Exact solution of Schrödinger equation in the case of reduction to *Riccati* type of ODE.

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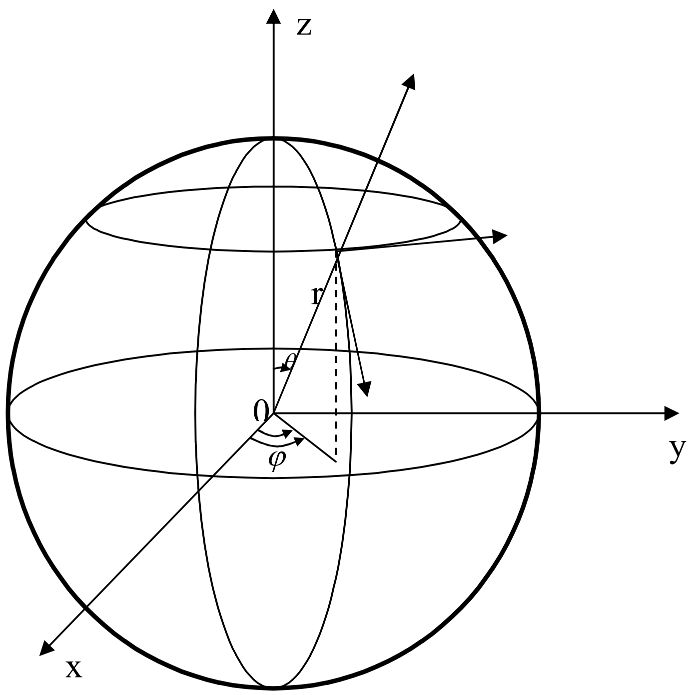
Here is presented a new type of exact solution of Schrödinger equation in the case of it's reduction to Riccati type of ordinary differential equations.

Due to a very special character of Riccati's type equation, it's general solution is proved to have *a proper gap* of components of the particle wavefunction (which is known to be determining a proper quantum state of the particle).

It means a possibility of sudden transformation *or transmutation* of quantum state of the particle (*from one meaning of wavefunction to another*), at definite moment of parametrical time.

Besides, in the case of *spherical symmetry* of particle potential V in position space, as well as *spherical symmetry* of quantum system E total energy, such a solution is proved to be a multiplying of *Bessel* function (for radial component) & *Legendre* spherical function (for angle component), in spherical coordinate system.

In accordance with [1-4], Schrödinger equation describes the quantum state of a physical system changing in time or how the wavefunction of a proper particle evolves over time. We observe below only the time-independent solutions ($\partial/\partial t = 0$) of Schrödinger equation [5-6], for the reason that in case of time-independent particle potential V in position space, the general solutions of Schrödinger equation could be obtained from time-independent solutions by multiplying it on a proper time-coefficient $e^{-\frac{i}{\hbar}\cdot E\cdot t}$, where E – is a total energy of quantum system, $i = \sqrt{-1}$ - is the imaginary unit, $\hbar$ - is *h-bar*.



Pic.1. Spherical coordinate system $R,\ \theta,\ \varphi$.

Schrödinger equation for a single particle in the 3-dimensional case in the presence of a potential V, should be represented [5] in a spherical coordinate system $R,\ \theta,\ \varphi$ (Pic.1):

$$\frac{\hbar^2}{2m}\Delta\psi = U\psi\ , \qquad (1.1)$$

- here m - is the mass of particle, $m = const \neq 0$; ψ - the wavefunction of particle in position space, which is known to be the most complete description of particle quantum mechanical behavior, $\psi = \psi\ (R,\ \theta,\ \varphi)$; $U = (V - E)$, where V – is the potential of the particle in position space, E – is a quantum system total energy, $U = U\ (R,\ \theta,\ \varphi)$; $\hbar$ – is the reduced Planck constant.

Besides, in spherical coordinate system:

$$\Delta\psi \ = \frac{\partial^2\psi}{\partial R^2} + \frac{2}{R}\frac{\partial\psi}{\partial R} + \frac{1}{R^2 sin^2\theta}\frac{\partial^2\psi}{\partial\varphi^2} + \frac{1}{R^2}\frac{\partial^2\psi}{\partial\theta^2} + \frac{1}{R^2}ctg\,\theta\frac{\partial\psi}{\partial\theta} \quad .$$

Also it should be taken into consideration that the *square of wavefunction* ψ is known to be *a proper probability amplitude* to detect such a particle in position space (*per unit of volume of such a space*).

In accordance with [5-6], we should find an exact solution of above equation (*1.1*) in a form below:

$$\begin{aligned} &\psi(R,\theta,\varphi) = \psi(R)\cdot\psi(\theta)\cdot\psi(\varphi), \\ &U(R,\theta,\varphi) = U(R)\cdot U(\theta)\cdot U(\varphi) \quad . \end{aligned} \qquad (1.2)$$

Having substituted (*1.2*) into (*1.1*), we obtain:

$$\frac{1}{\psi(R)}\cdot\frac{d^2\psi(R)}{dR^2} + \frac{2}{R}\cdot\left(\frac{1}{\psi(R)}\right)\cdot\frac{d\psi(R)}{dR} + \frac{1}{R^2 sin^2\theta}\cdot\left(\frac{1}{\psi(\varphi)}\right)\cdot\frac{d^2\psi(\varphi)}{d\varphi^2} +$$

$$+ \frac{1}{R^2}\cdot\left(\frac{1}{\psi(\theta)}\right)\cdot\frac{d^2\psi(\theta)}{d\theta^2} + \frac{ctg\theta}{R^2}\cdot\left(\frac{1}{\psi(\theta)}\right)\cdot\frac{d\psi(\theta)}{d\theta} = \frac{2m}{\hbar^2}\cdot U(R)\cdot U(\theta)\cdot U(\varphi) \quad .$$

Let's make a proper replacement of variables $v = \psi'/\psi$ over all the arguments $R,\ \theta,\ \varphi$:

$$\left(v^2(R)+v'(R)\right) + \frac{2}{R}\cdot v(R) + \frac{1}{R^2\cdot sin^2\theta}\cdot\left(v^2(\varphi)+v'(\varphi)\right) + \qquad (1.3)$$

$$+ \frac{1}{R^2}\cdot\left(v^2(\theta)+v'(\theta)\right) + \frac{ctg\,\theta}{R^2}\cdot v(\theta) = \frac{2m}{\hbar^2}\cdot U(R)\cdot U(\theta)\cdot U(\varphi)$$

The main idea is to find the solution of (*1.1*) like *Riccati's type* [7], that's why let's make an additional assumption below:

$$\begin{aligned}
&\left(v^2(R) + v'(R)\right) = f_1(R), \\
&\left(v^2(\theta) + v'(\theta)\right) = f_2(\theta), \\
&\left(v^2(\varphi) + v'(\varphi)\right) = f_3(\varphi).
\end{aligned} \tag{1.4}$$

Having substituted (*1.4*) into (*1.3*), we obtain an equation below:

$$f_1(R) + \frac{2}{R}\cdot v(R) + \frac{1}{R^2\cdot \sin^2\theta}\cdot f_3(\varphi) +$$
$$+ \frac{1}{R^2}\cdot f_2(\theta) + \frac{ctg\theta}{R^2}\cdot v(\theta) = \frac{2m}{\hbar^2}\cdot U(R)\cdot U(\theta)\cdot U(\varphi) .$$

According to [6], we will present a function $U = (V - E)$ in the axe-symmetrical form. It means that $\partial U/\partial\varphi = 0$

$$U(\varphi) = const = C_0 \neq 0.$$

Thus, from the last equation we obtain (*in the axe-symmetrical case*):

$$f_1(R) + \frac{2}{R}\cdot v(R) + \frac{1}{R^2\cdot \sin^2\theta}\cdot f_3(\varphi) + \tag{1.5}$$
$$+ \frac{1}{R^2}\cdot f_2(\theta) + \frac{ctg\,\theta}{R^2}\cdot v(\theta) = \frac{2m\cdot C_0}{\hbar^2}\cdot U(R)\cdot U(\theta) .$$

To separate a proper variables in (*1.5*), we should make an assumption (*below* $C_1 \neq 0$):

$$f_3(\varphi) = const = C_1 \tag{1.6}$$

Above (*1.6*) leads us to equality below:

$$f_1(R) + \frac{2}{R}\cdot v(R) + \frac{C_1}{R^2\cdot \sin^2\theta} + \frac{1}{R^2}\cdot f_2(\theta) + \frac{ctg\theta}{R^2}\cdot v(\theta) = \frac{2m\cdot C_0}{\hbar^2}\cdot U(R)\cdot U(\theta) ,$$

- or:

$$U(\theta)\cdot\left(\frac{2m\cdot C_0}{\hbar^2}\right)\cdot U(R)\cdot R^2 - \left(R^2\cdot f_1(R) + 2R\cdot v(R)\right) = \frac{C_1}{\sin^2\theta} + f_2(\theta) + ctg\theta\cdot v(\theta) \ .$$

From equality above we conclude that variables in (*1.5*) could be separated only in cases:

1) $U(\theta) = const = C_2 \neq 0$;

2) $U(R)\cdot R^2 = const = C_3 \neq 0$.

Above *2)* case was considered in [5]. Besides, as for the expression $U(R) = C_3 / R^2$ itself: presense of it means that meaning of the particle potential V is proved to differ from a total energy of quantum system E (*at given locus of position space*) on a proper term ~ ($1/R^2$), which could be associated with a proper influence on a particle the field of centrifugal forces.

Let's consider above *1)* case: as for the condition $U(\theta) = const = C_2$, it means we explore the case of stationary, *spherical symmetry* of particle potential V, as well as case of stationary, *spherical symmetry* of total energy of quantum system E .

In such an assumptions, last equation could be represented in form below:

$$\frac{2m\cdot C_0\cdot C_2}{\hbar^2}\cdot U(R)\cdot R^2 - \left(R^2\cdot f_1(R) + 2R\cdot v(R)\right) = \frac{C_1}{\sin^2\theta} + f_2(\theta) + ctg\theta\cdot v(\theta),$$

- or

$$-R^2\cdot\left(-\frac{2m\cdot C_0\cdot C_2}{\hbar^2}\cdot U(R) + f_1(R)\right) - 2R\cdot v(R) = \frac{C_1}{\sin^2\theta} + f_2(\theta) + ctg\theta\cdot v(\theta) \ .$$

To separate a proper variables ($R, \ \theta$) in above equality, we should make a conclusion

$$-R^2\cdot\left(-\frac{2m\cdot C_0\cdot C_2}{\hbar^2}\cdot U(R) + f_1(R)\right) - 2R\cdot v(R) = const = \tau$$

$$\frac{C_1}{\sin^2\theta} + f_2(\theta) + ctg\,\theta\cdot v(\theta) = const = \tau \ . \qquad (1.7)$$

If we determine the functions $f_1(R)$ & $f_2(\theta)$ from an equalities (*1.7*), then substituting of such a functions into (*1.4*), we should obtain:

$$\left(v^2(R) + v'(R)\right) = f_1(R) = -\frac{\tau}{R^2} - \frac{2v(R)}{R} + \frac{2m \cdot C_0 \cdot C_2}{\hbar^2} \cdot U(R),$$

$$\left(v^2(\theta) + v'(\theta)\right) = f_2(\theta) = \tau - ctg\,\theta \cdot v(\theta) - \frac{C_1}{sin^2\theta},$$

- or

$$v'(R) = -v^2(R) - \frac{2}{R} \cdot v(R) - \left(\frac{\tau}{R^2} - \frac{2m \cdot C_0 \cdot C_2}{\hbar^2} \cdot U(R)\right),$$

$$v'(\theta) = -v^2(\theta) - ctg\,\theta \cdot v(\theta) + \left(\tau - \frac{C_1}{sin^2\theta}\right). \qquad (1.8)$$

In accordance with previous assumption, we obtain a proper system of *Riccati* equations [7] in regard to the functions $v = \psi'/\psi$ (*which are functions of* R, θ, *respectively*).

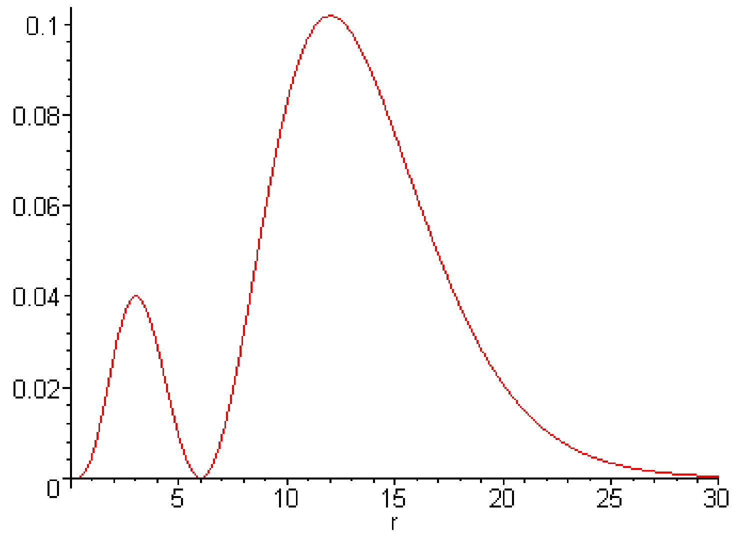
Due to a very special character of such an equations, it's general solution is known to have *a proper gap* of above components of such a functions $v = \psi'/\psi$ [7].
It means a possibility of sudden transformation *or transmutation* of quantum mechanical state of the particle (*from one meaning of wavefunction to another*) at definite moment of parametrical time, or existence of such a continuous solution only at some definite, restricted range of parameter *t* [8].

Expressing of functions ψ from equality $v = \psi'/\psi$ in regard to R, θ in (*1.8*), we obtain:

$$\psi''(R) + \frac{2}{R} \cdot \psi'(R) + \left(\frac{\tau}{R^2} - \frac{2m \cdot C_0 \cdot C_2}{\hbar^2} \cdot U(R)\right) \cdot \psi(R) = 0,$$

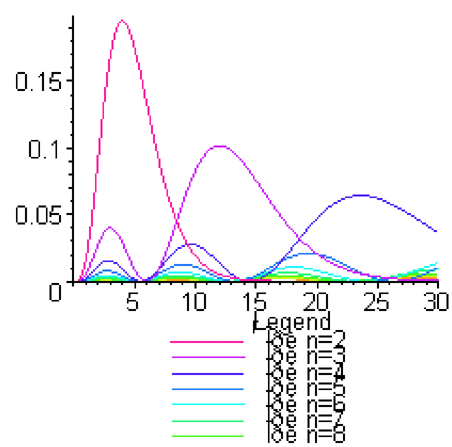
$$\psi''(\theta) + ctg\,\theta \cdot \psi'(\theta) + \left(\frac{C_1}{sin^2\theta} - \tau\right) \cdot \psi(\theta) = 0. \qquad (1.9)$$

The 1-st of equations (*1.9*) ($\tau \neq 0,\ U \neq 0$) – is known to be *the Bessel* ordinary differential equation, it's solutions are a proper *Bessel* functions (Pic.2; *besides, see for example* 2.162*,* 1a [7]):



Pic.2. Bessel functions.

The 2-nd equation of (*1.9*) – is known to be *the Legendre* ordinary differential equation, it's solutions are a proper *Legendre* spherical functions (Pic.3; *see for example* 2.240 [7]):



Pic.3. Legendre (spherical) functions.

Besides, we could make a conclusion that for function $U(R) = V(R) - E(R)$ are valid an equalities below:

$$V(R) \sim R^{m-2}, \quad E(R) \sim R^{-2}, \qquad (1.10)$$

- where $m \neq 0$. Such an assumption has a real physical sense: indeed, potential V very often has a form $V(R) \sim 1/R$ (*as in the case of gravitational field or case of electric charge field distribution*), but a total energy E corresponds to the case of a proper set of discrete energy levels, which are known to be reproduced by harmonic oscillator: $E(R) \sim 1/R^2$.

Besides, in accordance to [7] solutions of 1-st of (*1.9*) could be expressed in terms of *elementary transcendental* functions only in case if $U(R) = 0$ or when ($n = 0, 1, 2, \ldots$):

$$\tau = 1/4 - (1/4)\cdot(2n+1)^2$$

Let's choose as a basic model the case of accordance of total energy of quantum system E to a proper set of discrete energy levels, which are known to be reproduced by a harmonic oscillator (*in spherical coordinate system*):

$$E(R) = \frac{\left((2n+1)^2 - 1\right)}{4R^2},$$

- also let's choose a constants C_0 & C_2 as below

$$\left(\frac{2m\cdot C_0\cdot C_2}{\hbar^2}\right) = 1,$$

- then we obtain from (*1.9*)

$$\psi''(R) + \left(\frac{2}{R}\right)\cdot\psi'(R) - V(R)\cdot\psi(R) = 0,$$

- or, having chosen of a proper constant in representation of function $V(R)$, we obtain:

$$R^2\cdot\psi''(R) + 2R\cdot\psi'(R) + R^m\cdot\psi(R) = 0,$$

- if we take into consideration (*1.10*).

The last equation is a classical case of *Bessel* ordinary differential equation [7] ($m \neq 0$):

$$\psi(R) = \left(\frac{1}{R^{1/2}}\right) \cdot Z_\nu\left(\frac{2}{m} \cdot R^{\frac{m}{2}}\right), \quad \nu = \frac{1}{m} \ .$$

If $m = 1$, then from above we obtain: $V(R) \sim -1/R$.

In such a case, solution of the 1-st equation of (*1.9*) should be represented as below:

$$\psi(R) = \left(\frac{1}{R^{1/2}}\right) \cdot Z_1\left(2\sqrt{R}\right).$$

As for the function $v = \psi'/\psi$ in regard to the argument φ, taking into consideration (*1.4*) & (*1.6*), we obtain [5-6]:

$$v'(\varphi) + v^2(\varphi) = C_1 \ .$$

Besides, the requiring of limitation of such a solution for all $\varphi \in [0, 2\pi \cdot n], n \in N$, leads us to an equality below [5-6]:

$$\psi''(\varphi) - C_1 \cdot \psi(\varphi) = 0,$$
$$\psi(\varphi) = C_{01} \cdot \cos\left(\varphi \cdot \sqrt{|C_1|}\right) + C_{02} \cdot \sin\left(\varphi \cdot \sqrt{|C_1|}\right).$$

- where $C_1 < 0$.

As mentioned above, we should take into consideration a requiring that the *square of wavefunction* ψ is to be *a proper probability amplitude* to detect such a particle in position space (*per unit of volume*).

It means that calculating of square of wavefunction ψ over all the volume of position space *must be equal to unit* $= 1$. Such a requiring is used to restrict the choosing of a constants of solution to be constructed. Besides, in the case of *divergence* of meanings of wavefunction ψ in calculating over the variables R, θ, φ, such a wavefunction ψ should be normalized to a proper *Dirac* δ*-function* [6].

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