Derivation of the Electric and Magnetic Stresses and Forces from the Flux of Energy.

§ **19.** It will be observed that the convection of energy disappears by occurring twice oppositely signed; but as it comes necessarily into the expression for the stress flux of energy, I have preserved the cancelling terms in (132). A comparison of the stress flux with the POYNTING flux is interesting. Both are of the same form, viz., vector products of the electric and magnetic forces with convection terms; but whereas in the latter the forces in the vector product are those of the field (*i.e.*, only intrinsic forces deducted from **E** and **H**), in the former we have the motional forces **e** and **h** combined with the complete **E** and **H** of the fluxes. Thus the stress depends

does. For if we turn round an eolotropic portion of matter, keeping **E** unchanged, the value of **U** is altered by the rotation of the principal axes of c along with the matter, so that a torque is required.

In equation (132a), then, to produce (132b), we keep **E** constant, and let the six vectors, **i**, **j**, **k**, $\mathbf{c}_1$, $\mathbf{c}_2$, $\mathbf{c}_3$ rotate as a rigid body with the spin $\mathbf{a} = \frac{1}{2}\,\text{curl}\,\mathbf{q}$. But when a vector magnitude **i** is turned round in this way, its rate of time-change $\partial\mathbf{i}/\partial t$ is $\text{V}\mathbf{ai}$. Thus, for $\partial/\partial t$, we may put $\text{V}\mathbf{a}$ throughout. Therefore, by (132b),

$$\mathbf{E}\frac{\partial c}{\partial t}\mathbf{E} = \mathbf{E}(\text{V}\mathbf{ai}.\mathbf{c}_1 + \text{V}\mathbf{aj}.\mathbf{c}_2 + \text{V}\mathbf{ak}.\mathbf{c}_3)\mathbf{E} + \mathbf{E}(\mathbf{i}.\text{V}\mathbf{ac}_1 + \mathbf{j}.\text{V}\mathbf{ac}_2 + \mathbf{k}.\text{V}\mathbf{ac}_3)\mathbf{E}. \quad . \quad . \quad (132c)$$

In this use the parallelepipedal transformation (12), and it becomes

$$\begin{aligned}\mathbf{E}\frac{\partial c}{\partial t}\mathbf{E} &= \text{V}\mathbf{Ea}(\mathbf{i}.\mathbf{c}_1 + \mathbf{j}.\mathbf{c}_2 + \mathbf{k}.\mathbf{c}_3)\mathbf{E} + \mathbf{E}(\mathbf{i}.\mathbf{c}_1 + \mathbf{j}.\mathbf{c}_2 + \mathbf{k}.\mathbf{c}_3)\text{V}\mathbf{Ea}\\ &= (\text{V}\mathbf{Ea})c\mathbf{E} + \mathbf{E}c(\text{V}\mathbf{Ea}) = (\mathbf{D} + \mathbf{D}')\text{V}\mathbf{Ea}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (132d)\end{aligned}$$

by (132a), if $\mathbf{D}'$ is conjugate to $\mathbf{D}$; that is, $\mathbf{D}' = c'\mathbf{E} = \mathbf{E}c$. So, when $c = c'$, as in the electrical case, we have

$$\left.\begin{aligned}\frac{\partial \text{U}_c}{\partial t} &= \tfrac{1}{2}\mathbf{E}\frac{\partial c}{\partial t}\mathbf{E} = \mathbf{D}\text{V}\mathbf{Ea} = \mathbf{a}\text{V}\mathbf{DE},\\ \frac{\partial \text{T}_\mu}{\partial t} &= \tfrac{1}{2}\mathbf{H}\frac{\partial \mu}{\partial t}\mathbf{H} = \mathbf{B}\text{V}\mathbf{Ha} = \mathbf{a}\text{V}\mathbf{BH}.\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (132e)$$

and similarly (the second line above).

Now the torque arising from the stress is (see (139))

$$\mathbf{S} = \text{V}\mathbf{DE} + \text{V}\mathbf{BH},$$

so we have

$$\frac{\partial}{\partial t}(\text{U}_c + \text{T}_\mu) = \mathbf{Sa} = \text{torque} \times \text{spin}. \quad . \quad . \quad . \quad . \quad . \quad (132f)$$

The variation allowed to **i**, **j**, **k** may seem to conflict with their constancy (as reference vectors) in general. But they merely vary for a temporary purpose, being fixed in the matter instead of in space. But we may, perhaps better, discard **i**, **j**, **k** altogether, and use any independent vectors, **l**, **m**, **n** instead, making

$$\mathbf{D} = (\mathbf{l}.\mathbf{c}_1 + \mathbf{m}.\mathbf{c}_2 + \mathbf{n}.\mathbf{c}_3)\mathbf{E}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (132g)$$

wherein the **c**'s are properly chosen to suit the new axes. The calculation then proceeds as before, half entirely on the fluxes, however they be produced, in this respect resembling the electric and magnetic energies.

To exhibit the stress, we have (131), or

$$\mathbf{Q}_1 q_1 + \mathbf{Q}_2 q_2 + \mathbf{Q}_3 q_3 = \mathrm{V}\mathbf{eH} + \mathrm{V}\mathbf{Eh} + \mathbf{q}\,(\mathrm{U} + \mathrm{T}). \qquad (133)$$

In this use the expressions for $\mathbf{e}$ and $\mathbf{h}$, giving

$$\begin{aligned} \Sigma \mathbf{Q} q &= \mathrm{V}\mathbf{H}\mathrm{V}\mathbf{B}\mathbf{q} + \mathrm{V}\mathbf{E}\mathrm{V}\mathbf{D}\mathbf{q} + \mathbf{q}\,(\mathrm{U} + \mathrm{T}) \\ &= \mathbf{B}.\mathbf{Hq} - \mathbf{q}.\mathbf{HB} + \mathbf{D}.\mathbf{Eq} - \mathbf{q}.\mathbf{ED} + \mathbf{q}\,(\mathrm{U} + \mathrm{T}) \\ &= (\mathbf{B}.\mathbf{Hq} - \mathbf{q}\mathrm{T}) + (\mathbf{D}.\mathbf{Eq} - \mathbf{q}\mathrm{U})\,; \qquad (134) \end{aligned}$$

where observe the singularity that $\mathbf{q}\,(\mathrm{U} + \mathrm{T})$ has changed its sign. The first set belongs to the magnetic, the second to the electric stress, since we see that the complete stress is thus divisible.

The divergence of $\Sigma \mathbf{Q} q$ being the activity of the stress-variation per unit volume, its $\mathbf{N}$ component is the activity of the stress per unit surface, that is,

$$(\mathbf{NB}.\mathbf{Hq} - \mathbf{Nq}.\mathrm{T}) + (\mathbf{ND}.\mathbf{Eq} - \mathbf{Nq}.\mathrm{U}) = \mathbf{q}\,(\mathbf{H}.\mathbf{BN} + \mathbf{E}.\mathbf{DN} - \mathbf{N}\mathrm{U} - \mathbf{N}\mathrm{T}) = \mathbf{P}_{\mathrm{N}}\mathbf{q}. \qquad (135)$$

The stress itself is therefore

the value of $\partial \mathrm{U}_c/\partial t$ arising from the variation of $\mathbf{l}, \mathbf{m}, \mathbf{n}$, and the other half from the $\mathbf{c}$'s, provided c is irrotational.

Or we may choose the three principal axes of c in the body, when $\mathbf{l}, \mathbf{m}, \mathbf{n}$ will coincide with, and therefore move with them.

Lastly, we may proceed thus :—

$$\mathbf{E}\frac{\partial c}{\partial t}\mathbf{E} = \mathbf{E}\frac{\partial \mathbf{D}}{\partial t} - \mathbf{D}\frac{\partial \mathbf{E}}{\partial t} = \mathbf{E}\nabla\mathbf{a}\mathbf{D} - \mathbf{D}\nabla\mathbf{a}\mathbf{E} = 2\mathbf{a}\mathrm{V}\mathbf{DE}. \qquad (132h)$$

That is, replace $\partial/\partial t$ by $\nabla\mathbf{a}$ when the operands are $\mathbf{E}$ and $\mathbf{D}$. This is the correct result, but it is not easy to justify the process directly and plainly; although the clue is given by observing that what we do is to take a difference, from which the time-variation of $\mathbf{E}$ disappears.

If it is $\mathbf{D}$ that is kept constant, the result is $2\mathbf{a}\mathrm{V}\mathbf{ED}$, the negative of the above.

It is also worth noticing that if we split up $\mathbf{E}$ into $\mathbf{E}_1 + \mathbf{E}_2$ we shall have

$$\left.\begin{aligned} \mathbf{E}_1\frac{\partial c}{\partial t}\mathbf{E}_2 &= \mathbf{a}\,[\mathrm{V}(\mathbf{E}_1 c)\mathbf{E}_2 - \mathrm{V}\mathbf{E}_1(c\mathbf{E}_2)], \\ \mathbf{E}_2\frac{\partial c}{\partial t}\mathbf{E}_1 &= \mathbf{a}\,[\mathrm{V}(\mathbf{E}_2 c)\mathbf{E}_1 - \mathrm{V}_2(c\mathbf{E}_1)]. \end{aligned}\right\} \qquad (132i)$$

These are only equal when $c = c'$, or $\mathbf{E}c = c\mathbf{E}$; so that, in the expansion of the torque,

$$\mathrm{V}\mathbf{DE} = \mathrm{V}\mathbf{D}_1\mathbf{E}_1 + \mathrm{V}\mathbf{D}_2\mathbf{E}_2 + \mathrm{V}\mathbf{D}_2\mathbf{E}_1 + \mathrm{V}\mathbf{D}_1\mathbf{E}_2,$$

the cross-torques are not $\mathrm{V}\mathbf{D}_2\mathbf{E}_1$ and $\mathrm{V}\mathbf{D}_1\mathbf{E}_2$, which are unequal, but are each equal to half the sum of these vector-products.

$$\mathbf{P}_N = (\mathbf{E}.\mathbf{DN} - \mathbf{N}U) + (\mathbf{H}.\mathbf{BN} - \mathbf{N}T), \quad . \quad . \quad . \quad . \quad (136)$$

divided into electric and magnetic portions. This is with restriction to symmetrical μ and c, and with persistence of their forms as a particle moves, but is otherwise unrestricted.

Neither stress is of the symmetrical or irrotational type in case of eolotropy, and there appears to be no getting an irrotational stress save by arbitrary assumptions which destroy the validity of the stress as a correct deduction from the electromagnetic equations. But, in case of isotropy, with consequent directional identity of **E** and **D**, and of **H** and **B**, we see, by taking **N** in turns parallel to, or perpendicular to **E** in the electric case, and to **H** in the magnetic case, that the electric stress consists of a tension U parallel to **E** combined with an equal lateral pressure, whilst the magnetic stress consists of a tension T parallel to **H** combined with an equal lateral pressure. There are, in fact, MAXWELL's stresses in an isotropic medium homogeneous as regards μ and c. The difference from MAXWELL arises when μ and c are variable (including abrupt changes from one value to another of μ and c), and when there is intrinsic magnetisation, MAXWELL's stresses and forces being then different.

The stress on the plane whose normal is V**EH**, is

$$\frac{\mathbf{E}.\mathbf{D}V\mathbf{EH} + \mathbf{H}.\mathbf{B}V\mathbf{EH} - (U + T)\, V\mathbf{EH}}{V_0\mathbf{EH}},$$

$$= \frac{\mathbf{E}.\mathbf{H}V\mathbf{DE} + \mathbf{H}.\mathbf{E}V\mathbf{HB} - (U + T)\, V\mathbf{EH}}{V_0\mathbf{EH}}, \quad . \quad . \quad . \quad . \quad (137)$$

reducing simply to a pressure (U + T) in lines parallel to V**EH** in case of isotropy.

§ **20.** To find the force **F**, we have

$$\begin{aligned}
\mathbf{FN} &= \text{div}\, \mathbf{Q}_N = \text{div}\, (\mathbf{D}.\mathbf{EN} - \mathbf{N}U + \mathbf{B}.\mathbf{HN} - \mathbf{N}T) \\
&= \mathbf{EN}.\rho + \mathbf{D}\nabla.\mathbf{EN} - \tfrac{1}{2}\mathbf{E}.\mathbf{N}\nabla.\mathbf{D} - \tfrac{1}{2}\mathbf{D}.\mathbf{N}\nabla.\mathbf{E} + \&c. \\
&= \mathbf{EN}.\rho + \mathbf{D}\,(\nabla.\mathbf{EN} - \mathbf{N}\nabla.\mathbf{E}) + \tfrac{1}{2}\,(\mathbf{D}.\mathbf{N}\nabla.\mathbf{E} - \mathbf{E}.\mathbf{N}\nabla.\mathbf{D}) + \&c. \\
&= \mathbf{N}\left[\mathbf{E}\rho + V \text{ curl } \mathbf{E}.\mathbf{D} - \nabla U_c + \&c.\right], \quad . \quad . \quad . \quad . \quad (138)
\end{aligned}$$

where the unwritten terms are the similar magnetic terms. This being the **N** component of **F**, the force itself is given by (122), as is necessary.

It is V curl $\mathbf{h}_0 . \mathbf{B}$ that expresses the translational force on intrinsically magnetised matter, and this harmonises with the fact that the flux **B** due to any impressed force $\mathbf{h}_0$ depends solely upon curl $\mathbf{h}_0$.

Also, it is $-\nabla T_\mu$ that explains the forcive on elastically magnetised matter, *e.g.*, FARADAY's motion of matter to or away from the places of greatest intensity of the field, independent of its direction.

If **S** be the torque, it is given by

$$\mathbf{VSN} = \mathbf{P}_N - \mathbf{Q}_N = \mathbf{E.DN} - \mathbf{D.EN} + \&c.$$
$$\mathbf{VN}\,(\mathbf{VED} + \mathbf{VHB});$$

therefore

$$\mathbf{S} = \mathbf{VDE} + \mathbf{VBH} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (139)$$

But the matter is put more plainly by considering the convergence of the stress flux of energy and dividing it into translational and other parts. Thus

$$\text{div}\,\Sigma\mathbf{Q}q = \mathbf{Fq} + \big(\mathbf{E.D}\nabla.\mathbf{q} - \mathrm{U}\,\text{div}\,\mathbf{q}\big) + \big(\mathbf{H.B}\nabla.\mathbf{q} - \mathrm{T}\,\text{div}\,\mathbf{q}\big), \quad . \quad . \quad . \quad (140)$$

where the terms following **Fq** express the sum of the distortional and rotational activities.

Shorter Way of going from the Circuital Equations to the Flux of Energy, Stresses, and Forces.

§ **21.** I have given the investigation in §§ 17 to 19 in the form in which it occurred to me before I knew the precise nature of the results, being uncertain as regards the true measure of current, the proper form of the POYNTING flux, and how it worked in harmony with the stress flux of energy. But knowing the results, a short demonstration may be easily drawn up, though the former course is the most instructive. Thus, start now from

$$\left.\begin{aligned} \text{curl}\,(\mathbf{H} - \mathbf{h}_0) &= \mathbf{J}_0, \\ -\,\text{curl}\,(\mathbf{E} - \mathbf{e}_0) &= \mathbf{G}_0, \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (141)$$

on the understanding that $\mathbf{J}_0$ and $\mathbf{G}_0$ are the currents which make $\mathbf{e}_0\mathbf{J}_0$ and $\mathbf{h}_0\mathbf{G}_0$ the activities of $\mathbf{e}_0$ and $\mathbf{h}_0$ the intrinsic forces. Then

$$\mathbf{e}_0\mathbf{J}_0 + \mathbf{h}_0\mathbf{G}_0 = \mathbf{EJ}_0 + \mathbf{HG}_0 + \text{div}\,\mathbf{W}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (142)$$

where

$$\mathbf{W} = \mathrm{V}\,(\mathbf{E} - \mathbf{e}_0)\,(\mathbf{H} - \mathbf{h}_0); \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (143)$$

and we now take this to be the proper form of the POYNTING flux. Now develop $\mathbf{EJ}_0$ and $\mathbf{HG}_0$ thus:—

$$\begin{aligned}
\mathbf{EJ}_0 + \mathbf{HG}_0 &= \mathbf{E}\big(\mathbf{C} + \dot{\mathbf{D}} + \mathbf{q}\rho + \text{curl}\,\mathbf{h}\big) + \mathbf{H}\big(\mathbf{K} + \dot{\mathbf{B}} + \mathbf{q}\sigma - \text{curl}\,\mathbf{e}\big), \quad \text{by (93)};\\
&= \mathrm{Q}_1 + \dot{\mathrm{U}} + \dot{\mathrm{U}}_c + \mathbf{Eq}\rho + \mathbf{E}\,\text{curl}\,\mathbf{VDq}\\
&\quad + \mathrm{Q}_2 + \dot{\mathrm{T}} + \dot{\mathrm{T}}_\mu + \mathbf{Hq}\sigma + \mathbf{H}\,\text{curl}\,\mathbf{VBq}, \quad \text{by (88) and (91)};\\
&= \mathrm{Q}_1 + \dot{\mathrm{U}} + \dot{\mathrm{U}}_c + \mathbf{Eq}\rho + \mathbf{E}\,(\mathbf{D}\,\text{div}\,\mathbf{q} + \mathbf{q}\nabla.\mathbf{D} - \mathbf{q}\,\text{div}\,\mathbf{D} - \mathbf{D}\nabla.\mathbf{q})\\
&\quad + \mathrm{Q}_2 + \dot{\mathrm{T}} + \dot{\mathrm{T}}_\mu + \mathbf{Hq}\sigma + \mathbf{H}\big(\mathbf{B}\,\text{div}\,\mathbf{q} + \mathbf{q}\nabla.\mathbf{B} - \mathbf{q}\,\text{div}\,\mathbf{B} - \mathbf{B}\nabla.\mathbf{q}\big), \quad \text{by (26)},\\
&= \mathrm{Q}_1 + \dot{\mathrm{U}} + \dot{\mathrm{U}}_c + 2\mathrm{U}\,\text{div}\,\mathbf{q} + \mathbf{E.q}\nabla.\mathbf{D} - \mathbf{E.D}\nabla.\mathbf{q}\\
&\quad + \text{magnetic terms},\\
&= \big(\mathrm{Q}_1 + \dot{\mathrm{U}} + \text{div}\,\mathbf{q}\mathrm{U}\big) + \big(\mathrm{U}\,\text{div}\,\mathbf{q} - \mathbf{E.D}\nabla.\mathbf{q}\big) + \big(\dot{\mathrm{U}}_c - \mathbf{q}\nabla.\mathrm{U} + \mathbf{E.q}\nabla.\mathbf{D}\big)\\
&\quad + \text{magnetic terms.} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (144)
\end{aligned}$$

Now here

$$\mathrm{q}\nabla.\mathrm{U} = \tfrac{1}{2}\mathbf{E}.\mathrm{q}\nabla.\mathbf{D} + \tfrac{1}{2}\mathbf{D}.\mathrm{q}\nabla.\mathbf{E},$$

so that the terms in the third pair of brackets in (144) represent

$$\dot{\mathrm{U}}_c + \mathrm{q}\nabla.\mathrm{U}_c = \frac{\partial \mathrm{U}_c}{\partial t} = \tfrac{1}{2}\mathbf{E}\frac{\partial c}{\partial t}\mathbf{E},$$

with the generalised meaning before explained. So finally

$$\mathbf{EJ}_0 + \mathbf{HG}_0 = \mathrm{Q} + \dot{\mathrm{U}} + \dot{\mathrm{T}} + \operatorname{div}\mathrm{q}(\mathrm{U}+\mathrm{T}) + \frac{\partial}{\partial t}(\mathrm{U}_c + \mathrm{T}_\mu)$$
$$+ \left(\mathrm{U}\operatorname{div}\mathrm{q} - \mathbf{E.D}\nabla.\mathrm{q}\right) + \left(\mathrm{T}\operatorname{div}\mathrm{q} - \mathbf{H.B}\nabla.\mathrm{q}\right), \quad . \; . \; . \quad (145)$$

which brings (142) to

$$\mathbf{e}_0\mathbf{J}_0 + \mathbf{h}_0\mathbf{G}_0 = \mathrm{Q} + \dot{\mathrm{U}} + \dot{\mathrm{T}} + \operatorname{div}\{\mathbf{W} + \mathrm{q}(\mathrm{U}+\mathrm{T})\}$$
$$+ \frac{\partial}{\partial t}(\mathrm{U}_c + \mathrm{T}_\mu) + \left(\mathrm{U}\operatorname{div}\mathrm{q} - \mathbf{E.D}\nabla.\mathrm{q}\right) + \left(\mathrm{T}\operatorname{div}\mathrm{q} - \mathbf{H.B}\nabla.\mathrm{q}\right), \quad . \; . \; . \quad (146)$$

which has to be interpreted in accordance with the principle of continuity of energy.

Use the form (127), first, however, eliminating $\mathbf{Fq}$ by means of

$$\operatorname{div}\Sigma\mathbf{Q}q = \mathbf{Fq} + \Sigma\mathbf{Q}\nabla\mathrm{q},$$

which brings (127) to

$$\mathbf{e}_0\mathbf{J}_0 + \mathbf{h}_0\mathbf{G}_0 = \mathbf{Q} + \dot{\mathrm{U}} + \dot{\mathrm{T}} + \operatorname{div}\{\mathbf{W} + \mathrm{q}(\mathrm{U}+\mathrm{T})\} - \Sigma\mathbf{Q}\nabla q + \mathbf{Sa}\,; \quad . \; . \; . \; . \; . \quad (147)$$

and now, by comparison of (147) with (146) we see that

$$-\mathbf{Sa} + \Sigma\mathbf{Q}\nabla q = \left(\mathbf{E.D}\nabla.\mathrm{q} - \mathrm{U}\operatorname{div}\mathrm{q}\right) - \frac{\partial \mathrm{U}_c}{\partial t}$$
$$+ \left(\mathbf{H.B}\nabla.\mathrm{q} - \mathrm{T}\operatorname{div}\mathrm{q}\right) - \frac{\partial \mathrm{T}_\mu}{\partial t}\,; \quad . \; . \; . \; . \; . \; . \; . \; . \quad (148)$$

from which, when μ and c do not change intrinsically, we conclude that

$$\left.\begin{aligned}\mathbf{Q}_\mathrm{N} &= \mathbf{B.HN} - \mathbf{N}\mathrm{T} + \mathbf{D.EN} - \mathbf{N}\mathrm{U},\\ \mathbf{P}_\mathrm{N} &= \mathbf{H.BN} - \mathbf{N}\mathrm{T} + \mathbf{E.DN} - \mathbf{N}\mathrm{U},\end{aligned}\right\} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (149)$$

as before. In this method we lose sight altogether of the translational force which formed so prominent an object in the former method as a guide.

Some Remarks on HERTZ'S *investigation relating to the Stresses.*

§ **22.** Variations of c and μ in the same portion of matter may occur in different ways, and altogether independently of the strain variations. Equation (146) shows how their influence affects the energy transformations; but if we consider only such changes as depend on the strain, *i.e.*, the small changes of value which μ and c undergo as the strain changes, we may express them by thirty-six new coefficients each (there being six distortion elements, and six elements in μ, and six in c), and so reduce the expressions for $\partial U_c/\partial t$ and $\partial T_\mu/\partial t$ in (148) to the form suitable for exhibiting the corresponding change in $\mathbf{Q}_N$ and in the stress function $\mathbf{P}_N$. As is usual in such cases of secondary corrections, the magnitude of the resulting formula is out of all proportion to the importance of the correction terms in relation to the primary formula to which they are added.

Professor H. HERTZ* has considered this question, and also refers to VON HELMOLTZ's previous investigation relating to a fluid. The c and μ can then only depend on the density, or on the compression, so that a single coefficient takes the place of the thirty-six. But I cannot quite follow HERTZ's stress investigation. First, I would remark that in developing the expression for the distortional (*plus* rotational) activity, he assumes that all the coefficients of the spin vanish identically; this is done in order to make the stress be of the irrotational type. But it may easily be seen that the assumption is inadmissible by examining its consequence, for which we need only take the case of c and μ intrinsically constant. By (139) we see that it makes $\mathbf{S} = 0$, and therefore (since the electric and magnetic stress are separable), $\mathrm{V}\mathbf{HB} = 0$, and $\mathrm{V}\mathbf{ED} = 0$; that is, it produces directional identity of the force $\mathbf{E}$ and the flux $\mathbf{D}$, and of the force $\mathbf{H}$ and the flux $\mathbf{B}$. This means isotropy, and, therefore, breaks down the investigation so far as the eolotropic application, with six μ and six c coefficients, goes. Abolish the assumption made, and the stress will become that used by me above.

Another point deserving of close attention in HERTZ's investigation, relates to the principle to be followed in deducing the stress from the electromagnetic equations. Translating into my notation it would appear to amount to this, the *à priori* assumption that the quantity

$$\frac{1}{v}\frac{\partial}{\partial t}\left(\mathrm{T}v\right), \quad \ldots \ldots \ldots \ldots \quad (150)$$

where v indicates the volume of a moving unit element undergoing distortion, may be taken to represent the distortional (*plus* rotational) activity of the magnetic stress. Similarly as regards the electric stress.

Expanding (150) we obtain

$$\frac{\partial \mathrm{T}}{\partial t} + \frac{\mathrm{T}}{v}\frac{\partial v}{\partial t} = \mathbf{H}\frac{\partial \mathbf{B}}{\partial t} + \mathrm{T}\,\mathrm{div}\,\mathbf{q} - \frac{\partial \mathrm{T}_\mu}{\partial t}. \quad \ldots \ldots \ldots \quad (151)$$

Now the second circuital law (90) may be written

$$-\,\mathrm{curl}\,(\mathbf{E} - \mathbf{e}_0) = \mathbf{K} + \frac{\partial \mathbf{B}}{\partial t} + (\mathbf{B}\,\mathrm{div}\,\mathbf{q} - \mathbf{B}\nabla.\mathbf{q}). \quad \ldots \ldots \quad (152)$$

* 'WIEDEMANN'S Annalen,' v. 41, p. 369.

Here ignore $\mathbf{e}_0$, $\mathbf{K}$, *and ignore the curl of the electric force*, and we obtain, by using (152) in (151),

$$\mathbf{H}.\mathbf{B}\nabla.\mathbf{q} - \mathbf{HB}\,\mathrm{div}\,\mathbf{q} + \mathrm{T}\,\mathrm{div}\,\mathbf{q} - \frac{\partial \mathrm{T}_\mu}{\partial t} = \mathbf{H}.\mathbf{B}\nabla.\mathbf{q} - \mathrm{T}\,\mathrm{div}\,\mathbf{q} - \frac{\partial \mathrm{T}_\mu}{\partial t}, \quad . \; . \; . \quad (153)$$

which represents the distortional activity (my form, not equating to zero the coefficients of curl $\mathbf{q}$ in its development). We *can*, therefore, derive the magnetic stress in the manner indicated, that is, from (150), with the special meaning of $\partial\mathbf{B}/\partial t$ later stated, and the ignorations or nullifications.

In a similar manner, from the first circuital law (89), which may be written

$$\mathrm{curl}\,(\mathbf{H} - \mathbf{h}_0) = \mathbf{C} + \frac{\partial \mathbf{D}}{\partial t} + (\mathbf{D}\,\mathrm{div}\,\mathbf{q} - \mathbf{D}\nabla.\mathbf{q}), \quad . \; . \; . \; . \; . \; . \; . \quad (154)$$

we can, by ignoring the conduction current and the curl of the magnetic force, obtain

$$\frac{1}{v}\frac{\partial}{\partial t}(v\mathrm{U}) = \mathbf{E}.\mathbf{D}\nabla.\mathbf{q} - \mathrm{U}\,\mathrm{div}\,\mathbf{q} - \frac{\partial \mathrm{U}_c}{\partial t}, \quad . \; . \; . \; . \; . \; . \; . \; . \quad (155)$$

which represents the distortional activity of the electric stress.

The difficulty here seems to me to make it evident *à priori* that (150), with the special reckoning of $\partial\mathbf{B}/\partial t$ *should* represent the distortional activity (*plus* rotational understood); this interesting property should, perhaps, rather be derived from the magnetic stress when obtained by a safe method. The same remark applies to the electric stress. Also, in (150) to (155) we overlook the POYNTING flux. I am not sure how far this is intentional on Professor HERTZ's part, but its neglect does not seem to give a sufficiently comprehensive view of the subject.

The complete expansion of the magnetic distortional activity is, in fact,

$$\mathbf{H}.\mathbf{B}\nabla.\mathbf{q} - \mathrm{T}\,\mathrm{div}\,\mathbf{q} - \frac{\partial \mathrm{T}_\mu}{\partial t} = \mathrm{Q}_2 + \dot{\mathrm{T}} + \mathrm{div}\,\mathbf{q}\mathrm{T} - \mathbf{HG}_0; \quad . \; . \; . \; . \; . \; . \quad (156)$$

and similarly, that of the electric stress is

$$\mathbf{E}.\mathbf{D}\nabla.\mathbf{q} - \mathrm{U}\,\mathrm{div}\,\mathbf{q} - \frac{\partial \mathrm{U}_c}{\partial t} = \mathrm{Q}_1 + \dot{\mathrm{U}} + \mathrm{div}\,\mathbf{q}\mathrm{U} - \mathbf{EJ}_0. \quad . \; . \; . \; . \; . \; . \; . \quad (157)$$

It is the last term of (156) and the last term of (157), together, which bring in the POYNTING flux. Thus, adding these equations,

$$\Sigma\mathbf{Q}\nabla q - \frac{\partial}{\partial t}(\mathrm{U}_c + \mathrm{T}_\mu) = \mathrm{Q} + \dot{\mathrm{U}} + \dot{\mathrm{T}} + \mathrm{div}\,\mathbf{q}\,(\mathrm{U} + \mathrm{T}) - (\mathbf{EJ}_0 + \mathbf{HG}_0), \quad . \; . \; . \quad (158)$$

where

$$(\mathbf{EJ}_0 + \mathbf{HG}_0) = (\mathbf{e}_0\mathbf{J}_0 + \mathbf{h}_0\mathbf{G}_0) - \mathrm{div}\,\mathbf{W}; \quad . \; . \; . \; . \; . \; . \; . \; . \quad (159)$$

and so we come round to the equation of activity again, in the form (146), by using (159) in (158).

Modified Form of Stress-vector, and Application to the Surface separating two Regions.

§ **23.** The electromagnetic stress, $\mathbf{P}_N$ of (149) and (136) may be put into another interesting form. We may write it

$$\mathbf{P}_N = \tfrac{1}{2}(\mathbf{E}.\mathbf{ND} + \mathrm{V}.\mathrm{V}\mathbf{NE}.\mathbf{D}) + \tfrac{1}{2}(\mathbf{H}.\mathbf{NB} + \mathrm{V}.\mathrm{V}\mathbf{NH}.\mathbf{B}). \quad \ldots \ldots \quad (160)$$

Now, $\mathbf{ND}$ is the surface equivalent of div $\mathbf{D}$ and $\mathbf{NB}$ of div $\mathbf{B}$; whilst $\mathrm{V}\mathbf{NE}$ and $\mathrm{V}\mathbf{NH}$ are the surface equivalents of curl $\mathbf{E}$ and curl $\mathbf{H}$. We may, therefore, write

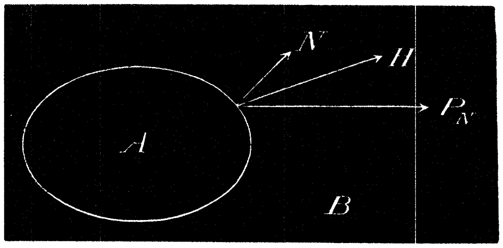
$$\mathbf{P}_N = \tfrac{1}{2}(\mathbf{E}\rho' + \mathrm{V}\mathbf{DG}') + \tfrac{1}{2}(\mathbf{H}\sigma' + \mathrm{V}\mathbf{J}'\mathbf{B}), \quad \ldots \ldots \quad (161)$$

and this is the force, reckoned as a pull, on unit area of the surface whose normal is $\mathbf{N}$. Here the accented letters are the surface equivalents of the same quantities unaccented, which have reference to unit volume.

Comparing with (122) we see that the type is preserved, except as regards the terms in $\mathbf{F}$ due to variation of c and μ in space. That is, the stress is represented in (101) as the translational force, due to $\mathbf{E}$ and $\mathbf{H}$, on the fictitious electrification, magnetification, electric current, and magnetic current produced by imagining $\mathbf{E}$ and $\mathbf{H}$ to terminate at the surface across which $\mathbf{P}_N$ is the stress.

The coefficient $\frac{1}{2}$ which occurs in (161) is understandable by supposing the fictitious quantities ("matter" and "current") to be distributed uniformly within a very thin layer, so that the forces $\mathbf{E}$ and $\mathbf{H}$ which act upon them do not then terminate quite abruptly, but fall off gradually through the layer from their full values on one side to zero on the other. The mean values of $\mathbf{E}$ and $\mathbf{H}$ through the layer, that is, $\frac{1}{2}\mathbf{E}$ and $\frac{1}{2}\mathbf{H}$ are thus the effective electric and magnetic forces on the layer as a whole, per unit volume density of matter or current; or $\frac{1}{2}\mathbf{E}$ and $\frac{1}{2}\mathbf{H}$ per unit surface density when the layer is indefinitely reduced in thickness.

Considering the electric field only, the quantities concerned are electrification and magnetic current. In the magnetic field only they are magnetification and electric current. Imagine the medium divided into two regions A and B, of which A is internal, B external, and let $\mathbf{N}$ be the unit normal from the surface into the external region. The mechanical action between the two regions is fully represented by the stress $\mathbf{P}_N$ over their interface, and the forcive of B upon A is fully represented by the $\mathbf{E}$ and $\mathbf{H}$ in B acting upon the fictitious matter and current produced on the boundary of B, on the assumption that $\mathbf{E}$ and $\mathbf{H}$ terminate there. If the normal and $\mathbf{P}_N$ be drawn the other way, thus negativing them both, as well as the fictitious matter and current on the interface, then it is the forcive of A on B that is represented by the action of **E** and **H** in A on the new interfacial matter and current. That is, the **E** and **H** in the region A may be done away with altogether, because their abolition will immediately introduce the fictitious matter and current equivalent, so far as B is concerned, to the influence of the region A. Similarly **E** and **H** in B may be abolished without altering them in A. And, generally, any portion of the medium may be taken by itself and regarded as being subjected to an equilibrating system of forces, when treated as a rigid body.



§ **24.** When c and μ do not vary in space, we do away with the forces $-\frac{1}{2}E^2\nabla c$ and $-\frac{1}{2}H^2\nabla\mu$, and make the form of the surface and volume translational forces agree. We may then regard every element of ρ or of σ as a source sending out from itself displacement and induction isotropically, and every element of **J** or **G** as causing induction or displacement according to AMPÈRE's rule for electric current and its analogue for magnetic current. Thus

$$\mathbf{E} = \Sigma \frac{\rho/c + \mathrm{V}\mathbf{r}_1\mathbf{G}}{4\pi r^2}, \qquad (162)$$

$$\mathbf{H} = \Sigma \frac{\sigma/\mu + \mathrm{V}\mathbf{J}\mathbf{r}_1}{4\pi r^2}, \qquad (163)$$

where $\mathbf{r}_1$ is a unit vector drawn from the infinitesimal unit volume in the summation to the point at distance r where **E** or **H** is reckoned. Or, introducing potentials,

$$\mathbf{E} = -\nabla\Sigma \frac{\rho/c}{4\pi r} - \mathrm{curl}\,\Sigma \frac{\mathbf{G}}{4\pi r}, \qquad (164)$$

$$\mathbf{H} = -\nabla\Sigma \frac{\sigma/\mu}{4\pi r} + \mathrm{curl}\,\Sigma \frac{\mathbf{J}}{4\pi r}. \qquad (165)$$

These apply to the whole medium, or to any portion of the same, with, in the latter case, the surface matter and current included, there being no **E** or **H** outside the region, whilst within it **E** and **H** are the same as due to the matter and current in the whole region ("matter," ρ and σ; "current," **J** and **G**). But there is no known general method of finding the potentials when c and μ vary.

We may also divide **E** and **H** into two parts each, say $\mathbf{E}_1$ and $\mathbf{H}_1$ due to matter and current in the region A, and $\mathbf{E}_2$, $\mathbf{H}_2$ due to matter and current in the region B surrounding it, determinable in the isotropic homogeneous case by the above formulæ. Then we may ignore $\mathbf{E}_1$ and $\mathbf{H}_1$ in estimating the forcive on the matter and current in the region A; thus,

$$\Sigma(\mathbf{H}_2\sigma_1 + \mathrm{V}\mathbf{J}_1\mathbf{B}_2) + \Sigma(\mathbf{E}_2\rho_1 + \mathrm{V}\mathbf{D}_2\mathbf{G}_1), \qquad (166)$$

where $\sigma_1 = \operatorname{div} \mathbf{B}_1 = \operatorname{div} \mathbf{B}$, and $\mathbf{J}_1 = \operatorname{curl} \mathbf{H}_1 = \operatorname{curl} \mathbf{H}$ in region A, is the resultant force on the region A, and

$$\Sigma(\mathbf{H}_1\sigma_2 + \mathrm{V}\mathbf{J}_2\mathbf{B}_1) + \Sigma(\mathbf{E}_1\rho_2 + \mathrm{V}\mathbf{D}_1\mathbf{G}_2), \qquad (167)$$

is the resultant force on the region B; the resultant force on A due to its own $\mathbf{E}$ and $\mathbf{H}$ being zero, and similarly for B. These resultant forces are equal and opposite, and so are the equivalent surface-integrals

$$\Sigma(\mathbf{H}_2\sigma_1' + \mathrm{V}\mathbf{J}_1'\mathbf{B}_2) + \Sigma(\mathbf{E}_2\rho_1' + \mathrm{V}\mathbf{D}_2\mathbf{G}_1'), \qquad (168)$$

and

$$\Sigma(\mathbf{H}_1\sigma_2' + \mathrm{V}\mathbf{J}_2'\mathbf{B}_1) + \Sigma(\mathbf{E}_1\rho_2' + \mathrm{V}\mathbf{D}_1\mathbf{G}_2'), \qquad (169)$$

taken over the interface. The quantity summed is that part of the stress-vector, $\mathbf{P}_{\mathrm{N}}$, which depends upon products of the $\mathbf{H}$ of one region and the $\mathbf{B}$ of the other, &c. Thus, for the magnetic stress only,

$$\begin{aligned}\mathbf{H}.\mathbf{BN} - \mathbf{N}.\tfrac{1}{2}\mathbf{HB} &= (\mathbf{H}_1.\mathbf{B}_1\mathbf{N} - \mathbf{N}.\tfrac{1}{2}\mathbf{H}_1\mathbf{B}_1) + (\mathbf{H}_1.\mathbf{B}_2\mathbf{N} - \mathbf{N}.\tfrac{1}{2}\mathbf{H}_1\mathbf{B}_2)\\ &\quad + (\mathbf{H}_2.\mathbf{B}_2\mathbf{N} - \mathbf{N}.\tfrac{1}{2}\mathbf{H}_2\mathbf{B}_2) + (\mathbf{H}_2.\mathbf{B}_1\mathbf{N} - \mathbf{N}.\tfrac{1}{2}\mathbf{H}_2\mathbf{B}_1), \qquad (170)\end{aligned}$$

and it is the terms in the second and fourth brackets (which, be it observed, are not equal) which together make up the magnetic part of (168) and (169) or their negatives, according to the direction taken for the normal; that is, since $\mathbf{H}_1\mathbf{B}_2 = \mathbf{H}_2\mathbf{B}_1$,

$$\begin{aligned}\Sigma\mathbf{P}_{\mathrm{N}} &= \Sigma(\mathbf{H}_1.\mathbf{B}_2\mathbf{N} + \mathbf{H}_2.\mathbf{B}_1\mathbf{N} - \mathbf{N}.\mathbf{H}_1\mathbf{B}_2) = \Sigma(\mathbf{H}.\mathbf{BN} - \mathbf{N}.\tfrac{1}{2}\mathbf{HB})\\ &= \Sigma(\mathbf{H}_1\sigma_2' + \mathrm{V}\mathbf{J}_2'\mathbf{B}_1) = \Sigma(\mathbf{H}_2\sigma_1' + \mathrm{V}\mathbf{J}_1'\mathbf{B}_2) = \Sigma(\mathbf{H}\sigma' + \mathrm{V}\mathbf{J}'\mathbf{B})\\ &= \Sigma\mathbf{F} = \Sigma(\mathbf{H}_1\sigma_2 + \mathrm{V}\mathbf{J}_2\mathbf{B}_1) = \Sigma(\mathbf{H}_2\sigma_1 + \mathrm{V}\mathbf{J}_1\mathbf{B}_2) = \Sigma(\mathbf{H}\sigma + \mathrm{V}\mathbf{JB}), \qquad (171)\end{aligned}$$

where the first six expressions are interfacial summations, and the four last summations throughout one or the other region, the last summation applying to either region. No special reckoning of the sign to be prefixed has been made. The notation is such that $\mathbf{H} = \mathbf{H}_1 + \mathbf{H}_2$, $\sigma = \sigma_1 + \sigma_2$, &c., &c.

The comparison of the two aspects of electromagnetic theory is exceedingly curious; namely, the precise mathematical equivalence of "explanation" by means of instantaneous action at a distance between the different elements of matter and current, each according to its kind, and by propagation through a medium in time at a finite velocity. But the day has gone by for any serious consideration of the former view other than as a mathematical curiosity.

Quaternionic Form of Stress-Vector.

§ **25.** We may also notice the Quaternion form for the stress function, which is so vital a part of the mathematics of forces varying as the inverse square of the distance, and of potential theory. Isotropy being understood, the electric stress may be written

$$\mathbf{P}_{\mathrm{N}} = \tfrac{1}{2} c\,[\mathbf{E}\mathbf{N}^{-1}\mathbf{E}], \qquad (172)$$

where the quantity in the square brackets is to be understood quaternionically. It is, however, a pure vector. Or,

$$\left[\frac{\mathbf{P}_{\mathrm{N}}}{\mathbf{E}}\right] = \frac{c}{2}\left[\frac{\mathbf{E}}{\mathbf{N}}\right], \qquad (173)$$

that is, not counting the factor $\frac{1}{2}c$, the quaternion $\left[\frac{\mathbf{P}_{\mathrm{N}}}{\mathbf{E}}\right]$ is the same as the quaternion $\left[\frac{\mathbf{N}}{\mathbf{E}}\right]$; or the same operation which turns **N** to **E** also turns **E** to $\mathbf{P}_{\mathrm{N}}$. Thus, **N**, **E**, and $\mathbf{P}_{\mathrm{N}}$ are in the same plane, and the angle between **N** and **E** equals that between **E** and $\mathbf{P}_{\mathrm{N}}$; and **E** and $\mathbf{P}_{\mathrm{N}}$ are on the same side of **N** when **E** makes an acute angle with **N**. Also, the tensor of $\mathbf{P}_{\mathrm{N}}$ is U, so that its normal and tangential components are U cos 2θ and U sin 2θ, if $\theta = \widehat{\mathbf{NE}}$.

Otherwise

$$\mathbf{P}_{\mathrm{N}} = -\tfrac{1}{2} c\,[\mathbf{E}\mathbf{N}\mathbf{E}], \qquad (174)$$

since the quaternionic reciprocal of a vector has the reverse direction. The corresponding volume translational force is

$$\mathbf{F} = -c\mathrm{V}\,[\mathbf{E}\nabla\mathbf{E}], \qquad (175)$$

which is also to be understood quaternionically, and expanded, and separated into parts to become physically significant. I only use the square brackets in this paragraph to emphasise the difference in notation. It rarely occurs that any advantage is gained by the use of the quaternion, in saying which, I merely repeat what Professor Willard Gibbs has been lately telling us; and I further believe the disadvantages usually far outweigh the advantages. Nevertheless, apart from practical application, and looking at it from the purely quaternionic point of view, I ought to also add that the invention of quaternions must be regarded as a most remarkable feat of human ingenuity. Vector analysis, without quaternions, could have been found by any mathematician by carefully examining the mechanics of the Cartesian mathematics; but to find out quaternions required a genius.

Remarks on the Translational Force in Free Ether.

§ **26.** The little vector **Veh**, which has an important influence in the activity equation, where **e** and **h** are the motional forces

$$\mathbf{e} = \mathrm{V}\mathbf{qB}, \qquad \mathbf{h} = \mathrm{V}\mathbf{Dq},$$

has an interesting form, viz., by expansion,

$$\mathrm{V}\mathbf{eh} = \mathbf{q}.\mathbf{q}\mathrm{V}\mathbf{DB} = \frac{\mathbf{q}}{v^2}\cdot\mathbf{q}\mathrm{V}\mathbf{EH}, \qquad (176)$$

if v be the speed of propagation of disturbances. We also have, in connection therewith, the equivalence

$$\mathbf{eD} = \mathbf{hB}, \qquad (177)$$

always.

The translational force in a non-conducting dielectric, free from electrification and intrinsic force, is

$$\mathbf{F} = \mathrm{V}\mathbf{JB} + \mathrm{V}\mathbf{DG} + \mathrm{V}\mathbf{jB} + \mathrm{V}\mathbf{Dg},$$

or, approximately,

$$= \mathrm{V}\dot{\mathbf{D}}\mathbf{B} + \mathrm{V}\mathbf{D}\dot{\mathbf{B}} = \frac{d}{dt}\mathrm{V}\mathbf{DB} = \frac{1}{v^2}\frac{d}{dt}\mathrm{V}\mathbf{EH} = \frac{\dot{\mathbf{W}}}{v^2}. \qquad (178)$$

The vector **VDB**, or the flux of energy divided by the square of the speed of propagation, is, therefore, the momentum (translational, not magnetic, which is quite a different thing), provided the force **F** *is* the complete force from all causes acting, and we neglect the small terms **VjB** and **VDg**.

But have we any right to safely write

$$\mathbf{F} = m\frac{\partial\mathbf{q}}{\partial t}, \qquad (179)$$

where m is the density of the ether? To do so is to assume that **F** is the only force acting, and, therefore, equivalent to the time-variation of the momentum of a moving particle.*

Now, if we say that there is a certain forcive upon a conductor supporting electric current; or, equivalently, that there is a certain distribution of stress, the magnetic stress, acting upon the same, we do not at all mean that the accelerations of momentum of the different parts are represented by the translational force, the "electromagnetic force." It is, on the other hand, a dynamical problem in which the electromagnetic force plays the part of an impressed force, and similarly as regards the magnetic

* Professor J. J. THOMSON has endeavoured to make practical use of the idea, 'Phil. Mag.,' March, 1891. See also my article, 'The Electrician,' January 15, 1886.

stress; the actual forces and stresses being only determinable from a knowledge of the mechanical conditions of the conductor, as its density, elastic constants, and the way it is constrained. Now, if there is any dynamical meaning at all in the electromagnetic equations, we must treat the ether in precisely the same way. But we do not know, and have not formularised, the equations of motion of the ether, but only the way it propagates disturbance through itself, with due allowance made for the effect thereon of given motions, and with formularisation of the reaction between the electromagnetic effects and the motion. Thus the theory of the stresses and forces in the ether and its motions is an unsolved problem, only a portion of it being known so far, *i.e.*, assuming that the Maxwellian equations do express the known part.

When we assume the ether to be motionless, there is a partial similarity to the theory of the propagation of vibrations of infinitely small range in elastic bodies, when the effect thereon of the actual translation of the matter is neglected.

But in ordinary electromagnetic phenomena, it does not seem that the ignoration of $\mathbf{q}$ can make any sensible difference, because the speed of propagation of disturbances through the ether is so enormous, that if the ether were stirred about round a magnet, for example, there would be an almost instantaneous adjustment of the magnetic induction to what it would be were the ether at rest.

Static Consideration of the Stresses.—Indeterminateness.

§ **27.** In the following the stresses are considered from the static point of view, principally to examine the results produced by changing the form of the stress function. Either the electric or the magnetic stress alone may be taken in hand. Start then, from a knowledge that the force on a magnetic pole of strength m is $\mathbf{R}m$, where $\mathbf{R}$ is the polar force of any distribution of intrinsic magnetisation in a medium, the whole of which has unit inductivity, so that

$$\text{div } \mathbf{R} = m = \text{conv } \mathbf{h}_0 \quad \ldots \quad (180)$$

measures the density of the fictitious "magnetic" matter; $\mathbf{h}_0$ being the intrinsic force, or, since here $\mu = 1$, the intensity of magnetisation. The induction is $\mathbf{B} = \mathbf{h} + \mathbf{R}$. This rudimentary theory locates the force on a magnet at its poles, superficial or internal, by

$$\mathbf{F} = \mathbf{R} \text{ div } \mathbf{R}. \quad \ldots \quad (181)$$

The $\mathbf{N}$ component of $\mathbf{F}$ is

$$\mathbf{FN} = \mathbf{RN}. \text{ div } \mathbf{R} = \text{div } \{\mathbf{R}.\mathbf{RN} - \mathbf{N}.\tfrac{1}{2}\mathbf{R}^2\}, \quad \ldots \quad (182)$$

because curl $\mathbf{R} = 0$. Therefore

$$\mathbf{P}_{\mathrm{N}} = \mathbf{R}.\mathbf{RN} - \mathbf{N}.\tfrac{1}{2}\mathbf{R}^2 \quad \ldots \quad (183)$$

is the appropriate stress, of irrotational type. Now, however uncertain we may be about the stress in the interior of a magnet, there can be no question as to the possible validity of this stress in the air outside our magnet, for we know that the force $\mathbf{R}$ is then a polar force, and that is all that is wanted, m and $\mathbf{h}$ being merely auxiliaries, derived from $\mathbf{R}$.

Now consider a region A, containing magnets of this kind, enclosed in B, the rest of space, also containing magnets. The mutual force between the two regions is expressed by $\Sigma\mathbf{P}_N$ over the interface, which we may exchange for $\Sigma\mathbf{R}m$ through either region A or B, still on the assumption that $\mathbf{R}$ remains polar.

But if we remove this restriction upon the nature of $\mathbf{R}$, and allow it to be arbitrary, say in region B or in any portion thereof, we find

$$\mathbf{NF} = \text{div } \mathbf{P}_N = \mathbf{RN} \text{ div } \mathbf{R} + \mathbf{N}\text{V curl } \mathbf{R} \, . \, \mathbf{R};$$

or

$$\mathbf{F} = \mathbf{R}m + \text{V}\mathbf{JR},$$

if $\mathbf{J} = \text{curl } \mathbf{R}$. This gives us, from a knowledge of the external magnetic field of polar magnets only, the mechanical force exerted by a magnet on a region containing $\mathbf{J}$, whatever that may be, provided it be measurable as above; and without any experimental knowledge of electric currents, we could now predict their mechanical effects in every respect by the principle of the equality of action and reaction, not merely as regards the mutual influence of a magnet and a closed current, but as regards the mutual influence of the closed currents themselves; the magnetic force of a closed current, for instance, being the force on unit of m, is equivalently the force exerted by m on the closed current, which, by the above, we know. Also, we see that according to this magnetic notion of electric current, it is necessarily circuital.

At the same time, it is to be remarked that our real knowledge must cease at the boundary of the region containing electric current, a metallic conductor for instance; the surface over which $\mathbf{P}_N$ is reckoned, on one side of which is the magnet, on the other side electric current, can only be pushed up as far as the conductor. The stress $\mathbf{P}_N$ may therefore cease altogether on reaching the conductor, where it forms a distribution of surface force fully representing the action of the magnet on the conductor. Similarly, we need not continue the stress into the interior of the magnet. Then, so far as the resultant force on the magnet as a whole, in translating or rotating it, and, similarly, so far as the action on the conductor, is concerned, the simple stress $\mathbf{P}_N$ of constant tensor $\frac{1}{2}\mathbf{R}^2$, varying from a tension parallel to $\mathbf{R}$ to an equal pressure laterally, acting in the medium between the magnet and conductor, accounts, by its terminal pulls or pushes, for the mechanical forces on them. The lateral pressure is especially prominent in the case of conductors, whilst the tension goes more or less out of sight, as the immediate cause of motion. Thus, when parallel currents appear to attract one another, the conductors are really pushed together by the lateral pressure on each conductor being greater on the side remote from the other than on the near side: whilst if the currents are oppositely directed, the pressure on the near sides is greater than on the remote sides, and they appear to repel one another.

The effect of continuing the stress into the interior of a conductor of unit inductivity, according to the same law, instead of stopping it on its boundary, is to distribute the translational force bodily, according to the formula $\Sigma \mathrm{V}\mathbf{JR}$, instead of superficially, according to $\Sigma \mathbf{P}_N$. In either case, of course, the conductor must be strained by the magnetic stress, with the consequent production of a mechanical stress. But the strain (and associated stress) will be different in the two cases, the applied forces being differently localised. The effect of the stress on a straight portion of a wire supporting current, due to its own field only, is to compress it laterally, and to lengthen it. Besides this, there will be resultant force on it arising from the different pressures on its opposite sides due to the proximity of the return conductor or rest of the circuit, tending to move it so as to increase the induction through the circuit per unit current, that is, the inductance of the circuit.

§ **28.** If now, we bring an elastically magnetisable body into a magnetic field, it modifies the field by its presence, causing more or less induction to go through it than passed previously in the air it replaces, according as its inductivity exceeds or is less than that of the air. The forcive on it, considered as a rigid body, is completely accounted for by the simple stress $\mathbf{P}_N$ in the air outside it, reckoned according to the changed field, and supposed to terminate on the surface of the disturbing body. This is true whether the body be isotropic or heterotropic in its inductivity; nor need the induction be a linear function of the magnetic force. It is also true when the body is intrinsically magnetised; or is the seat of electric current. In short, since the external stress depends upon the magnetic force outside the body, when we take the external field as we may find it, that is, as modified by any known or unknown causes within the body, the corresponding stress, terminated upon its boundary, fully represents the forcive on the body, as a whole, due to magnetic causes. This follows from the equality of action and reaction; the force on the body due to a unit pole is the opposite of that of the body on the pole.

If we wish to continue the stress into the interior of the body, surrounded on all sides by the unmagnetised medium of unit inductivity, as we must do if we wish to arrive ultimately at the mutual actions of its different parts, and how they are modified by variations of inductivity, by intrinsic magnetisation, and by electric current in the body, we may, so far as the resultant force and torque on it are concerned, do it in any way we please, provided we do not interfere with the stress outside. For the internal stress, of any type, will have no resultant force or torque on the body, and there is merely left the real external stress.

Practically, however, we should be guided by the known relations of magnetic force, induction, magnetisation, and current, and not go to work in a fanciful manner; furthermore, we should always choose the stress in such a way that if, in its expression, we take the inductivity to be unity, and the intrinsic magnetisation zero, it must reduce to the simple Maxwellian stress in air (assumed to represent ether here).

But as we do not know definitely the forcive arising from the magnetic stress in the interior of a magnet, there are several formulæ that suggest themselves as possible.

Special Kinds of Stress Formulæ statically suggested.

§ **29.** Thus, first we have the stress (183); let this be quite general, then

$$(1)\begin{cases}\mathbf{P}_N = \mathbf{R}.\mathbf{RN} - \mathbf{N}.\tfrac{1}{2}\mathbf{R}^2, & (184)\\ \mathbf{F} \;= \mathbf{R}\,\mathrm{div}\,\mathbf{R} + \mathrm{V}\mathbf{JR}. & (185)\end{cases}$$

Here **R** is the magnetic force of the field, not of the flux **B**. If $\mu = 1$, div **R** is the density of magnetic matter, the convergence of the intrinsic magnetisation, but not otherwise. In general, it is the density of the matter of the magnetic potential, calculated on the assumption $\mu = 1$. The force on a magnet is located in this system at its poles, whether the magnetisation be intrinsic or induced. The second term in (185) represents the force on matter bearing electric current (**J** = curl **R**), but has to be supplemented by the first term, unless div **R** = 0 at the place.

§ **30.** Next, let the stress be μ times as great for the same magnetic force, but be still of the same simple type, μ being the inductivity, which is unity outside the body, but having any positive value, which may be variable, within it. Then we shall have

$$(2)\begin{cases}\mathbf{P}_N = \mathbf{R}.\mathbf{N}\mu\mathbf{R} - \mathbf{N}.\tfrac{1}{2}\mathbf{R}\mu\mathbf{R}, & (186)\\ \mathbf{F} \;= \mathbf{R}m + \mathrm{V}\mathbf{J}\mu\mathbf{R} - \tfrac{1}{2}\mathbf{R}^2\nabla\mu, & (187)\end{cases}$$

where $m = \mathrm{conv}\;\mu\mathbf{h}_0 = \mathrm{div}\;\mu\mathbf{R}$ is the density of magnetic matter, $\mu\mathbf{h}_0$ being the intensity of intrinsic magnetisation.

The electromagnetic force is made μ times as great for the same magnetic force; the force on an intrinsic magnet is at its poles; and there is, in addition, a force wherever μ varies, including the intrinsic magnet, and not forgetting that a sudden change in μ, as at the boundary of a magnet, has to count. This force, the third term in (187), explains the force on inductively magnetised matter. It is in the direction of most rapid decrease of μ.

§ **31.** Thirdly, let the stress be of the same simple type, but taking **H** instead of **R**, **H** being the force of the flux $\mathbf{B} = \mu\mathbf{H} = \mu(\mathbf{R} + \mathbf{h}_0)$, where $\mathbf{h}_0$ is as before. We now have

$$(3)\begin{cases}\mathbf{P}_N = \mathbf{H}.\mathbf{NB} - \mathbf{N}.\tfrac{1}{2}\mathbf{HB}, & (188)\\ \mathbf{F} \;= \mathrm{V}\mathbf{JB} + \mathrm{V}\mathbf{j}_0\mathbf{B} - \tfrac{1}{2}\mathbf{H}^2\nabla\mu, & (189)\end{cases}$$

where $\mathbf{j}_0 = \mathrm{curl}\;\mathbf{h}_0$ is the distribution of fictitious electric current which produces the same induction as the intrinsic magnetisation $\mu\mathbf{h}_0$, and **J** is, as before, the real current.

It is now *quasi*-electromagnetic force that acts on an intrinsic magnet, with, however, the force due to $\nabla\mu$, since a magnet has usually large μ compared with air.

The above three stresses are all of the simple type (equal tension and perpendicular pressure), and are irrotational, unless μ be the eolotropic operator. No change is, in the latter case, needed in (186), (188), whilst in the force formulæ (187), (189), the only change needed is to give the generalised meaning to $\nabla\mu$. Thus, in (189), instead of $\mathbf{H}^2\nabla\mu$, use $2\nabla \mathrm{T}_\mu$,

or

$$\nabla_\mu (\mathbf{H}\mu\mathbf{H}),$$

or

$$\mathbf{i}\left(\mathbf{H}\frac{d\mu}{dx}\mathbf{H}\right) + \mathbf{j}\left(\mathbf{H}\frac{d\mu}{dy}\mathbf{H}\right) + \mathbf{k}\left(\mathbf{H}\frac{d\mu}{dz}\mathbf{H}\right),$$

or

$$(\nabla_{\mathrm{B}} - \nabla_{\mathrm{H}})\,\mathbf{HB},$$

or

$$\mathbf{i}\,(\mathbf{H}\nabla_1\mathbf{B} - \mathbf{B}\nabla_1\mathbf{H}) + \mathbf{j}\,(\mathbf{H}\nabla_2\mathbf{B} - \mathbf{B}\nabla_2\mathbf{H}) + \mathbf{k}\,(\mathbf{H}\nabla_3\mathbf{B} - \mathbf{B}\nabla_3\mathbf{H}),$$

showing the **i**, **j**, **k** components.

Similarly in the other cases occurring later.

The following stresses are not of the simple type, though all consist of a tension parallel to **R** or **H** combined with an isotropic pressure.

§ **32.** Alter the stress so as to locate the force on an intrinsic magnet bodily upon its magnetised elements. Add $\mathbf{R}.\mu\mathbf{h}_0\mathbf{N}$ to the stress (186), and therefore $\mu\mathbf{h}_0.\mathbf{RN}$ to its conjugate; then the divergence of the latter must be added to the **N**-component of the force (187). Thus we get, if $\mathbf{I} = \mu\mathbf{h}_0$,

$$(4)\ \begin{cases} \mathbf{P}_{\mathrm{N}} = \mathbf{R}.\mathbf{BN} - \mathbf{N}.\tfrac{1}{2}\,\mathbf{R}\mu\mathbf{R}, & \qquad (190) \\ \mathbf{F}\ \ = \mathbf{I}\nabla.\mathbf{R} + \mathrm{V}\mathbf{J}\mu\mathbf{R} - \tfrac{1}{2}\mathbf{R}^2\nabla\mu. & \qquad (191) \end{cases}$$

But here the sum of the first two terms in **F** may be put in a different form. Thus,

$$\begin{aligned} \mathbf{I}\nabla.\mathbf{R} &= \mathrm{I}_1\nabla_1\mathbf{R} + \mathrm{I}_2\nabla_2\mathbf{R} + \mathrm{I}_3\nabla_3\mathbf{R} \\ &= \mathbf{i}.\mathbf{I}\nabla\mathrm{R}_1 + \mathbf{j}.\mathbf{I}\nabla\mathrm{R}_2 + \mathbf{k}.\mathbf{I}\nabla\mathrm{R}_3. \end{aligned}$$

Also

$$\mathbf{I}\nabla\mathrm{R}_1 = \mathbf{I}\nabla_1\mathbf{R} + \mathbf{I}\,(\nabla\mathrm{R}_1 - \nabla_1\mathbf{R}) = \mathbf{I}\nabla_1\mathbf{R} + \mathbf{i}\mathrm{V}\mathbf{JI}.$$

These bring (191) to

$$\mathbf{F} = (\mathbf{i}.\mathbf{I}\nabla_1\mathbf{R} + \mathbf{j}.\mathbf{I}\nabla_2\mathbf{R} + \mathbf{k}.\mathbf{I}\nabla_3\mathbf{R}) + \mathrm{V}\mathbf{JB} - \tfrac{1}{2}\,\mathbf{R}^2\nabla\mu, \qquad (192)$$

where the first component (the bracketted part) is MAXWELL'S force on intrinsic magnetisation, and the second his electromagnetic force. The third, as before, is required where μ varies.

§ **33.** To the stress (190) add $-\,\mathbf{N}.\tfrac{1}{2}\,\mathbf{RI}$, without altering the conjugate stress making

$$(5)\begin{cases} \mathbf{P_N} = \mathbf{R.BN} - \mathbf{N}.\tfrac{1}{2}\mathbf{RB}, & (193) \\ \mathbf{F} = \mathbf{VJB} - \tfrac{1}{2}\{\mathbf{i}(\mathbf{R}\nabla_1\mathbf{B} - \mathbf{B}\nabla_1\mathbf{R}) + \mathbf{j}(\mathbf{R}\nabla_2\mathbf{B} - \mathbf{B}\nabla_2\mathbf{R}) + \mathbf{k}(\mathbf{R}\nabla_3\mathbf{B} - \mathbf{B}\nabla_3\mathbf{R})\}. & (194) \end{cases}$$
$$= \mathbf{VJB} - (\nabla_B - \nabla_R)\tfrac{1}{2}\mathbf{RB}.$$

This we need not discuss, as it is merely a transition to the next form.

§ **34.** To the stress (193) add $\mathbf{h_0.NB}$; we then get

$$(6)\begin{cases} \mathbf{P_N} = \mathbf{H.NB} - \mathbf{N}.\tfrac{1}{2}\mathbf{RB}, & (195) \\ \mathbf{F} = \mathbf{VJB} + \{\mathbf{i.B}\nabla h_1 + \mathbf{j.B}\nabla h_2 + \mathbf{k.B}\nabla h_3\} & \\ \quad - \tfrac{1}{2}\{\mathbf{i.}(\mathbf{R}\nabla_1\mathbf{B} - \mathbf{B}\nabla_1\mathbf{R}) + \mathbf{j.}(\mathbf{R}\nabla_2\mathbf{B} - \mathbf{B}\nabla_2\mathbf{R}) + \mathbf{k.}(\mathbf{R}\nabla_3\mathbf{B} - \mathbf{B}\nabla_3\mathbf{R})\}, & (196) \end{cases}$$
$$= \mathbf{VJB} + \mathbf{B}\nabla.\mathbf{h_0} - (\nabla_B - \nabla_R)\tfrac{1}{2}\mathbf{RB},$$

where h_1, h_2, h_3 are the components of $\mathbf{h_0}$.

Now if to this last stress (195) we add $-\mathbf{N}.\tfrac{1}{2}\mathbf{h_0B}$, we shall come back to the third stress, (188), of the simple type.

Perhaps the most instructive order in which to take the six stresses is (1), (2), (4), (5), (6), and (3); merely adding on to the force, in passing from one stress to the next, the new part which the alteration in the stress necessitates.

To the above we should add MAXWELL's general stress, which is

$$(7)\begin{cases} \mathbf{P_N} = \mathbf{R.NB} - \mathbf{N}.\tfrac{1}{2}\mathbf{R}^2, & (197) \\ \mathbf{F} = \mathbf{VJB} + \{\mathbf{i.I}\nabla_1\mathbf{R} + \mathbf{j.I}\nabla_2\mathbf{R} + \mathbf{k.I}\nabla_3\mathbf{R}\} & \\ \quad + \{\mathbf{i.M}\nabla_1\mathbf{R} + \mathbf{j.M}\nabla_2\mathbf{R} + \mathbf{k.M}\nabla_2\mathbf{R}\}, & (198) \end{cases}$$
$$= \mathbf{VJB} + \nabla_R[\mathbf{R}(\mathbf{I} + \mathbf{M})],$$

where $\mathbf{M} = (\mu - 1)\mathbf{R}$ = intensity of induced magnetisation. There is a good deal to be said against this stress; some of which later.

Remarks on MAXWELL'S *General Stress.*

§ **35.** All the above force formulæ refer to the unit volume; whenever, therefore, a discontinuity in the stress occurs at a surface, the corresponding expression per unit surface is needed; *i.e.*, in making a special application, for it is wasted labour else. It might be thought that as MAXWELL gives the force (198), and in his treatise usually gives surface expressions separately, so none is required in the case of this his force system (198). But this formula will give entirely erroneous results if carried out literally. It forms no exception to the rule that all the expressions require surface additions.

MAXWELL's general stress has the apparent advantage of simplicity. It merely requires an alteration in the tension parallel to $\mathbf{R}$, from $\mathbf{R}^2$ to $\mathbf{RB}$, whilst the lateral pressure remains $\tfrac{1}{2}\mathbf{R}^2$, when we pass from unmagnetised to magnetised matter. The force to which it gives rise is also apparently simple, being merely the sum of two forces, one the electromagnetic, **VJB**, the other a force on magnetised matter whose **i** component is $(\mathbf{I} + \mathbf{M})\nabla_1\mathbf{R}$, both per unit volume, the latter being accompanied (in case of eolotropy) by a torque. Now **I** is the intrinsic and **M** the induced magnetisation, so the force is made irrespective of the proportion in which the magnetisation exists as intrinsic or induced. In fact, MAXWELL'S "magnetisation" is the sum of the two without reservation or distinction. But to unite them is against the whole behaviour of induced and intrinsic magnetisation in the electromagnetic scheme of MAXWELL, as I interpret it. Intrinsic magnetisation (using Sir W. THOMSON'S term) should be regarded as impressed ($\mathbf{I} = \mu\mathbf{h}_0$, where $\mathbf{h}_0$ is the equivalent impressed magnetic force); on the other hand, "induced" magnetisation depends on the force of the field $\{\mathbf{M} = (\mu - 1)\,\mathbf{R}\}$. Intrinsic magnetisation keeps up a field of force. Induced magnetisation is kept up by the field. In the circuital law **I** and **M** therefore behave differently. There may be absolutely no difference whatever between the magnetisation of a molecule of iron in the two cases of being in a permanent or a temporary magnet. That, however, is not in question. We have no concern with molecules in a theory which ignores molecules, and whose element of volume must be large enough to contain so many molecules as to swamp the characteristics of individuals. It is the resultant magnetisation of the whole assembly that is in question, and there is a great difference between its nature according as it disappears on removal of an external cause, or is intrinsic. The complete amalgamation of the two in MAXWELL'S formula must certainly, I think, be regarded as a false step.

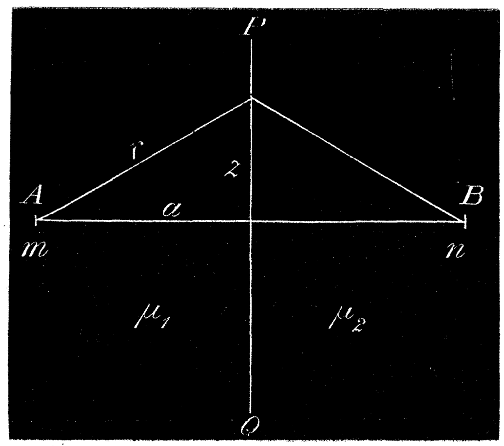
We may also argue thus against the probability of the formula. If we have a system of electric current in an unmagnetisable ($\mu = 1$) medium, and then change μ everywhere in the same ratio, we do not change the magnetic force at all, the induction is made μ times as great, and the magnetic energy μ times as great, and is similarly distributed. The mechanical forces are, therefore, μ times as great, and are similarly distributed. That is, the translational force in the $\mu = 1$ medium, or **VJR**, becomes $\mathbf{VJ}\mu\mathbf{R}$ in the second case in which the inductivity is μ, without other change. But there is no force brought in on magnetised matter *per se*.

Similarly, if in the $\mu = 1$ medium we have intrinsic magnetisation **I**, and then alter μ in any ratio everywhere alike, keeping **I** unchanged, it is now the induction that remains unaltered, the magnetic force becoming μ^{-1} times, and the energy μ^{-1} times the former values, without alteration in distribution (referring to permanent states, of course). Again, therefore, we see that there is no translational force brought in on magnetised matter merely because it is magnetised.

Whatever formula, therefore, we should select for the stress function, it would certainly not be MAXWELL'S, for cumulative reasons. When, some six years ago, I had occasion to examine the subject of the stresses, I was unable to arrive at any very definite results, except outside of magnets or conductors. It was a perfectly indeterminate problem to find the magnetic stress inside a body from the existence of a known, or highly probable, stress outside it. All one could do was to examine the consequences of assuming certain stresses, and to reject those which did not work well. After going into considerable detail, the only two which seemed possible were the second and third above (those of equations (186) and (188) above). As regards the seventh (MAXWELL's stress equation (198) above), the apparent simplicity produced by the union of intrinsic and induced magnetisation, turned out, when examined into its consequences, to lead to great complication and unnaturalness. This will be illustrated in the following example, a simple case in which we can readily and fully calculate all details by different methods, so as to be quite sure of the results we ought to obtain.

A worked-out Example to Exhibit the Forcives contained in Different Stresses.

§ **36**. Given a fluid medium of inductivity μ_1, in which is an intrinsic magnet of the same inductivity. Calculate the attraction between the magnet and a large solid mass of different inductivity μ_2. Here it is only needful to calculate the force on a single pole, so let the magnet be infinitely thin and long, with one pole of strength m at distance a from the medium μ_2, which may have an infinitely extended plane boundary. By placing a fictitious pole of suitable strength at the optical image in the second medium of the real pole in the first, we may readily obtain the solution.



Let PQ be the interface and the real pole be at A and its image at B. We have first to calculate the distribution of **R**, magnetic force, in both media due to the pole m, as disturbed by the change of inductivity. We have div $\mu_1\mathbf{R}_1 = m$ in the first medium, and div $\mu_2\mathbf{R}_2 = 0$ in the second, therefore **R** has divergence only on the interface. Let σ be the surface density of the fictitious interfacial matter to correspond; its force goes symmetrically both ways; the continuity of the normal induction therefore gives, at distance r from A, the condition

$$\mu_1\left(\frac{ma}{4\pi\mu_1 r^3} - \tfrac{1}{2}\sigma\right) = \mu_2\left(\frac{ma}{4\pi\mu_1 r^3} + \tfrac{1}{2}\sigma\right), \qquad (199)$$

because $m/4\pi\mu_1 r^2$ is the tensor of the magnetic force due to m in the μ_1 medium when of infinite extent. Therefore

$$\sigma = \frac{\mu_1 - \mu_2}{\mu_1 + \mu_2} \cdot \frac{ma}{2\pi\mu_1 r^3}. \qquad (200)$$

The magnetic potential Ω, such that $\mathbf{R} = -\nabla\Omega$ is the polar force in either region, is therefore the potential of m/μ_1 at A and of σ over the interface.

But if we put matter n at the image B, of amount

$$n = \frac{\mu_1 - \mu_2}{\mu_1 + \mu_2} \cdot \frac{m}{\mu_1}, \qquad (201)$$

the normal component of $\mathbf{R}_1$ on the μ_1 side due to n and the pole m will be

$$\frac{ma}{4\pi\mu_1 r^3} - \frac{na}{4\pi r^3} = \frac{ma}{4\pi\mu_1 r^3} - \tfrac{1}{2}\sigma, \qquad (202)$$

the same value as before; the force $\mathbf{R}_1$ on the μ_1 side is, therefore, the same as that due to matter m/μ_1 at A and matter n at B; whilst on the μ_2 side the force $\mathbf{R}_2$ is that due to matter m/μ_1 at A and matter n also at A, that is, to matter $\dfrac{2m}{\mu_1 + \mu_2}$ at A. Thus in the μ_2 medium the force $\mathbf{R}_2$ is radial from A as if there were no change of inductivity, though altered in intensity.

The repulsion between the pole m and the solid mass is not the repulsion between the matters m/μ_1 and n of the potential, but is

$= m \times$ magnetic force at A due to matter n at B,

$= n \times$ magnetic force at B due to matter m/μ_1 at A,

$$= \frac{mn}{4\pi(2a)^2} = \frac{\mu_1 - \mu_2}{\mu_1 + \mu_2} \cdot \frac{m^2}{4\pi\mu_1 (2a)^2}, \qquad (203)$$

becoming an attraction when $\mu_2 > \mu_1$, making n negative. When $\mu_2 = 0$, the repulsion is

$$\frac{m^2}{4\pi\mu_1 (2a)^2};$$

when $\mu_2 = \infty$, it is turned into an attraction of equal amount.

Similarly, if we consider the attraction to be the resultant force between m and the interfacial matter σ, we shall get the same result by

$$\Sigma \frac{\sigma m a}{4\pi r^3}, \qquad (204)$$

the quantity summed (over the interface) being $\sigma \times$ normal component of magnetic force due to matter m in a medium of unit inductivity, or the normal component of induction due to m in its own medium. For this is

$$\int \frac{ma}{4\pi r^3} \cdot \frac{\mu_1 - \mu_2}{\mu_1 + \mu_2} \cdot \frac{ma}{2\pi\mu_1 r^3}\, 2\pi r\, dr = \frac{m^2a^2}{4\pi\mu_1} \cdot \frac{\mu_1 - \mu_2}{\mu_1 + \mu_2} \int \frac{dr}{r^5} = (203) \text{ again.}$$

Another way is to calculate the variation of energy made by displacing either the pole m or the μ_2 mass. The potential energy is expressed by

$$\tfrac{1}{2}(\mathrm{P} + p)\, m = \tfrac{1}{2}\mathrm{P}m + \tfrac{1}{2}\Sigma \mathrm{P}\sigma\mu, \qquad (205)$$

where $\mathrm{P} = m/4\pi\mu_1 r$ and $p = \Sigma\, \sigma/4\pi r$, the potentials of matter m/μ_1 and σ, where r is the distance from m or from σ to the point where P and p are reckoned.

The value of the second part in (205), depending upon σ, comes to

$$\tfrac{1}{2}\frac{\mu_1 - \mu_2}{\mu_1 + \mu_2} \cdot \frac{m^2}{4\pi\mu_1 \,.\, 2a}, \qquad (206)$$

and its rate of decrease with respect to a expresses the repulsion between the pole and the μ_2 region. This gives (203) again.

A fourth way is by means of the *quasi*-electromagnetic force on fictitious interfacial electric current, instead of matter, the current being circular about the axis of symmetry AB. The formula for the attraction is

$$\Sigma \mathrm{V}\ \mathrm{curl}\ \mathbf{B}.\mathbf{R}_0, \qquad (207)$$

if $\mathbf{R}_0$ be the radial magnetic force from m in its own medium, tensor $m/4\pi\mu_1 r^2$. Here the curl of $\mathbf{B}$ is represented by the interfacial discontinuity in the tangential induction, or

$$\frac{2zm}{4\pi r^3}\frac{\mu_1 - \mu_2}{\mu_1 + \mu_2}.$$

Also the tangential component of $\mathbf{R}_0$ is $mz/4\pi\mu_1 r^3$. Therefore the repulsion is

$$\int \frac{2mz}{4\pi r^3} \cdot \frac{\mu_1 - \mu_2}{\mu_1 + \mu_2} \cdot \frac{mz}{4\pi\mu_1 r^3}\, 2\pi r\, dr = \frac{m^2}{4\pi\mu_1} \cdot \frac{\mu_1 - \mu_2}{\mu_1 + \mu_2} \int_a^\infty \frac{r^2 - a^2}{r^5}\, dr = \frac{m^2}{4\pi\mu_1} \cdot \frac{\mu_1 - \mu_2}{\mu_1 + \mu_2} \cdot \frac{1}{4a^2}, \qquad (208)$$

as before, equation (203). This method (207) is analogous to (204).

§ **37.** There are several other ways of representing the attraction, employing fictitious matter and current; but now let us change the method, and observe how the attraction between the magnetic pole and the iron mass is accounted for by a stress distribution, and its space-variation. The best stress is the third, equation (188), § 31. Applying this, we have simply a tension of magnitude $\frac{1}{2}\mu_1 \mathrm{R}_1{}^2 = \mathrm{T}_1$ in the first medium and $\frac{1}{2}\mu_2 R_2{}^2 = T_2$ in the second, parallel to $\mathbf{R}_1$ and $\mathbf{R}_2$ respectively, each combined with an equal lateral pressure, so that the tensor of the stress vector is constant.

But, so far as the attraction is concerned, we may ignore the stress in the second medium altogether, and consider it as the $\Sigma \mathbf{P}_N$ of the stress-vector in the first medium over the surface of the second medium. The tangential component summed has zero resultant; the attraction is therefore the sum of the normal components, or $\Sigma T_1 \cos 2\theta_1$, where θ_1 is the angle between $\mathbf{R}_1$ and the normal. This is the same as $\Sigma \frac{1}{2}\mu_1 (R_N{}^2 - R_T{}^2)$, if R_N and R_T are the normal and tangential components of $\mathbf{R}_1$; or

$$\int_a^\infty 2\pi r\, dr\, \tfrac{1}{2}\mu_1 \left[\left(\frac{ma}{4\pi\mu_1 r^3}\,\frac{2\mu_2}{\mu_1+\mu_2}\right)^2 - \left(\frac{mz}{4\pi\mu_1 r^3}\,\frac{2\mu_1}{(\mu_1+\mu_2)}\right)\right]; \quad . \quad . \quad . \quad . \quad . \quad . \quad (209)$$

which on evaluation gives the required result (203).

But this method does not give the true distribution of translational force due to the stresses. In the first medium there is no translational force, except on the magnet. Nor is there any translational force in the second μ_2 medium. But at the interface, where μ changes, there is the force $-\frac{1}{2}R^2\nabla\mu$ per unit volume, and this is represented by the stress-difference at the interface. It is easily seen that the tangential stress-difference is zero, because

$$T \sin 2\theta = \mu R_N R_T, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (210)$$

and both the normal induction and the tangential magnetic force are continuous. The real force is, therefore, the difference of the normal components of the stress-vectors, and is, therefore, normal to the interface. This we could conclude from the expression $-\frac{1}{2}R^2\nabla\mu$. But since the resultant of the interfacial stress in the second medium is zero, we need not reckon it, so far as the attraction of the pole is concerned. The normal traction on the interface, due to both stresses, is of amount

$$\frac{m^2}{8\pi^2 r^6}\,\frac{\mu_2-\mu_1}{(\mu_1+\mu_2)^2}\left(r^2 + a^2\,\frac{\mu_2-\mu_1}{\mu_1}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (211)$$

per unit area. Summed up, it gives (203) again.

That (211) properly represents the force $-\frac{1}{2}R^2\nabla\mu$ when μ is discontinuous, we may also verify by supposing μ to vary continuously in a very thin layer, and then proceed to the limit.

The change from an attraction to a repulsion as μ_2 changes from being greater to being less than μ_1, depends upon the relative importance of the tensions parallel to the magnetic force and the lateral pressures operative at different parts of the interface. In the extreme case of $\mu_2 = 0$, we have $\mathbf{R}_1$ tangential, with, therefore, a pressure everywhere. For the other extreme, $\mathbf{R}_1$ is normal, and there is a pull on the second medium everywhere. When μ_2 is finite there is a certain circular area on the interface within which the translational force due to the stress in the medium containing the pole m is towards that medium, whilst outside it the force is the other way. But when both stresses are allowed for, we see that when $\mu_2 > \mu_1$ the pull is towards the first medium in all parts of the interface, and that this becomes a push in all parts when $\mu_1 > \mu_2$.

A definite Stress only obtainable by Kinetic Consideration of the Circuital Equations and Storage and Flux of Energy.

§ **38**. We see that the stress considered in the last paragraph gives a rationally intelligible interpretation of the attraction or repulsion. The same may be said of other stresses than that chosen. But the use of MAXWELL's stress, or any stress leading to a force on inductively magnetised matter as this stress does, leads us into great difficulties. By (198) we see that there is first a bodily force on the whole of the μ_2 medium, because it is magnetised, unless $\mu_2 = 1$. When summed up, the resultant does not give the required attraction. For, secondly, the μ_1 medium is also magnetised, unless $\mu_1 = 1$, and there is a bodily force throughout the whole of it. When this is summed up (not counting the force on the magnet), its resultant added on to the former resultant still does not make up the attraction (*i.e.*, equivalently, the force on the magnet). For, thirdly, the stress is discontinuous at the interface (though not in the same manner as in the last paragraph). The resultant of this stress-discontinuity, added on to the former resultants, makes up the required attraction. It is unnecessary to give the details relating to so improbable a system of force.

Our preference must naturally be for a more simple system, such as the previously considered stress. But there is really no decisive settlement possible from the theoretical statical standpoint, and nothing short of actual experimental determination of the strains produced and their exhaustive analysis would be sufficient to determine the proper stress-function. But when the subject is attacked from the dynamical standpoint, the indeterminateness disappears. From the two circuital laws of variable states of electric and magnetic force in a moving medium, combined with certain distributions of stored energy, we are led to just one stress-vector, viz. (136). It is, in the magnetic case, the same as (188); that is, it reduces to the latter when the medium is kept at rest, so that $\mathbf{J}_0$ and $\mathbf{G}_0$ become $\mathbf{J}$ and $\mathbf{G}$.

It is of the simple type in case of isotropy (constant tensor), but is a rotational stress in general, as indeed are all the statically probable stresses that suggest themselves. The translational force due to it being divisible conveniently into (*a*) the electromagnetic force on electric current, (*b*) the ditto on the fictitious electric current taking the place of intrinsic magnetisation, (*c*) force depending upon space-variation of μ; we see that the really striking part is (*b*). Of all the various ways of representing the forcive on an intrinsic magnet it is the most extreme. The magnetic "matter" does not enter into it, nor does the distribution of magnetisation; it is where the intrinsic force $\mathbf{h}_0$ has curl that the translational force operates, usually on the sides of a magnet. From actual experiments with bar magnets, needles, &c., one would naturally prefer to regard the polar regions as the seat of translational force. But the equivalent forcive $\Sigma \mathbf{j}_0 \mathbf{B}$ has one striking recommendation (apart from the dynamical method of deducing it), viz., that the induction of an intrinsic magnet is determined by curl $\mathbf{h}_0$, not by $\mathbf{h}_0$ itself; and this, I have shown, is true when $\mathbf{h}_0$ is imagined to vary, the whole varying states of the fluxes **B**, **D**, **C** due to impressed force being determined by the curls of $\mathbf{e}_0$ and $\mathbf{h}_0$, which are the sources of the disturbances (though not of the energy).

The rotational peculiarity in eolotropic substances does not seem to be a very formidable objection. Are they not solid?

As regards the assumed constancy of μ, a more complete theory must, to be correct, reduce to one assuming constancy of μ, because, as Lord RAYLEIGH* has shown, the assumed law has a limited range of validity, and is therefore justifiable as a preparation for more complete views. Theoretical requirements are not identical with those of the practical engineer.

But, for quite other reasons, the dynamically determined stress might be entirely wrong. Electric and magnetic "force" and their energies are facts. But it is the total of the energies in concrete cases that should be regarded as the facts, rather then their distribution; for example, that, as Sir W. THOMSON proved, the "mechanical value" of a simple closed current C is $\frac{1}{2}LC^2$, where L is the inductance of the circuit (coefficient of electromagnetic capacity), rather than that its distribution in space is given by $\frac{1}{2}\mathbf{HB}$ per unit volume. Other distributions may give the same total amount of energy. For example, the energy of distortion of an elastic solid may be expressed in terms of the square of the rotation and the square of the expansion, if its boundary be held at rest; but this does not correctly localise the energy. If, then, we choose some other distribution of the energy for the same displacement and induction, we should find quite a different flux of energy. But I have not succeeded in making any other arrangement than MAXWELL's work practically, or without an immediate introduction of great obscurities. Perhaps the least certain part of MAXWELL's scheme, as modified by myself, is the estimation of magnetic energy as $\frac{1}{2}\mathbf{HB}$ in intrinsic magnets, as well as outside them, that is, by $\frac{1}{2}\mathbf{B}\mu^{-1}\mathbf{B}$, however **B** may be caused. Yet, only in this way are thoroughly consistent results apparently obtainable when the electromagnetic field is considered comprehensively and dynamically.

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APPENDIX.

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Extension of the Kinetic Method of arriving at the Stresses to cases of Non-linear Connection between the Electric and Magnetic Forces and the Fluxes. Preservation of Type of the Flux of Energy Formula.

§ **39**. It may be worth while to give the results to which we are led regarding the stress and flux of energy when the restriction of simple proportionality between "forces" and "fluxes," electric and magnetic respectively, is removed. The course to be followed, to obtain an interpretable form of the equation of activity, is sufficiently clear in the light of the experience gained in the case of proportionality.

First assume that the two circuital laws (89) and (90), or the two in (93), hold good generally, without any initially stated relation between the electric force $\mathbf{E}$ and its associated fluxes $\mathbf{C}$ and $\mathbf{D}$, or between the magnetic force $\mathbf{H}$ and its associated fluxes $\mathbf{K}$ and $\mathbf{B}$. When written in the form most convenient for the present application, these laws are

$$\text{curl}\,(\mathbf{H}-\mathbf{h}_0)=\mathbf{J}_0=\mathbf{C}+\frac{\partial\mathbf{D}}{\partial t}+(\mathbf{D}\,\text{div}\,\mathbf{q}-\mathbf{D}\nabla.\mathbf{q}), \qquad (212)$$

$$-\,\text{curl}\,(\mathbf{E}-\mathbf{e}_0)=\mathbf{G}_0=\mathbf{K}+\frac{\partial\mathbf{B}}{\partial t}+(\mathbf{B}\,\text{div}\,\mathbf{q}-\mathbf{B}\nabla.\mathbf{q}). \qquad (213)$$

Now derive the equation of activity in the manner previously followed, and arrange it in the particular form

$$\mathbf{e}_0\mathbf{J}_0+\mathbf{h}_0\mathbf{G}_0+\text{conv}\,\mathrm{V}\,(\mathbf{E}-\mathbf{e}_0)\,(\mathbf{H}-\mathbf{h}_0)$$
$$=(\mathbf{EC}+\mathbf{HK})+\left(\mathbf{E}\frac{\partial\mathbf{D}}{\partial t}+\mathbf{H}\frac{\partial\mathbf{B}}{\partial t}\right)+(\mathbf{E}.\mathbf{D}\nabla.\mathbf{q}-\mathbf{ED}\,\text{div}\,\mathbf{q})+(\mathbf{H}.\mathbf{B}\nabla.\mathbf{q}-\mathbf{HB}\,\text{div}\,\mathbf{q}), \qquad (214)$$

which will best facilitate interpretation.

Although independent of the relation between $\mathbf{E}$ and $\mathbf{D}$, &c., of course the dimensions must be suitably chosen so that this equation may really represent activity per unit volume in every term.

Now, guided by the previous investigation, we can assume that $(\mathbf{e}_0\mathbf{J}_0+\mathbf{h}_0\mathbf{G}_0)$ represents the rate of supply of energy from intrinsic sources, and also that $\mathrm{V}\,(\mathbf{E}-\mathbf{e}_0)\,(\mathbf{H}-\mathbf{h}_0)$, which is a flux of energy independent of $\mathbf{q}$, is the correct form in general. Also, if there be no other intrinsic sources of energy than $\mathbf{e}_0$, $\mathbf{h}_0$, and no other fluxes of energy besides that just mentioned except the convective flux and that due to the stress, the equation of activity should be representable by

$$
\begin{aligned}
&(\mathbf{e}_0\mathbf{J}_0 + \mathbf{h}_0\mathbf{G}_0) + \operatorname{conv}\left[\mathrm{V}\,(\mathbf{E} - \mathbf{e}_0)\,(\mathbf{H} - \mathbf{h}_0) + \mathbf{q}\,(\mathrm{U} + \mathrm{T})\right] \\
&= (\mathrm{Q} + \dot{\mathrm{U}} + \dot{\mathrm{T}}) + \mathbf{F}\mathbf{q} + \operatorname{conv} \mathbf{Q}_q q \\
&= (\mathrm{Q} + \dot{\mathrm{U}} + \dot{\mathrm{T}}) + \Sigma \mathbf{Q} \nabla q, \qquad (215)
\end{aligned}
$$

where $\mathbf{Q}$ is the conjugate of the stress vector, $\mathbf{F}$ the translational force, and Q, U, and T the rate of waste and the stored energies, whatever they may be.

Comparing with the preceding equation (214), we see that we require

$$
\begin{aligned}
\Sigma \mathbf{Q} \nabla q = (\mathrm{Q} - \mathbf{EC} - \mathbf{HK}) + \left(\frac{\partial \mathrm{U}}{\partial t} - \mathbf{E}\frac{\partial \mathbf{D}}{\partial t}\right) + \left(\frac{\partial \mathrm{T}}{\partial t} - \mathbf{H}\frac{\partial \mathbf{B}}{\partial t}\right) \\
+ \left[\mathbf{E}.\mathbf{D}\nabla.\mathbf{q} - (\mathbf{ED} - \mathrm{U}) \operatorname{div} \mathbf{q}\right] + \left[\mathbf{H}.\mathbf{B}\nabla.\mathbf{q} - (\mathbf{HB} - \mathrm{T}) \operatorname{div} \mathbf{q}\right]. \qquad (216)
\end{aligned}
$$

Now assume that there is no waste of energy except by conduction ; then

$$
\mathrm{Q} = \mathbf{EC} + \mathbf{HK}. \qquad (217a)
$$

Also assume that

$$
\frac{\partial \mathrm{U}}{\partial t} = \mathbf{E}\frac{\partial \mathbf{D}}{\partial t}, \qquad \frac{\partial \mathrm{T}}{\partial t} = \mathbf{H}\frac{\partial \mathbf{B}}{\partial t}. \qquad (217b)
$$

These imply that the relation between $\mathbf{E}$ and $\mathbf{D}$ is, for the same particle of matter, an invariable one, and that the stored electric energy is

$$
\mathrm{U} = \int_0^{\mathrm{D}} \mathbf{E}\, d\mathbf{D}, \qquad (218)
$$

where $\mathbf{E}$ is a function of $\mathbf{D}$. Similarly,

$$
\mathrm{T} = \int_0^{\mathrm{B}} \mathbf{H}\, d\mathbf{B}, \qquad (219)
$$

expresses the stored magnetic energy, and $\mathbf{H}$ must be a definite function of $\mathbf{B}$.

On these assumptions, (216) reduces to

$$
\Sigma \mathbf{Q} \nabla q = \left[\mathbf{E}.\mathbf{D}\nabla.\mathbf{q} - (\mathbf{ED} - \mathrm{U}) \operatorname{div} \mathbf{q}\right] + \left[\mathbf{H}.\mathbf{B}\nabla.\mathbf{q} - (\mathbf{HB} - \mathrm{T}) \operatorname{div} \mathbf{q}\right], \qquad (220)
$$

from which the stress-vector follows, namely,

$$
\mathbf{P}_{\mathrm{N}} = \left[\mathbf{E}.\mathbf{DN} - \mathbf{N}\,(\mathbf{ED} - \mathrm{U})\right] + \left[\mathbf{H}.\mathbf{BN} - \mathbf{N}\,(\mathbf{HB} - \mathrm{T})\right]. \qquad (221)
$$

Or,

$$
\mathbf{P}_{\mathrm{N}} = (\mathbf{VDVEN} + \mathbf{N}\mathrm{U}) + (\mathbf{VBVHN} + \mathbf{N}\mathrm{T}). \qquad (222)
$$

Thus, in case of isotropy, the stress is a tension U parallel to $\mathbf{E}$ combined with a lateral pressure $(\mathbf{ED} - \mathrm{U})$; and a tension T parallel to $\mathbf{H}$ combined with a lateral pressure $(\mathbf{HB} - \mathrm{T})$.

The corresponding translational force is

$$
\begin{aligned}
\mathbf{F} = \mathbf{E} \operatorname{div} \mathbf{D} + \mathbf{D}\nabla.\mathbf{E} - \nabla\,(\mathbf{ED} - \mathrm{U}) \\
+ \mathbf{H} \operatorname{div} \mathbf{B} + \mathbf{B}\nabla.\mathbf{H} - \nabla\,(\mathbf{HB} - \mathrm{T}), \qquad (223)
\end{aligned}
$$

which it is unnecessary to put in terms of the currents.

Exchange **E** and **D**, and **H** and **B**, in (221) or (222) to obtain the conjugate vector $\mathbf{Q}_N$; from which we obtain the flux of energy due to the stress,

$$-q\mathbf{Q}_q = \mathbf{D.Eq} - \mathbf{q}(\mathbf{ED} - \mathrm{U}) + \mathbf{B.Hq} - \mathbf{q}(\mathbf{HB} - \mathrm{T})$$
$$= \mathrm{V}\mathbf{E}\mathrm{V}\mathbf{Dq} + \mathrm{V}\mathbf{H}\mathrm{V}\mathbf{Bq} + \mathbf{q}(\mathrm{U} + \mathrm{T}), \quad \ldots \ldots \ldots \quad (224)$$

or

$$-q\mathbf{Q}_q = \mathrm{V}\mathbf{eH} + \mathrm{V}\mathbf{Eh} + \mathbf{q}(\mathrm{U} + \mathrm{T}), \quad \ldots \ldots \ldots \quad (225)$$

where **e** and **h** are the motional electric and magnetic forces, of the same form as before (88) and (91); so that the complete form of the equation of activity, showing the fluxes of energy and their convergence, is

$$\mathbf{e}_0\mathbf{J}_0 + \mathbf{h}_0\mathbf{G}_0 + \mathrm{conv}\left[\mathrm{V}(\mathbf{E} - \mathbf{e}_0)(\mathbf{H} - \mathbf{h}_0) + \mathbf{q}(\mathrm{U} + \mathrm{T})\right] - \mathrm{conv}\left[\mathrm{V}\mathbf{eH} + \mathrm{V}\mathbf{Eh} + \mathbf{q}(\mathrm{U} + \mathrm{T})\right] = \mathbf{Fq} + (\mathrm{Q} + \dot{\mathrm{U}} + \dot{\mathrm{T}}), \quad (226)$$

where **F** has the above meaning.

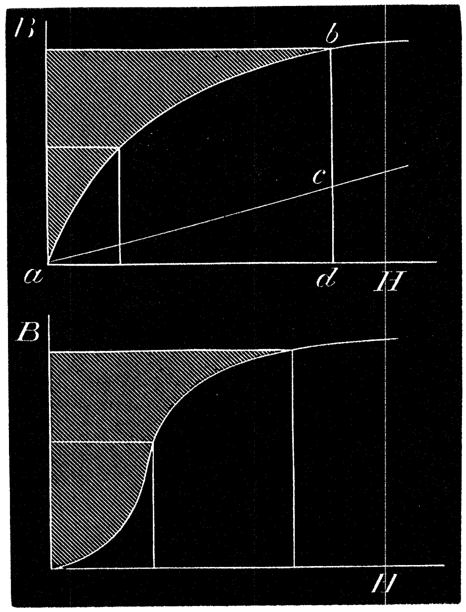
There is thus a remarkable preservation of form as compared with the corresponding formulæ when there is proportionality between force and flux. For we produce harmony by means of a POYNTING flux of identical expression and a flux due to the stress, which is also of identical expression, although U and T now have a more general meaning, of course.*

Example of the above, and Remarks on Intrinsic Magnetisation when there is Hysteresis.

§ **40.** In the stress-vector itself (for either the electric or the magnetic stress) the relative magnitude of the tension and the lateral pressure varies unless the curve

* As the investigation in this Appendix has some pretensions to generality, we should try to settle the amount it is fairly entitled to. No objection is likely to be raised to the use of the circuital equations (212), (213), with the restriction of strict proportionality between **E** and **H** and the fluxes **D** and **B**, or **C** and **K** entirely removed; nor to the estimation of $\mathbf{J}_0$ and $\mathbf{G}_0$ as the "true" currents; nor to the use of the same form of flux of electromagnetic energy when the medium is stationary. For these things are obviously suggested by the preceding investigations, and their justification is in their being found to continue to work, which is the case. But the use in the text of language appropriate to linear functions, which arose from the notation, &c., being the same as before, is unjustifiable. We may, however, remove this misuse of language, and make the equation (226), showing the flux of energy, rest entirely upon the two circuital equations. In fact, if we substitute in (226) the relations (217*a*), (217*b*), it becomes merely a particular way of writing (214).

It is, therefore, to (217*a*), (217*b*) that we should look for limitations. As regards (217*a*), there does not seem to be any limitation necessary. That is, there is no kind of relation imposed between **E** and **C**, and **H** and **K**. This seems to arise merely from Q meaning energy wasted for good, and having no further entry into the system. But as regards (217*b*), the case is different. For it seems necessary, in order to exclude terms corresponding to $\mathbf{E}(\partial c/\partial t)\mathbf{E}$ and $\mathbf{H}(\partial \mu/\partial t)\mathbf{H}$ in the linear theory, when there is connecting the force and the induction be a straight line. Thus, if the curve be of the type shown in the first figure, the shaded area will represent the stored energy and the tension, and the remainder of the rectangle will represent the lateral pressure. They are equal when **H** is small; later on the pressure preponderates, and more and more so the bigger **H** becomes.



But if the curve be of the type shown in the second figure, then, after initial equality the tension preponderates; though, later on, when **H** is very big the pressure preponderates.

To obtain an idea of the effect, take the concrete example of an infinitely long rod, uniformly axially inductized by a steady current in an overlapping solenoid, and consider the forcive on the rod. Here both **H** and **B** are axial or longitudinal; and so, by equation (223), the translational force would be a normal force on the surface of the rod, acting outwards, of amount

$$(\mathbf{HB} - \mathrm{T}) - \tfrac{1}{2}\mathbf{H}_0\mathbf{B}_0$$

per unit area; this being the excess of the lateral pressure in the rod over $\frac{1}{2}H_0B_0$, the lateral pressure just outside it.

In case of proportionality of force to flux, the first pressure is $\frac{1}{2}$HB, and if there is no intrinsic magnetisation H and H_0 are equal. The outward force is therefore positive for paramagnetic, and negative for diamagnetic substances, and the result would be lateral expansion or contraction, since the infinite length would prevent elongation.

rotation, that **E** and **D** should be parallel, and likewise **H** and **B**. At any rate, if such terms be allowed, some modification may be required in the subsequent reckoning of the mechanical force. In other respects, it is merely implied by (217*b*) that **E** and **D** are definitely connected, likewise **H** and **B**, so that there is no waste of energy other than that expressed by Q.

But if the curve in the rod be of the type of the first figure, and the straight line *ac* be the air curve to correspond, it is the area *abc* that now represents the outward force per unit area when the magnetic force has the value *ad*. If the straight line can cross the curve *ab*, we see that by sufficiently increasing H we can make the external air pressure preponderate, so that the rod, after initially expanding, would end by contracting.

If the rod be a ring of large diameter compared with its thickness, the forcive would be approximately the same, viz., an outward surface force equal to the difference of the lateral pressures in the rod and air. The result would then be elongation, with final retraction when the external pressure came to exceed the internal.

BIDWELL found a phenomenon of this kind in iron, but it does not seem possible that the above supposititious case is capable of explaining it, though of course the true explanation may be in some respects of a similar nature. But the circumstances are not the same as those supposed. The assumption of a definite connexion between H and B, and elastic storage of the energy T, is very inadequate to represent the facts of magnetisation of iron, save within a small range.

Magneticians usually plot the curve connecting $H-h_0$ and B, not between H and B, or which would be the same, between $H - h_0$ and $B - B_0$, where B_0 is the intrinsic magnetisation. Now when an iron ring is subjected to a given gaussage (or magneto motive force), going through a sequence of values, there is no definite curve connecting $H - h_0$ and B, on account of the intrinsic magnetisation. But, with proper allowance for h_0, it might be that the resulting curve connecting H and B in a given specimen, would be approximately definite, at any rate, far more so than that connecting $H - h_0$ and B. Granting perfect definiteness, however, there is still insufficient information to make a theory. The energy put into iron is not wholly stored; that is, in increasing the coil current we increase B_0 as well as B, and in doing so dissipate energy; and although we know, by EWING's experiments, the amount of waste in cyclical changes, it is not so clear what the rate of waste is at a given moment. There is also the further peculiarity that the energy of the intrinsic magnetisation at a given moment, though apparently locked up, and really locked up temporarily, however loosely it may be secured, is not wholly irrecoverable, but comes into play again when H is reversed. Now it may be that the energy of the intrinsic magnetisation plays, in relation to the stress, an entirely different part from that of the elastic magnetisation. It is easy to make up formulæ to express special phenomena, but very difficult to make a comprehensive theory.

But in any case, apart from the obscurities connected with iron, it is desirable to be apologetic in making any application of MAXWELL's stresses or similar ones to practice when the actual strains produced are in question, bearing in mind the difficulty of interpreting and harmonising with MAXWELL's theory the results of KERR, QUINCKE, and others.