U}(-t)$ and the right by $\mathrm{U}^*(-t)$ will yield

$$\mathrm{U}(-t)E\mathrm{U}^*(-t)\leqslant E \quad \text{for } all\ real\ t.$$

Together these two inequalities imply

$$E=\mathrm{U}(t)E\mathrm{U}^*(t) \quad \text{for } all\ real\ t$$

or, equivalently,

$$E\mathrm{U}(t)=\mathrm{U}(t)E \quad \text{for } all\ real\ t.$$

The proof of the corollary is thus reduced to the proof of (32) or (32′).

Since the operators $E\mathrm{U}(t)E$ are assumed to form a semigroup for $t\geqslant 0$ and $[E\mathrm{U}(t)E]^*=E\mathrm{U}(-t)E$, for all positive integers n and *all real* t we have

$$[E\mathrm{U}(t/n)E]^n=E\mathrm{U}(t)E.$$

Hence

$$\underset{n\to\infty}{\text{s-lim}}[EU(t/n)E]^n = EU(t)E \equiv T(t)$$

exists for *all* real t and all the assumptions of Theorem 1 are verified, except for (ii). But as we have pointed out, this assumption itself was needed for the sole purpose of guaranteeing the existence of $T(t)$ for *all real* t. Thus we can safely conclude that Theorem 1 applies in this case also and hence (32) holds for *all* real t and s.

4. CONCLUDING DISCUSSION

What conclusions must we draw from Zeno's paradox in quantum theory? Is it a curious but innocent mathematical result or does it have something to say about the foundation of quantum theory? Does it, for example, urge us to have a principle in the formulation of quantum theory that forbids the continuous observation of an observable that is not a constant of motion?

The answer to the first two questions appears to depend on whether it is operationally meaningful to seek the probability that the particle makes a transition from a preassigned subspace of states $E\mathcal{H}$ to the orthogonal subspace $E^{\perp}\mathcal{H}$ *sometime* during a given period of time. We have endeavored to present arguments that such probabilities possess operational meaning in terms of the outcome of successive (in the limit, continuous) measurements of an appropriate quantum mechanical observable. If this is accepted, it follows that to be a *complete* theory quantum mechanics must provide an algorithm for computing these probabilities. The quantum Zeno's paradox shows that the seemingly natural approach to this problem discussed in the preceding sections leads to bizarre and physically unacceptable answers. We thus lack a trustworthy quantum-theoretic algorithm for computing such probabilities. Until such a trustworthy algorithm is developed the *completeness* of quantum theory must remain in doubt.

The lack of a trustworthy quantum-theoretic algorithm for probabilities like $P(0,t;\rho)$ is intimately connected with the difficulties involved in defining an operator of "arrival time" (or, more generally, "time of transition") in quantum theory.[4] Let us briefly discuss this problem in the context of the "time of decay" of an unstable particle.

From the definition of $P(0,t;\rho)$ it must have the following properties:

$$\begin{aligned}
&\text{(i) } P(0,t;\rho) \geq 0 \quad \text{for all } t \geq 0\\
&\text{(ii) } P(0,t;\rho) \geq P(0,s;\rho) \quad \text{for } t \geq s\\
&\text{(iii) } P(0,t;\rho) \to 1 \quad \text{for } t \to \infty\\
&\text{(iv) } P(0,t;\rho) \to \mathrm{Tr}[\rho E^{\perp}] \quad \text{for } t \to 0_{+}.
\end{aligned} \tag{33}$$

In addition, $P(0,t;\rho)$ may be assumed to be continuous as a function of t. If we were to succeed in finding a formula

$$P(0,t;\rho) = \mathrm{Tr}[\rho B(0,t)], \tag{34}$$

then the operator $B(0,t)$ would have the following properties:

(i) $B(0,t) \geq 0$,

(ii) $B(0,t) \geq B(0,s), \quad t \geq s$,

(iii) $B(0,t) \to I$ (strongly), $\quad t \to \infty$,

(iv) $B(0,t) \to E^{\perp}$ (strongly), $\quad t \to 0_{+}$,

(v) $B(0,t)$ is a strongly continuous function of t.

The family $B'(0,t) \equiv EB(0,t)E$ restricted to the subspace $E\mathcal{H}$ of the (unstable) undecayed states will then form a "generalized resolution of the identity" (GRI). Unlike the more familiar (projection-valued) resolution of the identity, a GRI does not necessarily determine a densely defined operator, but under some mild additional assumptions (which we need not specify explicitly here) the GRI $B'(0,t)$ will determine a Hermitian (though not necessarily self-adjoint) operator τ so that

$$(\psi, \tau\psi) = \int t\, d(\psi, B'(0,t)\psi) \quad \text{for all } \psi \in \mathcal{D}(\tau). \tag{35}$$

The operator τ thus defined may then be interpreted as the operator of "time of decay."

Conversely, if there exists a positive Hermitian "time of decay" operator τ associated with the subspace $E\mathcal{H}$ of undecayed states and $B'(0,t)$ denotes a GRI associated with it, then through (33) we may define the probability $P(0,t;\rho)$ which may be interpreted as the probability that the system prepared initially in the (undecayed) state ρ will be found to decay *sometime* during the interval $[0,t]$.

Looked at from this point of view, the Zeno's paradox thus strengthens and sharpens the pessimistic conclusion of Allcock[4] and others concerning the possibility of introducing an observable of "arrival time" in quantum theory. We must emphasize that in our study here, the conclusion is *not* based on certain *a priori*, but questionable, assumptions about τ; such, for instance, as the assumption that τ be "canonically conjugate" to the Hamiltonian, or that τ be a *self-adjoint* operator in the Hilbert space. In the literature such requirements were implicitly or explicitly placed on τ.

We have so far supposed that it is operationally meaningful to ask about probabilities such as $P(0,t;\rho)$ and $Q(0,t;\rho)$. We have also taken the stance that the observed tracks of unstable particles in a bubble chamber or photographic emulsion is in contradiction with the conclusion we have called Zeno's paradox in quantum theory. It is, however, possible to adopt one of the following attitudes:

(1) Probabilities such as $P(0,t;\rho)$ have no operational meaning: There is a fundamental principle in quantum theory that denies the possibility of continuous observation.

Since so far no such principle has been derived from or incorporated into quantum theory, this is not a satisfactory way of resolving the paradox at the present time.

(2) Zeno's paradox is based on the assumption that the continuous measurements are ideal measurements. But measurements (or, observations) involved in the recording of the track of an unstable particle in a detecting medium are nonideal in the sense of (3).

This is a tenable view and it would deny the validity of Theorem 1 as stated and proved in this paper. It has the somewhat unsettling side effect that $P(0, t; \rho)$ and hence the "observed lifetime" of an unstable particle is not a property of the particle (and its Hamiltonian) only, but depends on the details of the observation process. At the present time we have no indication that this is so.

(3) The record of the track of a particle is not a *continuous* observation that the particle has not decayed, but only a discrete sequence of such observations; while Zeno's paradox obtains only in the limit of continuous observations.

While this is tenable, the sufficiently repeated monitoring of the particle should again lengthen the lifetime. There is, however, no indication that the lifetime of a (charged) unstable particle (say, a muon) is appreciably increased in the process of its track formation through bubble chamber. To shed additional light on this question a quantitative investigation of the effect of repeated monitoring on the lifetime of particles (in specific models) is in progress.[13]

(4) Natural though it seems, it is wrong to assume that the temporal evolution of a quantum system under continuing observation can be described by a linear operator of time-evolution such as $T(t)$. It can be described only in terms of a persistent interaction between the quantum system and the *classical measuring apparatus*. When this is done the quantum Zeno's paradox will either disappear or if it survives, at least, it will be understandable as the drastic change in the behavior of the quantum system caused by its continuous interaction with a classical measuring apparatus.

This point of view is at present only a program since there is no standard and detailed theory for the actual coupling between quantum systems with *classical measuring apparatus*. A beginning in this direction is made in a forthcoming paper.[14]

Having been forced into such unusual points of view by the quantum Zeno's paradox one is prompted to draw also some parallels between it and certain empirical findings in the study of human awareness. We shall present such close parallels between the quantum Zeno's paradox and the findings of sensory deprivation and other experiments pertaining to the study of consciousness in a separate publication.

In conclusion, it seems to us that the problems posed by Zeno's paradox have no clean cut resolution at the present time and deserve further discussions. It may also be reemphasized that the probabilities such as $Q(\Delta; \eta, \rho)$ [or $Q(\Delta; \rho)$] that pertain to the outcomes of successive measurements (or continuous measurements) depend on the law according to which "state collapses" occur at the time of measurements. Thus one may say that the "collapse of state vector" caused by measurement, which has haunted the foundation of quantum mechanics like an invisible ghost becomes visible through probabilities such as $Q(\Delta; \eta, \rho)$, etc. The probabilities pertaining to the outcomes of several successive (as well as continuous) measurements therefore deserve further theoretical as well as experimental study than they have received so far.

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[1]Another possible suggestion is to interpret

$$\frac{1}{t}\int_0^t \mathrm{Tr}[\rho U^*(s) E U(s)]\, ds$$

as the desired probability $Q(0, t; \rho)$. Apart from the fact that there is really no convincing reason for this interpretation this expression is *not* generally a monotone (decreasing) function of time t, a property which $Q(0, t; \rho)$ should possess.

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