moisture, or condensed gases, or to combinations of these causes. And it affords an explanation of the details of reflection, which is rigid, and at least as good as the representation given by the empirical formulæ of Cauchy, even as modified by Quincke.

VI. "On the Transformation of Optical Wave-Surfaces by Homogeneous Strain." By Oliver Heaviside, F.R.S.

Received December 20, 1893.

Simplex Eolotropy.

1. All explanations of double refraction (proximate, not ultimate) rest upon the hypothesis that the medium in which it occurs is so structured as to impart eolotropy to one of the two properties, associated with potential and kinetic energy, with which the ether is endowed in order to account for the transmission of waves through it in the simplest manner. It may be elastic eolotropy, or it may be something equivalent to eolotropy as regards the density. In Maxwell's electromagnetic theory the two properties are those connecting the electric force with the displacement, and the magnetic force with the induction, say the permittivity and the inductivity, or c and μ. These are, in the simplest case, constants corresponding to isotropy. The existence of eolotropy as regards either of them will cause double refraction. Then either c or μ is a symmetrical linear operator, or dyadic, as Willard Gibbs calls it. In either case the optical wave-surface is of the Fresnel type. In either case the fluxes displacement and induction are perpendicular to one another and in a wave-front, whilst the electric and magnetic forces are also perpendicular to one another. But it is the magnetic force that is in the wave-front, coincident with the induction, in case of magnetic isotropy and electric eolotropy, the electric force being then out of the wave-front, though in the plane of the normal and the displacement. And in the other extreme case of electric isotropy and magnetic eolotropy, the electric force is in the wave-front, coincident with the displacement, whilst the magnetic force is out of the wave-front, though in the plane of the normal and the induction. Now, as a matter of fact, crystals may be strongly eolotropic electrically, whilst their magnetic eolotropy, if existent, is insignificant. This, of course, justifies Maxwell's ascription of double refraction to electric eolotropy.

Properties connected with Duplex Eolotropy.

2. When duplex eolotropy, electric and magnetic, is admitted, we obtain a more general kind of wave-surface, including the former two as extreme cases. It is almost a pity that magnetic eolotropy should be insensible, because the investigation of the conditions regulating plane waves in media possessing duplex eolotropy, and the wave-surface associated therewith, possesses many points of interest. The chief attraction lies in the perfectly symmetrical manner in which the subject may be displayed, as regards the two eolotropies. This brings out clearly properties which are not always easily visible in the case of simplex eolotropy, when there is a one-sided and imperfect development of the analysis concerned.

In general, the fluxes displacement and induction, although in the wave-front, are not coperpendicular. Corresponding to this, the two forces electric and magnetic, which are always in the plane perpendicular to the ray, or the flux of energy, are not coperpendicular. Nor are the positions of the fluxes in the wave-front conditioned by the effective components in that plane of the forces being made to coincide with the fluxes. There are two waves with a given normal, and it would be impossible to satisfy this requirement for both. But there is a sort of balance of skewness, inasmuch as the positions of the fluxes in the wave-front are such that the angle through which the plane containing the normal and the displacement (in either wave) must be turned, round the normal as axis, to reach the electric force, is equal (though in the opposite sense) to the angle through which the plane containing the normal and the induction must be turned to reach the magnetic force. These are merely rudimentary properties. I have investigated the wave-surface and associated matters in my paper "On the Electromagnetic Wave-surface" ('Phil. Mag.,' June, 1885; or 'Electrical Papers,' vol. 2, p. 1).

Effects of straining a Duplex Wave-surface.

3. The connexion between the simplex and duplex types of wave-surface has been interestingly illustrated lately by Dr. J. Larmor in his paper "On the Singularities of the Optical Wave-surface," ('Proc. London Math. Soc.,' vol. 24, 1893). He points out, incidentally, that a simplex wave-surface, when subjected to a particular sort of homogeneous strain, becomes a duplex wave-surface of a special kind. To more precisely state the connexion, let there be electric eolotropy, say c, with magnetic isotropy. Then, if the strainer, or strain operator, applied to the simplex wave-surface, be homologous with c, given by $c^{-\frac{1}{2}} \times$ constant, the result is to turn it into a duplex wave-surface whose two eolotropies are also homologous with the original c; that is to say, the principal axes are parallel. This duplex wave-surface is, of course, of a specially simplified kind, though not the simplest. That occurs when the two eolotropies are not merely homologous, but are in constant ratio. The wave-surface then reduces to a single ellipsoid.

Conversely, therefore, if we start with the duplex wave-surface corresponding to homologous permittivity and inductivity, and homogeneously strain it, the strainer being proportional to $c^{\frac{1}{2}}$, we convert it to a simplex wave-surface whose one eolotropy is homologous with the former two.

Remembering that the equation of the duplex wave-surface is symmetrical with respect to the two eolotropies, so that they may be interchanged without altering the surface, it struck me on reading Dr. Larmor's remarks that a similar reduction to a simplex wave-surface could be effected by a strainer proportional to $\mu^{\frac{1}{2}}$. This was verified on examination, and some more general transformations presented themselves. The results are briefly these:—

Any duplex wave-surface (irrespective of homology of eolotropies), when subjected to homogeneous strain (not necessarily pure), usually remains a duplex wave-surface. That is, the transformed surface is of the same type, though with different inductivity and permittivity operators.

But in special cases it becomes a simplex wave-surface. In one way the strainer is $c^{\frac{1}{2}}/[c^{\frac{1}{2}}]$, where the square brackets indicate the determinant of the enclosed operator. In another the strainer is $\mu^{\frac{1}{2}}/[\mu^{\frac{1}{2}}]$. These indicate the strain operator to be applied to the vector of the old surface to produce that of the new one.

Now, these simplex wave-surfaces may be strained anew to their reciprocals with respect to the unit sphere, or the corresponding index-surfaces, which are surfaces of the same type. So we have at least four ways of straining any duplex wave-surface to a simplex one.

Furthermore, any duplex wave-surface may be homogeneously strained to its reciprocal, the corresponding index-surface, of the same duplex type. The strain is pure, but is complicated, as it involves both c and μ. The strainer is $c^{-1}(c\mu^{-1})^{\frac{1}{2}}$, divided by the determinant of the same. This transformation is practically the generalization for the duplex wave-surface of Plücker's theorem relating to the Fresnel surface, for that also involves straining the wave-surface to its reciprocal.

Instead of the single strain above mentioned, we may employ three successive pure strains. Thus, first strain the duplex wave-surface to a simplex surface. Secondly, strain the latter to its reciprocal. Thirdly, strain the last to the reciprocal of the original duplex wave-surface. There are at least two sets of three successive strains which effect the desired transformation. The investigation follows.

Forms of the Index- and Wave-surface Equations, and the Properties of Inversion and Interchangeability of Operators.

4. Let the electric and magnetic forces be **E** and **H**, and the corresponding fluxes, the displacement and induction, be **D** and **B**, then

$$\mathbf{D} = c\mathbf{E}, \qquad \mathbf{B} = \mu\mathbf{H}, \tag{1}$$

where c is the permittivity and μ the inductivity, to be symmetrical linear operators in general. We have also the circuital laws

$$\text{curl } \mathbf{H} = c\dot{\mathbf{E}}, \qquad -\text{curl } \mathbf{E} = \mu\dot{\mathbf{H}}. \tag{2}$$

Now, if we assume the existence of a plane wave, whose unit normal is **N**, propagated at speed v without change of type, and apply these equations, we find that **D** and **B** are in the wave-front, **E** and **H** are out of it, and that there are two waves possible. We are led directly to the velocity equation, a quadratic in v^2, giving the two values of v^2 belonging to a given **N**. Next, if we put $\mathbf{s} = \mathbf{N}/v$, then **s** is the vector of the index-surface, and its equation is

$$\mathbf{s}\frac{\mathbf{s}}{c^{-1} - \dfrac{\mu^{-1}}{[\mu^{-1}]\,(\mathbf{s}\mu\mathbf{s})}} = 0 = \mathbf{s}\frac{\mathbf{s}}{\mu^{-1} - \dfrac{c^{-1}}{[c^{-1}]\,(\mathbf{s}c\mathbf{s})}}, \tag{3}$$

which are, of course, equivalent to the velocity equation ('El. Pa.,' vol. 2, p. 11, equations (41)). Two forms are given, for a reason that will appear later. I employ the vector algebra and notation of the paper referred to, and others. Sufficient to say here that c^{-1} and μ^{-1} are the reciprocals of c and μ; and that $\mathbf{s}c\mathbf{s}$ means the scalar product of **s** and $c\mathbf{s}$; for example, if referred to the principal axes of c,

$$\mathbf{s}c\mathbf{s} = c_1 s_1^2 + c_2 s_2^2 + c_3 s_3^2, \tag{4}$$

if c_1, c_2, c_3 be the principal c's (positive scalars, to ensure positivity of the energy), and s_1, s_2, s_3 be the components of **s**. Also, $[c^{-1}]$ denotes the determinant* of c^{-1}, that is, $(c_1c_2c_3)^{-1}$.

The operators in the denominators of (3) may be treated, for our purpose, as linear operators themselves. But it is their reciprocals that occur. For example, the first form of (3) may be written

$$\mathbf{s}\left[c^{-1} - \frac{\mu^{-1}}{[\mu^{-1}]\,(\mathbf{s}\mu\mathbf{s})}\right]^{-1}\mathbf{s} = 0, \tag{5}$$

asserting that the vectors **s** and $[\ldots]^{-1}\mathbf{s}$ are perpendicular. The expansion of (3) to Cartesian form may be done immediately if c and μ are homologous, for then we may take the reference axes **i, j, k** parallel to those of c and μ, and at once produce

* It occurs to me in reading the proof that the use of $[c]$ to denote the determinant of c, which is plainer to read in combination with other symbols than $|c|$, is in conflict with the ordinary use of square brackets, as in (5) and some equations near the end. But there will be no confusion on this account in the present paper.

$$\frac{s_1^2}{\frac{1}{c_1}-\frac{\mu_2\mu_3}{\mathbf{s}\mu\mathbf{s}}}+\frac{s_2^2}{\frac{1}{c_2}-\frac{\mu_3\mu_1}{\mathbf{s}\mu\mathbf{s}}}+\frac{s_3^2}{\frac{1}{c_3}-\frac{\mu_1\mu_2}{\mathbf{s}\mu\mathbf{s}}}=0, \qquad (6)$$

where $\mathbf{s}\mu\mathbf{s}$ is as in (4), with μ written for c. Similarly as regards the second form of (3). When the operators are not homologous, the complication of the form of the constituents of the inverse operators makes the expansion less easy.

As regards the second form of (3), it is obtained from the first form by interchanging μ and c. It represents the same surface. The transformation from one form to the other, if done by ordinary algebra, without the use of vectors and linear operators, is very troublesome in the general case. But in the electromagnetic theory the equivalence can be seen to be true and predicted beforehand. For consider the circuital equations (2). If we eliminate **H**, we obtain

$$-\text{curl }\mu^{-1}\text{ curl }\mathbf{E}=c\ddot{\mathbf{E}}, \qquad (7)$$

whilst if we eliminate **E**, we obtain

$$-\text{curl }c^{-1}\text{ curl }\mathbf{H}=\mu\ddot{\mathbf{H}}. \qquad (8)$$

These are the characteristic equations of **E** and **H** respectively in a dielectric with duplex eolotropy, and we see that they only differ in the interchange of c and μ. When, therefore, we apply one of them, say that of **E**, to a plane wave to make the velocity equation, in which process **E** is eliminated, we can see that a precisely similar investigation applies to the **H** equation, provided μ and c be interchanged. So, if the **E** equation leads to the first form in (3), the **H** equation must lead to the second form. They therefore represent the same surface. The same property applies to any equation obtained from the circuital equations with the electrical variables eliminated, the equation of the wave-surface, for example. If we have obtained one special form, a second is got by interchanging the eolotropies.

The index equation being what we are naturally led to from the characteristic equation, it is merely a matter of mathematical work to derive the corresponding wave-surface. For $\mathbf{s}$ is the reciprocal of the perpendicular upon the tangent plane to the wave-surface, so that

$$\mathbf{rs}=1, \qquad (9)$$

if $\mathbf{r}$ is the vector of the wave-surface; and from the equation of $\mathbf{s}$ and its connexion with $\mathbf{r}$, we may derive the equation of $\mathbf{r}$ itself. I have shown (*loc. cit.*, vol. 2, pp. 12—16) that the result is expressed by simply inverting the operators in the index equation. Thus, the equation of the wave-surface is

$$\frac{\mathbf{r}}{c - \dfrac{\mu}{[\mu]\,(\mathbf{r}\mu^{-1}\mathbf{r})}} = 0 = \mathbf{r}\frac{\mathbf{r}}{\mu - \dfrac{c}{[c]\,(\mathbf{r}c^{-1}\mathbf{r})}}, \tag{10}$$

where, as before, two forms are given. Now, the final equivalence of this transition from the index to wave-equation to mere inversion of the two eolotropic operators is such a simple result that one would think there should be a very simple way of exhibiting how the transition comes about. Nevertheless, I am not aware of any simple investigation, and, in fact, found the transition rather difficult, and by no means obvious at first. I effected the transformation by taking advantage of symmetrical relations between the forces and fluxes; in particular proving, first, that $\mathbf{rE} = 0 = \mathbf{rH}$, or that the ray is perpendicular to the electric and magnetic forces, comparing this with the analogous property $\mathbf{sD} = 0 = \mathbf{sB}$, and constructing a process for leading from the former to the wave-equation analogous to that leading from the latter to the index equation. It then goes easily. However, we are not concerned with these details here.

A caution is necessary regarding the interchangeability of μ and c. They should be fully operative as linear operators. If one of them be a constant initially, and therefore all through, we may not then interchange them in the simplified equations which result. For example, let μ be constant in (10). We have now

$$\mathbf{r}\frac{\mathbf{r}}{c - \dfrac{1}{\mu\mathbf{r}^2}} = 0 = \mathbf{r}\frac{\mathbf{r}}{\mu - \dfrac{c}{[c]\,(\mathbf{r}c^{-1}\mathbf{r})}}. \tag{11}$$

The first form is what we are naturally led to by initial assumption of constancy of μ. Now observe that the interchange of μ and c in the second form gives us the first form, after a little reduction, remembering that $[\mu]$ is now μ^3. But the same interchange in the first form does not produce the second, because it is more general. So we have gained a relative simplicity of form at the cost of generality. The extra complication of the duplex wave-surface is accompanied by general analytical extensions which make the working operations more powerful. The equivalence of the two forms in (11) may be established by the use of Hamilton's general cubic equation of a linear operator, as done in Tait's work. Though not difficult to carry out, the operations are rather recondite. On the other hand, the much more general equivalence (10) is, as we saw for the reason following (7) and (8), obviously true. This suggests that some other transformations involving the general cubic may be made plainer by generalizing it, employing a pair of linear operators.

General Transformation of Wave-surface by Homogeneous Strain.

5. Now apply a homogeneous strain to the wave-surface. Let

$$\mathbf{q} = \frac{\phi}{[\phi]}\mathbf{r}. \tag{12}$$

We need not suppose that the strain is pure. Use (12) in the first of (10). It becomes

$$\phi^{-1}\mathbf{q}\frac{\phi^{-1}\mathbf{q}}{c-\dfrac{\mu}{[\mu][\phi]^2(\phi^{-1}\mathbf{q}\mu^{-1}\phi^{-1}\mathbf{q})}} = 0. \tag{13}$$

Now the use of vectors and linear operators produces such a concise exhibition of the essentially significant properties, freed from the artificial elaboration of coordinates, that a practised worker may readily see his way to the following results by mere inspection of equation (13), or with little more. I give, however, much of the detailed work that would then be done silently, believing that the spread of vector analysis is not encouraged by the quaternionist's practice of leaving out too many of the steps.

In the first place, $\phi^{-1}\mathbf{q}$ is the same as $\mathbf{q}\phi'^{-1}$, if ϕ' is the conjugate of ϕ. So

$$\phi^{-1}\mathbf{q}\mu^{-1}\phi^{-1}\mathbf{q} = \mathbf{q}\phi'^{-1}\mu^{-1}\phi^{-1}\mathbf{q} \tag{14}$$

in the denominator. Also, the first $\phi^{-1}\mathbf{q}$ in (13) may be written $\mathbf{q}\phi'^{-1}$, and the postfactor ϕ'^{-1} may then be transferred to the denominator. To do this, it must be inverted, of course, and then brought in as a postfactor. Similarly, the ϕ^{-1} in the numerator may be merged in the denominator by inversion first, and then bringing it in as a prefactor. We may see why this is to be done by the elementary formula

$$a^{-1}b^{-1}c^{-1} = (cba)^{-1}, \tag{15}$$

where a, b, c are any linear operators. So (13) becomes

$$\mathbf{q}\frac{\mathbf{q}}{\phi c\phi'-\dfrac{\phi\mu\phi'}{[\mu][\phi]^2(\mathbf{q}\phi'^{-1}\mu^{-1}\phi^{-1}\mathbf{q})}} = 0. \tag{16}$$

Now introduce some simplifications of form. Let

$$\phi c\phi' = b, \qquad \phi\mu\phi' = \lambda. \tag{17}$$

It follows from the second, and by (15), that

$$\phi'^{-1}\mu^{-1}\phi^{-1} = (\phi\mu\phi')^{-1} = \lambda^{-1}. \tag{18}$$

We also have

$$[\lambda] = [\mu][\phi]^2. \tag{19}$$

These three, (17) to (19), reduce (16) to

$$\mathbf{q}\frac{\mathbf{q}}{b-\dfrac{\lambda}{[\lambda](\mathbf{q}\lambda^{-1}\mathbf{q})}} = 0 = \mathbf{q}\frac{\mathbf{q}}{\lambda-\dfrac{b}{[b](\mathbf{q}b^{-1}\mathbf{q})}}, \tag{20}$$

where the second form is got from the first by interchanging λ and b, which is permissible on account of the interchangeability of μ and c.

Comparing (20) with (10), we see that there is identity of form. Consequently (20) represents a duplex wave-surface whose operators are b and λ, provided they are self-conjugate. They are, for, by the elementary formula

$$(abc)' = c'b'a', \tag{21}$$

it follows that

$$\phi c\phi' = (\phi c\phi')', \tag{22}$$

and similarly for the other one.

In case the strain is a pure rotation, we may take the form of ϕ (following Gibbs) as

$$\phi = \mathbf{I}.\mathbf{i}+\mathbf{J}.\mathbf{j}+\mathbf{K}.\mathbf{k}, \tag{23}$$

where $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ is one, and $\mathbf{I}$, $\mathbf{J}$, $\mathbf{K}$ another set of coperpendicular unit vectors. For, obviously, this makes

$$\phi\mathbf{r} = \mathbf{I}.\mathbf{i}\mathbf{r}+\mathbf{J}.\mathbf{j}\mathbf{r}+\mathbf{K}.\mathbf{k}\mathbf{r} = \mathbf{I}x+\mathbf{J}y+\mathbf{K}z. \tag{24}$$

Special Cases of Reduction to a Simplex Wave-surface.

6. Now take some special forms of ϕ. We see, by inspection of (17), that we can reduce either of b or λ to a constant. Thus, first,

$$\phi = \mu^{-\frac{1}{2}}, \qquad \lambda = 1, \qquad b = \mu^{-\frac{1}{2}}c\mu^{-\frac{1}{2}}. \tag{25}$$

Then (20) reduces to

$$\mathbf{q}\frac{\mathbf{q}}{b-\dfrac{1}{\mathbf{q}^2}} = 0 = \mathbf{q}\frac{\mathbf{q}}{1-\dfrac{b}{[b](\mathbf{q}b^{-1}\mathbf{q})}}, \tag{26}$$

showing that the original duplex wave-surface is reduced to a simplex one involving eolotropy b, given by (25).

Similarly, a second way is

$$\phi = c^{-\frac{1}{2}}, \qquad b = 1, \qquad \lambda = c^{-\frac{1}{2}}\mu c^{-\frac{1}{2}}, \tag{27}$$

which reduces (20) to the simplex wave-surface

$$\mathbf{q}\frac{\mathbf{q}}{1-\dfrac{\lambda}{[\lambda](\mathbf{q}\lambda^{-1}\mathbf{q})}} = 0 = \mathbf{q}\frac{\mathbf{q}}{\lambda-\dfrac{1}{\mathbf{q}^2}}, \tag{28}$$

involving the eolotropy λ.

The new surfaces (26), (28) may now be strained to their reciprocals. Thus, take the first of (26), and put

$$\mathbf{p} = \frac{b^{-\frac{1}{2}}}{[b^{-\frac{1}{2}}]}\mathbf{q}. \tag{29}$$

This makes

$$b^{\frac{1}{2}}\mathbf{p}\frac{b^{\frac{1}{2}}\mathbf{p}}{b-\dfrac{[b^{\frac{1}{2}}]^2}{(b^{\frac{1}{2}}\mathbf{p})^2}} = 0. \tag{30}$$

Here the initial and final $b^{\frac{1}{2}}$'s may be removed to the denominator, and, since we also have

$$(b^{\frac{1}{2}}\mathbf{p})^2 = b^{\frac{1}{2}}\mathbf{p}b^{\frac{1}{2}}\mathbf{p} = \mathbf{p}b\mathbf{p}, \tag{31}$$

we bring the first of (26) to

$$\mathbf{p}\frac{\mathbf{p}}{1-\dfrac{b^{-1}}{[b^{-1}](\mathbf{p}b\mathbf{p})}} = 0. \tag{32}$$

Now compare this with the second form of the same (26). They are identical, except that b is now inverted. Consequently (32) represents the index-surface corresponding to the wave-surface represented by the second of (26), and therefore by the first, since they are the same. In a similar manner the strain (29) applied to the second of (26) leads to the reciprocal of the first form.

In like manner the simplex surface (28) is strained to its reciprocal by

$$\mathbf{p} = \frac{\lambda^{-\frac{1}{2}}}{[\lambda^{-\frac{1}{2}}]}\mathbf{q}. \tag{33}$$

Applied to the first form of (28), we get the second form with λ inverted; and, applied to the second form, we get the first, with λ inverted. These inversions of simplex wave-surfaces by homogeneous strain are equivalent to Plücker's theorem showing that the Fresnel wave-surface is its own reciprocal with respect to a certain ellipsoid (Tait, 'Quaternions,' 3rd Ed., p. 342).

Transformation from Duplex Wave- to Index-surface by a Pure Strain.

7. What is of greater interest here is the generalization of this property for the duplex wave-surface itself. Take

$$\phi = c^{-1}(c\mu^{-1})^{\frac{1}{2}}. \tag{34}$$

Then we obtain

$$\phi c \phi = c^{-1}(c\mu^{-1})^{\frac{1}{2}} c c^{-1}(c\mu^{-1})^{\frac{1}{2}} = \mu^{-1}, \tag{35}$$

$$\phi\mu\phi = c^{-1}(c\mu^{-1})^{\frac{1}{2}}\mu c^{-1}(c\mu^{-1})^{\frac{1}{2}} = c^{-1}, \tag{36}$$

the first of which is obvious, whilst in the second we make use of

$$\mu c^{-1} = (c\mu^{-1})^{-1}. \tag{37}$$

There are other ways in which this ϕ may be expressed, viz.,

$$\phi = c^{-1}(c\mu^{-1})^{\frac{1}{2}} = \mu^{-1}(\mu c^{-1})^{\frac{1}{2}} = (\mu^{-1}c)^{\frac{1}{2}}c^{-1} = (c^{-1}\mu)^{\frac{1}{2}}\mu^{-1}, \tag{38}$$

all of which lead to $\qquad \mu\phi c\phi = 1. \qquad$ (39)

If this ϕ is self-conjugate, we see, by (17) and (35), that its use in (20) brings us to

$$\mathbf{q}\frac{\mathbf{q}}{\mu^{-1}-\dfrac{c^{-1}}{[c^{-1}](\mathbf{q}c\mathbf{q})}} = 0 = \mathbf{q}\frac{\mathbf{q}}{c^{-1}-\dfrac{\mu^{-1}}{[\mu^{-1}](\mathbf{q}\mu\mathbf{q})}}. \tag{40}$$

That is, the strain converts the first of (10) to the first of (40), and the second of (10) to the second of (40). But the first of (40) is the same as the second of (10) with μ and c inverted, and the second of (40) is the same as the first of (10) with the same inversions. In other words, the strain has converted the duplex wave-surface to its corresponding index-surface. Observe that the crossing over from first to second form is an essential part of the demonstration, which is the reason I have employed two forms.

In full, the strainer to be applied to $\mathbf{r}$ of the wave-surface to produce the vector $\mathbf{s}$ of the index-surface (or $\mathbf{q}$ in (40)) is

$$\frac{\phi}{[\phi]} = [c^{\frac{1}{2}}][\mu^{\frac{1}{2}}]c^{-1}(c\mu^{-1})^{\frac{1}{2}}. \tag{41}$$

But to complete the demonstration it should be shown that this strain is pure, because we have just assumed $\phi = \phi'$ in equation (20) to obtain (40). Now the purity of this strain is not obvious in the form (41), nor in any of the similar forms in (38). But we may change the expression for ϕ to such a form as will explicitly show its purity. Thus, we have

$$c\mu^{-1} = c^{\frac{1}{2}} \,.\, c^{\frac{1}{2}}\mu^{-1}c^{\frac{1}{2}} \,.\, c^{-\frac{1}{2}},$$

identically, and this may be expanded to

$$c\mu^{-1} = c^{\frac{1}{2}}(c^{\frac{1}{2}}\mu^{-1}c^{\frac{1}{2}})^{\frac{1}{2}}c^{-\frac{1}{2}}c^{\frac{1}{2}}(c^{\frac{1}{2}}\mu^{-1}c^{\frac{1}{2}})^{\frac{1}{2}}c^{-\frac{1}{2}},$$

the right member reducing to the left by obvious cancellations. Therefore

$$(c\mu^{-1})^{\frac{1}{2}} = c^{\frac{1}{2}}(c^{\frac{1}{2}}\mu^{-1}c^{\frac{1}{2}})^{\frac{1}{2}}c^{-\frac{1}{2}},$$

by taking the square root. So, finally,

$$\phi = c^{-1}(c\mu^{-1})^{\frac{1}{2}} = c^{-\frac{1}{2}}(c^{\frac{1}{2}}\mu^{-1}c^{\frac{1}{2}})^{\frac{1}{2}}c^{-\frac{1}{2}}. \tag{42}$$

This is of the form $\phi_1\phi_2\phi_1$, where ϕ_1 is pure. Its conjugate is therefore $\phi_1\phi'_2\phi_1$. This reduces to ϕ itself if ϕ_2 is pure. But ϕ_2 is pure, because it is also of the form $\theta_1\theta_2\theta_1$, where θ_1 and θ_2 are both pure. So our single strain depending on ϕ is pure.

Substitution of three successive Pure Strains for one. Two ways.

8. This is dry mathematics. But it is at once endowed with interest if we consider the meaning of the expression of the strain ϕ as equivalent to the three successive strains ϕ_1, ϕ_2, and ϕ_1. First, the strain

$$\mathbf{q} = \frac{\phi_1}{[\phi_1]}\mathbf{r} = \frac{c^{-\frac{1}{2}}}{[c^{-\frac{1}{2}}]}\mathbf{r} \tag{43}$$

converts the duplex wave-surface to a simplex surface. This was done before, equation (28). Next, the strain

$$\mathbf{p} = \frac{\phi_2}{[\phi_2]}\mathbf{q} = \frac{(c^{\frac{1}{2}}\mu^{-1}c^{\frac{1}{2}})^{\frac{1}{2}}}{[c^{\frac{1}{2}}][\mu^{-\frac{1}{2}}]}\mathbf{q} \tag{44}$$

converts the simplex surface $\mathbf{q}$ to another simplex surface whose vector is $\mathbf{p}$, and which is the index-surface corresponding to the wave-surface $\mathbf{q}$. This strain (44) is, in fact, the same as (33), and the result is

$$\mathbf{p}\frac{\mathbf{p}}{1-\dfrac{\lambda^{-1}}{[\lambda^{-1}](\mathbf{p}\lambda\mathbf{p})}} = 0 = \mathbf{p}\frac{\mathbf{p}}{\lambda^{-1}-\dfrac{1}{\mathbf{p}^2}}, \tag{45}$$

where $\lambda = c^{-\frac{1}{2}}\mu c^{-\frac{1}{2}}$. Finally, the strain

$$\mathbf{s} = \frac{\phi_1}{[\phi_1]}\mathbf{p} = \frac{c^{-\frac{1}{2}}}{[c^{-\frac{1}{2}}]}\mathbf{p} \tag{46}$$

converts the simplex surface $\mathbf{p}$ to a duplex surface $\mathbf{s}$, which is the reciprocal of the original duplex wave-surface, the result being (40).

The interchangeability of μ and c shows that we may also strain from $\mathbf{r}$ to $\mathbf{s}$ by a second set of three successive pure strains, thus,

$$\phi = \mu^{-\frac{1}{2}}(\mu^{\frac{1}{2}}c^{-1}\mu^{\frac{1}{2}})^{\frac{1}{2}}\mu^{-\frac{1}{2}}. \tag{47}$$

This is the same as first straining the surface $\mathbf{r}$ to the simplex surface (26); then inverting the latter, which brings us to the simplex surface (32); and finally straining the last to the duplex surface $\mathbf{s}$.

Transformation of Characteristic Equation by Strain.

9. In connexion with the above transformations, it may be worth while to show how they work out when applied to the characteristic equation itself of $\mathbf{E}$ or $\mathbf{H}$. Thus, take the form (7), or

$$-c\ddot{\mathbf{E}} = \mathrm{V}\nabla\mu^{-1}\mathrm{V}\nabla\mathbf{E}, \tag{48}$$

and let $\mathbf{r} = f\mathbf{r}', \quad \nabla = f^{-1}\nabla', \quad \mathbf{E} = f^{-1}\mathbf{E}',$ (49)

so that (48) becomes

$$-cf^{-1}\ddot{\mathbf{E}}' = \mathrm{V}f^{-1}\nabla'\mu^{-1}\mathrm{V}f^{-1}\nabla'f^{-1}\mathbf{E}'. \tag{50}$$

Now employ Hamilton's formula

$$\mathrm{V}\mathbf{mn} = \frac{\phi\mathrm{V}\phi\mathbf{m}\phi\mathbf{n}}{[\phi]}, \tag{51}$$

ϕ being here any self-conjugate operator. Take $\phi = f^{-1}$, and we transform (50) to

$$-cf^{-1}\mathbf{E}' = \mathrm{V}f^{-1}\nabla'\mu^{-1}f\mathrm{V}\nabla'\mathbf{E}' \times [f^{-1}] \tag{52}$$

$$= \mathrm{V}f^{-1}\nabla'f^{-1}(f\mu^{-1}f)\,\mathrm{V}\nabla'\mathbf{E}' \times [f^{-1}]. \tag{53}$$

In this use Hamilton's formula again, with $\phi = f^{-1}$, and we obtain

$$= f\mathrm{V}\nabla'(f\mu^{-1}f)\,\mathrm{V}\nabla'\mathbf{E}' \times [f^{-1}]^2. \tag{54}$$

Or, more conveniently written,

$$-\frac{(f^{-1}cf^{-1})}{[f^{-1}]}\ddot{\mathbf{E}}' = \mathrm{V}\nabla'\frac{(f\mu^{-1}f)}{[f]}\mathrm{V}\nabla'\mathbf{E}'. \tag{55}$$

So far, f is any pure strainer; we can now make various specializations. For example, to get rid of μ^{-1} from the right side of (48), and substitute c. Take

$$\frac{f\mu^{-1}f}{[f]} = c, \quad \text{then} \quad \frac{f^{-1}cf^{-1}}{[f^{-1}]} = \mu^{-1}, \tag{56}$$

which brings (55) to the form

$$-\mu^{-1}\ddot{\mathbf{E}}' = \mathrm{V}\nabla'c\mathrm{V}\nabla'\mathbf{E}', \tag{57}$$

which should be compared with the other characteristic, that of $\mathbf{H}$, which is (8), or

$$-\mu\ddot{\mathbf{H}} = \mathrm{V}\nabla c^{-1}\mathrm{V}\nabla\mathbf{H}. \tag{58}$$

The above process is analogous to our transformation from the duplex wave-surface to its reciprocal. As then, we have an inversion of operators and also a crossing over from one form to another.

Derivation of Index Equation from Characteristic.

10. We may also, in conclusion, exhibit how the index-surface arises from the characteristic, when done in terms of ∇ up to the last moment. Start from the last equation (58). Hamilton's formula (51) makes it become

$$-[c]\mu\ddot{\mathbf{H}} = \mathrm{V}\nabla\mathrm{V}c\nabla c\mathbf{H}. \tag{59}$$

The elementary formula in vector algebra,

$$\mathrm{V}\mathbf{a}\mathrm{V}\mathbf{bc} = \mathbf{b}\,(\mathbf{ca}) - \mathbf{c}\,(\mathbf{ab}), \tag{60}$$

transforms (59) to

$$-[c]\mu\ddot{\mathbf{H}} = c\nabla\,(\nabla c\mathbf{H}) - (\nabla c\nabla)\,c\mathbf{H}, \tag{61}$$

or

$$\left[(\nabla c\nabla)\,c - [c]\,\mu\frac{d^2}{dt}\right]\mathbf{H} = c\nabla\,(\nabla c\mathbf{H}), \tag{62}$$

from which

$$\mu\mathbf{H} = \mu\left[(\nabla c\nabla)\,c - [c]\,\mu\frac{d^2}{dt^2}\right]^{-1} c\nabla\,(\nabla c\mathbf{H}). \tag{63}$$

So far we have merely a changed form of the characteristic. But the induction $\mu\mathbf{H}$ is circuital. Therefore, taking the divergence of (63), we obtain

$$0 = \nabla\mu\left[(\nabla c\nabla)\,c - [c]\,\mu\frac{d^2}{dt^2}\right]^{-1} c\nabla\,(\nabla c\mathbf{H}), \tag{64}$$

or, which is the same,

$$0 = \nabla\left[(\nabla c\nabla)\mu^{-1} - [c]\,c^{-1}\frac{d^2}{dt^2}\right]^{-1}\nabla\,(\nabla c\mathbf{H}). \tag{65}$$

Here $\nabla c\mathbf{H}$ is the divergence of $c\mathbf{H}$. It is the same as $(c\nabla)\mathbf{H}$.

Now (65) only differs from the velocity equation (for plane waves) in containing ∇ instead of the unit normal $\mathbf{N}$ and d^2/dt^2 instead of v^2, v being the wave-velocity. Thus, let

$$\mathbf{H} = f\,(z - vt),$$

then we shall have

$$v^2\nabla^2\mathbf{H} = \frac{d^2}{dt^2}\,\mathbf{H},$$

where, however, ∇^2 is specialized, being only ∇_3^2 or d^2/dz^2. We therefore put $v^2\nabla_3^2$ for d^2/dt^2 and $\mathbf{N}\nabla_3$ for ∇ in equation (65), thus making

$$0 = \mathbf{N}\nabla_3\left[(\mathbf{N}\nabla_3 c\mathbf{N}\nabla_3)\,\mu^{-1} - [c]c^{-1}v^2\nabla_3^2\right]^{-1}\mathbf{N}\nabla_3\,(\mathbf{N}\nabla_3 c\mathbf{H}) \tag{66}$$

We may now cancel out all the ∇_3's except the last, making

$$0 = \mathbf{N}\left[(\mathbf{N}c\mathbf{N})\,\mu^{-1} - [c]c^{-1}v^2\right]^{-1}\mathbf{N}(\mathbf{N}\nabla_3 c\mathbf{H}). \tag{67}$$

Now throw away the operand $\mathbf{N}\nabla_3 c\mathbf{H}$, and we get the velocity equation pure and simple, and the index equation (3) then comes by $\mathbf{s} = \mathbf{N}v^{-1}$.

But, although the above manipulation of the characteristic equation has some analytical interest, the process cannot be always recommended on the score of simplicity. It is, on the contrary, usually easier and simpler to work upon the component equations upon which the characteristic is founded.

Presents, January 18, 1894.

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