

$$1 - n + n \cdot \frac{n-1}{2} - n \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} + \dots (r+1) \text{ terms}$$

$$= (-1)^r \cdot \frac{(n-1)(n-2)(n-3) \dots (n-r)}{1 \cdot 2 \cdot 3 \dots r}$$

Hence  $\int_0^\pi \cos^{n-1} \theta d\theta \left\{ \cos (n+1)\theta + (-1)^r \cos (n-2r+1)\theta \right\}$

$$= \frac{\pi}{2^n} (-1)^r \frac{(n-1)(n-2) \dots (n-r)}{1 \cdot 2 \cdot 3 \dots r}$$

“The Vortex Ring Theory of Gases. On the Law of the Distribution of Energy among the Molecules.” By J. J. THOMSON, M.A., F.R.S., Fellow of Trinity College, Cavendish Professor of Experimental Physics in the University of Cambridge. Received June 4, 1885.

In any kinetic theory of gases the statistical method of investigation must be used, and since the separate molecules of the gas are supposed to possess some properties to very different extents, it is necessary to know how many molecules there are which have the measure of any given property between certain limits. Thus the question of the distribution of configuration and velocity amongst the molecules is one of the most important problems in any theory of gases.

This problem has been solved for the ordinary solid particle theory by Maxwell and Boltzmann, and their researches are the more valuable as the results do not depend on any assumption about the law of force between the molecules.

In this paper I shall attempt to solve the same problem for the vortex atom theory of gases. In this case the question is a little more complicated, as the radii of the vortex rings can vary as well as their velocities. This is one of the most striking differences between the two theories; according to the ordinary theory all the molecules of a gas are of the same size, according to the vortex atom theory the molecules of the same gas vary in size. If this be true, a porous plate of the requisite degree of fineness might play in this theory the part which Maxwell's demons play in the ordinary theory. For let us suppose that we have two chambers, A and B, separated by a porous plate, and that A is filled with gas initially while B is empty, then if the pores in the porous plate are so fine that only the smaller molecules can get through from A to B, then, though some of the molecules will recross the plate, some gas will remain in B, and the

molecules in B will, on the whole, be moving faster than those in A, and so may be supposed to be at a higher temperature, since the smaller the radius of a vortex ring the greater its velocity. Thus B and A might be the hot and cold chambers respectively of a heat engine, and in this way work might be derived from the gas which was originally at a uniform temperature, so that this arrangement would not obey the second law of thermodynamics.

If the molecule on the vortex atom theory of matter consisted of a single ring its velocity of translation would be a function only of its radius. It is, however, for several reasons advisable to take a more general case, and to suppose that the molecule consists of several rings linked through each other, the rings being nearly equal in radius, and also nearly coincident in position; or what is perhaps better, we may suppose that the vortex core forms an endless chain, but that instead of being a single loop like the simple ring, it is looped into a great many coils nearly equal in radius and nearly coincident in position. We may realise this way of arranging the vortex core if we take a cylindrical rod whose length is great compared with its radius, and describe on its surface a screw with  $n$  threads so that the threads make  $m/n$  turns in the length of the rod, where  $m$  is an integer not divisible by  $n$ . Then bend the rod into a circle and join the ends, the threads of the screw will form an endless chain with  $n$  loops, and we may suppose that this represents the way in which the vortex rings are arranged; it is shewn, however, in my "Treatise on the Motion of Vortex Rings" that this way of arranging the vortex core is unstable if  $n$  be greater than six. When the vortex core is arranged in the way just described, the velocity of translation is no longer a function of the size of the ring alone; at the same time when a vortex ring of this kind moves about in a fluid where the velocity is not uniform, the change in the velocity of the ring will be due chiefly to the change in its radius. For the velocity at a small distance  $d$  from the circular axis of a vortex ring whose radius is  $a$  and strength  $m$  is—

$$\frac{m}{2\pi a} \log \frac{8a}{d},$$

so that as  $8a/d$  is very large, a change  $\delta a$  in the radius of the ring produces a change in the velocity approximately equal to—

$$-\frac{\delta a}{a} \cdot \frac{m}{2\pi a} \log \frac{8a}{d},$$

while a change  $\delta d$  in the distance of the point from the circular axis of the ring produces a change in the velocity equal to—

$$-\frac{\delta d}{d} \cdot \frac{m}{2\pi a}.$$

Thus for the same relative changes of  $a$  and  $d$  the changes in the velocities are in the ratio of  $\log 8a/d$  to 1, and as  $\log 8a/d$  is very great, we may neglect the change in the velocity of the ring produced by the alteration in the distance between the loops in comparison with that produced by the alteration in the size of the ring.

The kinetic energy of a quantity of fluid containing vortex rings of this kind may conveniently be divided into several parts. The first part consists of the kinetic energy of the irrotationally moving fluid surrounding the ring, the second part of the kinetic energy of the rotationally moving fluid; this again may conveniently be divided into two parts, one part being the kinetic energy due to the rotation in the core, and the other that due to the translational velocity of the vortex core.

The kinetic energy of the irrotationally moving liquid surrounding the ring may be expressed in several ways; it is equal to the strength of the ring multiplied by the rate of flow of the fluid through it; the most convenient expression for our purpose, however, is

$$Ava^2,$$

where  $v$  is the velocity of translation of the vortex ring resolved along the normal to its plane,  $a$  is the radius of the ring and  $A$  a constant.

(See p. 12 of my "Treatise on the Motion of Vortex Rings.")

The energy due to the rotation of the vortex core is

$$\frac{1}{2}n\pi^2\rho m^2a,$$

where  $n$  is the number of loops in the ring,  $\rho$  the density of the fluid, and  $m$  the strength of the ring.

The kinetic energy due to the translational velocity of the ring is

$$\frac{1}{2}M(u^2 + v^2 + w^2),$$

where  $M$  is the mass of fluid in the ring and  $u^2 + v^2 + w^2$  the square of the velocity of the ring.

Thus if  $T$  be the whole kinetic energy due to the ring—

$$T = Ava^2 + \frac{1}{2}n\pi^2\rho m^2a + \frac{1}{2}M(u^2 + v^2 + w^2).$$

Let us consider a vortex ring placed in a fluid where there is a velocity potential  $\Omega$  independent of that due to the vortex ring itself, the value of  $\Omega$  is supposed to be known at every point of the fluid.

We have to fix the position, size, and motion of the ring. We can do this if we know the coordinates  $(x, y, z)$  of its centre, its radius ( $a$ ), the direction cosines  $(l, m, n)$  of its plane, and  $V$  that part of the velocity at the ring which is due to the ring itself.  $V$  is not necessarily the actual velocity of the ring, for this latter quantity is the

resultant of  $V$ , and the velocity whose components are  $d\Omega/dx$ ,  $d\Omega/dy$ ,  $d\Omega/dz$ .

$$\text{Let} \quad \xi = la^p, \quad \eta = ma^p, \quad \zeta = na^p, \quad \omega = V^q.$$

Then we shall prove that it is possible to determine  $p$  and  $q$  so that the number of molecules which have the values of  $x, y, z, \xi, \eta, \zeta, \omega$ , between  $x, y, z, \xi, \eta, \zeta, \omega$ , and  $x+dx, y+dy, z+dz, \xi+d\xi, \eta+d\eta, \zeta+d\zeta, \omega+d\omega$ , and for which the kinetic energy of the molecule and the surrounding fluid is  $T$ , is when the gas is in a uniform and steady state—

$$Ce^{-hT} dx dy dz d\xi d\eta d\zeta d\omega,$$

where  $C$  is some constant determined by the number of molecules in the gas.

We shall first prove that this represents a possible distribution among the molecules of the quantities denoted by  $\xi, \eta, \zeta, \omega$ , when the vortex rings are moving in a fluid whose velocity varies from point to point; we disregard for the present the effects of any collisions which may take place among the vortex rings themselves. In this case the rings are supposed to be so far apart that they do not influence each other, so that the velocity of any ring is the same as if the others did not exist.  $T$  represents the kinetic energy due to the ring and the distribution of velocity potential  $\Omega$  on this supposition.

We have to prove that if the distribution be represented by this expression at any time, it will continue to be represented by it. This will be the case if the expression

$$Ce^{-hT} dx dy dz d\xi d\eta d\zeta d\omega,$$

remains constant as the molecules move about. Now  $T$ , the kinetic energy, remains constant, so that we have to prove that

$$dx dy dz d\xi d\eta d\zeta d\omega$$

also remains constant.

Since  $\Omega$  is the part of the velocity potential which is not due to the rings themselves, by the equations on pages 65 and 66 of my "Treatise on the Motion of Vortex Rings," we have—

$$\frac{da}{dt} = -\frac{1}{2} a \frac{d^2\Omega}{dh^2},$$

$$\frac{dl}{dt} = l \frac{d^2\Omega}{dh^2} - \frac{d^2\Omega}{dh dx},$$

$$\frac{dm}{dt} = m \frac{d^2\Omega}{dh^2} - \frac{d^2\Omega}{dh dy},$$

$$\frac{dn}{dt} = n \frac{d^2\Omega}{dh^2} = \frac{d^2\Omega}{dh dz};$$

where

$$\frac{d}{dh} = l \frac{d}{dx} + m \frac{d}{dy} + n \frac{d}{dz}.$$

so that if  $\delta a$ ,  $\delta l$ ,  $\delta m$ ,  $\delta n$  be the changes in  $l$ ,  $m$ ,  $n$  respectively in the small time  $\tau$  then—

$$\delta a = -\frac{1}{2} a \frac{d^2 \Omega}{dh^2} \tau,$$

$$\delta l = \left( l \frac{d^2 \Omega}{dh^2} - \frac{d^2 \Omega}{dh dx} \right) \tau,$$

$$\delta m = \left( m \frac{d^2 \Omega}{dh^2} - \frac{d^2 \Omega}{dh dy} \right) \tau,$$

$$\delta n = \left( n \frac{d^2 \Omega}{dh^2} - \frac{d^2 \Omega}{dh dz} \right) \tau,$$

and if  $\delta x$ ,  $\delta y$ ,  $\delta z$  are the changes in  $x$ ,  $y$ ,  $z$  in the time  $\tau$ ,

$$\delta x = u\tau,$$

$$\delta y = v\tau,$$

$$\delta z = w\tau,$$

where  $u$ ,  $v$ ,  $w$  are the component velocities of the centre of the vortex ring.

Let  $x'$ ,  $y'$ ,  $z'$ ,  $\omega'$ ,  $\xi'$ ,  $\eta'$ ,  $\zeta'$  be the values of  $x$ ,  $y$ ,  $z$ ,  $\omega$ ,  $\xi$ ,  $\eta$ ,  $\zeta$  respectively after the time  $\tau$ , then—

$$x' = x + u\tau,$$

$$y' = y + v\tau,$$

$$z' = z + w\tau,$$

$$\omega' = \omega + \delta\omega,$$

but since

$$\omega = \nabla a,$$

$$\delta\omega = q \nabla a^{-1} \delta a.$$

Now the change in  $\nabla$  will be due to the change in the shape and size of the ring, and as we saw before that this is due almost entirely to the change in the radius, thus the change in  $\nabla$  will be  $-\nabla \delta a / a$ , or substituting for  $\delta a$  its value—

$$\delta \nabla = \frac{1}{2} \nabla \frac{d^2 \Omega}{dh^2} \tau,$$

thus

$$\omega' = \omega + \frac{1}{2} q \omega \frac{d^2 \Omega}{dh^2} \tau,$$

$$\begin{aligned}\xi' &= \xi + \delta\xi \\ &= \xi + pa^{p-1}\delta a + a^p\delta l.\end{aligned}$$

Substituting for  $\delta a$  and  $\delta l$ , their values, we get

$$\xi' = \xi - \frac{1}{2}p\xi \frac{d^2\Omega}{dh^2} \tau + \xi \frac{d^2\Omega}{dh^2} \tau - \left( \xi \frac{d^2\Omega}{dx^2} + \eta \frac{d^2\Omega}{dxdy} + \zeta \frac{d^2\Omega}{dxdz} \right) \tau.$$

Similarly,

$$\eta' = \eta - \frac{1}{2}p\eta \frac{d^2\Omega}{dh^2} \tau + \eta \frac{d^2\Omega}{dh^2} \tau - \left( \xi \frac{d^2\Omega}{dxdy} + \eta \frac{d^2\Omega}{dy^2} + \zeta \frac{d^2\Omega}{dxdz} \right) \tau,$$

$$\zeta' = \zeta - \frac{1}{2}p\zeta \frac{d^2\Omega}{dh^2} \tau + \zeta \frac{d^2\Omega}{dh^2} \tau - \left( \xi \frac{d^2\Omega}{dxdz} + \eta \frac{d^2\Omega}{dydz} + \zeta \frac{d^2\Omega}{dz^2} \right) \tau.$$

If we neglect the squares of  $\delta x$ ,  $\delta y$ ,  $\delta z$ ,  $\delta w$ ,  $\delta\xi$ ,  $\delta\eta$ ,  $\delta\zeta$ ,

$$\begin{aligned}dx'dy'dz'dw'd\xi'd\eta'd\zeta' &= dx'dy'dz'dw'd\xi'd\eta'd\zeta' \left\{ 1 + \left( \frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} \right) \tau \right. \\ &\quad \left. + \frac{d\delta w}{dw} + \frac{d\delta\xi}{d\xi} + \frac{d\delta\eta}{d\eta} + \frac{d\delta\zeta}{d\zeta} \right\}.\end{aligned}$$

Now

$$\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} = 0.$$

And by the equations written above

$$\frac{d\delta w}{dw} = \frac{1}{2}d \frac{d^2\Omega}{dh^2} \tau,$$

and

$$\begin{aligned}\frac{d\delta\xi}{d\xi} + \frac{d\delta\eta}{d\eta} + \frac{d\delta\zeta}{d\zeta} &= \\ \tau \left\{ 3(1 - \frac{1}{2}p) \frac{d^2\Omega}{dh^2} + (1 - \frac{1}{2}p) \left( \xi \frac{d}{d\xi} + \eta \frac{d}{d\eta} + \zeta \frac{d}{d\zeta} \right) \frac{d^2\Omega}{dh^2} \right. \\ &\quad \left. - \left\{ \frac{d^2\Omega}{dx^2} + \frac{d^2\Omega}{dy^2} + \frac{d^2\Omega}{dz^2} \right\} \right\}.\end{aligned}$$

But

$$\frac{d^2\Omega}{dx^2} + \frac{d^2\Omega}{dy^2} + \frac{d^2\Omega}{dz^2} = 0,$$

and since

$$\frac{d^2\Omega}{dh^2} = \left\{ \frac{\xi}{\{\xi^2 + \eta^2 + \zeta^2\}^{\frac{1}{2}}} \frac{d}{dx} + \frac{\eta}{\{\xi^2 + \eta^2 + \zeta^2\}^{\frac{1}{2}}} \frac{d}{dy} + \frac{\zeta}{\{\xi^2 + \eta^2 + \zeta^2\}^{\frac{1}{2}}} \frac{d}{dz} \right\}^2 \Omega,$$

we see that  $\frac{d^2\Omega}{dh^2}$  is a homogeneous function of  $\xi$ ,  $\eta$ ,  $\zeta$  of zero dimensions, and therefore by Euler's theorem

$$\left\{ \xi \frac{d}{d\xi} + \eta \frac{d}{d\eta} + \zeta \frac{d}{d\zeta} \right\} \frac{d^2\Omega}{dh^2} = 0,$$

hence

$$dx'dy'dz'dw'd\xi'd\eta'd\zeta'$$

$$= dx'dy'dz'dw'd\xi'd\eta'd\zeta' \left( 1 + \left\{ \frac{1}{2}q + 3\left(1 - \frac{1}{2}p\right) \right\} \frac{d^2\Omega}{dh^2} \right),$$

so that if

$$q = 3(p - 2)$$

$$dx'dy'dz'dw'd\xi'd\eta'd\zeta' = dx'dy'dz'dw'd\xi'd\eta'd\zeta',$$

and therefore the distribution represented by the expression

$$Ce^{-h^2} dx'dy'dz'dw'd\xi'd\eta'd\zeta',$$

will be permanent if there are no collisions between the molecules.

We shall now go on to shew that this expression will represent the distribution of coordinates and momenta among the molecules even when collisions take place, at any rate if the collisions are not very violent.

Let us call the group of molecules which have the quantities  $x, y, z, \xi, \eta, \zeta, \omega$  between  $x, y, z, \xi, \eta, \zeta, \omega$ , and  $x+dx, y+dy, z+dz, \xi+d\xi, \eta+d\eta, \zeta+d\zeta, \omega+dw$  the group A. The number of molecules in this group is

$$Ce^{-h^2} dx'dy'dz'dw'd\xi'd\eta'd\zeta',$$

all the symbols having the same meaning as before.

Let us consider another group of molecules which have their co-ordinates between  $x_1, y_1, z_1, \xi_1, \eta_1, \zeta_1, \omega_1$ , and  $x_1+dx_1, y_1+dy_1, z_1+dz_1, \xi_1+d\xi_1, \eta_1+d\eta_1, \zeta_1+d\zeta_1, \omega_1+d\omega_1$ . We shall call this group B. The number of molecules in this group is

$$De^{-h^2} dx_1 dy_1 dz_1 d\xi_1 d\eta_1 d\zeta_1 d\omega_1.$$

We shall suppose that the molecules of the A group come into collision with those of the B group, and that the values of the co-ordinates after the collision are denoted by putting dashes to the letters which denoted the corresponding coordinates before the collision.

In my "Treatise on the Motion of Vortex Rings" it is proved that the effects of a collision depend on, in addition to the quantities already specified, the angle which the line joining the centres of the rings when they are nearest together makes with the shortest distance between the directions of motion of the rings; let us call this angle  $\phi$ .  $\phi$  is positive for the ring which first passes through the shortest distance between the directions of motion of the ring, negative for the other ring, and it may have any value between  $-\pi/2$  and  $\pi/2$ .

We may suppose that a collision takes place when the shortest distance between the centres of the two rings is less than some assigned value; it is not, however, necessary to limit ourselves to any particular way of defining a collision.

Let  $\gamma d\phi$  be the fraction of the number of pairs of molecules which come into collision in the unit of time and contain one molecule from the group A and another from the group B, and for which  $\phi$  is between  $\phi$  and  $\phi + d\phi$ . Then if the states in which the A and B molecules are in after the collision be called A' and B' respectively, the number of pairs of molecules which in the unit of time leave the state {AB} and enter the state {A'B'} is—

$$CD \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\phi \{ e^{-h(T+T_1)} \gamma dx dy dz d\xi d\eta d\xi d\omega dx_1 dy_1 dz_1 d\xi_1 d\eta_1 d\xi_1 d\omega_1 \}.$$

Now the distribution will be steady if this equals the number of molecules which leave the state {A'B'} in the unit of time, but this number is—

$$CD \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\phi \{ e^{-h(T+T_1)} \gamma' dx' dy' dz' d\xi' d\eta' d\xi' d\omega' dx'_1 dy'_1 dz'_1 d\xi'_1 d\eta'_1 d\xi'_1 d\omega'_1 \}.$$

We can see that if the gas does not exhibit vector properties  $\gamma$  must equal  $\gamma'$ . For since the motion is reversible, if any two molecules whose coordinates are  $\xi, \eta, \zeta, \omega; \xi_1, \eta_1, \zeta_1, \omega_1$ , come into collision, then the molecules whose coordinates are  $-\xi, -\eta, -\zeta, \omega; -\xi_1, -\eta_1, -\zeta_1, \omega_1$ , will also collide. Let these latter molecules be said to be in the states  $(-A'), (-B')$  respectively. Thus the percentage of collision for the states (AB), that is for collisions between two molecules in the states A and B, respectively is the same as for the state  $(-A', -B')$ . But since as many molecules are moving in any direction as in the opposite, the number of molecules in the state  $-A'$  will equal the number in the state  $A'$ , and similarly the number of molecules in the state  $-B'$  is the same as the number in the state  $B'$ , and since the gas exhibits no vector properties, the mean path between the collisions between the molecules in the states  $A'$  and  $B'$  must equal the mean path between the collisions between the molecules in the states  $-A'$  and  $-B'$ ; and thus the percentage of collisions must be the same. So that the percentage for the state  $(A'B')$  equals the percentage for the state  $(-A', -B')$ , but this, as we saw, equals the percentage for the state AB; and, therefore, the percentage for the state AB equals the percentage for the state  $(A'B')$ ; or  $\gamma = \gamma'$ . Since the collision may be fixed with regard to either molecule, and since  $\phi$  is positive for one molecule, negative for the other, we see that  $\gamma$  cannot change sign with  $\phi$ , so that if  $\gamma$  is a function of  $\phi$  it must be one of the form—

$$\gamma = \gamma_0 + \gamma_1 \cos \phi + \gamma_2 \cos 2\phi + \dots$$



Again,

$$dx dy dz = d\omega' d\eta' dz'$$

$$dx_1 dy_1 dz_1 = d\omega'_1 dy'_1 dz'_1$$

and none of the quantities are functions of  $\phi$ .

We have also since the total kinetic energy is not changed by the collision  $T + T_1 = T' + T'_1$ , and neither of these quantities is a function of  $\phi$ . Since this is so, we see that the expressions we have assumed will represent a steady distribution if—

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \gamma d\phi \{ d\xi d\eta d\zeta d\omega d\xi_1 d\eta_1 d\zeta_1 d\omega_1 \} = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \gamma d\phi (d\xi' d\eta' d\zeta' d\omega' d\xi'_1 d\eta'_1 d\zeta'_1 d\omega'_1).$$

Let us suppose that the collisions are not violent enough to make the vortex rings deviate greatly from their circular forms, and let us consider the effect produced on an A molecule by collision with a B molecule. Let  $\Omega'$  be the potential due to the B molecule, then just as before we have—

$$\omega' = \omega + \frac{1}{2} q \omega \int_{-\infty}^{+\infty} \frac{d^2 \Omega}{dh^2} dt$$

$$\xi' = \xi + (1 - \frac{1}{2} p) \xi \int_{-\infty}^{+\infty} \frac{d^2 \Omega}{dh^2} dt - \left\{ \xi \int_{-\infty}^{+\infty} \frac{d^2 \Omega}{dx^2} dt + \eta \int_{-\infty}^{+\infty} \frac{d^2 \Omega}{dx dy} dt + \zeta \int_{-\infty}^{+\infty} \frac{d^2 \Omega}{dx dz} dt \right\},$$

with similar expressions for  $\eta'$  and  $\zeta'$ . Here  $h$  is drawn along the normal to the A molecule, and the coordinates are supposed to be changed by the collision by only a small fraction of their values.

Now the only thing that makes any difference between this case and the former one is that now  $\int_{-\infty}^{+\infty} \frac{d^2 \Omega}{dh^2} dt$  is a function of  $v$ , and therefore of  $\omega$ . If therefore we assume that  $3(p-2) = q$ , we have

$$d\xi' d\eta' d\zeta' d\omega' = d\xi d\eta d\zeta d\omega \left\{ 1 + \frac{1}{2} q \omega \frac{d}{d\omega} \int_{-\infty}^{+\infty} \frac{d^2 \Omega}{dh^2} dt \right\}$$

Now  $\int_{-\infty}^{+\infty} \frac{d^2 \Omega}{dh^2} dt$  is proportional to the change in  $\omega$ , and therefore, by § 29 of my "Treatise on Vortex Motion" is of the form  $f \sin 3\phi$ , where  $f$  is a function of  $\omega$  but not of  $\phi$ , thus:—

$$d\xi' d\eta' d\zeta' d\omega' = d\xi d\eta d\zeta d\omega \left\{ 1 + \frac{1}{2} q \omega \frac{df}{d\omega} \sin 3\phi \right\}.$$

Similarly

$$d\xi'_1 d\eta'_1 d\zeta'_1 d\omega'_1 = d\xi_1 d\eta_1 d\zeta_1 d\omega_1 \left\{ 1 + \frac{1}{2} q \omega_1 \frac{df'}{d\omega_1} \sin 3\phi \right\},$$

so that neglecting the squares of small quantities

$$d\xi' d\eta' d\xi' d\omega' d\xi'_1 d\eta'_1 d\omega'_1 = d\xi d\eta d\xi d\omega d\xi_1 d\eta_1 d\omega_1 \left\{ 1 + \frac{1}{2} q \omega \frac{df}{d\omega} \sin 3\phi \right. \\ \left. + \frac{1}{2} q \omega' \frac{df'}{d\omega'} \sin 3\phi \right\},$$

and therefore

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \gamma d\phi \{ d\xi' d\eta' d\xi' d\omega' d\xi'_1 d\eta'_1 d\omega'_1 \} = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \gamma d\phi \{ d\xi d\eta d\xi d\omega d\xi_1 d\eta_1 d\omega_1 \}.$$

Since we see from the form of  $\gamma$  that—

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin 3\phi \gamma d\phi = 0.$$

Thus the condition for a steady distribution is satisfied, and we therefore conclude that a possible distribution of the values of the coordinates among the molecules of the gas is represented by the expression—

$$C e^{-hT} d\xi d\eta d\xi d\omega d\xi_1 d\eta_1 d\omega_1,$$

where  $\xi = la^p$ ,  $\eta = ma^p$ ,  $\zeta = na^p$ ,  $\omega = v^q$

and  $3(p-2) = q$ .

Let us consider the case when there is no external disturbance in the fluid containing the vortex rings; the distribution will be uniform in all parts of the fluid, so that the number of molecules which have the quantities  $\xi$ ,  $\eta$ ,  $\zeta$ ,  $\omega$ , between  $\xi$ ,  $\eta$ ,  $\zeta$ ,  $\omega$  and  $\xi + d\xi$ ,  $\eta + d\eta$ ,  $\zeta + d\zeta$ ,  $\omega + d\omega$  is independent of  $x$ ,  $y$ ,  $z$ , and so by the above formula will be proportional to—

$$e^{-hT} d\xi d\eta d\xi d\omega,$$

or if the normals to the planes of the vortex rings point uniformly in all directions the number of molecules which have  $a$  between  $a$  and  $a + da$ ,  $v$  between  $v$  and  $v + dv$  is proportional to—

$$e^{-hT} a^{3p-1} v^{q-1} da dv,$$

or substituting for  $q$  the value  $3(p-2)$

$$e^{-hT} a^{3p-1} v^{3p-7} da dv.$$

Though in the kind of molecule we are considering,  $a$  and  $v$  may be treated as independent variables, still the limits of  $v$  depend upon the value of  $a$ . For suppose the molecule to consist of  $n$  rings

linked together, then for a given value of  $a$  the velocity of the molecule will be least when the links are so far apart that they do not greatly affect each other's velocity ; in this case  $v$  will equal—

$$\frac{m}{2\pi a} \log \frac{8a}{e},$$

the velocity of the molecule will be greatest when the  $n$  rings are close together ; in this case  $v$  will equal—

$$\frac{nm}{2\pi a} \log \frac{8a}{e}.$$

So that if we integrate first with respect to  $v$  we must do so between these limits.

Since, however—

$$T = \Lambda a^2 v + \frac{1}{2} n m^2 \pi^2 \rho a + \frac{1}{2} M v^2,$$

we cannot perform the integration except between the limits zero and infinity for both  $a$  and  $v$  ; if, however,  $n$  be large, or the molecule complicated, the results got by integration between the limits—

$$\frac{m}{2\pi a} \log \frac{8a}{e} \text{ and } \frac{nm}{2\pi a} \log \frac{8a}{e}$$

for  $v$  and zero and infinity for  $a$  will not differ much from those got by integrating between zero and infinity for both  $a$  and  $v$ .

The second term in the expression for the kinetic energy is very small compared with the first, so that it may be neglected without causing sensible error. We shall find it convenient to take as new variables the two remaining terms in the expression for the kinetic energy ; we shall call these new variables  $\alpha$  and  $\beta$  respectively, where  $\alpha$  denotes the energy in the fluid surrounding the ring,  $\beta$  the energy due to the translational velocity of the ring, so that—

$$\Lambda a^2 v = \alpha$$

$$\frac{1}{2} M v^2 = \beta,$$

and therefore

$$dadv = \frac{1}{M \Lambda a v^2} d\alpha d\beta,$$

so that

$$C e^{-kT} \alpha^{3p-1} v^{3p-7} dadv = C' e^{-h(\alpha+\beta)} \alpha^{\frac{3p-2}{2}} \beta^{\frac{3p-16}{4}} d\alpha d\beta,$$

where  $C'$  is a new constant.

Thus the number of molecules which have the energy in the fluid surrounding them between  $\alpha$  and  $\alpha + \delta\alpha$ , and also the energy due to the translational velocity of the ring between  $\beta$  and  $\beta + \delta\beta$  is—

$$C' e^{-h(\alpha+\beta)} \alpha^{\frac{3p-2}{2}} \beta^{\frac{3p-16}{4}} d\alpha d\beta,$$

and if the molecule is so complex that  $\alpha$  and  $\beta$  may be regarded as independent, then the limits of  $\alpha$  and  $\beta$  are zero and infinity.

The quantity  $p$  is at present undetermined.

Let us apply this result to find the pressure of a gas on the sides of the vessel which contains it. To do this we must consider what takes place at the sides of the vessel. The general nature of this action was described by Sir William Thomson ("Nature," vol. xxiv, p. 47). As the vortex rings move up to the sides of the vessel they swell out and move slowly up the bounding surface, where they form a layer of swollen vortices sticking to the sides of the vessel. A vortex ring coming up to the surface tends to wash off the vortex rings attached to the surface on either side of it, so that when things have got into a state of equilibrium there is a vortex ring washed off for each one that comes up. Thus the pressure on the surface of the vessel will be the same as if the vortex ring struck against the surface and was reflected away again with its velocity reversed, if we assume, as seems natural, that the average velocity of the rings leaving the surface is the same as of those approaching it. Thus each ring that comes up may be looked upon as communicating twice its momentum to the surface, and we can explain the pressure of a gas, just as in the ordinary theory. We have to remark here, however, that the phrase momentum of the vortex ring is ambiguous, as there are two different momenta connected with the ring; there is (1) the momentum of the ring and the fluid surrounding it; and (2) the momentum of the fluid forming the ring alone; this is proportional to the velocity of the ring, while (1) is not only not proportional to the velocity, but in the single ring decreases as the velocity of the ring increases; in a very complex ring it does not necessarily do this, but even in this case it is not proportional to the velocity.

Now, when a vortex ring gets stopped by a surface the question arises whether the momentum communicated to the surface is the momentum (1) or (2). The answer to this question depends on what we consider the nature of the surface to be. If the surface stops the fluid as well as the ring, then no doubt (1) is the momentum which is communicated to the surface. If, however, the surface stops the ring but allows the greater part of the fluid to flow on, then the momentum communicated to the surface is evidently approximately equal to (2). If we consider that the surface is formed of vortex rings the latter supposition seems the more probable, as the fluid in which the rings move can hardly be supposed to be stopped by such a porous surface. We may illustrate this by a mechanical analogy. Let us suppose that we have a number of anchor rings with circulation

established round them moving about in water, and striking against a grating immersed in it. The momentum of the anchor ring will consist of two parts, one due to the circulation, the other due to the translational velocity of the ring. If the grating is so fine that the openings are only a small fraction of the whole area, then the momentum communicated to the grating will be the whole momentum; if, however, the grating is a coarse one, so that the openings form the larger portion of the area, then the momentum communicated to the grating will only be the momentum of the ring itself. And this seems to correspond to the case of vortex motion.

Thus if  $a$  be the velocity of the gas resolved along the normal to the boundary surface, the pressure on the surface per unit of area or the momentum communicated to it per unit of time is—

$$\begin{aligned} & 2\Sigma Ma^2 \\ & = \frac{2}{3}\Sigma Mv^2 \\ & = \frac{4}{3}\Sigma\beta, \end{aligned}$$

using the same notation as before.

Now, the number of molecules which have the quantities  $\alpha$  and  $\beta$  between  $\alpha$ ,  $\beta$  and  $\alpha+d\alpha$ ,  $\beta+d\beta$  is proportional to—

$$e^{-h(\alpha+\beta)} \frac{3p-2}{\alpha^2} \beta^{\frac{3p-16}{4}} d\alpha d\beta,$$

so that if  $N$  be the number of molecules—

$$\frac{\Sigma\beta}{N} = \frac{\Gamma\left(\frac{3p-8}{4}\right)}{h\Gamma\left(\frac{3p-12}{4}\right)} = \frac{1}{h} \frac{3p-12}{4}$$

where  $\Gamma(n)$  is written for

$$\int_0^\infty e^{-x} x^{n-1} dx$$

and the molecule is supposed to be so complex that we may, without sensible error, suppose the limits of  $\alpha$  and  $\beta$  to be zero and infinity.

We may take  $1/h$  as proportional to the temperature  $\theta$  of the gas, since it is the same for each of two gases which are in contact with each other, and is also proportional to the mean kinetic energy of the rings themselves.

Substituting the above value for  $\epsilon\beta$ , we see that the pressure equals

$$(p-4)\frac{N}{h},$$

and thus varies as  $N\theta$ .

Thus Boyle and Gay-Lussac's laws follow from the vortex atom theory.

At present the quantity  $p$  is quite undetermined. It could be determined by comparing the coefficients of viscosity and the conductivity for heat of the gas, since these depend on the mean values of different powers of  $\beta$ , and the ratio of such quantities evidently depends on  $p$ . In this paper I shall not consider the theory of the conduction of heat. If it were the same on the vortex atom theory of gases as on the ordinary theory, then the distribution of velocities would follow Maxwell's law, as the values of the ratio of the coefficient of viscosity to the conductivity deduced from this law agree fairly well with experiment. For this to be the case—

$$\frac{3p-16}{4} = \frac{1}{2},$$

or

$$p=6,$$

and the number of molecules which have the quantities  $\alpha$  and  $\beta$  between  $\alpha$  and  $\alpha + \delta\alpha$ ,  $\beta$  and  $\beta + \delta\beta$  would be proportional to—

$$e^{-h(\alpha+\beta)} \alpha^3 \beta^3 d\alpha d\beta.$$