before the travellers reached any considerable body of water, nocturnal dews were abundant, and they were deposited from the air, and did not rise out of the ground.

Mr. Aitken also remarks that my notice of the Florentine Academicians, of Robert Boyle, and Le Roi have no bearing on the subject. The bearing is that these early observers proved that the moisture which forms dew and hoar frost exists in the air, and does not exhale from the ground.

Mr. Aitken is also "puzzled to understand" what bearing Pictet's observation has on the subject. In the abstract in 'Nature' of Mr. Aitken's memoir, it appears as an original discovery that "these observations made at night showed the ground at a short distance below the surface to be always hotter than the air over it." Pictet observed the same fact in 1779. So also in my account of the weighed turf, I certainly did not wilfully form a "misconception of the essential features of the experiment," when I compared it to objects which, when exposed on Patrick Wilson's scale-board, gained weight, while in Mr. Aitken's case the turf lost weight. It is true that my observations were founded on the abstract of the memoir contained in 'Nature.' In January last I wrote for a copy of the memoir, which was promised as soon as the 'Edinburgh Transactions' were published. I waited until May and did not receive it. I inquired for it at the Royal Society in June, but it had not arrived, nor have I yet had the privilege of reading it. Mr. Aitken is therefore entitled to any advantage that may arise from my use of the abstract instead of the original memoir.

As I do not intend to write again on this subject, I conclude by assuring Mr. Aitken that I have no unfriendly feeling towards him; but on the contrary freely admit that he has achieved much good scientific work, which I cannot but admire; but as regards his new theory of Dew I think he has gone astray, and in the interests of scientific truth I have ventured to criticise it. The subject is one that has occupied a great deal of my attention, and there is no doubt in my mind that, if this theory be accepted, a large amount of excellent work on the part of first-rate observers must be set aside as false.

Highgate, N., August 9, 1886.

XXXIII. *On the Self-induction of Wires.*—Part II.

By OLIVER HEAVISIDE*.

IN Part I. (p. 118) the inner conductor was solid. Let now the central portion be removed, making it a hollow tube of outer radius a_1 and inner a_0. The reason for this modification is that the theory of a tube is not the same when the return-conductor is outside as when it is inside it; that is to say, it depends upon the position of the dielectric, the primary seat of the transfer of energy. The expression for H_1, the magnetic force at distance r from the axis, will now be

$$H_1 = \{J_1(s_1r) - (J_1/K_1)(s_1a_0)K_1(s_1r)\}A_1; \quad . \quad . \quad (49)$$

instead of the former $A_1J_1(s_1r)$, of the first of equations (18); if we impose the condition $H_1=0$ at the inner boundary of the wire (as we may still call the inner tube). This means that there is to be no current from $r=0$ to $r=a_0$; we therefore ignore the minute longitudinal dielectric current in this space, just as we ignored that beyond $r=a_3$ previously. If we wish to necessitate that this shall be rigidly true, we may suppose that within $r=a_0$ and beyond $r=a_3$ we have not merely $k=0$, but also $c=0$, thus preventing current, either conducting or dielectric. In any case, with only $k=0$, the dielectric disturbance must be exceedingly small. On this point I may mention that my brother, Mr. A. W. Heaviside, experimenting with a wire and outer tube for the return, using a (for telegraphic purposes) very strong current, rapidly interrupted, and a sensitive telephone in circuit with a parallel outer wire, could not detect the least sign of any inductive action outside the tube, at least when the source of energy (the battery) was kept at a distance from the telephone. In explanation of the last remark, we need only consider that, although the transfer of energy is from the battery along the tubular space between the wire and return, yet, before getting to this confined space, there is a spreading out of the disturbances, so that in the neighbourhood of the battery the disk of a telephone may be strongly influenced by the variations of the magnetic field. On the other hand, the induction between parallel wires whose circuits are completed through the earth, is perceptible with the telephone at hundreds of miles distance, or practically at any distance, if the proper means be taken which theory points out. His direct experiments have, so far, only gone as far as forty miles, quite recently; but this may easily be extended.

* Communicated by the Author.

Corresponding to (49) we shall have

$$4\pi\Gamma_1 = s_1\{J_0(s_1r) - (J_1/K_1)(s_1a_0)K_0(s_1r)\}\,A_1\,; \quad . \quad (50)$$

omitting, in both, the z and t factors. Now, to obtain the corresponding development of the general equation (22), we have only to change the $J_0(s_1a_1)$ in it to the quantity in the {} in (50) and the $J_1(s_1a_1)$ to that in the {} in (49), with $r=a_1$ in both cases.

The method by which (22) was got was the simplest possible, reducing to mere algebra the work that would otherwise involve much thinking out; and, in particular, avoiding some extremely difficult reasoning relatively to potentials, scalar and vector, that would occur were they considered *ab initio*. But, having got (22), the interpretation is comparatively easy. Starting with the inner tube, (49) is the general solution of (14), with the limitation $H_1=0$ at $r=a_0$, if, in s, given by

$$-s_1^2 = 4\pi\mu_1 k_1 p + m^2,$$

we let p mean d/dt and m^2 mean $-d^2/dz^2$, instead of the constants in a normal system of subsidence, and let A_1 be an arbitrary function of z and t. Similarly, (50) gives us the connection between Γ and A_1. From it we may see what A_1 means. For, put $r=a_0$ in (50); then, since

$$(J_0K_1 - J_1K_0)(x) = -x^{-1},$$

we see that $A_1 = -4\pi a_0 K_1(s_1a_0)\Gamma_0$, if Γ_0 is the current-density at $r=a_0$. When the tube is solid, $A_1 = 4\pi\Gamma_0/s_1$. But, without knowing A_1, (49) and (50) connect H_1 and Γ_1 directly, when A_1 is eliminated by division. Also $H_1 = C_1 \times (2/r)$, if C_1 be the total longitudinal current from $r=a_0$ to r; hence

$$\Gamma_1 = \frac{s_1}{2\pi r}\,\frac{J_0(s_1r) - (J_1/K_1)(s_1a_0)K_0(s_1r)}{J_1\,.\,.\,.\, - \,.\,.\,.\,.\,.\,.\,.\,.\,.\, K_1\,.\,.\,.}\,C_1 \quad . \quad (51)$$

connects the current-density and the integral current.

Now pass to the outer tube. Quite similarly, remembering that $H_3=0$ at $r=a_3$, we shall arrive at

$$\Gamma_3 = \frac{s_3}{2\pi r}\,\frac{J_0(s_3r) - (J_1/K_1)(s_3a_3)K_0(s_3r)}{J_1\,.\,.\,.\, - \,.\,.\,.\,.\,.\,.\,.\,.\, K_1\,.\,.\,.}\,C_3, \quad . \quad . \quad (52)$$

connecting Γ_3, the longitudinal current-density at distance r in the outer tube, with C_3, the current through the circle of radius r in the plane perpendicular to the axis.

Next, let there be longitudinal impressed electric forces in the wire and return, of uniform intensities e_1 and e_3, over the sections of the two conductors. We shall have

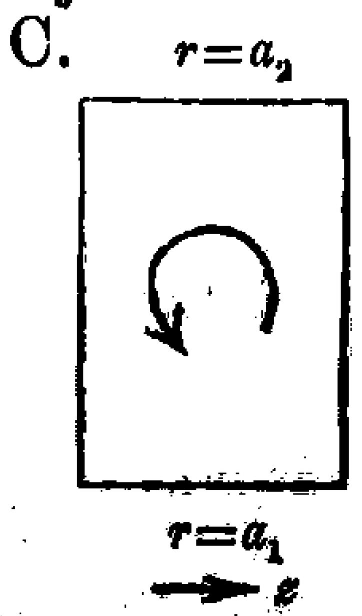
$$\rho_1\Gamma_1 = e_1 + E_1, \quad \rho_3\Gamma_3 = e_3 + E_3\,; \quad . \quad . \quad . \quad . \quad (53)$$

if E_1 and E_3 are the longitudinal electric forces "of the field." Therefore

$$e_1 - e_3 = e = \rho_1\Gamma_1 - \rho_3\Gamma_3 - (E_1 - E_3), \quad . \quad . \quad . \quad (54)$$

where e is the impressed force per unit length *in the circuit* at the place considered; the positive direction in the circuit being along the wire in the direction of increasing z, and oppositely in the return.

If in (51) we take $r=a_1$, and $r=a_2$ in (52), and use them in (54), then, since C_1 becomes C, the wire current, and C_3 becomes the same *plus* the longitudinal dielectric current, if we agree to ignore the latter, and can put E_1-E_3 in terms of C, (54) will become an equation between e and C.



To obtain the required $E_1 - E_3$, consider a rectangular circuit in a plane through the axis, two of whose sides are of unit length parallel to z at distances a_1 and a_2 from the axis, and the other two sides parallel to r, and calculate the E.M.F. of the field in this circuit in the direction of the circular arrow. If z be positive from left to right, the positive direction of the magnetic force through the circuit is upward through the paper. Therefore, if V be the line integral of the radial electric force from $r=a_1$ to $r=a_2$, so that dV/dz is the part of the E.M.F. in the rectangular circuit due to the radial force, we shall have

$$E_1 - E_3 + \frac{dV}{dz} = -\int_{a_1}^{a_2} \mu_2 \dot{H}_2\,dr,$$

by the Faraday law, or equation (7); H_2 being the magnetic force in the dielectric. This being $2C/r$, on account of our neglect of Γ_2, we get, on performing the integration, $-L_0\dot{C}$, on the right side, where L_0 is the previously used inductance of the dielectric per unit length. This brings (54) to

$$e - \frac{dV}{dz} = L_0pC + \frac{\rho_1 s_1}{2\pi a_1}\,\frac{J_0(s_1a_1) - (J_1/K_1)(s_1a_0)K_0(s_1a_1)}{J_1\,.\,.\,.\, - \,.\,.\,.\,.\,.\,.\,.\,.\, K_1\,.\,.\,.}\,C$$
$$- \frac{\rho_3 s_3}{2\pi a_2}\,\frac{J_0(s_3a_2) - (J_1/K_1)(s_3a_3)K_0(s_3a_2)}{J_1\,.\,.\,.\, - \,.\,.\,.\,.\,.\,.\,.\,.\, K_1\,.\,.\,.}\,C, \, (55)$$

which, for brevity, write thus,

$$e - \frac{dV}{dz} = L_0pC + R_1''C + R_2''C, \quad . \quad . \quad . \quad (56)$$

where R_1'' and R_2'' define themselves in (55). They are generalized resistances of wire and return respectively, per unit length. But of their structure, later. Equation (56) is what we get from (22) by treating s_2r as a small quantity and using (26); remembering also the extension from a solid to a hollow wire.

By more complex reasoning we may similarly put the right member of (54) in terms of C without the neglect of Γ_2, and arrive at (22) itself, in a form similar to (55) or (56). But we may get it from (22) at once by a proper arrangement of the terms, and introducing e. It becomes

$$e=\left(R_1''+R_2''\frac{R_{02}''}{R_{01}''}+R_{03}''+\frac{R_1''R_2''}{R_{01}''}\right)C. \quad (57)$$

Here R_1'' and R_2'' are as before, whilst R_{01}'' and R_{02}'' are similar expressions for the dielectric, on the assumption that $H=0$ at $r=a_1$ or at $r=a_2$ respectively; thus,

$$R_{01}''=+\frac{\rho_2 s_2}{2\pi a_2}\frac{J_0(s_2a_2)-(J_1/K_1)(s_2a_1)K_0(s_2a_2)}{J_1\ldots-\ldots\ldots K_1\ldots},$$

$$R_{02}''=-\frac{\rho_2 s_2}{2\pi a_1}\frac{J_0(s_2a_1)-(J_1/K_1)(s_2a_2)K_0(s_2a_1)}{J_1\ldots-\ldots\ldots K_1\ldots}.$$

R_{03}'' has a different structure, being given by

$$R_{03}''=-\frac{\rho_2 s_2}{2\pi a_1}\frac{J_0(s_2a_1)-(J_0/K_0)(s_2a_2)K_0(s_2a_1)}{J_1\ldots-\ldots\ldots K_1\ldots}.$$

In these take s_2r small; they will become

$$R_{01}''=R_{02}''=\frac{\rho_2}{\pi(a_2^2-a_1^2)};$$

that is, if ρ_2 be imagined to be resistivity, the steady flow resistance per unit length of the dielectric tube (fully, ρ_2 is the reciprocal of $k_2+c_2p/4\pi$); and, with $k_2=0$,

$$R_{03}''=-\frac{s_2^2}{cp}2\log\frac{a_2}{a_1}=L_0p+\frac{m^2}{Sp},$$

if S is the electrostatic capacity per unit length, such that $L_0S=\mu_2c_2$. Then (57) reduces to

$$e=(L_0p+m^2/Sp+R_1''+R_2'')C, \quad . \quad . \quad . \quad (58)$$

which is really the same as (56). For, by continuity, or by the second of (11),

$$-\frac{dC}{dz}=2\pi a_1\gamma_1=2\pi a_1p\sigma=SpV, \quad . \quad . \quad . \quad (59)$$

if σ is the time-integral of the radial current at $r=a_1$, or, in other words, the electrification surface-density there, when the conductors are non-dielectric. (There is equal $-\sigma$ at the $r=a_2$ surface). Therefore

$$-\frac{1}{Sp}\frac{d^2C}{dz^2}=\frac{m^2}{Sp}C=\frac{dV}{dz}, \quad . \quad . \quad . \quad . \quad (60)$$

which establishes the equivalence.

Particular attention to the meaning of the quantity V is needed. It is the line-integral of the radial force in the dielectric from $r=a_1$ to $r=a_2$. Or it may be defined by

$$SV=2\pi a_1\sigma=Q,$$

if Q be the charge per unit length of wire. But it is not the electric potential at the surface of the wire. It is not even the excess of the potential at the wire boundary over that at the inner boundary of the return. For, as it is the line-integral of the electric force from end to end of the tubes of displacement, it includes the line-integral of the electric force of inertia. It has, however, the obvious property of allowing us to express the electric energy in the dielectric in the form of a surface-integral, thus, $\frac{1}{2}V\sigma$ per unit area of wire surface, or $\frac{1}{2}VQ$ per unit length of wire, instead of by a volume integration throughout the dielectric. Hence the utility of V. The possibility of this property depends upon the comparative insignificance of the longitudinal current in the dielectric, which we ignore. It may happen, however, that the longitudinal displacement is far greater than the radial; but then it will be of so little moment that the problem could be taken to be a purely electromagnetic one. We need not use V at all, (58) being the equation between e and C without it. It is, however, useful in electrostatic problems, for the above-mentioned reason. Again, instead of V, we may use σ or Q, which are definitely localized.

The physical interpretation of the force $-dV/dz$, in terms of Maxwell's inimitable dielectric theory, a theory which is spoiled by the least amount of tinkering, confusion and bemuddlement immediately arising, is sufficiently clear, especially when we assist ourselves by imagining the dielectric displacement to be a real displacement, elastically resisted, or any similar elastically resisted generalized displacement of a vector character. When there is current from the wire into the dielectric there is necessarily a back electric force in it due to the elastic displacement; and if it vary in amount along the wire, its variation constitutes a longitudinal electric force.

(58) being a differential equation previously, let in it m^2 be a constant. Then R_1'' and R_2'' may be thus expressed:—

$$R_1''=R_1'+L_1'p, \quad R_2''=R_2'+L_2'p, \quad . \quad . \quad . \quad (61)$$

where R_1' and R_2', L_1' and L_2' are functions of p^2. The utility of this notation arises from R_1' &c. becoming mere constants in simple-harmonically vibrating systems. Let e_m, V_m, and C_m be the corresponding quantities for the particular m; then, by (56),

$$e_m - \frac{dV_m}{dz} = L_0 p C_m + (R'_{1m} + L'_{1m} p) C_m + (R'_{2m} + L'_{2m} p) C_m. \quad (62)$$

Or

$$e_m - \frac{dV_m}{dz} = (R'_m + L'_m p) C_m, \quad \ldots \quad (63)$$

where

$$R'_m = R'_{1m} + R'_{2m}; \quad L'_m = L_0 + L'_{1m} + L'_{2m}. \quad (64)$$

R'_m and L'_m are functions of p^2. Therefore, by (62), summing up,

$$e - \frac{dV}{dz} = \Sigma (R'_m + L'_m p) C_m. \quad \ldots \quad (65)$$

Now, although R'_m and L'_m are really different functions of p^2 for every different value of m, since they contain m^2, yet if, in changing from one m to another, through a great many m's, from $m=0$ upward, they should not materially change, we may regard R'_m and L'_m as having the $m=0$ expressions, as in the purely electromagnetic case, and denote them by R' and L' simply. Then (65) becomes

$$e - \frac{dV}{dz} = (R' + L'p) C \quad \ldots \quad (66)$$

simply. The equation of V is now

$$-\frac{de}{dz} + \frac{d^2V}{dz^2} = (R' + L'p) SpV; \quad \ldots \quad (67)$$

and that of C_m being

$$e_m = (R'_m + L'_m p + m^2/Sp) C_m \quad \ldots \quad (68)$$

in the m case, that of C becomes now simply

$$Spe + \frac{d^2C}{dz^2} = (R' + L'p) SpC. \quad \ldots \quad (69)$$

The assumption above made is, in general, justifiable.

Let us now compare these equations with the principal ways that have been previously employed to express the conditions of propagation of signals along wires. For simplicity, leave out the impressed force e. First, we have Ohm's system, which may be thus written :—

$$-\frac{dV}{dz} = RC, \quad -\frac{dC}{dz} = SpV, \quad \frac{d^2V}{dz^2} = RSpV. \quad . \quad (70)$$

Here the first equation expresses Ohm's law. C is the wire

current, R the resistance per unit length, and V is a quantity whose meaning is rather indistinct in Ohm's memoir, but which would be now called the potential. The second equation is of continuity. Misled by an entirely erroneous analogy, Ohm supposed electricity could accumulate in the wire in a manner expressed by the second of (70), wherein S therefore depends upon a specific quality *of the conductor*. The third equation results from the two previous, and shows that V, or C, or $Q=SV$ diffuse themselves through the wire as heat does by differences of temperature when there is no surface loss. This system has at present only historical interest. The most remarkable thing about it is the getting of equations correct in form, at least approximately, by entirely erroneous reasoning.

The matter was not set straight till a generation later, when Sir W. Thomson arrived at a system which is formally the same as (70), but in which V is precisely defined, whilst S changes its meaning entirely. V is now to be the electrostatic potential, and S is the electrostatic capacity *of the condenser* formed by the opposed surfaces of the wire and return with dielectric between. The continuity of the current in the wire is asserted; but it can be discontinuous at its surface, where electricity accumulates and charges the condenser. In short, we simply unite Ohm's law (with continuity of current in the conductor) and the similar condenser law. The return is supposed to be of no resistance, and $V=0$ at its boundary.

The next obvious step is to bring the electric force of inertia into the Ohm's law equation, and make the corresponding change in that of V; that is, if we decide to accept the law of quasi-incompressibility of electricity in the conductor, which is implied by the second of (70), when Sir W. Thomson's meanings of S and V are accepted. Kirchhoff seems to have been the first to take inertia into account, arriving at an equation of the form $d^2V/dz^2 = (R + Lp)SpV$.

I am, unfortunately, not acquainted with his views regarding the continuity of the current, so that, translated into physical ideas, his equation may not be conformable to Maxwell's ideas, even as regards the conductor. Also, as his estimation of the quantity L was founded upon Weber's hypothesis, it may possibly turn out to be different in value from that in the next following system. In ignorance of Kirchhoff's investigation, I made the necessary change of bringing in the electric force of inertia in a paper "On the Extra Current" (Phil. Mag. August 1876), getting this system,

$$-\frac{dV}{dz} = (R + Lp)C, \quad -\frac{dC}{dz} = SpV, \quad \frac{d^2V}{dz^2} = (R + Lp)SpV, \quad (71)$$

wherein everything is the same as in Sir W. Thomson's system, with the addition of the electric force of inertia $-LpC$, where L is the coefficient of self-induction, or, as I now prefer to call it, for brevity, the inductance, per unit length of the wire, according to Maxwell's system, being numerically equal to twice the energy, per unit length of wire, of the unit current in the wire, uniformly distributed. Coming after Maxwell's treatise, there is of course no question of any important step in advance here, except perhaps in the clearing away of hypotheses involved in Kirchhoff's investigation.

The system (71) is amply sufficient for all ordinary purposes, with exceptions to be later mentioned. It applies to short lines as well as to long ones; whereas the omission of L, reducing (71) to (70), renders the system quite inapplicable to lines of moderate length, as the influence of S tends to diminish as the line is shortened, relatively to that of L. An easily made extension of (71) is to regard R as the sum of the steady-flow resistances of wire and return, and V as the quantity Q/S, Q being the charge per unit length of wire. Nor are we, in this approximate system (71), obliged to have the return equidistant from the wire. It may, for instance, be the earth, or a parallel wire, with the corresponding changes in the formulæ for the electrostatic capacity and inductance.

But there are extreme cases when (71) is not sufficient. For example, an iron wire, unless very fine, by reason of its high inductivity; a very thick copper wire, by reason of thickness and high conductivity; or, a very close return current, in which case, no matter how fine a wire may be, there is extreme departure from uniformity of current distribution in the variable period; or, extremely rapid reversals of current, for, no matter what the conductors may be, by sufficiently increasing the frequency we approximate to surface conduction.

We must then, in the system (71), with the extension of meaning of R and V just mentioned, change R and L to R′ and L′, as in (67), and other equations. In a S.H. problem, this simply changes R and L from certain constants to others, depending on the frequency. But, in general, it would I imagine be of no use developing R_1'' &c. in powers of p, so that we must regard $(R_1'+L_1'p)$ &c. merely as a convenient abbreviation for the R_1'' &c. defined by (56) and (55).

A further refinement is to recognise the differences between R′ and L′ in one m system and another, instead of assuming $m=0$ in R''_m. And lastly, to obtain a complete development, and exact solutions of Maxwell's equations, so as to be able to fully trace the transfer of energy from source to sink, fall back upon (57), or (22), and the normal systems (18) of Part I.

Now, as regards our obtaining the expansions of R_1' &c. in powers of p^2, we have to expand the numerators and the denominators of R_1'' and R_2'' in powers of p, perform the divisions, and then separate into odd and even powers. When the wire is solid, the division is merely of $\frac{1}{2}xJ_0(x)$ by $J_1(x)$, a comparatively easy matter. The solid wire R′ and L′ expansions were given by Lord Rayleigh (Phil. Mag. May 1886). I should mention that my abbreviated notation was suggested by his. But in the tubular case, the work is very heavy, so, on account of possible mistakes, I go only as far as p^2, or three terms in the quotient. The work does not need to be done separately for the inner and the outer tube, as a simple change converts one R′ or L′ into the other. Thus, in the case of the inner tube, we shall have

$$R_1'=R_1\Big[1+n^2(\mu_1k_1\pi a_1^2)^2\Big\{\tfrac{1}{12}-\tfrac{2}{3}\frac{a_0^2}{a_1^2}+\tfrac{7}{12}\frac{a_0^4}{a_1^4}-\frac{2a_0^4\log(a_1/a_0)}{a_1^2(a_1^2-a_0^2)}+\frac{4a_0^6\{\log(a_1/a_0)\}^2}{a_1^2(a_1^2-a_0^2)^2}\Big\}\Big], \quad (72)$$

$$L_1'=R_1(\mu_1k_1\pi a_1^2)\Big\{\tfrac{1}{2}-\tfrac{3}{2}\frac{a_0^2}{a_1^2}+2\log\frac{a_1}{a_0}\cdot\frac{a_0^4}{a_1^2(a_1^2-a_0^2)}\Big\}, \quad (73)$$

where n^2 is written for $-p^2$, for the S.H. application.

As for L_1', it is simply the inductance of the tube per unit length (of the tube only), as may be at once verified by the square of force method. The first correction depends upon p^2. But R_1' gives us the first correction to R_1, which is the steady-flow resistance, so it is of some use. To obtain R_2' and L_2' from these, change R_1 to R_2, μ_1 and k_1 to μ_3 and k_3, a_0 to a_3, and a_1 to a_2. Or, more simply, (72) and (73) being the tube-formulæ when the return is outside it, if we simply exchange a_0 and a_1 we shall get the formulæ for the *same* tube when the return is inside it.

If the tube is thin, there is little change made by thus shifting the locality of the return. But if a_1/a_0 be large, there is a large change. This will be readily understood by considering the case of a wire whose return is outside it, and of great bulk. Although the steady resistance of the return may be very low, yet the percentage correction will be very large, compared with that for the wire.

Taking $a_1/a_0=2$ only, we shall find

$$R_1'=R_1[1+(\pi k_1a_1^2\mu_1n)^2\times\cdot 012]$$

when the return is outside, and

$$R_1' = R_1[1 + (\pi k_1 a_0^2 \mu_1 n)^2 \times \cdot 503]$$
$$= R_1[1 + (\pi k_1 a_1^2 \mu_1 n)^2 \times \cdot 031]$$

when the return is inside. In the case of a solid wire, the decimals are ·083, so that whilst the correction is reduced, in this $a_1/a_0 = 2$ example, the reduction is far greater when the return is outside than when it is inside.

The high-speed tube formulæ are readily obtained. Those for the inner tube are the same as for a solid wire, and that for the outer tube depends not on its bulk, but on its inner radius. That is, in both cases it is the extent of surface that is in question, next the dielectric, from which the current is transmitted into the conductors. Let $G_0(x) = (2/\pi)K_0(x)$, and $G_1(x) = (2/\pi)K_1(x)$; then, when x is very large,

$$\left.\begin{aligned} J_0(x) &= -G_1(x) = (\sin x + \cos x) \div (\pi x)^{\frac{1}{2}} \\ J_1(x) &= \quad G_0(x) = (\sin x - \cos x) \div (\pi x)^{\frac{1}{2}} \end{aligned}\right\} \quad . \quad (74)$$

Use these in the R_1'' fraction, and put in the exponential form. We shall obtain

$$R_1'' = (\rho_1 s_1 i)/(2\pi a_1).$$

But

$$\tfrac{1}{2} s_1 a_1 i = (\pi k_1 \mu_1 p)^{\frac{1}{2}} a_1,$$

therefore

$$R_1'' = (\mu_1 \rho_1 p / \pi a_1^2)^{\frac{1}{2}}.$$

But

$$p^2 = -n^2,$$

therefore

$$p^{\frac{1}{2}} = (\tfrac{1}{2}n)^{\frac{1}{2}}(1+i) = (\tfrac{1}{2}n)^{\frac{1}{2}} + p(\tfrac{1}{2}n^{-1})^{\frac{1}{2}},$$

so that, finally,

$$R_1' = \frac{(\mu_1 \rho_1 q)^{\frac{1}{2}}}{a_1}, \quad L_1' = \frac{R_1'}{n}, \quad . \quad . \quad . \quad (75)$$

$q = n/2\pi =$ the frequency. To get R_2' and L_2', change the μ and ρ of course, and also a_1 to a_2.

It is clear that the thinner the tube, the greater must be the frequency before these formulæ can be applicable. For the steady-flow resistance is increased indefinitely by reducing the thickness of the tube, whilst the high-speed resistance is independent of the steady-flow resistance, and must be much greater than it. In (75) then, q must be great enough to make R' several times R, itself very large when the tube is very thin. Consequently thin tubes, as is otherwise clear,

may be treated as linear conductors, subject to the equations (71), with no corrections, except under extreme circumstances. The L may be taken as L_0, except in the case of iron.

I will now give the S.H. solution in the general case, subject to (58). Let there be any distribution of e (longitudinal, and of uniform intensity over cross sections). Expand it in the Fourier series appropriate to the terminal conditions at $z=0$ and l. For definiteness, let wire and return be joined direct, without any terminal resistances. Then, $e_0 \sin nt$ being e at distance z, the proper expansion is

$$e_0 = e_{00} + e_{01} \cos m_1 z + e_{02} \cos m_2 z + \dots,$$

where $m_1 = \pi/l$, $m_2 = 2\pi/l$, &c. [It should be remembered that e is the $e_1 - e_2$ of (54) and (53). Shifting impressed force from the wire to the return, with a simultaneous reversal of its direction, makes no difference in e. Thus two e's directed the same way in space, of equal amounts, and in the same plane z = constant, one in the inner, the other in the outer conductor, cancel. This will clearly become departed from as the distance of the return from the wire is increased.] Then, in the equation

$$e_m = (R'_m + L'_m p)C_m + (m^2/Sp)C_m$$
$$= R'_m C_m + (L'_m - m^2/Sn^2)pC_m,$$

we know e_m; whilst R'_m and L'_m are constants. The complete solution is obtained by adding together the separate solutions for e_{00}, e_{01}, &c., and is

$$C = \frac{1}{l}\left\{\frac{e_{00}\sin(nt-\theta_0)}{(R'^2 + L'^2 n^2)^{\frac{1}{2}}} + 2\Sigma \frac{e_{0m}\sin(nt-\theta_m)\cos mz}{[R'^2_m + (L'_m - m^2/Sn^2)^2 n^2]^{\frac{1}{2}}}\right\}, (76)$$

where the summation includes all the m's, and

$$\tan\theta_m = (L'_m - m^2/Sn^2)n \div R'_m.$$

A practical case is, no impressed force anywhere except at $z=0$, one end of the line, where it is $V_0 \sin nt$. Then, imagining it to be V_0/z_1 from $z=0$ to $z=z_1$, and zero elsewhere, and diminishing z_1 indefinitely, the expansion required is

$$V_0 z_1 = (V_0/l)(1 + 2\Sigma \cos j\pi z/l),$$

j going from 1, 2, ... to ∞. This makes the current solution become

$$C = \frac{V_0}{l}\left\{\frac{\sin(nt-\theta_0)}{(R'^2 + L'^2 n^2)^{\frac{1}{2}}} + 2\Sigma \frac{\sin(nt-\theta_m)\cos mz}{\{R'^2_m + (L'_m - m^2/Sn^2)^2 n^2\}^{\frac{1}{2}}}\right\}. (77)$$

If the line is short, neglect the summation altogether, unless the speed is excessive. Now (77) may perhaps be put in a finite form when R'_m is allowed to be different from R', though I do not see how to do it. But when $R'_m=R'$ and $L'_m=L'$ it can of course be done, for we may then use the finite solutions of (66) and (67). Thus, given $V=V_0\sin nt$ at $z=0$, and no impressed force elsewhere, find V and C everywhere subject to (66) and (67) with $e=0$, and $V=0$ at $z=l$.

Let

$$\left.\begin{aligned} P&=(\tfrac{1}{2}Sn)^{\frac{1}{2}}\{(R'^2+L'^2n^2)^{\frac{1}{2}}-L'n\}^{\frac{1}{2}},\\ Q&=\ldots\ldots\{(\ldots\ldots\ldots)^{\frac{1}{2}}+\ldots\}^{\frac{1}{2}},\end{aligned}\right\} \quad . \quad . \quad (78)$$

$$\left.\begin{aligned} \tan\theta_2&=\sin 2Ql\div(\epsilon^{-2Pl}-\cos 2Ql),\\ \tan\theta_1&=(L'nP-R'Q)\div(R'P+L'nQ);\end{aligned}\right\} \quad . \quad (79)$$

then the finite V and C solutions are

$$V=V_0\epsilon^{-Pz}\sin(nt-Qz)$$
$$+V_0\frac{\epsilon^{Pz}\sin(nt+Qz+\theta_2)-\epsilon^{-Pz}\sin(nt-Qz-\theta_2)}{\epsilon^{Pl}(\epsilon^{2Pl}+\epsilon^{-2Pl}-2\cos 2Ql)^{\frac{1}{2}}}, \quad (80)$$

$$C=V_0\frac{(Sn)^{\frac{1}{2}}}{(R'^2+L'^2n^2)^{\frac{1}{4}}}\Big[\epsilon^{-Pz}\sin(nt-Qz-\theta_1)$$
$$-\frac{\epsilon^{Pz}\sin(nt+Qz-\theta_1+\theta_2)+\epsilon^{-Pz}\sin(nt-Qz-\theta_1+\theta_2)}{\epsilon^{Pl}(\epsilon^{2Pl}+\epsilon^{-2Pl}-2\cos 2Ql)^{\frac{1}{2}}}\Big]. \quad (81)$$

If we expand this last in cosines of mz we shall obtain (77), with $R'_m=R'$. There are three waves; the first is what would represent the solution if the line were of infinite length; being of finite length there is a reflected wave (the ϵ^{Pz} term), and another reflected at $z=0$, the third and least important.

The amplitude of C anywhere is

$$V_0\frac{(Sn)^{\frac{1}{2}}}{(R'^2+L'^2n^2)^{\frac{1}{4}}}\left[\frac{\epsilon^{2P(l-z)}+\epsilon^{-2P(l-z)}+2\cos 2Q(l-z)}{\epsilon^{2Pl}+\epsilon^{-2Pl}-2\cos 2Ql}\right]^{\frac{1}{2}}.$$

At the distant $z=l$ end it is

$$C_0=2V_0\frac{(Sn)^{\frac{1}{2}}}{(R'^2+L'^2n^2)^{\frac{1}{4}}}(\epsilon^{2Pl}+\epsilon^{-2Pl}-2\cos 2Ql)^{-\frac{1}{2}}. \quad (82)$$

I have already spoken of the apparent resistance of a line as its impedance (from impedo). The steady flow impedance is the resistance. The short line impedance is $(R^2+L^2n^2)^{\frac{1}{2}}l$ or $(R'^2+L'^2n^2)^{\frac{1}{2}}l$, at the frequency $n/2\pi$, according as current density differences are, or are not, ignorable. The impedance according to the latter formula increases with the speed, but is greater or less than that of the former formula (linear theory) according as the speed is below or above a certain speed.

But if the speed is sufficiently increased, even on a short line, the formula ceases to represent the impedance, whilst, if the line be long it will not do so at any speed except zero. According to (82) we have

$$V_0/C_0=\frac{(R'^2+L'^2n^2)^{\frac{1}{4}}}{2(Sn)^{\frac{1}{2}}}(\epsilon^{2Pl}+\epsilon^{-2Pl}-2\cos 2Ql)^{\frac{1}{2}}, \quad (83)$$

as the distant end impedance of the line. That is, we have extended the meaning of impedance, as we must (or else have a new word), since the current-amplitude varies as we pass from beginning to end of the line. (83) will, roughly speaking, on the average, give the greatest value of the impedance. It is what the resistance of the line would have to be in order that when an S.H. impressed force acts at one end, the current-amplitude at the distant end should be, without any electromagnetic and electrostatic induction, what it really is. The distant end impedance may easily be less than the impedance according to the electromagnetic reckoning. What is more remarkable, however, is that it may be much less than the steady-flow resistance of the line. This is due to the to-and-fro reflection of the dielectric waves, which is a phenomenon similar to resonance.

To show this, take $R'=0$ in the first place, which requires the conductors to be of infinite conductivity. Then $L'=L_0$, the dielectric inductance. We shall have, by (83) and (78),

$$V_0/C_0=L_0v\sin(nl/v), \quad . \quad . \quad . \quad . \quad . \quad (84)$$

where $v=(L_0S)^{-\frac{1}{2}}=(\mu_2c_2)^{-\frac{1}{2}}$, the speed of waves through the dielectric when undissipated. The sine is to be taken positive always. If $nl/v=\pi$, 2π, &c., the impedance is zero, and the current-amplitude infinite. Here $nl/v=\pi$ means that the period of a wave equals the time taken to travel to the distant end and back again, which accounts for the infinite accumulation, which is, of course, quite unrealizable.

Now, giving resistance to the line, it is clear that although the impedance can never vanish, it will be subject to maxima and minima values as the speed increases continuously, itself increasing, on the whole. We may transform (83) to

$$V_0/C_0=(R'^2+L'^2n^2)^{\frac{1}{2}}l\Big[\Big(\frac{v'}{nl}\Big)^2\sin^2\Big(\frac{nl}{v'}\Big)+\Big(\frac{nl}{v'}\Big)^4\frac{h}{90}\Big\{1-\tfrac{1}{7}\Big(\frac{nl}{v'}\Big)^2$$
$$+\tfrac{1}{105}\Big(\frac{nl}{v'}\Big)^4(1+\tfrac{1}{12}h)-\frac{4}{105.99}\Big(\frac{nl}{v'}\Big)^6(1+\tfrac{3}{10}h)$$
$$+\frac{10}{105.99.91}\Big(\frac{nl}{v'}\Big)^8(1+\tfrac{3}{10}h+\tfrac{1}{80}h^2)-\dots\Big\}\Big]^{\frac{1}{2}}, \quad (85)$$

where

$$v'=(L'S)^{-\frac{1}{2}}, \text{ and } h=(R'/L'n)^2.$$

The factor outside the [ ] is the electromagnetic impedance; and, if we take only the first term within the [ ], we shall obtain the former infinite conductivity formula (84). The effect of resistance is shown by the terms containing h.

With this v' and h notation (83) becomes

$$V_0/C_0=\tfrac{1}{2}L'v'(1+h)^{\frac{1}{2}}\{\epsilon^{2Pl}+\epsilon^{-2Pl}-2\cos 2Ql\}^{\frac{1}{2}}; \quad (86)$$

where

$$Ql=(nl/v')(\sqrt{1+h}+1)^{\frac{1}{2}}\div\sqrt{2},$$
$$Pl=(nl/v')(\sqrt{1+h}-1)^{\frac{1}{2}}\div\sqrt{2}.$$

Choose Q so that $2Ql=2\pi$, and let $h=1$. This requires $nl/v'=2\cdot85$. Then

$$V_0/C_0=\tfrac{1}{2}L'v'.2^{\frac{1}{2}}[\epsilon^{\cdot8284\pi}+\epsilon^{-\cdots}-2]^{\frac{1}{2}},$$
$$=60\cdot6\,L' \text{ ohms},$$

if we take $v=30^{10}$ cm. $=30$ ohms. This implies $L'=L_0$, and the dielectric air. Without making use of current-density differences, we may suppose that the conductors are thin tubes. Therefore,

$$\frac{\text{Impedance}}{\text{Resistance}}=\frac{60\cdot6\,L'.10^9}{R'l}=\text{about }\tfrac{202}{285},$$

by making use of the above values of h and nl/v'.

But take $2Ql=\frac{1}{2}\pi$, or one fourth of the above value. Then

$$V_0/C_0=28\,L' \text{ ohms},$$

and

$$\frac{\text{Impedance}}{\text{Resistance}}=\text{about }\tfrac{1}{3}.$$

Thus the amplitude of the current, from being less than the steady-flow strength in the last case, becomes 42 per cent. greater than the steady-flow current by quadrupling nl/v', and keeping $h=1$. We have evidently ranged from somewhere near the first maximum to the first minimum value of the impedance. These figures suit lines of any length, if we

choose the resistances &c. properly. The following will show how the above apply practically. Remember that 1 ohm per kilom. $=10^4$ per cm. Then, if $l_1=$length of line in kilom.,

If	$R'=10^3$,	and	$L'=1$,	$\therefore$	$n=10^3$,	and	$l_1=856$,
„	$R'=10^3$,	„	$L'=10$,	„	$n=10^2$,	„	$l_1=8568$,
„	$R'=10^4$,	„	$L'=1$,	„	$n=10^4$,	„	$l_1=85$,
„	$R'=10^4$,	„	$L'=10$,	„	$n=10^3$,	„	$l_1=856$,
„	$R'=10^4$,	„	$L'=100$,	„	$n=10^2$,	„	$l_1=8568$,
„	$R'=10^5$,	„	$L'=1$,	„	$n=10^5$,	„	$l_1=8\cdot5$,
„	$R'=10^5$,	„	$L'=10$,	„	$n=10^4$,	„	$l_1=85$,
„	$R'=10^5$,	„	$L'=100$,	„	$n=10^3$,	„	$l_1=856$,
„	$R'=10^6$,	„	$L'=10$,	„	$n=10^5$,	„	$l_1=8\cdot5$.

The resistances vary from $\frac{1}{10}$ to 100 ohms per kilom., the inductances from 1 to 100 per cm., the frequencies from $10^2/2\pi$ to $10^5/2\pi$, and the lengths from 8·5 to 8568 kilom. In all cases $\frac{2}{3}$ is the ratio of the distant end impedance to the resistance. The common value of nl_1 is 856800.

In the other case, nl/v' has one fourth of the value just used, so that, with the same R′ and L′, l_1 has values one fourth of those in the above series.

Telephonic currents are so rapidly undulatory (it is the upper tones that go to make good articulation, and convert mumblings and murmurs into something like human speech) that it is evident there must be a considerable amount of this dielectric resonance, if a tone last through the time of several wave periods.

Having got the solution for C, the wire current, we may obtain those for H, Γ, and γ from it. Thus, H_r being the same as $(2/r)C_r$, where C_r is the longitudinal current through the circle of radius r, we may first derive C_r or H_r from C, and then derive Γ and γ from either by (11). Thus, make use of (49) and (50), and the value of A_1 there given. Then we shall obtain

$$C_r=\frac{r}{a_1}\frac{J_1(s_1r)-(J_1/K_1)(s_1a_0)K_1(s_1r)}{J_1(s_1a_1)-(J_1/K_1)(s_1a_0)K_1(s_1a_1)}C, \quad (87)$$

where, in the s_1, p and m^2 are to be d/dt and $-d^2/dz^2$. Similarly for the return tube.

In a comprehensive investigation, the C solution would be only a special result; as this special result is more easily got by itself, it might appear that there would be some saving of labour by first getting the C solution and then deriving from it the general. But this does not stand examination; the work has to be done, whether we derive the special results from the general, or conversely.

In the solid wire case

$$C_r = \frac{r}{a_1}\frac{J_1(s_1 r)}{J_1(s_1 a_1)}\,C,$$

or

$$C_r = \frac{r^2}{a_1^2}\Big\{1 + \tfrac{1}{2}(\pi\mu_1 k_1 p + \tfrac{1}{4}m^2)(r^2 - a_1^2)$$

$$+ \tfrac{1}{3}\frac{1}{1^2 2^2}(\pi\mu_1 k_1 p + \tfrac{1}{4}m^2)^2(r^2 - a_1^2)(r^2 - 2a_1^2)$$

$$+ \tfrac{1}{4}\frac{1}{1^2 2^2 3^2}(\pi\mu_1 k_1 p + \tfrac{1}{4}m^2)^3(r^2 - a_1^2)(r^4 - 5r^2 a_1^2 + 7a_1^4)\} + \dots\Big\}\,C.$$

Or, use the M and N functions of Part I., equations (42). For we have

$$J_0(s_1 r) = (M + iN)(s_1 r i^{\frac{3}{2}}),$$

where $s_1 r i^{\frac{3}{2}}$ takes the place of the y in those equations. M contains the even and N the odd powers of $(p + m^2/4\pi\mu_1 k_1)$.

We have also

$$\Gamma_r = J_0(s_1 r)\Gamma_0 = \frac{s_1}{2\pi a_1}\frac{J_0(s_1 r)}{J_1(s_1 a_1)}\,C,$$

Γ_0 being Γ at $r=0$; and, since by the first of these

$$\Gamma_{a_1} = J_0(s_1 a_1)\Gamma_0$$

connects the boundary and axial current-densities, we see that the ratio of their amplitudes in the S.H. case is

$$(M^2 + N^2)^{\frac{1}{2}},$$

using the $r=a_1$ expressions, with $m=0$.

I hope to be able to conclude this paper in a third part.

XXXIV. *Further Notes on the Formulæ of the Electromagnet and the Equations of the Dynamo.* By Professor SILVANUS P. THOMPSON, *D.Sc., B.A.**

1. *The Lamont-Frölich Formula.*

DR. O. FRÖLICH has done me the honour of replying† to a certain point in my former communication to the Physical Society, "On the Law of the Electromagnet and the Law of the Dynamo"‡. In that communication I pointed

* Communicated by the Physical Society: read June 26, 1886.

† *Elektrotechnische Zeitschrift*, vii. p. 163, May 1886.

‡ Phil. Mag. vol. xxi. p. 1, January 1886.

out that Lamont had in 1867 published a rational theory of the electromagnet, based upon the assumption that the permeability of the iron was at every stage of the magnetization proportional to the deficit of saturation, leading him to an exponential expression,

$$m = M(1 - \epsilon^{-kx}),$$

where m is the magnetism present at any stage, M its maximum value, k the ratio of the permeability to the deficit of saturation, and x the magnetizing force proportional (approximately) to the number of ampere-turns of the magnetizing current. This formula more correctly expressed the facts than either of the commoner formulæ of Lenz and Jacobi and of Müller.

I further pointed out that Lamont had himself* given, as a sufficient approximation to the formula, the simpler expression,

$$m = \frac{aMx}{M + ax},$$

which formula is mathematically identical with that now commonly attributed to Dr. Frölich. For, writing $a = kM$, we get at once

$$m = M\frac{kx}{1 + kx},$$

which is the formula claimed by Frölich.

Lamont having developed his exponential expression in a series of ascending powers of kx, I did the same for the simpler formula for the purpose of comparison, and showed that, *neglecting the fourth and higher terms* of each series, the expansions are very nearly equal for all values of kx except for very large ones, and are identical for the value $kx = \frac{3}{2}$. Dr. Frölich, overlooking the words I have above italicized, commits the mistake of supposing that I had said that Lamont's exponential expression is identical in value with the simpler formula when $kx = \frac{3}{2}$. I have said nothing of the kind.

Further, when Dr. Frölich says, "Hiernach ist die Aussicht vorhanden dass nicht die Lamontsche sondern die von mir benutzte Formel die wahre Gesetz der Elektromagnete enthält," he is forgetting that the formula used by him is also Lamont's. He has proved, in his most recent communication, that the differences between the calculated and the observed values are about half as great when calculated by the simpler formula. The second and simpler formula suggested by Lamont appears therefore to be better than the first and more

* Lamont, *Magnetismus*, p. 41.