IV. *On some Applications of Dynamical Principles to Physical Phenomena.*

By J. J. THOMSON, *M.A., F.R.S., Fellow of Trinity College, Cambridge, and Cavendish Professor of Experimental Physics.*

Received December 16, 1884,—Read January 8, 1885.

§ 1. THE tendency to apply dynamical principles and methods to explain physical phenomena has steadily increased ever since the discovery of the principle of the Conservation of Energy. This discovery called attention to the ready conversion of the energy of visible motion into such apparently dissimilar things as heat and electric currents, and led almost irresistibly to the conclusion that these too are forms of kinetic energy, though the moving bodies must be infinitesimally small in comparison with the bodies which form the moving pieces of any of the structures or machines with which we are acquainted. As soon as this conception of heat and electricity was reached mathematicians began to apply to them the dynamical method of the Conservation of Energy, and many physical phenomena were shown to be related to each other, and others predicted by the use of this principle; thus, to take an example, the induction of electric currents by a moving magnet was shown by VON HELMHOLTZ to be a necessary consequence of the fact that an electric current produces a magnetic field. Of late years things have been carried still further; thus Sir WILLIAM THOMSON in many of his later papers, and especially in his address to the British Association at Montreal on "Steps towards a Kinetic Theory of Matter," has devoted a good deal of attention to the description of machines capable of producing effects analogous to some physical phenomenon, such, for example, as the rotation of the plane of polarisation of light by quartz and other crystals. For these reasons the view (which we owe to the principle of the Conservation of Energy) that every physical phenomenon admits of a dynamical explanation is one that will hardly be questioned at the present time. We may look on the matter (including, if necessary, the ether) which plays a part in any physical phenomenon as forming a material system and study the dynamics of this system by means of any of the methods which we apply to the ordinary systems in the Dynamics of Rigid Bodies. As we do not know much about the structure of the systems we can only hope to obtain useful results by using methods which do not require an exact knowledge of the mechanism of the system. The method of the Conservation of Energy is such a method, but there are others which hardly require a greater knowledge of the structure of the system and yet are capable of giving us more definite information than that principle when used in the ordinary way. LAGRANGE'S equations and HAMILTON'S method of Varying Action are methods of this kind, and it is the object of this paper to apply these methods to study the transformations of some of the forms of energy, and to show how useful they are for coordinating results of very different kinds as well as for suggesting new phenomena. A good many of the results which we shall get have been or can be got by the use of the ordinary principle of Thermodynamics, and it is obvious that this principle must have close relations with any method based on considerations about energy.

LAGRANGE'S equations were used with great success by MAXWELL in his 'Treatise on Electricity and Magnetism,' vol. ii., chaps. 6, 7, 8, to find the equations of the electromagnetic field.

In order to confine this paper to a reasonable size I shall limit myself to the consideration of the relations existing between various phenomena in elasticity, heat, electricity and magnetism, but even with this limitation it will only be possible to consider a few of the more prominent out of the many phenomena to which the method can be applied.

When we apply LAGRANGE'S or HAMILTON'S methods to discuss the motion of any material system we have first to choose coordinates which can fix the configuration of the system, and then to find an expression for the kinetic energy in terms of these coordinates and their differential coefficients with respect to the time. Before applying these methods therefore to any physical phenomenon we must have coordinates which are sufficient to fix the configuration of the system which takes part in the phenomenon we are considering. The notation which we shall use will be as follows:—

To fix the geometrical configuration of the system, *i.e.*, to fix the position in space of any bodies of finite size which may be in the system, we shall use coordinates denoted by the letters $x_1, x_2 \ldots, x_n$, and when we want to denote a geometrical coordinate generally without reference to any one in particular we shall use the letter x.

To fix the configuration of the strains in the system we shall in any particular case use the ordinary strain components a, b, c, e, f, g (THOMSON and TAIT'S 'Natural Philosophy,' vol. ii., § 669), but as it will be convenient to have a letter typifying the coordinates generally we shall use the letter w for this purpose.

To fix the electrical configuration of the system we shall use coordinates denoted by the letters $y_1, y_2 \ldots$, and y for the typical coordinate, where y in a dielectric is what MAXWELL calls an electric displacement and in a conductor the time integral of a current.

To fix the configuration of the magnetic field by coordinates in such a way that LAGRANGE'S equations can be used I have found it necessary, for reasons which will be given later, to introduce two coordinates to fix the magnitude of the intensity of magnetisation at a point: one of these is what we may call a kinosthenic* coordinate,

* I am indebted to Professor J. P. POSTGATE, Fellow of Trinity College, Cambridge, for this word.

i.e., one which only enters into the expression for the kinetic energy through its differential coefficient with respect to the time, so that the kinetic energy is not an explicit function of this coordinate, but only of its rate of change; the other coordinate is of a geometrical kind. This way of fixing the intensity of magnetisation is the mathematical analogue of AMPÈRE'S theory of magnetism, which supposes that electrical currents flow round the molecules of bodies, and that a body is magnetised when its molecules are so arranged that the normals to the planes of the currents which flow round them are not distributed uniformly in all directions. Thus the kinosthenic coordinates would be the ones fixing the molecular currents, and the geometrical ones those fixing the arrangement in space of the normals to the planes of the currents. We shall call the kinosthenic coordinate ζ, and the geometrical one η, and suppose they are chosen so that the intensity of magnetisation is $\eta\xi$ where ξ is the generalised component of momentum of the type ζ.

We must now consider how to represent the temperature of a body so as to bring it within the power of dynamical methods. In the ordinary kinetic theory of gases the temperature is taken to be the mean kinetic energy of translation of the molecules of the gas, and there are reasons for believing that a similar thing may be true for bodies in the solid and liquid as well as in the gaseous state. We shall therefore suppose that the temperature is represented by that part of the kinetic energy in unit of volume which involves the squares and products of the velocities of a system of coordinates denoted by the symbols $u_1, u_2 \ldots, u_m$. A moment's consideration will show that the u's must be kinostenic coordinates, *i.e.*, that the kinetic energy is a function of the differential coefficients of these quantities with respect to the time and not of the quantities themselves; and since the kinetic energy cannot be altered if we reverse the motion of the system whose kinetic energy is supposed to be measured by the temperature, the expression for the kinetic energy cannot contain any terms which involve the product of a "u" velocity with one of the type x, y, z, or w. We may therefore take $u_1, u_2 \ldots, u_m$ to be principal coordinates, and suppose that that part of the kinetic energy per unit of volume which depends upon the squares of their velocities, and which we shall call Θ, is given by the equation

$$\Theta=\tfrac{1}{2}\{[u_1u_1]\dot{u}_1^2+[u_2u_2]\dot{u}_2^2+\ldots\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

or if $v_1, v_2 \ldots$ be the momenta corresponding to $u_1, u_2 \ldots$

$$\Theta=\tfrac{1}{2}\left\{\frac{v_1^2}{[u_1u_1]}+\frac{v_2^2}{[u_2u_2]}+\ldots\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

and this form is more convenient for some purposes than the preceding one. Since the temperature can be fixed by one coordinate when it is uniform, we must suppose, at any rate when things are in a steady state, that there are $(m-1)$ linear relations between the m quantities $\dot{u}_1, \dot{u}_2 \ldots$. We might easily alter the equations of motion so as to allow for this relation, or we might eliminate $(m-1)$ of the quantities $\dot{u}_1, \dot{u}_2 \ldots$, but it is more convenient to use the expressions for Θ given in equations (1) and (2): the reason for this is that when any property of a body depends upon the temperature it does not depend upon the value of one of the $\dot{u}$'s more than another, but only on the value of Θ: the $\dot{u}$'s never occur except as parts of the expression Θ.

We have at present only considered kinetic energy and have not said anything about potential energy, but we shall see that if we give a sufficiently wide meaning to the term material system we can explain all the effects produced by potential energy by considerations about kinetic energy alone. There is an advantage gained by doing this, because kinetic energy appears to be a much more fundamental conception than potential energy. When we can explain any phenomenon as a property of the motion of bodies, we have got what may be called a physical explanation of the phenomenon, and any further explanation must be rather metaphysical than physical; it is not so, however, with regard to potential energy, the use of this quantity cannot in any ordinary sense of the word be said to explain any physical phenomenon, it does little more than embody the results of experiments in a form well adapted to mathematical investigation. An investigation similar to that given in THOMSON and TAIT's 'Treatise on Natural Philosophy,' vol. i., p. 320, 2nd edition, shows that the effects produced by the potential energy of a system (A) can be explained by changes in the kinetic energy of another system (B) connected with (A). We must suppose that there are portions of matter connected with the system (A) and capable of motion which are not fixed by any of the coordinates which we have already spoken about, viz., the geometric, electric, magnetic, elastic, and temperature coordinates, and that the coordinates fixing the position of these portions of matter enter the expression for the kinetic energy only through their different coefficients. An analogous case is that of a sphere surrounded by water; in order to fix the configuration of the water it would be necessary to use an infinite number of coordinates, but these would enter the expression for the kinetic energy of the sphere and water only through their differential coefficients. For the sake of brevity we have called coordinates of this kind kinosthenic coordinates.

The coordinates fixing the configuration of the portions of matter mentioned above are kinosthenic coordinates. We shall denote them by the letters $\chi_1, \chi_2 \ldots, \chi_n$. We shall suppose for the sake of simplicity that there are no terms containing the product of a differential coefficient of one of the χ's with a differential coefficient of one of the geometric, elastic, electric, magnetic, or temperature coordinates.

Since $\chi_1, \chi_2 \ldots$ do not enter into the expression for the kinetic energy T of the systems (A) and (B), we have by LAGRANGE's equations

$$\left.\begin{aligned} \frac{d\mathrm{T}}{d\dot{\chi}_1}&=c_1 \\ \frac{d\mathrm{T}}{d\dot{\chi}_2}&=c_2 \\ \cdots& \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

where $c_1, c_2 \ldots, c_n$ are constants.

We can find the values of $\dot{\chi}_1, \dot{\chi}_2 \ldots$ from these equations in terms of $c_1, c_2 \ldots$, and substitute them in the expression for T, which will then be of the form

$$\mathfrak{T}+\mathrm{K}$$

where $\mathfrak{T}$ is a homogeneous quadratic function of the differential coefficients of the x, y, z and w coordinates; in fact, the kinetic energy of the system (A) and K is a quadratic function of the c's involving it may be the coordinates x, y, z and w, but not their differential coefficients; K is evidently equal in magnitude to the kinetic energy of the system (B). Equations (19) on page 323 of THOMSON and TAIT'S 'Natural Philosophy,' vol. i., 2nd ed., show that if we consider only the kinetic energy, LAGRANGE'S equation takes the form

$$\frac{d}{dt}\frac{d\mathfrak{T}}{d\dot{x}}-\frac{d\mathfrak{T}}{dx}+\frac{d\mathrm{K}}{dx}=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

but if the system A had possessed potential energy equal to V the equation (considering A alone) would have been

$$\frac{d}{dt}\frac{d\mathfrak{T}}{d\dot{x}}-\frac{d\mathfrak{T}}{dx}+\frac{d\mathrm{V}}{dx}=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Thus the effect of the system (B) on (A) is the same as if (A) possessed potential energy equal to K, which is, as we saw, the kinetic energy possessed by the system B, which is fixed by the kinosthenic coordinates. Thus we may look on the potential energy of any system (A) as being the kinetic energy of a kinosthenic system (B) connected with A; and so we may regard all energy as kinetic. If we do this it will simplify some of the dynamical principles very much. We may take as our fundamental principle HAMILTON'S Principle of Varying Action, as we can very readily deduce from this principle the ordinary dynamical methods, such as LAGRANGE'S and HAMILTON'S equations. But if all the energy is kinetic, then by the principle of the conservation of energy the magnitude of the kinetic energy remains constant, and the principle of least action takes the very simple form that, with a given quantity of energy, any material system will by its unguided motion pass from one configuration to another in the least possible time, where, of course, in the phrase material system we include the kinosthenic systems whose motion produce the same effects as the potential energy of the original system, and two configurations are not supposed to coincide unless the configuration of these systems coincide also.

We shall, however, in the following investigations sometimes make use of the potential energy in the usual way, as the analysis is the same, and the use of the ordinary method saves a good deal of explanation.

§ 2. Let us now go on to consider the various terms in the expression for the kinetic energy of a material system, taking into account the geometric, electric, magnetic, and thermal conditions of the system. We are dealing with five sets of coordinates—the sets we have denoted by the letters x, y, z, u, w.

The kinetic energy T will be of the form

$$
\begin{aligned}
T=\tfrac{1}{2}\{&[x_1x_1]\dot{x}_1^2+2[x_1x_2]\dot{x}_1\dot{x}_2+\ldots \\
&+[y_1y_1]\dot{y}_1^2+2[y_1y_2]\dot{y}_1\dot{y}_2 \\
&+2[xy]\dot{x}\dot{y}+\ldots \\
&+\ldots \qquad \}
\end{aligned}
$$

where the quantities denoted by $[x_1x_1]$, $[y_1y_1]$, $[x_1y_1]$, may be functions of the coordinates x, y, z, u, w.

The various terms in T can be divided into fifteen types; there are five sets, one corresponding to each of the coordinates x, y, z, u, w, where the terms are of the same character as

$$[x_1x_1]\dot{x}_1^2+2[x_1x_2]\dot{x}_1\dot{x}_2$$

where each term involves the squares of the velocities of the coordinates of one kind, or the product of two velocities of the same kind: it is evident that each of these five types can exist in actual dynamical systems.

There are ten sets of the type

$$[xy]\dot{x}\dot{y}$$

involving the product of the differential coefficients of two coordinates of different kinds: we can see, however, that some of these can not exist in any actual material system. Thus, for example, since the u coordinates only enter through the temperature, there can be no terms involving the product of the differential coefficient of u and the differential coefficient of any of the other coordinates: this consideration reduces the ten sets to six. To determine whether terms of the type of any particular set exist or not we must determine what the consequences would be if terms of this type did exist; if these are contrary to experience we conclude that terms of this type do not exist. We can determine these consequences in the following way. Suppose we have a term in the kinetic energy equal to

$$(\lambda\mu)\dot{\lambda}\dot{\mu}$$

where λ and μ may be any of the five kinds of coordinates which we are considering. We have by LAGRANGE'S equations

$$\frac{d}{dt}\frac{dT}{d\dot{\lambda}}-\frac{dT}{d\lambda}= \text{external force of the type } \lambda \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

If instead of T we consider only the term $(\lambda\mu)\dot{\lambda}\dot{\mu}$ we see that it requires the existence of a force of the type λ equal to

$$-\left[(\lambda\mu)\ddot{\mu}+\frac{d}{d\mu}(\lambda\mu)\dot{\mu}^2+\Sigma\frac{d}{d\nu}(\lambda\mu)\dot{\mu}\dot{\nu}\right] \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

and a force of the type μ equal to

$$-\left[(\lambda\mu)\ddot{\lambda}+\frac{d}{d\lambda}(\lambda\mu)\dot{\lambda}^2+\Sigma\frac{d}{d\nu}(\lambda\mu)\dot{\lambda}\dot{\nu}\right] \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

when ν is a coordinate of any other type.

Thus as it is clearer to take a definite case, suppose that λ is the geometrical coordinate x, and μ then electrical coordinate y; then if the term $(xy)\,\dot{x}\dot{y}$ occurred in the expression for the kinetic energy, the mechanical force produced by a varying current would be different from that produced by a steady one; this is shown by the existence of the term $(xy)\ddot{y}$ in equation (7), which now represents the force of type x, *i.e.*, the mechanical force. If $[xy]$ were a function of y, the term $\frac{d}{dy}[xy]\dot{y}^2$, which occurs in the expression for the mechanical force, shows that a current would produce a mechanical force proportional to the square of the intensity of the current, and which therefore would not be reversed if the direction of the current were reversed. Again, if we consider the expression for the force of type y, that is, the electromotive force, we see that the existence of this term implies the production of an electromotive force by a body whose velocity is changing depending on the acceleration of the body; this is indicated by the term $[xy]\ddot{x}$: if $[xy]$ were a function of x, equation (8) shows that a moving body would produce an electromotive force proportional to the square of its velocity. As none of these effects have been observed we conclude that this term does not exist.

Another point worthy of attention is that the existence of a force of any type ξ say, in any region, implies the existence in the expression for the kinetic energy of that region of terms involving ξ or $\dot{\xi}$, so that if we alter the values of ξ or $\dot{\xi}$ by external means there will be either an absorption or an emission of energy in the region which is the seat of the force of type ξ. This is one method of determining the seat of a force. The kind of energy absorbed or emitted will depend on the coordinates with which ξ and $\dot{\xi}$ are associated in the expression for the kinetic energy.

§ 3. We shall now go through the various types of the terms which involve the products of the differential coefficients of coordinates of different kinds, and see whether they exist or not.

1. Terms of the type $\{xy\}\dot{x}\dot{y}$: we have just seen that these terms do not exist. See also MAXWELL'S 'Electricity and Magnetism,' vol. ii., chap. 7.

2. Terms of the type $\{xz\}\dot{x}\dot{z}$: z is a coordinate fixing the magnetic configuration; we can see just as in the last case that the existence of this term would require the production of a magnetic force by a body whose velocity is changing depending on the rate of change of the velocity, as well as other similar effects. As these have not been observed, we conclude that this term does not exist.

3. Terms of the type $\{xu\}\dot{x}\dot{u}$: u is a coordinate helping to fix the temperature; as the "u's" only enter the expression for the kinetic energy through the temperature, *i.e.*, through terms of the type $\{uu\}\dot{u}^2$, we see that these terms do not exist.

4. Terms of the type $\{xw\}\dot{x}\dot{w}$: w is a coordinate helping to fix the strain configuration of the system; these may exist in a vibrating solid which has also got a motion of translation, for the velocity of any point in a small element of the solid equals the velocity of the centre of gravity of that element plus the velocity of the point relatively to the centre of gravity. This latter velocity will involve $\dot{w}$, so that the square of the velocity, and therefore the kinetic energy, may involve the product $\dot{x}\dot{w}$.

5. Terms of the type $\{yz\}\dot{y}\dot{z}$: the production of a magnetic field by an electric current such that the direction of magnetic force is reversed when the direction of the current is reversed shows that terms of this type must exist.

6. Terms of the type $\{yu\}\dot{y}\dot{u}$: the reasoning given when we were discussing (3) terms of the type $\{xu\}\dot{x}\dot{u}$ shows that terms of this type do not exist.

7. Terms of the type $\{yw\}\dot{y}\dot{w}$: by reasoning in the same way as we did about the general term $[\lambda\mu]\dot{\lambda}\dot{\mu}$, we can see that if this term exists a varying current will produce strains differing from those produced by a steady current, and also an electromotive force could be produced by merely altering the state of strain of a body. As these effects are not known to occur, we conclude that terms of this type do not exist.

8. Terms of the type $\{zu\}\dot{z}\dot{u}$: the reasoning given when we considered the term (3) shows that terms of this type do not exist.

9. Terms of the type $\{zw\}\dot{z}\dot{w}$: terms of this type would correspond to effects analogous in character to those due to the terms of the type $\{yw\}\dot{y}\dot{w}$, and as these have not been observed, we conclude that terms of this type do not exist.

10. Terms of the type $\{uw\}\dot{u}\dot{w}$: the reasoning given when we considered the term (3) shows that terms of this type do not exist.

We thus see that all the terms which occur in the expression for the kinetic energy of any real material system are of one or other of the following types :—

$$\{xx\}\dot{x}^2;$$

$$\{yy\}\dot{y}^2;$$

$$\{zz\}\dot{z}^2;$$

$$\{uu\}\dot{u}^2;$$

$$\{ww\}\dot{w}^2;$$

$$\{xw\}\dot{x}\dot{w};$$

$$\{yz\}\dot{y}\dot{z}.$$

§ 4. We must now proceed to examine the terms of these types in greater detail, and see what coordinates the coefficients $\{xx\}$, $\{yy\}$, &c., involve.

Let us commence with the term $\{xx\}\dot{x}^2$, which corresponds to the expression for the kinetic energy in the ordinary dynamics of a rigid body. We have to consider what coordinates the quantity $\{xx\}$ can be a function of. We know that it may be a function of the geometrical coordinates x, but we need not consider here the consequences of this, as they are fully investigated in treatises on ordinary dynamics. Next, $\{xx\}$ may be a function of the electrical coordinates y, for in a paper published in the Philosophical Magazine for April, 1881, I have shown that the kinetic energy of a small sphere of mass m, charged with a quantity e of electricity, and moving with a velocity v, is

$$\left\{\tfrac{1}{2}m+\frac{2}{15}\frac{\mu e^2}{a}\right\}v^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

where a is the radius of the sphere, and μ the magnetic permeability of the medium surrounding the sphere. Thus $\{xx\}$ may be a function of the electrical coordinates. The easiest way of finding the effects of the electrification on the motion of the body is to notice that by equation (9) these effects are exactly the same as those due to an increase $4\mu e^2/15a$ in the mass of the sphere, so that whenever an electric charge is communicated to a moving sphere its velocity will be impulsively changed. If the sphere is in air there will, of course, be a limit to the quantity of electricity which can be accumulated on it, and so a limit to the apparent increase in mass. Taking Dr. MACFARLANE'S value for the electric strength of air (Phil. Mag., Dec., 1880), viz., 75, as the intensity in electrostatic measure in C.G.S. units of the greatest force which a fairly thick layer of air can bear, it is easy to see from equation (9) that the ratio of the greatest apparent increase in mass to the mass of the sphere is of the order $3 \cdot 10^{-19}/\rho$ where ρ is the density, measured in C.G.S. units, of the sphere which is supposed to be solid. If the charge is on the surface of a thin spherical shell, the ratio will be greater in the proportion of the radius of the shell to three times its thickness.

Let us now go on to consider the electrical effects of this term. Since the potential energy of a charged sphere is $e^2/2c$, where c is the capacity of the sphere, we see from equation (9) that the motion of the sphere will alter the coefficient of e^2 in the expression for the energy, and will therefore alter the capacity. To find this alteration let us suppose that the charge on the sphere is increased by δe, and that Q is the external electromotive force acting on the sphere, the energy of the system is

$$\left(\frac{m}{2}+\frac{2}{15}\frac{\mu e^2}{a}\right)v^2+\frac{1}{2}\frac{e^2}{\mathrm{K}a}, \quad \ldots \ldots \ldots \quad (10)$$

if K be the specific inductive capacity of the medium surrounding the sphere, if the increment in v be δv, and in e δe, the increment in the energy

$$=\left(m+\frac{4}{15}\frac{\mu e^2}{a}\right)v\delta v+\frac{4}{15}\frac{\mu v^2}{a}e\delta e+\frac{e\delta e}{\mathrm{K}a} \quad \ldots \ldots \ldots \quad (11)$$

but by the conservation of energy this must equal

$$\mathrm{Q}\delta e$$

Now since the momentum of the sphere is not altered

$$\left(m+\frac{4}{15}\frac{\mu e^2}{a}\right)\delta v+\frac{8}{15}\frac{\mu e}{a}v\delta e=0 \quad \ldots \ldots \ldots \quad (12)$$

eliminating δv between these equations we find

$$\frac{e}{\mathrm{K}a}\left\{1-\frac{4}{15}\mu\mathrm{K}v^2\right\}=\mathrm{Q} \quad \ldots \ldots \ldots \quad (13)$$

so that the capacity of the sphere is increased in the ratio of 1 to $1-\frac{4}{15}\mu\mathrm{K}v^2$; or since by the electromagnetic theory of light $\mu\mathrm{K}=1/u^2$ where u is the velocity of light, the capacity of the sphere is increased in the ratio of 1 to $1-\frac{4}{15}\frac{v^2}{u^2}$, and thus the alteration depends on the square of the ratio of the velocity of the sphere to the velocity of light, and will consequently be very small. If the earth does not carry the ether with it, the alteration in velocity which occurs at a point on the earth's surface will produce a diurnal variation in the capacity of a condenser at that point. If the condenser is a sphere, the maximum diurnal variation in its capacity, which will be when the direction of motion of the solar system, is parallel to the direction of motion of the earth in its orbit: is about 4×10^{-8} per cent. of the capacity of the condenser. This is much too small to observe, but it is remarkable that the capacity of a condenser should depend upon its velocity.

There seems nothing to show that $\{xx\}$ is a function of the magnetic coordinates z, and it cannot be a function of the temperature coordinates u because these coordinates are kinosthenic, *i.e.*, they only enter the expression for the kinetic energy through their differential coefficients.

In order to see that $\{xx\}$ may be a function of the strain coordinates w, it is convenient to notice that the kinetic energy of a system in ordinary dynamics may be written in the form

$$M\dot{\bar{x}}^2 + Mk^2\dot{\theta}^2 . \quad \quad (14)$$

where M is the mass, $\bar{x}$ a coordinate fixing the position of the centre of gravity, k a radius of gyration, and θ the angle made by a line fixed in the body with a line fixed in space. Terms of the type $M\dot{\bar{x}}^2$ evidently cannot involve the w coordinates, but terms of the type $Mk^2\dot{\theta}^2$ may; for take the simple case of a bar which is compressed at its middle and extended at its ends, rotating about an axis through its centre, it is easy to see that the moment of inertia of this rod about the axis of rotation is less than it would be if the rod were unstrained, and thus Mk^2 may be a function of the strain components. These components will in general only enter through the expression for the alteration in the density, *i.e.*, using THOMSON and TAIT's notation through the expression $d\alpha/dx + d\beta/dy + d\gamma/dz$, and this expression will only occur raised to the first power; if we employ the energy method of forming the equations of elasticity, we easily find that the presence of this term leads to the introduction of the so-called "centrifugal force" into the equations of elasticity for a rotating elastic solid.

Collecting our results we see that $\{xx\}$ may be a function of the coordinates x, y and w, but not of u and probably not of z.

§ 5. The next terms which we have to consider are the terms of the type $\{yy\}\dot{y}^2$, which is supposed to include terms of the form $\{y_1y_2\}\dot{y}_1\dot{y}_2$.

Now since the kinetic energy of a circuit carrying a current $\dot{y}$ is $\frac{1}{2}L\dot{y}^2$ if L be the coefficient of self-induction of the circuit, we see that $\{yy\}$ is a coefficient of self-induction; and since the coefficient of self-induction depends on the shape of the circuit $\{yy\}$ will be a function of the geometrical coordinates which fix the shape of the circuit: since there is a mechanical force between two circuits conveying electric currents, $\{y_1y_2\}$ must be a function of the geometrical coordinates which fix the relative position of the circuits, for if it were not so, we see by LAGRANGE's equations that there would be no mechanical action between the circuits. Let us suppose that we have two circuits, and that the current in one circuit is $\dot{y}_1$ and the current in the other $\dot{y}_2$. Let the kinetic energy of the system be

$$\tfrac{1}{2}L\dot{y}_1^2 + M\dot{y}_1\dot{y}_2 + \tfrac{1}{2}N\dot{y}_2^2 \quad \quad (15)$$

then if x be any coordinate fixing the position of one circuit with respect to the other, LAGRANGE's equation for x shows that there will be a force tending to increase x equal to

$$\tfrac{1}{2}\frac{d\mathrm{L}}{dx}\dot{y}_1^2+\frac{d\mathrm{M}}{dx}\dot{y}_1\dot{y}_2+\tfrac{1}{2}\frac{d\mathrm{N}}{dx}\dot{y}_2^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

Now we see that $d\mathrm{L}/dx$ and $d\mathrm{N}/dx$ must vanish, otherwise there would be a force between the two circuits when the current in one of them vanished, thus we see that the coefficients of self-induction are independent of the position of the other circuits in the neighbourhood. Thus the force tending to increase x reduces to

$$\frac{d\mathrm{M}}{dx}\dot{y}_1\dot{y}_2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

We see also from LAGRANGE's equation for the coordinate y_1 that

$$\frac{d}{dt}\frac{d\mathrm{T}}{d\dot{y}_1}-\frac{d\mathrm{T}}{dy_1}=\text{ external force of type } y_1. \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

But $\frac{d\mathrm{T}}{dy_1}=0$, as we shall see directly, and therefore

$$\frac{d}{dt}(\mathrm{L}\dot{y}_1+\mathrm{M}\dot{y}_2)=\text{ external force of type } y_1 \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

so that the term $d(\mathrm{L}\dot{y}_1+\mathrm{M}\dot{y}_2)/dt$ will produce the same effect as an external electromotive force equal to

$$-\frac{d}{dt}(\mathrm{L}\dot{y}_1+\mathrm{M}\dot{y}_2) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

and thus there is an electromotive force of this magnitude acting round the circuit through which the current $\dot{y}_1$ flows. This expression expresses the law of the induction of currents, either by the motion of neighbouring circuits which convey currents or by the alteration in the magnitude of the currents flowing through these circuits. This example is given in MAXWELL's 'Electricity and Magnetism,' vol. ii., chap. 7; it illustrates the power of the method very well, as the existence of a mechanical force between two circuits showed that there was a term of the form $\mathrm{M}\dot{y}_1\dot{y}_2$ in the expression for the kinetic energy, and from this the law of induction followed at once by LAGRANGE's equations. There is no experimental evidence to show that $\{yy\}$ is ever a function of y the electrical coordinates, and when the system consists of a lot of conducting circuits it certainly is not, for if it were the coefficients of self and mutual induction in a system of circuits would depend upon the length of time the current had been flowing through the circuits; in any case it would involve the existence of electromotive forces which would not be reversed if the directions of all the electric displacements in the field were reversed.

There seems also good reason for believing that $\{yy\}$ is not a function of the magnetic coordinates z, for if it were, a current $\dot{y}_1$ would produce a magnetic force proportional to $\dot{y}_1^2$, and thus the force would not be reversed if the direction of the current were reversed; as no such forces have been observed we conclude that $\{yy\}$ is not a function of z. It cannot be a function of the temperature coordinates u because these are kinosthenic coordinates.

The question as to whether $\{yy\}$ is a function of the elastic coordinates w or not is one where the experimental evidence is somewhat conflicting. Both WERTHEIM (Ann. de Chim. et de Phys. [3], 12, p. 160, or WIEDEMANN's 'Lehre von der Elektricität,' vol. ii., p. 403) and TOMLINSON (Phil. Trans., 1882) have observed that the elasticity of a wire is less when a current is passing along the wire than when it is not, and that this diminution in the elasticity is not due to the heat generated by the current. STREINTZ (Wien. Ber. [2], 67, p. 323, or WIEDEMANN's 'Lehre von der Elektricität,' vol. ii., p. 404), on the other hand, was unable to detect any such effect. Supposing that the passage of a current of electricity along a wire does alter the elasticity of it there must be terms in the kinetic energy of the form

$$\dot{y}^2\{A(e+f+g)^2+B(e^2+f^2+g^2-2ef-2ge-2fg+a^2+b^2+c^2)\} \quad . \quad . \quad (21)$$

where e, f, g, a, b, c denote as before the six strains; comparing this expression with that for the potential energy of a strained solid, and remembering that this energy is kinetic and not potential, we see that the rigidity is *diminished* by $B\dot{y}^2$ and the bulk modulus by $(A-B/3)\dot{y}^2$. Let us consider what electrical effects this term will indicate; since half the coefficient of $\dot{y}^2$ in the expression for the kinetic energy is the coefficient of self induction of the circuit conveying the current $\dot{y}$: we see that if this term exists the coefficient of self-induction of a circuit will be increased by straining the wire which forms the circuit; it will be increased because WERTHEIM's experiments show that the elasticity was diminished by the passage of a current, and therefore that B is positive; so that if a current be flowing along a wire it will be momentarily diminished if the wire be twisted. If the coefficients of induction are altered by straining the wire the force between two currents or two elements of current will be altered by straining the wires along which they flow, even although the intensity of the currents is kept constant; this is contrary to AMPÈRE's hypothesis that the force between two elements of a circuit conveying a current depends upon nothing but the strength of the currents and the position of the elements. If this term exists we shall see if we make a numerical calculation that the alteration in the coefficient of self-induction of a coil due to the straining of the wire of which it is made is likely to be much more easily detected than the corresponding alteration in the elasticity produced by the passage of a current of electricity. For using the C.G.S. system of units the coefficients of elasticity are quantities of the order 10^{11}; thus, taking copper as an example, YOUNG's modulus is $1{\cdot}234\times10^{12}$ and the coefficient of rigidity is $4{\cdot}47\times10^{11}$, so that we shall be under the mark if we take 10^{11} as the value of a coefficient of elasticity for copper. Thus, supposing that we can measure a coefficient of elasticity correct to one part in a thousand, in order that we may be able to detect the alteration in the coefficient of elasticity produced by a current of 10 ampères—whose value in absolute measure is unity—the quantities A and B must be at least $10^{11}/1000$ or 10^8. Now we can produce strains in a copper wire greater than 1/10000 without breaking the wire, and if we produce strains of this magnitude the alteration in the coefficient of self-induction will be between 1 and 10 per centimetre of wire in the coil; supposing A to be 10^8; this is a very large change in the coefficient of self-induction, and ought to be easily detected, much more easily than the corresponding changes in the elasticity of a copper wire produced by the passage of a current of electricity.

To sum up, we see that $\{yy\}$ is a function of the coordinates of the type x, but not a function of the y, z, or u coordinates, while it is doubtful whether it is a function of the strain coordinates w or not.

§ 6. Let us now consider the terms of the type $\{zz\}\dot{z}^2$, these are the terms which express the magnetic energy of the system. In § 1 we discussed a method of fixing the magnetic configuration by means of coordinates. To fix the intensity of magnetisation we use two coordinates, η and ζ, of which η is a coordinate of a geometrical kind, and ζ a kinosthenic coordinate, and they are chosen so that the intensity of magnetisation is $\eta\xi$, where ξ is the momentum of the type ζ. Since ζ is a kinosthenic coordinate ξ is constant, it is therefore convenient to use ξ instead of $\dot{\zeta}$ in the expression for the kinetic energy. Now ROUTH ('Stability of Motion,' p. 62) has shown that if θ be a coordinate and $d\mathrm{T}/d\dot{\theta}=a$, then if we eliminate θ' from the expression

$$\mathrm{T}'=\mathrm{T}-a\theta' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

we may use LAGRANGE's equations for all the other coordinates if we use the function T′ instead of T. Let us apply this theorem to the case we are considering: suppose we have a term of the form $\mathrm{A}\dot{\zeta}^2$ in the expression for the kinetic energy T, we must eliminate $\dot{\zeta}$ by the use of the equation

$$\frac{d\mathrm{T}}{a\dot{\zeta}}=\xi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

and use T′ instead of T where

$$\mathrm{T}'=\mathrm{T}-\dot{\zeta}\xi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

thus the term containing ξ^2 in T′ is

$$-\tfrac{1}{2}\frac{\xi^2}{\mathrm{A}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

this is of the same magnitude as the corresponding term in T but of opposite sign. [This change of sign is instructive if, as we have done in this paper, we regard potential energy as kinetic energy due to kinosthenic coordinates, for it explains at once why the Lagrangian function is T−V and not T+V; for as we saw above, the kinetic energy due to the kinosthenic coordinates enters with the negative sign into the equations of motion.] As we suppose that $\xi\eta$ is the intensity of magnetisation, it is convenient to write (25) as

$$-\frac{1}{2}\frac{\eta^2\xi^2}{B} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (26)$$

where $B=A\eta^2$.

If H be the external magnetic force, the work done when the intensity of magnetisation is increased by $\delta(\eta\xi)$ is $-H\delta(\eta\xi)$ or as ξ remains constant $-H\xi\delta\eta$, and thus LAGRANGE'S equations as modified by ROUTH give

$$\frac{d}{dt}\frac{dT'}{d\dot{\eta}}-\frac{dT'}{d\eta}=H\xi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (27)$$

or when things have settled down to a steady state

$$-\frac{dT'}{d\eta}=H\xi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (28)$$

applying this equation to the term $-\frac{1}{2}\eta^2\xi^2/B$ we get

$$\frac{\xi\eta}{B}\left\{1-\frac{1}{2}\eta\frac{\frac{dB}{d\eta}}{B}\right\}=H \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (29)$$

Thus, since $\xi\eta$ is the intensity of magnetisation and H the magnetising force, the coefficient of magnetic induction, which we shall denote by k, will be given by the equation

$$k=\frac{B}{1-\frac{1}{2}\frac{\eta}{B}\frac{dB}{d\eta}}=\frac{B}{1-\frac{1}{2}\frac{d\log B}{d\log\eta}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (30)$$

At present we shall only consider magnetic induction, and suppose that there is no permanent magnetisation in the part of the system which we are considering.

[When a piece of soft iron is placed in a magnetic field where the force is, for the sake of simplicity, supposed to be parallel to x and equal to X, the energy is $-$IX where I is the intensity of magnetisation, since $I=\xi\eta$, the energy $=-\xi\eta X$.

Now when the energy is expressed in terms of the velocities, LAGRANGE'S equation gives

$$\frac{d}{dt}\frac{dT}{d\dot{x}}-\frac{dT}{dx}=\text{ external force parallel to } x \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (31)$$

so that the existence of T implies a force parallel to x equal to $d\mathrm{T}/dx$, but if T′ is the energy expressed in terms of the momenta instead of the velocities,

$$\frac{d\mathrm{T}'}{dx} = -\frac{d\mathrm{T}}{dx},$$

so that the force parallel to $x = -d\mathrm{T}'/dx$. As in our case the energy is expressed in terms of the momenta, and not of the velocities, we see that the force parallel to x

$$\left.\begin{aligned} &= \xi\eta\frac{d\mathrm{X}}{dx} \\ &= \mathrm{I}\ \frac{d\mathrm{X}}{dx} \end{aligned}\right\} \quad \ldots \ldots \ldots \quad (32)$$

or since I=KX

$$= \tfrac{1}{2}\frac{d}{dx}\{\mathrm{KX}^2\} \quad \ldots \ldots \ldots \quad (33)$$

the usual expression, though (32) is preferable.*]

$\{zz\}$ cannot be a function of the electrical coordinates y, because if it were electromotive forces due to magnetised bodies would exist which would not be reversed when the magnetism of all the bodies in the field was reversed.

$\{zz\}$ cannot be a function of the quantity ξ which helps to fix the intensity of magnetisation, for if it were the kinetic energy would no longer be a quadratic function of ξ, but would involve higher powers. It may, however, be a function of η, and will be so if the coefficient of magnetic induction depends upon the intensity of magnetisation. As experiments show that this is the case we conclude that $\{zz\}$ is a function of η.

A large number of experiments have been made in order to find the value of the coefficient of magnetic induction for all values of the intensity of magnetisation; all these experiments agree in showing that the coefficient diminishes as the intensity of magnetisation increases, so much so indeed that there is a maximum value of the intensity of magnetisation which cannot be exceeded however intense the magnetising force may be. As the experiments are not sufficiently consistent to lead to the determination of the law connecting the intensity of magnetisation with the coefficient of magnetic induction, all that we can do is to take some empirical law which agrees with one set of experiments and deduce its consequences. It is probable that the results will in their main features agree with those which could be deduced from the true law. For our purpose we shall take the empirical law proposed by STEFAN (WIEDEMANN'S 'Lehre von der Elektricität,' vol. 3, p. 432), which is expressed by the equation

* This paragraph has been re-written since the paper was sent in to the Royal Society.

$$\frac{1}{k}=\frac{a\mathrm{I}_0}{\mathrm{I}^{1-n}(\mathrm{I}_0-\mathrm{I})^n}-b \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (34)$$

where k is the coefficient of magnetisation, I the intensity of magnetisation, I_0 the maximum value of this intensity, and a, b, n constants, n being less than unity. If we express the intensity of magnetisation in terms of ξ and η the corresponding law will be

$$\frac{1}{k}=\frac{a'}{\eta^{1-n}(\eta_0-\eta)^n}-b \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (35)$$

where a' is a new constant and η_0 the maximum value of η.

But by equation (32)

$$\frac{1}{k}=\frac{1}{2\eta}\frac{d}{d\eta}\left(\frac{\eta^2}{\mathrm{B}}\right)$$

so that

$$-\frac{1}{2\eta}\frac{d}{d\eta}\left(\frac{\eta^2}{\mathrm{B}}\right)=b-\frac{a'}{\eta^{1-n}(\eta_0-\eta)^n}$$

therefore

$$-\frac{\eta^2}{\mathrm{B}}=b\eta^2-2a'\int_0^\eta\frac{d\eta}{\eta^{-n}(\eta_0-\eta)^n}$$

or

$$-\frac{1}{\mathrm{B}}=b-\frac{2a'}{\eta^2}\int_0^\eta\frac{\eta^n}{(\eta_0-\eta)^n}d\eta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (36)$$

When $\eta=\eta_0$, *i.e.*, when η has its maximum value

$$-\frac{1}{\mathrm{B}}=b-\frac{2a'}{\eta_0}\frac{\Gamma(1+n)\Gamma(1-n)}{\Gamma(2)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (37)$$

where Γ denotes the ordinary gamma function. But

$$\Gamma(1+n)\Gamma(1-n)=\frac{n\pi}{\sin n\pi}$$

and

$$\Gamma(2)=1$$

so that in this case

$$-\frac{1}{\mathrm{B}}=b-\frac{2a'}{\eta_0}\frac{n\pi}{\sin n\pi} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (38)$$

this expression shows that $1/\mathrm{B}$ is finite when $\eta=\eta_0$, but equation (34) shows that $1/k$ is infinite when $\mathrm{I}=\mathrm{I}_0$, which corresponds to $\eta=\eta_0$, so that when the magnetic force is so great that the iron or other magnetisable substance is magnetised to saturation

$-$B and k will differ widely. According to equation (32) the force acting on a piece of soft iron placed in a magnetic field is

$$\tfrac{1}{2}\frac{d}{dx}\{\mathrm{IX}\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (39)$$

if I be the intensity of magnetisation, but according to the ordinary formula it is equal to the differential coefficient with respect to X of the energy per unit volume

$$=\frac{d}{dx}\left\{\frac{\mathrm{I}^2}{\mathrm{B}}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (40)$$

Now we have just seen that B and k differ widely when I approaches the value which it has when the iron is magnetised to saturation, so that if our view be correct the ordinary formula will not give correct results in this case.

$\{zz\}$ cannot be a function of the heat coordinates u, as these are kinosthenic.

JOULE's discovery that a bar of soft iron lengthens when magnetised in the longitudinal direction, and that the increment in length is proportional to the square of the intensity of the magnetisation, shows that $\{zz\}$ is a function of the strain coordinates w. We shall consider in some detail a few of the consequences of this result. Let us suppose that we longitudinally magnetise a soft iron bar, whose length is in the direction of the axis of x; let e, f, g be the dilatations at any point of the bar parallel to the axes of x, y, z respectively.

Then neglecting for the present any torsion there may be in the bar the potential energy V of the strained bar will in the usual notation be

$$\tfrac{1}{2}m(e+f+g)^2+\tfrac{1}{2}n(e^2+f^2+g^2-2ef-2eg-2fg) \quad . \quad . \quad . \quad . \quad . \quad (41)$$

But if T′ has the same meaning as in equation (22)

$$\mathrm{T}'=\text{terms not depending on the magnetic coordinate}-\tfrac{1}{2}\frac{\eta^2\xi^2}{\mathrm{B}}$$

$$= \quad ,, \qquad ,, \qquad ,, \qquad ,, \qquad -\tfrac{1}{2}\frac{\mathrm{I}^2}{\mathrm{B}} \quad . \quad . \quad . \quad (42)$$

if I be the intensity of magnetisation. JOULE's discovery shows that B is a function of the strain coordinates e, f, g. If there is no external force tending to strain the bar, the modified Lagrangian equation gives, when things are in a steady state,

$$-\frac{d\mathrm{T}'}{de}+\frac{d\mathrm{V}}{de}=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (43)$$

with similar equations for f and g.

or considering only the term $-I^2/2B$ in the kinetic energy we have

$$\tfrac{1}{2}I^2\frac{d}{de}\frac{1}{B}+m(e+f+g)+n(e-f-g)=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (44)$$

Now JOULE found that the volume of the bar was not altered, so that $e+f+g=0$ and the last equation becomes

$$\tfrac{1}{2}I^2\frac{d}{de}\left(\frac{1}{B}\right)+2ne=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (45)$$

$$\text{or } ne=\tfrac{1}{4}\frac{d}{de}\left(-\frac{I^2}{B}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (46)$$

as the bar lengthens when it is magnetised e is positive, and thus $-1/B$ increases with e.

Going back to equation (29) we have, if H be the external magnetic force,

$$\tfrac{1}{2}\frac{d}{d\eta}\left(\frac{\eta^2\xi}{B}\right)=H \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (47)$$

or as it may be written

$$I\frac{d}{dI^2}\left(\frac{I^2}{B}\right)=H \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (48)$$

Let us suppose that the magnetising force H remains constant, then I will increase if $-d(I^2/B)/dI^2$ diminishes, and diminish if this quantity increases. If the change in this quantity is due to an increase δe in the elongation, equation (48) shows that the corresponding increase δI in the intensity of magnetisation will be given by the equation

$$\delta I=-I\frac{\dfrac{d}{de}\dfrac{d}{dI^2}\left(\dfrac{I^2}{B}\right)}{\dfrac{d}{dI^2}\left(\dfrac{I^2}{B}\right)}\delta e \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (49)$$

but if k be the coefficient of magnetic induction

$$1/k=\frac{d}{dI^2}\left(\frac{I^2}{B}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (50)$$

by equation (48); so that

$$\delta I=-\frac{I}{k}\frac{dk}{de}\delta e \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (51)$$

but if E be the elongation of the bar produced when the bar is magnetised so that the intensity of magnetisation is I, equation (46) shows that

$$n\frac{dE}{dI^2}=-\tfrac{1}{2}\frac{d}{dI^2}\frac{d}{de}\left(\frac{I^2}{B}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (52)$$

$$=\frac{1}{4k^2}\frac{dk}{de} \text{ by equation (50)}$$

so that by equation (51)

$$\delta I=-\tfrac{1}{4}Ink\frac{dE}{dI^2}\delta e \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (53)$$

Now, since the elongation increases with the intensity of magnetisation—at any rate when the latter is strong—dE/dI^2 is positive, so that from equation (53) δe and δI will have the opposite sign, so that if the strain of a strongly magnetised soft iron bar be increased the intensity of its magnetisation will be diminished. The experiments of VILLARI and Sir WILLIAM THOMSON confirm this result, but each of these physicists found that if the intensity of magnetisation was below a certain critical value, an increase in the strain was accompanied by an increase in the intensity of magnetisation; equation (53) shows that when this is the case dE/dI^2 must be negative, so that a soft iron bar will contract on magnetisation when the magnetising force is small. I have not been able to find any experiments on the extension of soft iron bars under the action of small magnetising forces. Sir WILLIAM THOMSON found that the critical force was about thirty times the earth's vertical force at Glasgow, so that to test this point the magnetising force ought to be less than this value. The critical value of the magnetisation, *i.e.*, the intensity of magnetisation when it is not altered by slightly straining the bar, is given by the equation

$$\frac{d}{de}\frac{d}{dI^2}\frac{I^2}{B}=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (54)$$

Sir WILLIAM THOMSON'S experiments show that this value of I depends upon the state of strain of the bar; hence we see that equation (54) must involve e, so that if $1/B$ be expanded in powers of e it must contain powers of e about the first, but if $1/B$ involves the square of e there will be a term in the expression for the kinetic energy of the form AI^2e^2, and as the coefficients of elasticity depend upon the coefficients of e^2 in the expressions for the kinetic and potential energies, we see that in this case the elasticity of a soft iron wire will be altered by magnetisation. Many experiments have been made to detect this effect, but with negative results; an investigation similar to that on page 313 will show however that the effect on the critical value of magnetisation will be more easily detected than the effect on the elasticity. Since

1/B contains powers of e higher than the first, the terms which involve e may be written in the form

$$\alpha e + \beta e^2 + \ldots$$

Thus, if we neglect the terms containing higher powers of e than the second, $d(1/\mathrm{B})de$, which equals $\alpha + 2\beta e$, will change sign when $e = -\alpha/2\beta$. Equation (46) shows that a soft iron bar will expand or contract on magnetisation according as the strain e in it makes the expression $\alpha + 2\beta e$ negative or positive; this changes sign when e passes through the critical value $-\alpha/2\beta$, so that a soft iron bar when strained beyond a certain limit will contract instead of expanding when it is magnetised. Some experiments which JOULE made on the effect of magnetising soft iron bars which were under great tension confirm this result. JOULE found (STURGEON'S "Annals of Electricity, 1842," Phil. Mag., 1847) that when soft iron wires were stretched beyond a certain limit they became shorter instead of longer when they were magnetised; since the limiting strain is an extension, the limiting value of e, $-\alpha/2\beta$ must be positive, so that α and β must be of opposite signs; and as α is negative when the magnetisation is intense, β must in this case be positive, and if β be positive the coefficients of elasticity will be increased as we can see from equation (41), so that we conclude that the elasticity of a soft iron bar will be diminished by strong magnetisation, *i.e.*, the same force will not stretch it so much.

A great many relations exist between torsion and magnetism, but many of these are relations between permanent magnetism and permanent set, both of which are outside the scope of this paper. We shall confine ourselves to the relations existing between temporary magnetisation and twists which are not large enough to give the body any permanent set. Since the magnetisation of a soft iron bar is altered by twisting it, 1/B (using the same notation as before) must be a function of the torsion coordinates a, b, c: let us suppose as before that the length of the bar is parallel to the axis of x, then the coordinate a will fix the rotation of the bar round this axis.

The couple tending to twist the bar is by the modified Lagrangian equation for a, equal to $d\mathrm{T}'/da$, and this equals

$$-\tfrac{1}{2}\frac{d}{da}\left(\frac{\mathrm{I}^2}{\mathrm{B}}\right) \quad \ldots \ldots \ldots \ldots \quad (55)$$

if ω be the angle through which this couple twists the bar and n the coefficient of rigidity,

$$\omega = -\frac{1}{2n}\frac{d}{da}\left(\frac{\mathrm{I}^2}{\mathrm{B}}\right) \quad \ldots \ldots \ldots \ldots \quad (56)$$

When a twisted bar is magnetised it untwists to a certain extent (WIEDEMANN'S 'Lehre von der Elektricität,' vol. 3, p. 692), but if an untwisted bar be magnetised it does not twist at all; this shows that if 1/B be expanded in ascending powers of a, the first power must be absent, for if it were present an untwisted rod would become twisted when it was magnetised. By a similar investigation to that by which we established equation (53), we find that if δI be the change in the intensity of magnetisation when the twist in the bar is increased by an angle $\delta\omega$, and k the coefficient of magnetisation, then

$$\delta I = -2nIk \frac{d\omega}{dI^2} \delta\omega . \qquad (57)$$

Now since a soft iron wire untwists when the magnetisation is increased, $d\omega/dI^2$ is negative, so that δI and ϖ have the same sign if k be positive; hence we see that the magnetisation is increased by twisting the rod: this agrees with WIEDEMANN's result. He found, however, that when the twist exceeded a certain value the magnetisation was diminished instead of being increased by twisting. Equation (57) shows that in this case $d\omega/dI^2$ must be positive, or when the original twist is great the bar will no longer be untwisted when it is magnetised but twisted. Professor G. WIEDEMANN's experiments show that this is the case.* There are many other relations between magnetism and torsion which can be investigated in this way, but we have no space to consider them here.

§ 7. We must now go on to consider the part of the kinetic energy which depends upon the temperature, and which we denote by $\{uu\}\dot{u}^2$. We considered before a way of fixing the temperature by means of coordinates: we supposed that it was fixed by the coordinates $u_1, u_2, \ldots u_n$. Since the temperature depends upon the whole of the terms involving u, and not in a special way upon any one term in particular, we see that if one of the u's enters into any term it must be because the whole of that part of the kinetic energy which depends upon the differential coefficients of the u's enters into the term. Let us, as before, call this part of the energy Θ, then if $v_1, v_2, v_u, \ldots$ be the momenta of the type $u_1, u_2, \ldots u_u$, we have by equation (8)

$$\Theta = \tfrac{1}{2}\left\{\frac{v_1^2}{\{u_1, u_1\}} + \frac{v_2^2}{\{u_2, u_2\}} + \cdots\right\}$$

As Θ is the energy in unit of volume the total amount of energy of this kind is

$$\iiint \Theta\, dx dy dz \qquad (58)$$

As the consideration of the terms of the type $\{uu\}\dot{u}^2$ is somewhat difficult, in order to simplify it we shall neglect all the effects due to radiation, which is equivalent to supposing that the bodies are incapable of radiating or absorbing heat.

If this is so $\{uu\}$ cannot be a function of the geometrical coordinates x, for if it were it would be possible to alter the temperature of a body by merely moving it about.

* 'Die Lehre von der Elektricität.' Dritter Band. S. 688.

The phenomena of thermo-electricity show that $\{u, u\}$ must be a function of the electrical coordinates y, but it is difficult to determine the form of this function with certainty. The two most striking phenomena in thermo-electricity are (1) the PELTIER effect which shows that electromotive forces exist at the junction of different metals in the thermoelectric circuit, and (2) the THOMSON effect which shows that electromotive forces exist throughout an unequally heated conductor. We shall try to find a term whose presence in the expression for the kinetic energy would correspond to these effects. Let us consider a circuit formed by two metallic wires each parallel to the axis of x, let p, q, r denote the components of the electric displacement parallel to the axes of x, y, z respectively, let us suppose that we have in the expression for the kinetic energy per unit of volume the term

$$\kappa\left(\frac{dp}{dx}+\frac{dq}{dy}+\frac{dr}{dz}\right)\Theta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (59)$$

where κ is a quantity which depends upon the material of which the wire is made. Thus the expression for the energy of the wires will contain the term

$$\iiint\kappa\left(\frac{dp}{dx}+\frac{dq}{dy}+\frac{dr}{dz}\right)\Theta dxdydz \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (60)$$

In order to avoid considerations about discontinuity let us suppose that κ instead of changing abruptly as we go from one wire to another changes very rapidly but continuously throughout a small region enclosing the junction of the wires, when we wish to pass to the actual case we shall suppose the rate of variation to increase and the size of the region to diminish indefinitely.

Now

$$\iiint\kappa\left(\frac{dp}{dx}+\frac{dq}{dy}+\frac{dr}{dz}\right)\Theta dxdydz$$
$$=-\iint\kappa\Theta(lp+mq+nr)dS-\iiint\left(p\frac{d}{dx}(\kappa\Theta)+q\frac{d}{dy}(\kappa\Theta)+r\frac{d}{dz}(\kappa\Theta)\right)dxdydz \quad . \quad (61)$$

where dS is an element of the surface bounding the region through which we integrate, and l, m, n are the direction cosines of the normal to this surface drawn inwards. The case we are considering is that of two wires parallel to the axis of x, for by far the greater part of their length the bending which is necessary to make their ends join being supposed to occupy an indefinitely small space. Let us suppose that the electric displacement is parallel to the axis of x, then the surface integral vanishes and the term under consideration

$$=-\iiint\left(p\frac{d}{dx}(\kappa\Theta)+q\frac{d}{dy}(\kappa\Theta)+r\frac{d}{dz}(\kappa\Theta)\right)dxdydz \quad . \quad . \quad . \quad . \quad . \quad (62)$$

or since q and r both vanish

$$=-\iiint p\frac{d}{dx}(\kappa\Theta)dxdydz \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (63)$$

We see by LAGRANGE's equation for p that this corresponds to an electromotive force at each point of the bar equal to

$$-\frac{d}{dx}(\kappa\Theta) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (64)$$

and if we remember that $d\kappa/dx$ becomes infinite in the limit at the junction of the wires we see by integrating over this region that the potential close to the junction in the wire (2) will exceed that close to the junction in wire (1) by $-(\kappa_2-\kappa_1)\Theta$, where κ_2 and κ_1 are the values of κ in the wires (2) and (1) respectively.

The force $-d(\kappa\Theta)/dx$ is that which corresponds to the THOMSON effect; the difference of potentials at the junction is analogous to the PELTIER effect; it is, however, probable that other terms of the form

$$\iiint \rho\kappa\Theta dxdydz$$

may occur in the expression for the kinetic energy near the junction, and the effects due to these terms may add on to the others; indeed, if the other terms explain the THOMSON effect, there must be some additional terms of this kind required to explain the PELTIER effect, for otherwise the total electromotive force round the circuit would vanish.

Since

$$\frac{dp}{dx}+\frac{dq}{dy}+\frac{dr}{dz}=\rho$$

where ρ is the volume density of the free electricity, we may write

$$\iiint \kappa\left(\frac{dp}{dx}+\frac{dq}{dy}+\frac{dr}{dz}\right)\Theta dxdydz$$

as

$$\iiint \kappa\rho\Theta dxdydz$$

If we keep the term in this form, and consider the temperature effects to which it corresponds, we are led to suspect the existence of some very remarkable phenomena.

Writing Θ in full we see that in the expression for the energy per unit volume there is the term

$$\kappa e\{\{u_1u_1\}\dot{u}_1^2+\{u_2u_2\}\dot{u}_2^2+\ldots\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (65)$$

if e be the charge of electricity per unit volume, or if we write v_1, v_2 . . . for the momenta corresponding to u_1, u_2 . . . we see that the energy may be expressed as

$$\frac{1}{(1+\kappa e)}\left\{\frac{v_1^2}{\{u_1u_1\}}+\frac{v_2^2}{\{u_2u_2\}}+\ldots\right\} \quad . \; . \; . \; . \; . \; . \; . \; . \quad (66)$$

Then if e be increased by δe this part of the energy, which is measured by the temperature, will be diminished by

$$\left\{\frac{v_1^2}{\{u_1u_1\}}+\frac{v_2^2}{\{u_2u_2\}}+\ldots\right\}\frac{\kappa\delta e}{(1+\kappa e)^2} \quad . \; . \; . \; . \; . \; . \; . \; . \quad (67)$$

or if κe be small we may put $1/(1+\kappa e)^2$ equal to 1, and write Θ where Θ is the temperature for

$$\left\{\frac{v_1^2}{\{u_1u_1\}}+\frac{v_2^2}{\{u_2u_2\}}+\ldots\right\}$$

so that (67) becomes

$$\delta\Theta=-\Theta\kappa\delta e \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (68)$$

Thus if κ be positive the temperature of the body will be diminished by communicating to it a positive charge of electricity, or electricity will behave like a substance with real specific heat. Since the electromotive force in an unequally heated conductor $=-d(\kappa\theta)/dx$, when κ is positive the current goes from the hot to the cold parts of the wire, but when this is the case, what Sir William Thomson calls the "specific heat of electricity in the conductor," is negative. In this case we see that the temperature of the body will fall or rise according as a charge of positive or negative electricity is communicated to it; when κ is negative or the "specific heat of electricity in the conductor" positive, the temperature will rise when a positive and fall when a negative charge of electricity is communicated to the body. These effects have, as far as I know, never been observed.

Another relation between heat and electricity is afforded by the phenomenon of pyroelectricity. This phenomenon is well illustrated by a crystal of tourmaline which when warmed becomes positively electrified at one end, which we shall call the positive end, and negatively electrified at the other, which we shall call the negative end. Sir William Thomson has shown that this phenomenon can be explained by supposing that there is an electric displacement in the tourmaline depending upon the temperature. Now if $\{uu\}$ involves the coordinates which fix the electric displacement we can easily see that there must be such an effect. For take the case of a tourmaline crystal whose axis is taken as the axis of x, the positive direction of the axis being that drawn from the negative to the positive ends of the crystal; let p be the electric displacement parallel to this axis, and suppose that in the expression for the energy per unit volume there is the term

$$\phi(p)\Theta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (69)$$

where $\phi(p)$ merely denotes some function of p and Θ denotes the temperature as before. By LAGRANGE'S equation we see that there will be an electromotive force parallel to the axis of x and equal to

$$\phi'(p)\Theta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (70)$$

so that if K be the specific inductive capacity of the tourmaline the electric displacement will be

$$\frac{K}{4\pi}\phi'(p)\Theta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (71)$$

and this will depend upon the temperature, which is all that is required for Sir WILLIAM THOMSON'S explanation. Let us now consider the effect of the term $\phi(p)\Theta$ upon the temperature. We see, as in the last case, that the part of the energy on which the temperature depends may be expressed in the form

$$\left\{\frac{v_1^2}{\{u_1u_1\}}+\frac{v_2^2}{\{u_2u_2\}}+\ldots\right\}\frac{1}{1+\phi(p)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (72)$$

where $v_1, v_2, \ldots v_n$ will remain constant, so that if p be increased by δp the temperature will be diminished by

$$\left\{\frac{v_1^2}{\{u_1u_1\}}+\ldots\right\}\frac{\phi'(p)\delta p}{\{1+\phi(p)\}^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (73)$$

and we may write this as

$$\delta\Theta=-\Theta\frac{\phi'(p)}{1+\phi(p)}.\delta p \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (74)$$

As the positive direction of x is that drawn from the negative to the positive ends of the crystal the electric displacement in this direction must be positive, hence we see from equation (71) that $\phi'(p)$ is positive; but if $\phi'(p)$ be positive we see from equation (74) that when p is increased the temperature will fall. Thus if we take a tourmaline crystal and place it in an electric field so that the line joining the negative end of the crystal to the positive is in the direction of the electric force, the temperature of the tourmaline crystal will fall when the strength of the field is increased and rise when it is diminished.

§ 8. We know that when there are inequalities both of strain and temperature in a circuit made of one kind of substance there will be an electromotive force acting round the circuit, but if the strain be uniform this E.M.F. will vanish however the temperature may vary provided it remains continuous, while if the temperature be uniform the E.M.F. will vanish however the strain may vary from point to point.

Let us suppose that the circuit consists of a wire of which ds is an element of arc, and that e denotes the longitudinal extension; then if in the expression for the kinetic energy there is a term of the form

$$\mathrm{A}y\theta\frac{df(e)}{ds} \quad \ldots \quad (75)$$

where θ denotes the temperature, y the quantity of electricity which has crossed any section of the wire, and $f(e)$ denotes any function of e, LAGRANGE'S equation shows that the E.M.F. round the circuit equals

$$\int \mathrm{A}\theta\frac{df(e)}{ds}ds \quad \ldots \quad (76)$$

this vanishes when θ is constant and also when e is constant, so that the theory does not indicate the existence of a current when experiment shows that there is none. If we suppose that the circuit consists of two pieces of the same kind of wire, in one of which the longitudinal strain is constant and equal to e while the other piece is unstrained, and that one of the junctions of the strained and unstrained pieces is at a temperature θ_1 and the other at a lower temperature θ_2, then the E.M.F. round the circuit tending to make the current flow from the strained to the unstrained wire across the hot junction is

$$\mathrm{A}(\theta_1-\theta_2)\{f(o)-f(e)\} \quad \ldots \quad (77)$$

where $f(o)$ denotes the value of $f(e)$ when (e) is zero. VON TUNZELMANN (Phil. Mag. [5], 5, p. 339) has proved that the current is reversed when the strain exceeds a certain limit, this shows that $f(e)$ cannot be a linear function of e, but must involve powers of e above the first.

Since the term

$$\int \mathrm{A}y\theta\frac{d}{ds}f(e)ds$$

involves the coordinates which fix the state of strain of the wire, it indicates the existence of certain stresses in it. Let α be the displacement of a particle parallel to ds an element of the arc of the wire, then if α becomes $\alpha+\delta\alpha$, e is increased by $d.\delta\alpha/ds$, when this change occurs in e, the change in the term

$$\int \mathrm{A}y\theta\frac{d}{ds}f(e)ds$$

is

$$\int \mathrm{A}y\theta\left\{f'(e)\frac{d^2\delta\alpha}{ds^2}+f''(e)\frac{d.\delta\alpha}{ds}\frac{d\alpha}{ds}\right\}ds,$$

if we integrate this by parts we see that the coefficient of $\delta\alpha$ is

$$\int \mathrm{A}y\left\{\frac{d^2}{ds^2}\theta f'(e)-\frac{d}{ds}\left(\theta f''(e)\frac{d\alpha}{ds}\right)\right\}ds$$

since y is constant along the wire, the expression may be written

$$\int \mathrm{A}y\frac{d}{ds}\left\{\frac{d\theta}{ds}f'(e)\right\}ds \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (78)$$

so that at each point of the rod there is a force tending to increase α equal to

$$\mathrm{A}y\frac{d}{ds}\left\{\frac{d\theta}{ds}f'(e)\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (79)$$

From the form of this expression we see that the force should increase with y, which increases indefinitely with the time the current has been flowing, so that theoretically this expression, and therefore the stress, should increase indefinitely with the time; in this case, however, other considerations will come in to modify this result. If we imagine that the strain is constant along one of the wires the expression (79) becomes

$$\mathrm{A}yf'(e)\frac{d^2\theta}{ds^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (80)$$

but by the theory of the conduction of heat $d^2\theta/ds^2$ varies as $d\theta/dt$ if there is no loss by radiation, so that in this case the force per unit length of the wire varies as

$$yf'(e)\frac{d\theta}{dt} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (81)$$

so that if we have a current flowing through a wire, one half of which is heated to redness and if necessary strained, if the temperature of the wire be allowed to equalise itself by conduction there will be forces along the wire tending either to stretch or compress it, and if the parts which are losing heat are compressed, the parts where the temperature is rising will be extended.

§ 9. Effects of Heat upon Magnetism.

The magnetic properties of bodies are very much affected by heat, thus a very high temperature seems to destroy altogether both the magnetic susceptibility and the power of retaining magnetism. This shows that $\{uu\}$ the type of the coefficient of the squares of the differential coefficient of the u's must involve the coordinates which fix the magnetic configuration; these coordinates are of two kinds: one, which is of a geometrical type, we shall denote as before by η, the other, which is a kinosthenic coordinate, will, as before, be denoted by ζ, and the momentum corresponding to it by ξ; the coefficient of $\dot{u}^2$ cannot contain the kinosthenic coordinate, because if it did it would not be of the dimensions of kinetic energy, it must therefore contain the geometrical coordinate η only. Suppose that in the expression for the kinetic energy there is, using the same notation as before, a term of the form

$$\tfrac{1}{2}f(\eta)\left\{\frac{v_1^2}{\{u_1u_1\}}+\frac{v_2^2}{\{u_2u_2\}}+\dots\right\} \quad (82)$$

then T′ the function which, in the modified Lagrangian equations, takes the place of T will contain the term

$$-\tfrac{1}{2}f(\eta)\left\{\frac{v_1^2}{\{u_1u_1\}}+\frac{v_2^2}{\{u_2u_2\}}+\dots\right\} \quad (83)$$

but we saw before in equation (25) that T′ also contained the term

$$-\tfrac{1}{2}\frac{\eta^2\xi^2}{B}$$

so that if H be the external magnetising force we have, by the modified Lagrangian equations,

$$\tfrac{1}{2}\frac{d}{d\eta}\left(\frac{\eta^2\xi^2}{B}\right)+\tfrac{1}{2}f'(\eta)\left\{\frac{v_1^2}{\{u_1u_1\}}+\frac{v_2^2}{\{u_2u_2\}}+\dots\right\}=H\xi \quad (84)$$

so that if k be the coefficient of magnetic induction

$$\frac{1}{k}=\frac{1}{\xi^2}\frac{d}{d\eta^2}\left(\frac{\eta^2\xi^2}{B}\right)+\frac{1}{\xi^2}\frac{d}{d\eta^2}f(\eta)\Theta \quad (85)$$

where Θ is the temperature. It is convenient to write the terms in this way, because $f(\eta)$ must be an even function of η, otherwise the magnetic susceptibility would be altered by reversing the direction of magnetisation. If k_0 be the part of the magnetic induction which does not depend upon the temperature

$$\frac{1}{k}=\frac{1}{k_0}+\frac{1}{\xi^2}\frac{d}{d\eta^2}f(\eta)\Theta \quad (86)$$

so that, approximately

$$k=k_0\left\{1-\frac{k_0}{\xi^2}\frac{d}{d\eta^2}f(\eta)\Theta\right\} \quad (87)$$

if we suppose that the second term in the bracket is small compared with the first, thus

$$\frac{dk}{d\Theta}=-\frac{k_0^2}{\xi^2}\frac{d}{d\eta^2}f(\eta) \quad (88)$$

Now let us consider the effect of this term on the temperature. There is in the expression for the kinetic energy the term

$$\tfrac{1}{2}f(\eta)\left\{\frac{v_1^2}{\{u_1u_1\}}+\dots\right\} \quad (89)$$

Now, suppose that no heat is supplied to the body, but that η is increased by $d\eta$, the v's will remain constant, so that

$$\delta\Theta = \tfrac{1}{2}f'(\eta)\left\{\frac{v_1^2}{\{u_1u_1\}} + \ldots\right\}\delta\eta \quad \ldots \ldots \quad (90)$$

or if $f(\eta)$ be small compared with unity

$$= \tfrac{1}{2}f'(\eta)\Theta\delta\eta \quad \ldots \ldots \quad (91)$$

since in this case we may write

$$\Theta = \frac{v_1^2}{\{u_1u_1\}} + \ldots \text{ nearly}$$

thus

$$\frac{d\Theta}{d\eta} = \tfrac{1}{2}f'(\eta)\Theta \quad \ldots \ldots \quad (92)$$

$$= -\frac{\eta\xi^2}{k_0^2}\frac{dk}{d\Theta}\Theta \text{ by equation (87)}$$

or

$$\frac{d\Theta}{dI} = -\frac{\Theta}{k_0}\frac{dI}{d\Theta} \quad \ldots \ldots \quad (93)$$

if I be the intensity of magnetisation; if H be the magnetising force we may write this equation, since $I = kH$, as

$$\frac{d\Theta}{dH} = -\Theta\frac{dI}{d\Theta} \quad \ldots \ldots \quad (94)$$

where the magnetising force is supposed to remain constant in the differential coefficient on the right hand side of this equation and the coefficient of magnetic induction on the left.

Thus if the coefficient of magnetic induction increases with the temperature, the temperature of a soft iron bar will be lowered by magnetisation. It is difficult to tell from the experiments which have been made what the effect of temperature on the coefficient of magnetic induction really is; according to WIEDEMANN, when the substance has been repeatedly heated and cooled down again to its initial temperature, the coefficient of magnetic induction of soft iron diminishes as the temperature increases, while for hard steel it increases with the temperature provided the temperature does not exceed a certain limit. Thus the temperature of hard steel ought to fall when it is magnetised and the temperature of soft iron to rise. As far as is known the temperature of all bodies rises on magnetisation, but this may be due to the electric currents induced by the sudden starting of the magnetic field; unless the heat generated by these currents is allowed for the results tell us nothing as to whether the temperature is raised or lowered when the magnetisation is increased.

The effect of temperature depends, too, upon the magnitude of the magnetising force; thus BAUR (WIEDEMANN'S Annalen, xi., 1880) finds that for soft iron the coefficient of magnetic induction increases with the temperature if the magnetising force be less than about 3·6, while if the magnetising force be greater than this the coefficient of magnetic induction diminishes as the temperature increases. Equation (88) shows that if this is the case $df(\eta)/d\eta^2$ must be a function of η; it is probably of the form $\alpha\{1-\kappa\eta^2\}$, where the critical value of η is $1/\sqrt{\kappa}$.

Since the u's are kinosthenic coordinates $\{uu\}$ cannot be a function of them.

As the coefficients of elasticity depend upon the temperature $\{uu\}$ will be a function of the coordinates which fix the strain configuration of the system; now the coefficients of elasticity are the coefficients of the squares of the strain coordinates in the expression for the kinetic energy; so that using the ordinary notation for the strain coordinates we may suppose that the term

$$\tfrac{1}{2}\{\alpha(e+f+g)^2+\beta(e^2+f^2+g^2-2ef-2eg-2fg+a^2+b^2+c^2)\}\left\{\frac{v_1^2}{\{u_1u_1\}}+\ldots\right\} \quad (95)$$

occurs in the expression for the kinetic energy T; in T′, the function which takes the place of T in the modified Lagrangian equations, this term will occur with the opposite sign.

Since the equation for any of the coordinates a, b, c, e, f, g is of the type

$$-\frac{dT'}{de}+\frac{dV}{de}=\text{external force of type } e \quad \ldots\ldots \quad (96)$$

we see that the effect of the presence of the term (94) is the same as if the coefficients of elasticity denoted by THOMSON and TAIT in their treatise on Natural Philosophy by the letters m and n were increased by

$$\left.\begin{aligned}&\alpha\left\{\frac{v_1^2}{\{u_1u_1\}}+\frac{v_2^2}{\{u_2u_2\}}+\ldots\right\} \text{ or } \alpha\theta\\ &\beta\left\{\frac{v_1^2}{\{u_1u_1\}}+\frac{v_2^2}{\{u_2u_2\}}+\ldots\right\} \text{ or } \beta\theta\end{aligned}\right\} \quad \ldots\ldots \quad (97)$$

respectively, if θ be the temperature, since we may neglect the small part of it due to the terms we are considering. Let us now consider the effect of this term upon the temperature, and to illustrate the point in the simplest way let us suppose that all the strains but e vanish, then if e^2 be increased by δe^2 the corresponding increase $\delta\Theta$ in the temperature is by equation (95)

$$\tfrac{1}{2}(\alpha+\beta)\left(\frac{v_1^2}{\{u_1u_1\}}+\ldots\right)\delta e^2 \quad \ldots\ldots \quad (98)$$

or

$$\delta\theta=\tfrac{1}{2}(\alpha+\beta)\theta\delta e^2 \quad \ldots\ldots \quad (99)$$

but by equation (97) we see that

$$\alpha=\frac{dm}{d\theta}\ ;\ \beta=\frac{dn}{d\theta} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (100)$$

so that

$$\delta\theta=\tfrac{1}{2}\theta\frac{d}{d\theta}(m+n)\delta e^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (101)$$

so that if the coefficient of elasticity diminish with the temperature, as is the case for most substances, an increase in the strain will produce a lowering of temperature which can be calculated by equation (101); if, on the contrary, the coefficient of elasticity increase with the temperature so that $d(m+n)/d\theta$ is positive, an increase in the strain will cause the temperature to rise. These results were given by Sir WILLIAM THOMSON in his paper on the "Dynamical Theory of Heat" ('Mathematical and Physical Papers,' xlviii., § 202).

§ 10. The next term in order is the coefficient of the squares of the differential coefficients of the coordinates which fix the strain configuration. From the way this coefficient arises we can see that, in the present state of our knowledge, there is no reason to believe that these involve any of the coordinates which we are considering. For if ξ, η, ζ, are the coordinates of the centre of an element $dxdydz$ of an elastic solid, and if ρ be the density of the element, which is supposed to be so small that its density may be taken as uniform, then the kinetic energy of the strained solid equals

$$\iiint\rho(\dot{\xi}^2+\dot{\eta}^2+\dot{\zeta}^2)dxdydz \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (102)$$

Now $\rho dxdydz$ is not altered by moving the body about, so that when the integration is completed the coefficient of $\dot{w}_1{}^2$ will not involve the geometrical coordinates x. It is conceivable that it might depend upon the electrical coordinates, just as the coefficient of $\dot{x}^2$ may depend upon these quantities; but if this were so, the capacity of a condenser would be altered by making the air between the plates vibrate, and the time of vibration of a bar made of some dielectric would be altered by communicating a charge of electricity to it. This latter phenomenon actually takes place, but it is probably due to an alteration in the elasticity of the bar produced by the electricity, and not to an effect of the kind we are considering. As there is no experimental evidence that the coefficient of $\dot{w}^2$ does involve the electrical coordinates, we shall not consider what the effects of its doing so would be. The same thing applies to the magnetic coordinate η. The coefficient of $\dot{w}_1{}^2$ is evidently not a function of the temperature or strain coordinates.

§ 11. The most important term containing the product of differential coefficients of coordinates of different kinds is the term containing the product of the rate of changes of the electric and magnetic coordinates. In considering this term we shall use the same notation as before, but it will be convenient to resolve η into three components, ηl, ηm, ηn, parallel to the axes of x, y, z respectively. We are at liberty to do this, as η is evidently a vector. If we take AMPÈRE's theory of magnetism, ηl, ηm, ηn will be proportional to the excess of the number of normals to the planes of the currents round the molecules which point in the positive direction of the axes of x, y, z respectively over those which point in the corresponding negative directions. If ξ has the same meaning as before, $\eta l\xi$, $\eta m\xi$, $\eta n\xi$ will be the components of magnetisation in the directions x, y, z respectively. It is proved in MAXWELL's 'Electricity and Magnetism,' vol. ii., art. 634, that if we have currents whose components parallel to the axes of x, y, z respectively are u, v, w, placed in a magnetic field, the kinetic energy of the two

$$=\iiint(\mathrm{F}u+\mathrm{G}v+\mathrm{H}w)dxdydz \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (103)$$

where ('Electricity and Magnetism,' vol. ii., art. 405)

$$\left.\begin{aligned}\mathrm{F}&=\iiint\eta\xi\left(m\frac{dp}{dz}-n\frac{dp}{dy}\right)dxdydz\\ \mathrm{G}&=\iiint\eta\xi\left(n\frac{dp}{dx}-l\frac{dp}{dz}\right)dxdydz\\ \mathrm{H}&=\iiint\eta\xi\left(l\frac{dp}{dy}-m\frac{dp}{dx}\right)dxdydz\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (104)$$

where p is the reciprocal of the distance of the point where the components of magnetisation are $\eta\xi l$, $\eta\xi m$, $\eta\xi n$ from the point where the components of current are u, v, w. To apply LAGRANGE's equations we must see what T′, the function which takes the place of T in the modified Lagrangian equation, becomes in this case, as we saw before that some of the terms occurred with opposite signs in T and T′.

Suppose that in the expression for the kinetic energy we have the terms

$$\tfrac{1}{2}\mathrm{A}\dot{\phi}^2+\mathrm{B}\dot{\phi}\dot{\psi} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (105)$$

and that ϕ is a gyroscopic coordinate, then

$$\mathrm{T}'=\mathrm{T}-\dot{\phi}\Phi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (106)$$

where Φ is the momentum corresponding to ϕ, so that, considering these terms alone,

$$\Phi=\mathrm{A}\dot{\phi}+\mathrm{B}\dot{\psi} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (107)$$

Substituting for $\dot{\phi}$ we find

$$\mathrm{T}'=-\tfrac{1}{2}\frac{\Phi^2}{\mathrm{A}}-\tfrac{1}{2}\frac{\dot{\psi}^2\mathrm{B}^2}{\mathrm{A}}+\frac{\mathrm{B}\Phi\dot{\psi}}{\mathrm{A}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (108)$$

so that though the term in Φ^2 and $\dot{\psi}^2$ occur with opposite signs in T and T′, the term involving the product $\Phi.\dot{\psi}$ occurs with the same sign in both. On this account we may apply LAGRANGE'S equation to the term

$$\iiint(\mathrm{F}u+\mathrm{G}v+\mathrm{H}w)dxdydz$$

as it stands.

If we apply LAGRANGE'S equation to the x coordinate we see that the force per unit volume parallel to x on the element conveying the current u, v, w is

$$u\frac{d\mathrm{F}}{dx}+v\frac{d\mathrm{G}}{dx}+w\frac{d\mathrm{H}}{dx} \quad \ldots \ldots \ldots \ldots \quad (109)$$

or

$$v\left\{\frac{d\mathrm{G}}{dx}-\frac{d\mathrm{F}}{dy}\right\}-w\left\{\frac{d\mathrm{F}}{dz}-\frac{d\mathrm{H}}{dx}\right\}+u\frac{d\mathrm{F}}{dx}+v\frac{d\mathrm{F}}{dy}+w\frac{d\mathrm{F}}{dz}$$

this differs from MAXWELL'S expression for the force by the term $u\frac{d\mathrm{F}}{dx}+v\frac{d\mathrm{F}}{dy}+w\frac{d\mathrm{F}}{dz}$. This term vanishes when integrated over a closed circuit. There will be corresponding expressions for the forces parallel to y and z respectively. We have thus the forces given in MAXWELL plus the forces

$$u\frac{d\mathrm{F}}{dx}+v\frac{d\mathrm{F}}{dy}+w\frac{d\mathrm{F}}{dz}$$

$$u\frac{d\mathrm{G}}{dx}+v\frac{d\mathrm{G}}{dy}+w\frac{d\mathrm{G}}{dz}$$

$$u\frac{d\mathrm{H}}{dx}+v\frac{d\mathrm{H}}{dy}+w\frac{d\mathrm{H}}{dz}$$

parallel to the axes of x, y, z respectively. We can prove that if the circuits are closed these form a system of forces in equilibrium, so that the force on any closed circuit is the same as that given by MAXWELL'S theory.

If we apply LAGRANGE'S equations to the electrical coordinates we find that the electromotive forces parallel to the axes of x, y, z are respectively $-d\mathrm{F}/dt$, $-d\mathrm{G}/dt$, $-d\mathrm{H}/dt$. And if we apply them to the magnetic coordinates ηl, ηm, ηn, writing in the expression

$$\iiint(\mathrm{F}u+\mathrm{G}v+\mathrm{H}w)dxdydz$$

the values of F, G, H given in equations (104) we find that the magnetic forces parallel to the axes of x, y, z respectively are

$$\left.\begin{aligned}&\iiint\left(w\frac{d}{dy}\frac{1}{r}-v\frac{d}{dz}\frac{1}{r}\right)dxdydz\\&\iiint\left(u\frac{d}{dz}\frac{1}{r}-w\frac{d}{dx}\frac{1}{r}\right)dxdydz\\&\iiint\left(v\frac{d}{dx}\frac{1}{r}-u\frac{d}{dy}\frac{1}{r}\right)dxdydz\end{aligned}\right\} \quad . \; . \; . \; . \; . \; . \; . \; . \quad (110)$$

where r is the distance between the point where the magnetic force is required and the point where the components of the electric current are u, v, w. The expressions (110) agree with the ordinary expressions for these forces.

WIEDEMANN has shown that an electric current flowing through a longitudinally magnetised iron wire twists the wire. This shows that the coefficient of the term we are considering is a function of the strain coordinates. Let us suppose that we have a current flowing through a straight longitudinally magnetised wire coinciding with the axis of x; let a denote the twist of the wire round the axis of x, $\dot{y}$ the strength of the current, and $\eta\xi$ or I the intensity of magnetisation, then we have a term in the expression for the kinetic energy of the form

$$f(a)\eta\xi\dot{y} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (111)$$

where $f(a)$ denotes some function of a.

We see by LAGRANGE'S equation for the coordinate a that the couple Ω tending to twist the wire is given by the equation

$$\Omega=f'(a)\eta\xi\dot{y}=f'(a)\mathrm{I}\dot{y} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (112)$$

If we apply LAGRANGE'S equation to the electrical coordinate we see that this term corresponds to an electromotive force E given by the equation

$$\mathrm{E}=-\frac{d}{dt}\{f(a)\mathrm{I}\} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (113)$$

so that if we twist a magnetised iron wire we shall get an electric current. This phenomenon has been observed. If we suppose that the intensity of magnetisation remains constant, equation (113) becomes

$$\mathrm{E}=-f'(a)\frac{da}{dt}\mathrm{I} \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (114)$$

or from (112)

$$\mathrm{E}\dot{y}=-\Omega\dot{a} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (115)$$

If we apply LAGRANGE'S equation to the magnetic coordinate η we see that there is a magnetising force

$$=f(a)\dot{y} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (116)$$

so that when an electric current flows along a twisted wire it magnetises it. This phenomenon has also been observed.

§ 12. We may complete our survey of the terms in the kinetic energy with this example as the term which contains the product of the velocities of the geometrical and strain coordinates does not give rise to questions of much interest, and we saw that this was the only other product term in the expression for the kinetic energy.

We have hitherto made the very important restriction that the cases we considered were those where there were no resistances, frictional forces, or such things as electric resistance, &c. If we give a wide enough meaning to the term material system, we ought to be able to deduce such forces from the dynamics of such systems by the use of the ordinary dynamical methods. Forces of this kind are assumed to be proportional to the velocities of the corresponding coordinates, and a steady transformation from one kind of energy into another, generally heat, is supposed to go on without any reverse transformation taking place. It can, however, I think, be proved that such forces cannot be deduced from the dynamics of an ordinary system, supposing the arrangement of the system to remain continuous. What I believe the equations of motion with the frictional forces inserted in the usual way give, is a result which is true on the average taken over a time which depends upon the nature of the problem, but is not true at any particular instant. Thus, to take the case of electrical resistance as an example, we may look upon an electrical current flowing along a wire as the limit of what happens when we produce a succession of sparks across an air space by means of an electrical machine; if we suppose the interval between the sparks to diminish indefinitely, the discharge will behave like a continuous current, and we may suppose a continuous current to be a succession of discharges following each other at very short intervals. The ordinary electrical equations with the resistances inserted in the usual way will give the mean state of things when the mean is taken over a time which includes a good many discharges, but it does not represent the state of things at any particular instant. I hope in a future paper to return to the theory of this kind of average motion of dynamical systems.