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XLV. *On the Electromagnetic Wave-surface.* By OLIVER HEAVISIDE*.

MAXWELL showed (Electricity and Magnetism, vol. ii. art. 794) that his equations of electromagnetic disturbances, on the assumption that the electric capacity varies in different directions in a crystal, lead to the Fresnel form of wave-surface. There is no obscurity arising from the ignored wave of normal disturbance, because the very existence of a plane wave requires that there be none. In fact, the electric displacement and the magnetic induction are both in the wave-front, and are perpendicular to one another. The magnetic force and induction are parallel, on account of the constant permeability; whilst the electric force, though not parallel to the displacement, is yet perpendicular to the magnetic induction (and force); the normal to the wave-front, the electric force, and the displacement being in one plane. The ray is also in this plane, perpendicular to the electric force. There are of course two rays for (in general) every direction of wave-normal, each with separate electromagnetic variables to which the above remarks apply.

It is easily proved, and it may be legitimately inferred without a formal demonstration, from a consideration of the equations of induction, that if we consider the dielectric to be isotropic as regards capacity, but eolotropic as regards permeability, the same general results will follow, if we translate capacity to permeability, electric to magnetic force, and elec-

* Communicated by the Author.

tric displacement to magnetic induction. The three principal velocities will be $(c\mu_1)^{-\frac{1}{2}}$, $(c\mu_2)^{-\frac{1}{2}}$, and $(c\mu_3)^{-\frac{1}{2}}$, if c is the constant value of the capacity, and μ_1, μ_2, μ_3 are the three principal permeabilities. The wave-surface will be of the same character, only differing in the constants.

But a dielectric may be eolotropic both as regards capacity and permeability. The electric displacement is then a linear function of the electric force, and the magnetic induction another linear function of the magnetic force. The principal axes of capacity, or lines of parallelism of electric force and displacement, cannot, in the general case, be assumed to have any necessary relation to the principal axes of permeability, or lines of parallelism of magnetic force and induction. Disconnecting the matter altogether from the hypothesis that light consists of electromagnetic vibrations, we shall inquire into the conditions of propagation of plane electromagnetic waves in a dielectric which is eolotropic as regards both capacity and permeability, and determine the equation to the wave-surface.

For any direction of the normal (to the wave-front, understood) there are in general two normal velocities, *i. e.* there are two rays differently inclined to the normal whose ray-velocities and normal wave-velocities are different. And for any direction of ray there are in general two ray-velocities, *i. e.* two parallel rays having different velocities and wave-fronts.

In any wave (plane) the electric displacement and the magnetic induction must be always in the wave-front, *i. e.* perpendicular to the normal. But they are only exceptionally perpendicular to one another.

In any ray the electric force and the magnetic force are both perpendicular to the direction of the ray. But they are only exceptionally perpendicular to one another.

The magnetic force is always perpendicular to the electric displacement, and the electric force perpendicular to the magnetic induction. This of course applies to either wave. If we have to rotate the plane through the normal and the magnetic force through an angle θ to bring it to coincide with the magnetic induction, we must rotate the plane through the normal and the electric displacement through the same angle θ in the same direction to bring it to coincide with the electric force, the axis of rotation being the normal itself.

In the two waves having a common wave-normal, the displacement of either is parallel to the induction of the other. And in the two rays having a common direction, the magnetic force of either is parallel to the electric force of the other.

Nearly all our equations are symmetrical with respect to capacity and permeability; so that for every equation containing some electric variables there is a corresponding one to be got by exchanging electric force and magnetic force, &c. And when the forces, inductions, &c. are eliminated, leaving only capacities and permeabilities, these may be exchanged in any formula without altering its meaning, although its immediate Cartesian expansion after the exchange may be entirely different, and only convertible to the former expression by long processes.

If either μ or c be constant, we have the Fresnel wave-surface. Perhaps the most important case besides these is that in which the principal axes of permeability are parallel to those of capacity. There are then six principal velocities instead of only three, for the velocity of a wave depends upon the capacity in the direction of displacement as well as upon the permeability in the direction of induction. For instance, if μ_1, μ_2, μ_3 and c_1, c_2, c_3 are the principal permeabilities and capacities, and the wave-normal be parallel to the common axis of μ_1 and c_1, the other principal axes are the directions of induction and displacement, and the two normal velocities are $(c_2\mu_3)^{-\frac{1}{2}}$ and $(c_3\mu_2)^{-\frac{1}{2}}$.

The principal sections of the wave-surface in this case are all ellipses (instead of ellipses and circles, as in the one-sided Fresnel-wave); and two of these ellipses always cross, giving two axes of single-ray velocity. But should the ratio of the capacity to the permeability be the same for all the axes ($\mu_1/c_1=\mu_2/c_2=\mu_3/c_3$), the wave-surface reduces to a single ellipsoid, and any line is an optic axis. There is but one velocity, and no particular polarization. If the ratio is the same for two of the axes, the third is an optic axis.

Owing to the extraordinary complexity of the investigation when written out in Cartesian form (which I began doing, but gave up aghast), some abbreviated method of expression becomes desirable. I may also add, nearly indispensable, owing to the great difficulty in making out the meaning and mutual connections of very complex formulæ. In fact the transition from the velocity-equation to the wave-surface by proper elimination would, I think, baffle any ordinary algebraist, unassisted by some higher method, or at any rate by some kind of shorthand algebra. I therefore adopt, with some simplification, the method of vectors, which seems indeed the only proper method. But some of the principal results will be fully expanded in Cartesian form, which is easily done. And since all our equations will be either wholly scalar or wholly vector, the investigation is made independent of quaternions by simply defining a scalar product to be so and so, and a vector product so and so. The investigation is thus a Cartesian one modified by certain simple abbreviated modes of expression.

I have long been of opinion that the sooner the much needed introduction of quaternion methods into practical mathematical investigations in Physics takes place the better. In fact every analyst to a certain extent adopts them: first, by writing only one of the three Cartesian scalar equations corresponding to the single vector equation, leaving the others to be inferred; and next, by writing the first only of the three products which occur in the scalar product of two vectors. This, systematized, is I think the proper and natural way in which quaternion methods should be gradually brought in. If to this we further add the use of the vector product of two vectors, immensely increased power is given, and we have just what is wanted in the three dimensional analytical investigations of electromagnetism, with its numerous vector magnitudes.

It is a matter of great practical importance that the notation should be such as to harmonize with Cartesian formulæ, so that we can pass from one to the other readily, as is often required in mixed investigations, without changing notation. This condition does not appear to me to be attained by Professor Tait's notation, with its numerous letter prefixes, and especially by the $-S$ before every scalar product, the negative sign being the cause of the greatest inconvenience in transitions. I further think that Quaternions, as applied to Physics, should be established more by definition than at present; that scalar and vector products should be defined to mean such or such operations, thus avoiding some extremely obscure and quasi-metaphysical reasoning, which is quite unnecessary.

The first three sections of the following preliminary contain all we want as regards definitions; most of the rest of the preliminary consists of developments and reference-formulæ, which, were they given later, in the electromagnetic problem, would inconveniently interrupt the argument, and much lengthen the work.

Scalars and Vectors.—In a scalar equation every term is a scalar, or algebraic quantity, a mere magnitude; and $+$ and $-$ have the ordinary signification. But in a vector equation every term stands for a vector, or directed magnitude, and $+$ and $-$ are to be understood as compounding like velocities, forces, &c. Putting all vectors upon one side, we have the general form

$$\mathrm{A}+\mathrm{B}+\mathrm{C}+\mathrm{D}+\ldots=0;$$

where A, B,..., are any vectors, which, if n in number, may be represented, since their sum is zero, by the n sides of a polygon. Let A_1, A_2, A_3 be the three ordinary scalar components of A referred to any set of three rectangular axes, and similarly for the other vectors. This notation saves multiplication of letters. Then the above equation stands for the three scalar equations

$$\left.\begin{array}{l}\mathrm{A}_1+\mathrm{B}_1+\mathrm{C}_1+\mathrm{D}_1+\ldots=0,\\ \mathrm{A}_2+\mathrm{B}_2+\mathrm{C}_2+\mathrm{D}_2+\ldots=0,\\ \mathrm{A}_3+\mathrm{B}_3+\mathrm{C}_3+\mathrm{D}_3+\ldots=0.\end{array}\right\}$$

The $-$ sign before a vector simply reverses its direction—that is, negatives its three components.

According to the above, if i, j, k be rectangular vectors of unit length, we have

$$\mathrm{A}=i\mathrm{A}_1+j\mathrm{A}_2+k\mathrm{A}_3 \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

&c.; if A_1, A_2, A_3 be the components of A referred to the axes of i, j, k. That is, A is the sum of the three vectors $i\mathrm{A}_1$, $j\mathrm{A}_2$, $k\mathrm{A}_3$, of lengths A_1, A_2, A_3 parallel to i, j, k respectively.

Scalar Product.—We define AB thus,

$$\mathrm{AB}=\mathrm{A}_1\mathrm{B}_1+\mathrm{A}_2\mathrm{B}_2+\mathrm{A}_3\mathrm{B}_3, \quad . \quad . \quad . \quad . \quad (2)$$

and call it the scalar product of the vectors A and B. Its magnitude is that of A $\times$ that of B $\times$ the cosine of the angle between them. Thus, by (1) and (2),

$$\mathrm{A}_1=\mathrm{A}i, \quad \mathrm{A}_2=\mathrm{A}j, \quad \mathrm{A}_3=\mathrm{A}k;$$

and in general, N being any unit vector, AN is the scalar component of A parallel to N, or, briefly, the N component of A. Similarly,

$$i^2=1, \quad j^2=1, \quad k^2=1,$$

because i and i are parallel and of length unity, &c. And

$$ij=0, \quad jk=0, \quad ki=0,$$

because i and j, for instance, are perpendicular. Notice that AB=BA.

We have also

$$\mathrm{A}=\frac{\mathrm{A}^2}{\mathrm{A}}=\frac{\mathrm{A}^3}{\mathrm{A}^2}=\&\text{c}.,$$

and

$$\frac{1}{\mathrm{A}} \text{ or } \mathrm{A}^{-1}=\frac{\mathrm{A}}{\mathrm{A}^2}=\frac{\mathrm{A}^2}{\mathrm{A}^3}=\&\text{c}.$$

Thus A^{-1} has the same direction as A; its length is the reciprocal of that of A.

Vector Product.—We define VAB thus,

$$\mathrm{VAB}=i(\mathrm{A}_2\mathrm{B}_3-\mathrm{A}_3\mathrm{B}_2)+j(\mathrm{A}_3\mathrm{B}_1-\mathrm{A}_1\mathrm{B}_3)+k(\mathrm{A}_1\mathrm{B}_2-\mathrm{A}_2\mathrm{B}_1), \quad (3)$$

and call VAB the vector product of A and B. Its magnitude is that of A × that of B × the sine of the angle between them. Its direction is perpendicular to A and to B with the usual conventional relation between positive directions of translation and of rotation (the vine system). Thus, $\mathrm{V}ij=k$; $\mathrm{V}jk=i$; $\mathrm{V}ki=j$. Notice that $\mathrm{VAB}=-\mathrm{VBA}$, the direction being reversed by reversing the order of the letters; or by exchanging A and B in (3) we negative each term.

Hamilton's ∇. The operator

$$\nabla=i\frac{d}{dx}+j\frac{d}{dy}+k\frac{d}{dz} \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

may, since the differentiations are scalar, be treated as a vector, of course with either a scalar or a vector to follow it. If it operate on a scalar P we have the vector

$$\nabla\mathrm{P}=i\frac{d\mathrm{P}}{dx}+j\frac{d\mathrm{P}}{dy}+k\frac{d\mathrm{P}}{dz}, \quad . \quad (5)$$

whose three components are $d\mathrm{P}/dx$, &c. If it operate on a vector A, we have, by (2), the scalar product

$$\nabla\mathrm{A}=\frac{d\mathrm{A}_1}{dx}+\frac{d\mathrm{A}_2}{dy}+\frac{d\mathrm{A}_3}{dz}, \quad . \quad . \quad . \quad . \quad . \quad (6)$$

and, by (3), the vector product

$$\mathrm{V}\nabla\mathrm{A}=i\left(\frac{d\mathrm{A}_3}{dy}-\frac{d\mathrm{A}_2}{dz}\right)+j\left(\frac{d\mathrm{A}_1}{dz}-\frac{d\mathrm{A}_3}{dx}\right)+k\left(\frac{d\mathrm{A}_2}{dx}-\frac{d\mathrm{A}_1}{dy}\right). \quad (7)$$

The scalar product ∇A is the divergence of the vector A, the amount leaving the unit volume, if it be a flux. The vector product (7) is the curl of A, which will occur below. There are three remarkable theorems relating to ∇, viz.

$$\mathrm{P}_2-\mathrm{P}_1=\int_1^2\nabla\mathrm{P}ds, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

$$\int\mathrm{A}ds=\iint\mathrm{B}d\mathrm{S}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

$$\iint\mathrm{C}d\mathrm{S}=\iiint\nabla\mathrm{C}dv. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

Starting with P, a single-valued scalar function of position, the rise in its value from any point to another is expressed in (8) as the line-integral, along any line joining the points, of $\nabla\mathrm{P}ds$, the scalar product of ∇P, and ds the vector element of the curve.

Then passing from an unclosed to a closed curve, let A be any vector function of position (single-valued of course). Its line-integral round the closed curve is expressed in (9) as the surface-integral over any surface bounded by the curve of another vector B, which $=\mathrm{V}\nabla\mathrm{A}$. $\mathrm{B}d\mathrm{S}$ is the scalar product of B and the vector element of surface $d\mathrm{S}$, whose direction is defined by its unit normal.

Finally, passing from an unclosed to a closed surface, (10) expresses the surface-integral of any vector C over the closed surface (normal positive outward), as the volume-integral of its divergence within the included space.

Linear Vector Operators.—If H be the magnetic force at a point, B the induction, E the electric force, and D the displacement, all vectors, then

$$\mathrm{B}=\mu\mathrm{H}, \text{ and } \mathrm{D}=c\mathrm{E}/4\pi \quad . \quad . \quad . \quad . \quad (11)$$

express the relation of B to H and of D to E in a dielectric medium. If it be isotropic as regards displacement, c is the electric capacity; and if it be isotropic as regards induction, μ is the magnetic permeability; c and μ are then constants, if the medium be homogeneous, or scalar functions of position, if it be heterogeneous.

We shall not alter the form of the above equations in the case of eolotropy, when c and μ become linear operators. For instance, the induction will always be $\mu\mathrm{H}$, to be understood as a definite vector, got from H another vector, in a manner fully defined by (in case we want the developments) the following equations (not otherwise needed). Let $\mathrm{H}_1,\ldots$, and $\mathrm{B}_1,\ldots$, be the components of H and B referred to any rectangular axes. Then

$$\left.\begin{array}{l}\mathrm{B}_1=\mu_{11}\mathrm{H}_1+\mu_{12}\mathrm{H}_2+\mu_{13}\mathrm{H}_3,\\ \mathrm{B}_2=\mu_{21}\mathrm{H}_1+\mu_{22}\mathrm{H}_2+\mu_{23}\mathrm{H}_3,\\ \mathrm{B}_3=\mu_{31}\mathrm{H}_1+\mu_{32}\mathrm{H}_2+\mu_{33}\mathrm{H}_3,\end{array}\right\} \quad . \quad . \quad . \quad . \quad (12)$$

where μ_{11} &c. are constants, which may have any values not making HB negative; with the identities $\mu_{12}=\mu_{21}$, &c.

Or,

$$\mathrm{B}_1=\mu_1\mathrm{H}_1, \quad \mathrm{B}_2=\mu_2\mathrm{H}_2, \quad \mathrm{B}_3=\mu_3\mathrm{H}_3; \quad . \quad . \quad . \quad (13)$$

when the components are those referred to the principal axes of permeability, μ_1, μ_2, μ_3 being the principal permeabilities, all positive.

Inverse Operators.—Since $\mathrm{B}=\mu\mathrm{H}$, we have $\mathrm{H}=\mu^{-1}\mathrm{B}$, where μ^{-1} is the operator inverse to μ. When referred to the principal axes, we have

$$\mu'_1=\frac{1}{\mu_1}, \quad \mu'_2=\frac{1}{\mu_2}, \quad \mu'_3=\frac{1}{\mu_3}. \quad . \quad . \quad . \quad . \quad (14)$$

But when referred to any rectangular axes, we have

$$\mu'_{11}=\frac{\mu_{11}\mu_{22}-\mu_{12}^2}{\mu_1\mu_2\mu_3},\quad \mu'_{12}=\frac{\mu_{13}\mu_{23}-\mu_{12}\mu_{33}}{\mu_1\mu_2\mu_3},\ \&c. \quad . \quad (15)$$

by solution of (12). The accents belong to the inverse coefficients. The rest may be written down symmetrically, by cyclical changes of the figures. In the index surface the operators are inverse to those in the wave-surface.

Conjugate Property.—The following property will occur frequently. A and B being any vectors

$$A\mu B=B\mu A, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

or the scalar product of A and μB equals that of B and μA. It only requires writing out the full scalar products to see its truth, which results from the identities $\mu_{12}=\mu_{21}$, &c. Similarly,

$$A\mu cB=\mu AcB=c\mu AB,\ \&c.,$$
$$AB=A\mu\mu^{-1}B=\mu A\mu^{-1}B,\ \&c.,$$

where in the first line c is another self-conjugate operator.

D is expressed in terms of E similarly to (12) by coefficients c_{11}, c_{12}, &c.; or, as in (13), by the principal capacities c_1, c_2, c_3.

Theorem.—The following important theorem will be required. A and B being any vectors,

$$\mu_1\mu_2\mu_3VAB=\mu V\mu A\mu B. \quad . \quad . \quad . \quad . \quad (17)$$

For completeness a proof is now inserted, adapted from that given by Tait. Since VAB is perpendicular to A and B, by definition of a vector product, therefore

$$AVAB=0, \text{ and } BVAB=0,$$

by definition of a scalar product. Therefore

$$A\mu\mu^{-1}VAB=0, \text{ and } B\mu\mu^{-1}VAB=0,$$

by introducing $\mu\mu^{-1}=1$. Hence

$$\mu A\mu^{-1}VAB=0, \text{ and } \mu B\mu^{-1}VAB=0$$

by the conjugate property; that is, $\mu^{-1}VAB$ is perpendicular to μA and to μB. Or

$$h\mu^{-1}VAB=V\mu A\mu B,$$

where h is a scalar. Or

$$hVAB=\mu V\mu A\mu B$$

by operating by μ. To find h, multiply by any third vector C (not to be in the same plane as A and B), giving

$$hCVAB=C\mu V\mu A\mu B;$$

therefore

$$h=\frac{\mu CV\mu A\mu B}{CVAB}$$

by the conjugate property. Now expand this quotient of two scalar products, and it will be found to be independent of what vectors A, B, C may be. Choose them then to be i, j, k, three unit vectors parallel to the principal axes of μ. Then

$$h=\frac{\mu_3 kV\mu_1 i\mu_2 j}{kVij}=\mu_1\mu_2\mu_3,$$

by the $i\,j\,k$ properties before mentioned. This proves (17).

Transformation-Formula.—The following is very useful. A, B, C being any vectors,

$$VAVBC=B(CA)-C(AB). \quad . \quad . \quad . \quad (18)$$

Here CA and AB are scalar products, merely set in brackets to separate distinctly from the vectors B and C they multiply. This formula is evident on expansion.

The Equations of Induction.—E and H being the electric and magnetic forces at a point in a dielectric, the two equations of induction are

$$\text{curl } H=c\dot{E}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$
$$-\text{curl } E=\mu\dot{H}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

c and μ being the capacity and permeability operators, and curl standing for V∇ as defined in equation (7). Let Γ and G be the electric and the magnetic current, then

$$\Gamma=c\dot{E}/4\pi,\quad G=\mu\dot{H}/4\pi. \quad . \quad . \quad . \quad . \quad (21)$$

The dot, as usual, signifies differentiation to the time. The electric energy is EcE$/8\pi$ per unit volume, and the magnetic energy HμH$/8\pi$ per unit volume. If A is Maxwell's vector potential of the electric current, we have also

$$\text{curl } A=\mu H,\quad E=-\dot{A}. \quad . \quad . \quad . \quad . \quad (21\,a)$$

Similarly, we may make a vector Z the vector potential of the magnetic current, such that

$$-\text{curl } Z=cE,\quad H=-\dot{Z}. \quad . \quad . \quad . \quad . \quad (22)$$

The complete magnetic energy, by a well-known transformation, of any current system may be expressed in the two ways,

$$T=\Sigma H\mu H/8\pi=\tfrac{1}{2}\Sigma A\Gamma,$$

the Σ indicating summation through all space. Similarly, the electric energy, if there be no electrification, may be written in the two ways,

$$U = \Sigma E c E / 8\pi = \tfrac{1}{2} \Sigma Z G.$$

If there be electrification, we have also another term to add, the real electro*static* energy, in terms of the scalar potential and electrification. And if there be impressed electric force in the dielectric, part of G will be imaginary magnetic current, analogous to the imaginary electric current which may replace a system of intrinsic magnetization.

Plane Wave.—Let there be a plane wave in the medium. Its direction is defined by its normal. Let then N be the vector normal of unit length, and z be distance measured along the normal. If v be the velocity of the wave-front, the rate the disturbance travels along the normal, or the component parallel to the normal of the actual velocity of propagation of the disturbance, we have

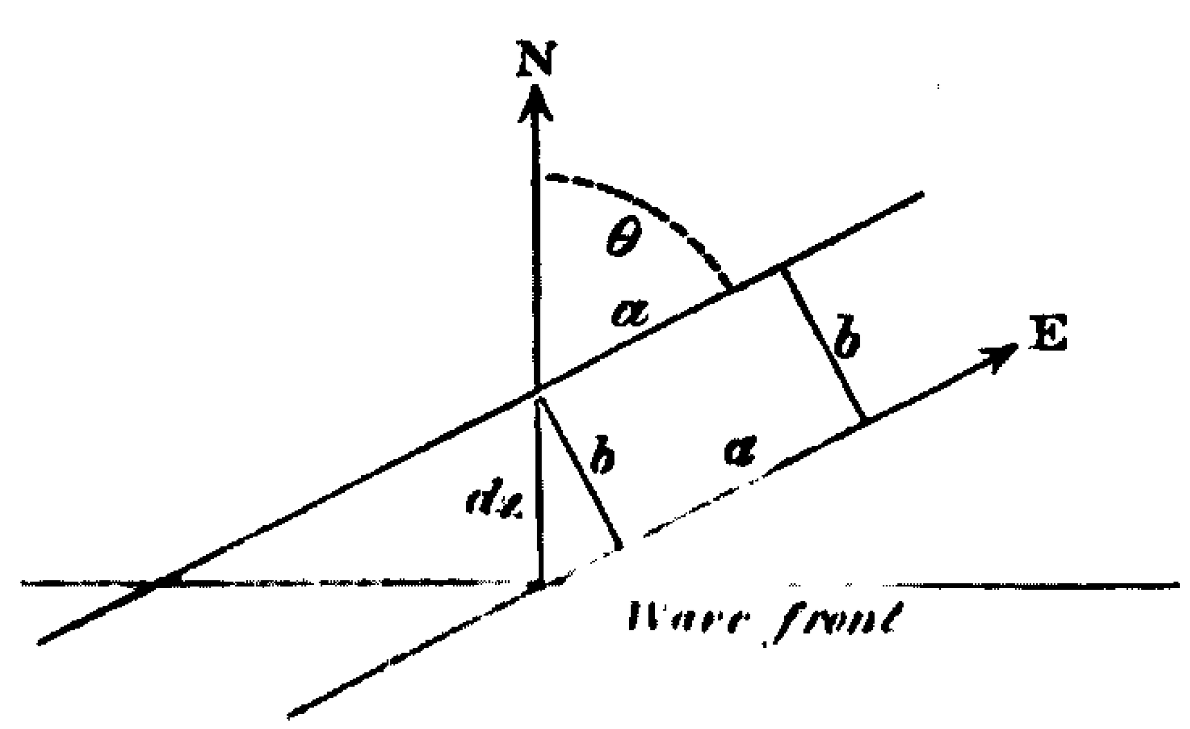
$$H = f(z - vt)$$

if the wave be a positive one, as we shall suppose, giving

$$-v \frac{d}{dz} = \frac{d}{dt}, \quad \dots \dots \dots \quad (23)$$

applied to H or E.

Next, examine what the operator $V\nabla$ or curl becomes when, as at present, the disturbance is assumed not to change direction, but only magnitude, as we pass along the normal. Apply the theorem of Version (9) to the elementary rectangular area bounded by two sides parallel to E of length a, and



two sides of length b perpendicular to E and in the same plane as E and the normal N. Since its area is ab, and $b = dz \sin\theta$, and the two sides b contribute nothing to the line-integral, we find that

$$\text{curl} = VN \frac{d}{dz}, \quad \dots \dots \dots \quad (24)$$

applied to E or H or other vectors, in the case of a plane

wave. Using this, and (23), in the equations of induction (19), (20), they become

$$VN \frac{dH}{dz} = -vc \frac{dE}{dz},$$

$$VN \frac{dE}{dz} = v\mu \frac{dH}{dz}.$$

Here, since the z differentiation is scalar, and occurs on both sides, it may be dropped, giving us

$$VNH = -vcE, \quad \dots \dots \dots \quad (25)$$

$$VNE = v\mu H. \quad \dots \dots \dots \quad (26)$$

The induction and the displacement are therefore necessarily in the wave-front, by the definition of a vector product, being perpendicular to N. Also the displacement is perpendicular to the magnetic force, and the induction is perpendicular to the electric force.

Index Surface.—Let

$$\sigma = \frac{N}{v}, \quad \dots \dots \dots \quad (27)$$

be a vector parallel to the normal, whose length is the reciprocal of the normal velocity v. It is the vector of the index surface. By (25) and (26) we have

$$cE = -V\sigma H, \text{ therefore } -E = c^{-1} V\sigma H; \quad \dots \quad (28)$$

and

$$\mu H = V\sigma E, \text{ therefore } H = \mu^{-1} V\sigma E. \quad \dots \quad (29)$$

Now use the theorem (17). Then, if

$$m = \mu_1 \mu_2 \mu_3, \quad n = c_1 c_2 c_3 \quad \dots \dots \quad (30)$$

be the products of the principal permeabilities and capacities, the theorem gives, applied to (28) and (29),

$$-nE = Vc\sigma cH, \quad \dots \dots \dots \quad (31)$$

$$mH = V\mu\sigma\mu E. \quad \dots \dots \dots \quad (32)$$

Putting the value of H given by (32) in (28) first, and then the value of E given by (31) in (29), we have

$$-mE = c^{-1} V\sigma V\mu\sigma\mu E, \quad \dots \dots \quad (33)$$

$$-nH = \mu^{-1} V\sigma V c\sigma c H. \quad \dots \dots \quad (34)$$

To these apply the transformation-formula (18), giving

$$-mcE = \mu\sigma(\sigma\mu E) - \mu E(\sigma\mu\sigma) \quad \dots \quad (33a)$$

and

$$-n\mu H = c\sigma(\sigma c H) - cH(\sigma c\sigma), \quad \dots \quad (34a)$$

where the bracketed quantities are scalar products. Put in this form

$$\{(\sigma\mu\sigma)\mu - mc\}\mathrm{E} = \mu\sigma(\sigma\mu\mathrm{E}), \quad . \quad . \quad . \quad . \quad (35)$$

$$\{(\sigma c\sigma)c - n\mu\}\mathrm{H} = c\sigma(\sigma c\mathrm{H}), \quad . \quad . \quad . \quad . \quad (36)$$

and perform on them the inverse operations to those contained in the { }'s, dividing also by the scalar products on the right sides. Then

$$\frac{\mathrm{E}}{\sigma\mu\mathrm{E}} = \frac{\mu\sigma}{(\sigma\mu\sigma)\mu - mc}, \quad . \quad . \quad . \quad . \quad . \quad (37)$$

$$\frac{\mathrm{H}}{\sigma c\mathrm{H}} = \frac{c\sigma}{(\sigma c\sigma)c - n\mu}. \quad . \quad . \quad . \quad . \quad . \quad (38)$$

Operate by c on (37) and by μ on (38), and transfer all operators to the denominators on the right. Then

$$\frac{c\mathrm{E}}{\sigma\mu\mathrm{E}} = \frac{\sigma}{(\sigma\mu\sigma)c^{-1} - m\mu^{-1}} = \beta_1 \text{ say}, \quad . \quad . \quad (39)$$

$$\frac{\mu\mathrm{H}}{\sigma c\mathrm{H}} = \frac{\sigma}{(\sigma c\sigma)\mu^{-1} - nc^{-1}} = \beta_2 \text{ say}. \quad . \quad . \quad (40)$$

[It should be noted that, in thus transferring operators, care should be taken to do it properly, otherwise it had better not be done at all. Thus, we have by (37),

$$\beta_1 = c\frac{\mu\sigma}{(\sigma\mu\sigma)\mu - mc}, \text{ or } \beta_1 = c\{(\sigma\mu\sigma)\mu - mc\}^{-1}\mu\sigma,$$

and the left c and the right μ are to go inside the { }. Operate by c^{-1} and then again by $\{\}^{+1}$, thus cancelling the $\{\}^{-1}$, giving

$$\mu\sigma = \{(\sigma\mu\sigma)\mu - mc\}c^{-1}\beta_1.$$

Here we can move c^{-1} inside, giving

$$\mu\sigma = \{(\sigma\mu\sigma)\mu c^{-1} - m\}\beta_1;$$

and now operating by μ^{-1}, it may be moved inside, giving

$$\sigma = \{(\sigma\mu\sigma)c^{-1} - m\mu^{-1}\}\beta_1,$$

as in (39).]

We can now, by (39) and (40), get as many forms of the index equation as we please. We know that the displacement is perpendicular to the normal, and so is the induction. Hence

$$\sigma\beta_1 = 0, \quad \sigma\beta_2 = 0; \quad . \quad . \quad . \quad . \quad . \quad . \quad (41)$$

where β_1 and β_2 are the above vectors, [(39) and (40)], are two equivalent equations of the index surface.

Also, operate on (39) by $\sigma\mu c^{-1}$, and on (40) by $\sigma c\mu^{-1}$, and

the left members become unity, by the conjugate property; hence

$$\mu\sigma c^{-1}\beta_1 = 1, \quad c\sigma\mu^{-1}\beta_2 = 1 \quad . \quad . \quad . \quad . \quad (42)$$

are two other forms of the index equation. (41) and (42) are the simplest forms. More complex forms are created with that surprising ease which is characteristic of these operators; but we do not want any more. When expanded, the different forms look very different, and no one would think they represented the same surface. This is also true of the corresponding Fresnel surface, which is comparatively simple in expression. In any equation we may exchange the operators μ and c.

Put $\sigma = \mathrm{N}v^{-1}$ in any form of index equation, and we have the velocity equation, a quadratic in v^2 giving the two velocities of the wave-front. And if we put $\mathrm{N}v = p$, making thus p a vector parallel to the normal of length equal to the velocity, it will be the vector of the surface which is the locus of the foot of the perpendicular from the origin upon the tangent-plane to the wave-surface.

By (33 *a*), remembering that σ is parallel to the normal, we see that

$$\left.\begin{array}{l} c\mathrm{E},\ \mu\mathrm{E}, \text{ and } \mu\mathrm{N} \text{ are in one plane;} \\ \text{or,} \quad \mathrm{E},\ \mathrm{N}, \text{ and } \mu^{-1}c\mathrm{E} \text{ are in one plane.} \end{array}\right\} \quad . \quad . \quad (43)$$

And by (34 *a*),

$$\left.\begin{array}{l} \mu\mathrm{H},\ c\mathrm{N}, \text{ and } c\mathrm{H} \text{ are in one plane;} \\ \text{or,} \quad \mathrm{H},\ \mathrm{N}, \text{ and } c^{-1}\mu\mathrm{H} \text{ are in one plane.} \end{array}\right\} \quad . \quad . \quad (44)$$

These conditions expanded, give us the directions of the electric force and displacement, the magnetic force and induction, for a given normal. We may write the second of (43) thus,

$$\mathrm{NV}\frac{\mathrm{D}}{c}\frac{\mathrm{D}}{\mu} = 0; \quad . \quad . \quad . \quad . \quad . \quad . \quad (45)$$

and the second of (44) thus,

$$\mathrm{NV}\frac{\mathrm{B}}{c}\frac{\mathrm{B}}{\mu} = 0; \quad . \quad . \quad . \quad . \quad . \quad . \quad (46)$$

and as these differ only in the substitution of B for D, we see that the induction of either ray is parallel to the displacement of the other; that is, the two directions of induction in the wave-front are the two directions of displacement.

The Wave-Surface.—Since the velocity-surface with the vector $p = v\mathrm{N}$ is the locus of the foot of the perpendicular on the tangent-plane to the wave-surface, we have, if ρ be the vector of the wave-surface,

$$\rho\rho = p^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (47)$$

But σ the vector of the index-surface being $=\mathrm{N}v^{-1}=pv^{-2}$, we have by (47), dividing it by v^2,

$$\sigma\rho = 1. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (48)$$

To find the wave-surface, we must therefore let σ be variable and eliminate it between (48) and any one of the index equations. This is not so easy as it may appear.

General considerations may lead us to the conclusion that the equation to the wave-surface and that to the index-surface may be turned one into the other by the simple process of inverting the operators, turning c into c^{-1} and μ into μ^{-1}. Although this will be verified later, any form of index equation giving a corresponding form of wave by inversion of operators, yet it must be admitted that this requires proof. That it is true when one of the operators c or μ is a constant does not prove that it is also true when we have the inverse compound operator $\{(\sigma c\sigma)\mu^{-1}-nc^{-1}\}^{-1}$ containing both c and μ, neither being constant. I have not found an easy proof. This will not be wondered at when the similar investigations of the Fresnel surface are referred to. Professor Tait, in his 'Quaternions,' gives two methods of finding the wave-surface; one from the velocity equation, the other from the index equation. The latter is rather the easier, but cannot be said to be very obvious, nor does either of them admit of much simplification. The difficulty is of course considerably multiplied when we have the two operators to reckon with. I believe the following transition from index to wave cannot be made more direct, or shorter, except of course by omission of steps, which is not a real shortening.

Given
$$\frac{c\mathrm{E}}{\sigma\mu\mathrm{E}} = \beta_1 = \frac{\sigma}{(\sigma\mu\sigma)c^{-1}-m\mu^{-1}}, \quad (49)=(39)\ bis,$$

$$\sigma\beta_1 = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad (50)=(41)\ bis,$$

$$\rho\sigma = 1, \quad . \quad . \quad . \quad . \quad . \quad . \quad (51)=(48)\ bis.$$

Eliminate σ and get an equation in ρ. We have also

$$\mu\sigma c^{-1}\beta_1 = 1, \quad . \quad . \quad . \quad . \quad . \quad (52)=(42)\ bis$$

which will assist later.

By (49) we have

$$\sigma = (\sigma\mu\sigma)c^{-1}\beta_1 - m\mu^{-1}\beta_1, \quad . \quad . \quad . \quad . \quad . \quad . \quad (53)$$

Multiply by β_1 and use (50); then

$$0 = (\sigma\mu\sigma)(\beta_1 c^{-1}\beta_1) - m(\beta_1\mu^{-1}\beta_1). \quad . \quad . \quad . \quad . \quad (54)$$

By differentiation, σ being variable, and therefore β_1 also,

$$0 = 2(d\sigma\mu\sigma)(\beta_1 c^{-1}\beta_1) + 2(\sigma\mu\sigma)(d\beta_1 c^{-1}\beta_1) - 2m(d\beta_1\mu^{-1}\beta_1). \quad (55)$$

Also, differentiating (53),

$$d\sigma = 2(d\sigma\mu\sigma)c^{-1}\beta_1 + (\sigma\mu\sigma)dc^{-1}\beta_1 - md\mu^{-1}\beta_1;$$

and, multiplying this by $2\beta_1$, gives

$$2\beta_1 d\sigma = 4(d\sigma\mu\sigma)\beta_1 c^{-1}\beta_1 + 2(\sigma\mu\sigma)(d\beta_1 c^{-1}\beta_1) - 2m(d\beta_1\mu^{-1}\beta_1). \quad (56)$$

Subtract (55) from (56) and halve the result; thus obtaining

$$\beta_1 d\sigma = (d\sigma\mu\sigma)(\beta_1 c^{-1}\beta_1),$$

or

$$\{\beta_1 - (\beta_1 c^{-1}\beta_1)\mu\sigma\}d\sigma = 0. \quad . \quad . \quad . \quad . \quad (57)$$

In the last five equations it will be understood that $d\sigma$ and $d\beta_1$ are differential vectors, and that $d\sigma\mu\sigma$ is the scalar product of $d\sigma$ and $\mu\sigma$, &c.; also in getting (56) from the preceding equation we have $\beta_1 dc^{-1}\beta_1 = \beta_1 c^{-1}d\beta_1 = d\beta_1 c^{-1}\beta_1$, &c. Equation (57) is the expression of the result of differentiating (50),

$$d(\sigma\beta_1) = d\sigma\beta_1 + \sigma d\beta_1 = 0,$$

with $d\beta_1$ eliminated.

Now (57) shows that the vector in the $\{\}$ is perpendicular to $d\sigma$ the variation of σ. But by (51) we also have, on differentiation,

$$\rho d\sigma = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (58)$$

Hence ρ and the $\{\}$ vector in (57) must be parallel. This gives

$$h\rho = \beta_1 - (\beta_1 c^{-1}\beta_1)\mu\sigma, \quad . \quad . \quad . \quad . \quad . \quad (59)$$

where h is a scalar. If we multiply this by $c^{-1}\beta_1$ and use (52), we obtain

$$\rho c^{-1}\beta_1 = 0; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (60)$$

or, by (49), giving β_1 in terms of $c\mathrm{E}$,

$$\rho\mathrm{E} = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (61)$$

a very important landmark. The ray is perpendicular to the electric force.

Similarly, if we had started from, (instead of (49), (50), and (52)), the corresponding H equations, viz.,

$$\frac{\mu\mathrm{H}}{\sigma c\mathrm{H}} = \beta_2 = \frac{\sigma}{(\sigma c\sigma)\mu^{-1}-nc^{-1}},$$

$$\sigma\beta_2 = 0, \qquad c\sigma\mu^{-1}\beta_2 = 1,$$

with of course the same equation (51) connecting ρ and σ, we should have arrived at

$$h'\rho=\beta_2-(\beta_2\mu^{-1}\beta_2)c\sigma\,; \qquad (62)$$

h' being a constant, corresponding to (59); of this no separate proof is needed, as it amounts to exchanging μ and c and turning E into H, to make (39) become (40). And from (62), multiplying it by $\mu^{-1}\beta_2$, we arrive at

$$\rho\mu^{-1}\beta_2=0, \text{ or } \rho\mathrm{H}=0, \qquad (63)$$

corresponding to (61). The ray is thus perpendicular both to the electric and to the magnetic force. The first half of the demonstration is now completed, but before giving the second half we may notice some other properties.

Thus, to determine the values of the scalar constants h and h'. Multiply (59) by σ, and use (50) and (51); then

$$h=-(\beta_1 c^{-1}\beta_1)(\sigma\mu\sigma)=-m(\beta_1\mu^{-1}\beta_1),$$

the second form following from (54). Insert in (59), then

$$\rho=\frac{\mu\sigma}{\sigma\mu\sigma}-\frac{\beta_1}{m(\beta_1\mu^{-1}\beta_1)} \qquad (64)$$

gives ρ explicitly in terms of $\mu\sigma$ and β_1, the latter of which is known in terms of the former by (49). Multiply this by $\mu^{-1}\beta_1$, using (50); then

$$\rho\mu^{-1}\beta_1=-\frac{1}{m}. \qquad (65)$$

Similarly we shall find

$$h'=-n(\beta_2 c^{-1}\beta_2), \qquad (66)$$

giving

$$\rho=\frac{c\sigma}{\sigma c\sigma}-\frac{\beta_2}{n(\beta_2 c^{-1}\beta_2)}; \qquad (67)$$

and, corresponding to (65), we shall have

$$\rho c^{-1}\beta_2=-\frac{1}{n}. \qquad (68)$$

Now to resume the argument, stopped at equation (63). Up to equation (59) the work is plain and straightforward, according to rule in fact, being merely the elimination of the differentials, and the getting of an equation between ρ and σ. What to do next is not at all obvious. From (59), or from (64), the same with h eliminated, we may obtain all sorts of scalar products containing ρ and β_1, and if we could put β_1 explicitly in terms of ρ, (60) or (65) would be forms of the wave-surface equation. From the purely mathematical point of view no direct way presents itself; but (61) and (63), considered physically as well as mathematically, guide us at once to the second half of the transformation from the index

to the wave equation. As, at the commencement, we found the induction and the displacement to be perpendicular to the normal, so now we find that the corresponding forces are perpendicular to the ray. There was no difficulty in reaching the index equation before, when we had a single normal with two values of v the normal velocity, and two rays differently inclined to the normal. There should then be no difficulty, by parallel reasoning, in arriving at the wave-surface equation from analogous equations which express that the ray is perpendicular to the magnetic and electric forces, considering two parallel rays travelling with different ray-velocities with two differently inclined wave-fronts.

Now, as we got the index equation from

$$\mathrm{VNH}=-vc\mathrm{E}, \qquad (25)\,bis$$

$$\mathrm{VNE}=\ \ v\mu\mathrm{H}, \qquad (26)\,bis$$

we must have two corresponding equations for one ray-direction. Let M be a unit vector defining the direction of the ray, and w be the ray-velocity, so that

$$\rho=w\mathrm{M}. \qquad (69)$$

Operate on (25) and (26) by VM, giving

$$\mathrm{VMVNH}=-v\mathrm{VM}c\mathrm{E},$$

$$\mathrm{VMVNE}=\ \ v\mathrm{VM}\mu\mathrm{H}.$$

Now use the formula of transformation (18), giving

$$\mathrm{N(HM)}-\mathrm{H(MN)}=-v\mathrm{VM}c\mathrm{E},$$

$$\mathrm{N(EM)}-\mathrm{E(MN)}=\ \ v\mathrm{VM}\mu\mathrm{H}.$$

But HM=0 and EM=0, as proved before. Also $v=w(\mathrm{MN})$, or the wave-velocity is the normal component of the ray-velocity. Hence

$$\mathrm{H}=w\mathrm{VM}c\mathrm{E}, \qquad (70)$$

$$-\mathrm{E}=w\mathrm{VM}\mu\mathrm{H}, \qquad (71)$$

which are the required analogues of (25) and (26). Or, by (69),

$$\mathrm{H}=\mathrm{V}\rho c\mathrm{E}, \qquad (72)$$

$$-\mathrm{E}=\mathrm{V}\rho\mu\mathrm{H} \qquad (73)$$

are the analogues of (28) and (29). The rest of the work is plain. Eliminating E and H successively, we obtain

$$0=\mathrm{E}+\mathrm{V}\rho\mu\mathrm{V}\rho c\mathrm{E},$$

$$0=\mathrm{H}+\mathrm{V}\rho c\mathrm{V}\rho\mu\mathrm{H}\,;$$

and, using the theorem (17), these give

$$0=\mathrm{E}+m\mathrm{V}\rho\mathrm{V}\mu^{-1}\rho\mu^{-1}c\mathrm{E},$$
$$0=\mathrm{H}+n\mathrm{V}\rho\mathrm{V}c^{-1}\rho c^{-1}\mu\mathrm{H};$$

which, using the transformation-formula (18), become

$$0=\mathrm{E}+m\mu^{-1}\rho(\mu^{-1}\rho c\mathrm{E})-\mu^{-1}c\mathrm{E}(\rho\mu^{-1}\rho)m,$$
$$0=\mathrm{H}+nc^{-1}\rho(c^{-1}\rho\mu\mathrm{H})-c^{-1}\mu\mathrm{H}(\rho c^{-1}\rho)n;$$

or, rearranging, after operating by μ and c respectively,

$$\{(\rho\mu^{-1}\rho)mc-\mu\}\mathrm{E}=m\rho(\mu^{-1}\rho c\mathrm{E}),$$
$$\{(\rho c^{-1}\rho)n\mu-c\}\mathrm{H}=n\rho(c^{-1}\rho\mu\mathrm{H}).$$

Or

$$\frac{\mathrm{E}}{\mu^{-1}\rho c\mathrm{E}}=\frac{\rho}{(\rho\mu^{-1}\rho)c-m^{-1}\mu}=\gamma_1,\text{ say}, \quad . \quad . \quad (74)$$

$$\frac{\mathrm{H}}{c^{-1}\rho\mu\mathrm{H}}=\frac{\rho}{(\rho c^{-1}\rho)\mu-n^{-1}c}=\gamma_2,\text{ say}. \quad . \quad . \quad . \quad (75)$$

These give us the four simplest forms of equation to the wave. For, since $\rho\mathrm{E}=0=\rho\mathrm{H}$, we have

$$\rho\gamma_1=0, \quad \rho\gamma_2=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (76)$$

Also, operating on (74) by $\mu^{-1}\rho c$ and on (75) by $c^{-1}\rho\mu$ we get

$$\mu^{-1}\rho c\gamma_1=1, \quad c^{-1}\rho\mu\gamma_2=1, \quad . \quad . \quad . \quad (77)$$

two other forms.

γ_1 and γ_2 differ from β_1 and β_2 merely in the change from σ to ρ, and in the inversion of the operators. The two forms of wave (76) are analogous to (41), and the two forms (77) analogous to (42), inverting operators and putting ρ for σ.

Similarly, if the wave-surface equation be given and we require that of the index-surface, we must impose the same condition $\rho\sigma=1$ as before, and eliminate ρ. This will lead us to

$$\sigma c\gamma_1=0, \quad \sigma\mu\gamma_1=-m, \quad . \quad . \quad . \quad . \quad (78)$$

corresponding to (60) and (65); and

$$\sigma\mu\gamma_2=0, \quad \sigma c\gamma_2=-n, \quad . \quad . \quad . \quad . \quad (79)$$

corresponding to (63) and (68); and the firsts of (78) and (79) are equivalent to

$$\sigma c\mathrm{E}=0, \quad \sigma\mu\mathrm{H}=0;$$

or the displacement and the induction are perpendicular to the normal. This completes the first half of the process; the second part would be the repetition of the already given investigation of the index equation.

The vector rate of transfer of energy being $\mathrm{VEH}/4\pi$ in general, when a ray is solitary, its direction is that of the transfer of energy. It seems reasonable, then, to define the direction of a ray, whether the wave is plane or not, as perpendicular to the electric and the magnetic forces. On this understanding, we do not need the preliminary investigation of the index-surface, but may proceed at once to the wave-surface by the investigation (69) to (77), following equations (25) and (26).

The following additional useful relations are easily deducible :—From (25) and (26) we get

$$\sigma=\frac{\mathrm{V}c\mathrm{E}\mu\mathrm{H}}{\mathrm{E}c\mathrm{E}}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (80)$$

and from (72) and (73),

$$\rho=\frac{\mathrm{VEH}}{\mathrm{E}c\mathrm{E}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (81)$$

Also, from either set,

$$\mathrm{E}c\mathrm{E}=\mathrm{H}\mu\mathrm{H}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (82)$$

expressing the equality of the electric to the magnetic energy per unit volume (strictly, at a point).

Some Cartesian Expansions.—In the important case of parallelism of the principal axes of capacity and permeability, the full expressions for the index or the wave-surface equations may be written down at once from the scalar product abbreviated expressions. Thus, taking any equation to the wave, as the first of (76), for example, $\rho\gamma_1=0$, γ being given in (74), take the axes of coordinates parallel to the common principal axes of c and μ; so that we can employ c_1, c_2, c_3, the principal capacities, and μ_1, μ_2, μ_3 the principal permeabilities in the three components of γ_1. We then have, x, y, z being the coordinates of ρ,

$$\frac{x^2}{(\rho\mu^{-1}\rho)c_1-m\mu_1}+\frac{y^2}{(\rho\mu^{-1}\rho)c_2-m\mu_2}+\frac{z^2}{(\rho\mu^{-1}\rho)c_3-m\mu_3}=0, \quad (83)$$

where

$$\rho\mu^{-1}\rho=\frac{x^2}{\mu_1}+\frac{y^2}{\mu_2}+\frac{z^2}{\mu_3}.$$

In (83) we may exchange the c's and μ's, getting the second of (76). Similarly the first of (77) gives

$$\frac{\mu_1^{-1}c_1x^2}{(\rho\mu^{-1}\rho)c_1-m\mu_1}+\frac{\mu_2^{-1}c_2y^2}{(\rho\mu^{-1}\rho)c_2-m\mu_2}+\frac{\mu_3^{-1}c_3z^2}{(\rho\mu^{-1}\rho)c_3-m\mu_3}=1 \quad (84)$$

as another form, in which, again, the μ's and c's may be exchanged (not forgetting to change m into n) to give a fourth form.

These reduce to the Fresnel surface if either $\mu_1=\mu_2=\mu_3$ or $c_1=c_2=c_3$.

Let $x=0$ to find the sections in the plane yz. The first denominator in (83) gives

$$\left(\frac{y^2}{\mu_2}+\frac{z^2}{\mu_3}\right)c_1-\frac{1}{\mu_2\mu_3}=0,\quad\text{or}\quad y^2c_1\mu_3+z^2c_1\mu_2=1,$$

representing an ellipse, semiaxes

$$v_{13}=(c_1\mu_3)^{-\frac{1}{2}}\quad\text{and}\quad v_{12}=(c_1\mu_2)^{-\frac{1}{2}}.$$

The other terms give

$$\left(\frac{y^2}{\mu_2}+\frac{z^2}{\mu_3}\right)(c_3y^2+c_2z^2)=\frac{y^2}{\mu_1\mu_2}+\frac{z^2}{\mu_1\mu_3}.$$

Or

$$y^2\mu_1c_3+z^2\mu_1c_2=1.$$

An ellipse, semiaxes $v_{31}=(c_3\mu_1)^{-\frac{1}{2}}$ and $v_{21}=(c_2\mu_1)^{-\frac{1}{2}}$. Similarly, in the plane zx the sections are ellipses whose semiaxes are v_{21}, v_{23}, and v_{12}, v_{32}, where for brevity $v_{rs}=(c_r\mu_s)^{-\frac{1}{2}}$; and in the plane xy, the ellipses have semiaxes v_{31}, v_{32}, and v_{13}, v_{12}.

In one of the principal planes two of the ellipses intersect, giving four places where the two members of the double surface unite.

If $c_1/\mu_1=c_2/\mu_2=c_3/\mu_3$, we have a single ellipsoidal wave-surface whose equation is

$$\frac{x^2}{v_{23}^2}+\frac{y^2}{v_{31}^2}+\frac{z^2}{v_{12}^2}=1. \quad . \quad . \quad . \quad . \quad (85)$$

Now, of course, $v_{12}=v_{21}$, &c.

When the μ and c axes are not parallel, we cannot immediately write down the full expansion of the wave-surface equation. Proceed thus:—Taking $\rho\gamma_1=0$ as the equation, let

$$R=m(\rho\mu^{-1}\rho),\quad\text{and}\quad \alpha=m^{-1}\gamma_1;$$

then, by (74) and (76),

$$\rho\frac{\rho}{Rc-\mu}=0,\quad\text{or}\quad \rho\alpha=0,$$

where

$$\rho=(Rc-\mu)\alpha. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (86)$$

R is a scalar. If α_1, α_2, α_3 are the three components of α referred to any rectangular axes, and x, y, z the components of ρ, we have, by (86) and (12),

$$x=(Rc_{11}-\mu_{11})\alpha_1+(Rc_{12}-\mu_{12})\alpha_2+(Rc_{13}-\mu_{13})\alpha_3,$$
$$y=(Rc_{21}-\mu_{21})\alpha_1+(Rc_{22}-\mu_{22})\alpha_2+(Rc_{23}-\mu_{23})\alpha_3,$$
$$z=(Rc_{31}-\mu_{31})\alpha_1+(Rc_{32}-\mu_{32})\alpha_2+(Rc_{33}-\mu_{33})\alpha_3;$$

from which α_1, α_2, α_3 may be solved in terms of x, y, z thus

$$\alpha_1=a_{11}x+a_{12}y+a_{13}z,$$
$$\alpha_2=a_{21}x+a_{22}y+a_{23}z,$$
$$\alpha_3=a_{31}x+a_{32}y+a_{33}z;$$

where, by using (15),

$$a_{11}=\frac{(Rc_{22}-\mu_{22})(Rc_{33}-\mu_{33})-(Rc_{23}-\mu_{23})^2}{\Delta},$$

$$a_{12}=\frac{(Rc_{13}-\mu_{13})(Rc_{23}-\mu_{23})-(Rc_{12}-\mu_{12})(Rc_{33}-\mu_{33})}{\Delta};$$

and the rest by symmetry. Then, since

$$\rho\alpha=x\alpha_1+y\alpha_2+z\alpha_3=0,$$

we get the full expansion. Δ need not be written fully, as it goes out. The equation may be written symmetrically, thus,

$$0=1+mn(\rho\mu^{-1}\rho)(\rho c^{-1}\rho)-\{x^2(c_{22}\mu_{33}+c_{33}\mu_{22}-2c_{23}\mu_{23})+\ldots$$
$$+2xy(c_{13}\mu_{23}+c_{23}\mu_{13}-c_{12}\mu_{33}-c_{33}\mu_{12})+\ldots\}, \quad . \quad (87)$$

where the coefficients of y^2, z^2, yz, and zx are omitted. Here $m=\mu_1\mu_2\mu_3$ and $n=c_1c_2c_3$; whilst

$$\rho c^{-1}\rho=c'_{11}x^2+c'_{22}y^2+c'_{33}z^2+2c'_{12}xy+2c'_{23}yz+2c'_{31}zx,$$

where $c'_{11}, \ldots$, are the inverse coefficients. See equation (15). The expansion of $\rho\mu^{-1}\rho$ is exactly similar, using the inverse μ coefficients.

If in (87) we for every c or μ write the reciprocal coefficients, we obtain the equation to the index-surface; that is, supposing x, y, z then to be the components of σ instead of ρ. And, since $\sigma v=\mathrm{N}$, the unit wave normal, we have the velocity equation as follows, in the general case,

$$0=\frac{\mathrm{N}\mu\mathrm{N}}{m}\cdot\frac{\mathrm{N}c\mathrm{N}}{n}+v^4-v^2\{\mathrm{N}_1^2(c'_{22}\mu'_{33}+c'_{33}\mu'_{22}-2c'_{23}\mu'_{23})+\ldots$$
$$+2\mathrm{N}_1\mathrm{N}_2(c'_{13}\mu'_{23}+c'_{23}\mu'_{13}-c'_{12}\mu'_{33}-c'_{33}\mu'_{12})+\ldots\} \quad . \quad (88)$$

in which N_1, N_2, N_3 are the components of N, or the direction-cosines of the normal. To show the dependence of v^2 upon the capacity and permeability perpendicular to N, take $\mathrm{N}_1=1$, $\mathrm{N}_2=0$, $\mathrm{N}_3=0$, which does not destroy generality, because in (88) the axes of reference are arbitrary. Then (88) reduces to

$$v^4-(c'_{22}\mu'_{33}+c'_{33}\mu'_{22}-2c'_{23}\mu'_{23})v^2$$
$$+(c'_{22}c'_{33}-c'^2_{23})(\mu'_{22}\mu'_{33}-\mu'^2_{23})=0.$$

When the μ and c axes are parallel, and their principal axes are those of reference, we have

$$0=\frac{N\mu N}{m}\cdot\frac{NcN}{n}+v^4-v^2\{N_1^2(v_{23}^2+v_{32}^2)+N_2^2(v_{31}^2+v_{13}^2)+N_3^2(v_{12}^2+v_{21}^2)\}, \quad . \quad . \quad . \quad (89)$$

where

$$N\mu N=\mu_1N_1^2+\mu_2N_2^2+\mu_3N_3^2,$$

with a similar expression for NcN, and $v_{23}=(c_2\mu_3)^{-\frac{1}{2}}$, &c., as before. The solution is

$$v^2=\tfrac{1}{2}N_1^2(v_{23}^2+v_{32}^2)+\tfrac{1}{2}N_2^2(v_{31}^2+v_{13}^2)+\tfrac{1}{2}N_3^2(v_{12}^2+v_{21}^2)\pm\tfrac{1}{2}\sqrt{X}, \quad (90)$$

where

$$X=N_1^4u_1^2+N_2^4u_2^2+N_3^4u_3^2-2(N_1^2N_2^2u_1u_2+N_2^2N_3^2u_2u_3+N_3^2N_1^2u_1u_3),$$

in which

$$u_1=v_{23}^2-v_{32}^2, \quad u_2=v_{31}^2-v_{13}^2, \quad u_3=v_{12}^2-v_{21}^2. \quad . \quad (91)$$

Take $u_1=0$, or $c_2/\mu_2=c_3/\mu_3$; the two velocities are then

$$N_1^2v_{23}^2+N_2^2v_{31}^2+N_3^2v_{12}^2, \text{ and } N_1^2v_{23}^2+N_2^2v_{13}^2+N_3^2v_{21}^2,$$

reducing to one velocity v_{23} when $N_1=1$.

If, further, $u_2=0$, or $u_3=0$, making $c_1/\mu_1=c_2/\mu_2=c_3/\mu_3$, $X=0$ always, and

$$v^2=N_1^2v_{23}^2+N_2^2v_{31}^2+N_3^2v_{12}^2, \quad . \quad . \quad . \quad . \quad (92)$$

is the single value of the square of velocity of wave-front.

Directions of E, H, D, *and* B.—We may expand (45) to obtain an equation for the two directions of the induction and displacement. Thus, since

$$\frac{D}{c}=i(c'_{11}D_1+c'_{12}D_2+c'_{13}D_3)+j(c'_{21}D_1+c'_{22}D_2+c'_{23}D_3)+k(c'_{31}D_1+c'_{32}D_2+c'_{33}D_3),$$

$$\frac{D}{\mu}=i(\mu'_{11}D_1+\mu'_{12}D_2+\mu'_{13}D_3)+j(\mu'_{21}D_1+\mu'_{22}D_2+\mu'_{23}D_3)+k(\mu'_{31}D_1+\mu'_{32}D_2+\mu'_{33}D_3),$$

$$N=iN_1+jN_2+kN_3,$$

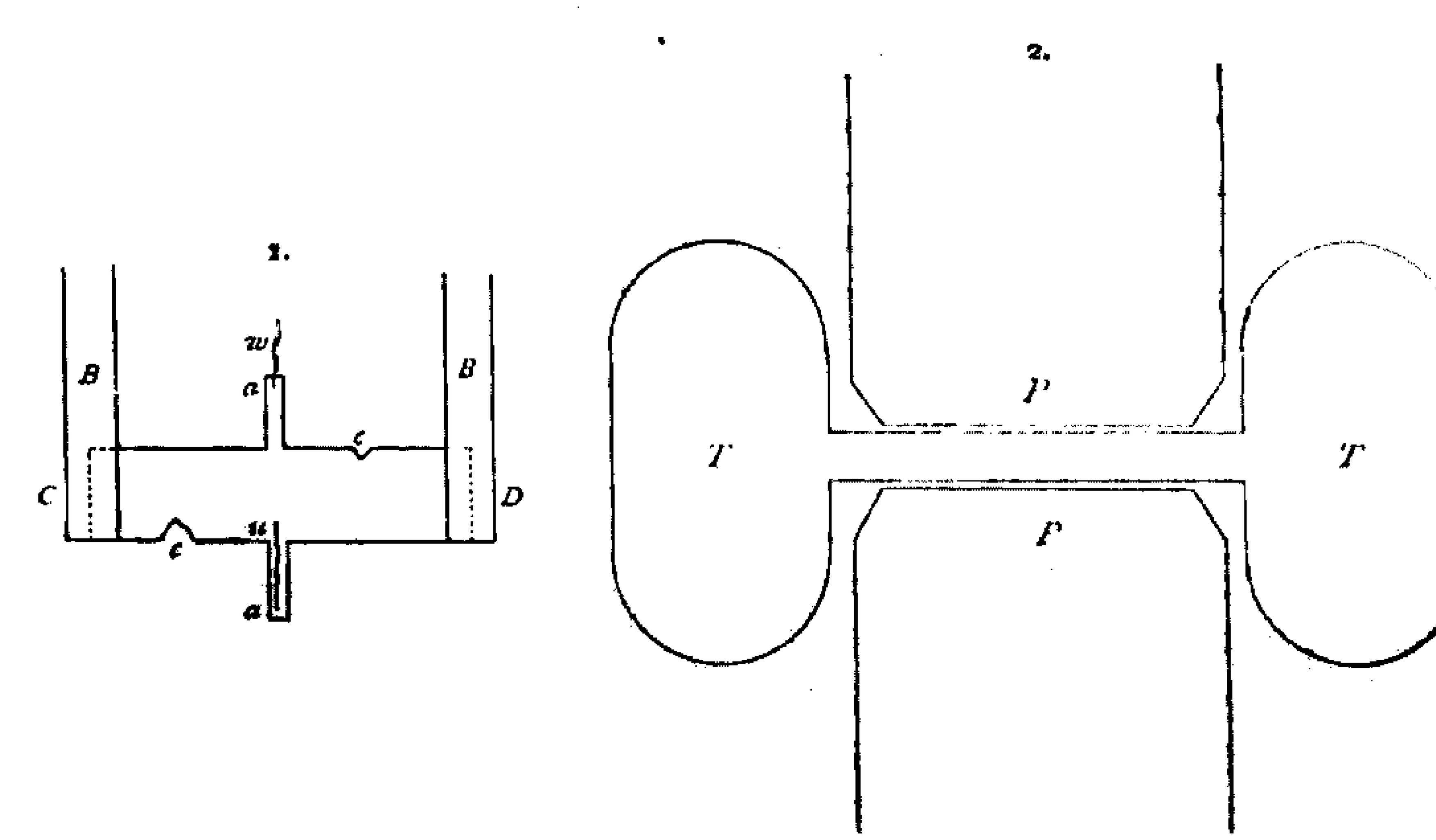
the determinant of the coefficients of i, j, k equated to zero gives the required equation. When the principal axes of μ and c are parallel, the equation greatly simplifies, being then

$$0=\frac{N_1u_1}{D_1}+\frac{N_2u_2}{D_2}+\frac{N_3u_3}{D_3}, \quad . \quad . \quad . \quad . \quad (93)$$

where $u_1, \ldots$, are the same differences of squares of principal velocities, as in (91). For D_1 &c. write B_1 &c.; and we have the same equations for the induction directions. For D_1, &c., write c_1E_1, &c., and the resulting equation gives the directions of E. For D_1, &c., write μ_1H_1, &c., and the resulting equation gives the directions of H.

XLVI. *On the Rotation of the Equipotential Lines of an Electric Current by Magnetic Action.* *By* E. H. Hall, *Instructor in Physics at Harvard College**.

In this article the results will be given of experiments made during the month of August 1883, and at intervals since, in the Physical Laboratory of Harvard College. The substances which have been chiefly examined are copper, zinc, certain of their alloys, and iron and steel. Some mention will be made also of gold, cobalt, nickel, bismuth, and antimony. In most cases when possible the metal was used in the form of a thin strip, about 1·1 centim. wide and about 3 centim. long, between the two pieces of brass B, B (fig. 1), which, soldered to the ends of the strip, served as electrodes for the entrance and escape of the main current. To the arms a, a, about 2 millim. wide and perhaps 7 millim. long, were soldered



the wires w, w, which led to a Thomson galvanometer. The notches c, c, show how adjustment was secured. The strip thus prepared was fastened to a plate of glass by means of a cement of beeswax and rosin, all the parts shown in the figure being imbedded in and covered by this cement, which was so

* From Silliman's American Journal, February 1885.