XI. *On the Forces, Stresses, and Fluxes of Energy in the Electromagnetic Field.*

By OLIVER HEAVISIDE, *F.R.S.*

Received June 9,—Read June 18, 1891.*

General Remarks, especially on the Flux of Energy.

§ **1.** THE remarkable experimental work of late years has inaugurated a new era in the development of the Faraday-Maxwellian theory of the ether, considered as the primary medium concerned in electrical phenomena—electric, magnetic, and electromagnetic. MAXWELL'S theory is no longer entirely a paper theory, bristling with unproved possibilities. The reality of electromagnetic waves has been thoroughly demonstrated by the experiments of HERTZ and LODGE, FITZGERALD and TROUTON, J. J. THOMSON, and others; and it appears to follow that, although MAXWELL'S theory may not be fully correct, even as regards the ether (as it is certainly not fully comprehensive as regards material bodies), yet the true theory must be one of the same type, and may probably be merely an extended form of MAXWELL'S.

No excuse is therefore now needed for investigations tending to exhibit and elucidate this theory, or to extend it, even though they be of a very abstract nature. Every part of so important a theory deserves to be thoroughly examined, if only to see what is in it, and to take note of its unintelligible parts, with a view to their future explanation or elimination.

§ **2.** Perhaps the simplest view to take of the medium which plays such a necessary part, as the recipient of energy, in this theory, is to regard it as continuously filling all space, and possessing the mobility of a fluid rather than the rigidity of a solid. If whatever possess the property of inertia be matter, then the medium is a form of matter. But away from ordinary matter it is, for obvious reasons, best to call it as usual by a separate name, the ether. Now, a really difficult and highly speculative question, at present, is the connection between matter (in the ordinary sense) and ether. When the medium transmitting the electrical disturbances consists of ether and matter, do they move together, or does the matter only partially carry forward the ether which immediately surrounds it? Optical reasons may lead us to conclude, though only tentatively, that the latter may be the case; but at present, for the purpose of fixing the data, and in the pursuit of investigations not having specially

* Typographical troubles have delayed the publication of this paper. The footnotes are of date May 11, 1892.

11.8 92

optical bearing, it is convenient to assume that the matter and the ether in contact with it move together. This is the working hypothesis made by H. HERTZ in his recent treatment of the electrodynamics of moving bodies; it is, in fact, what we tacitly assume in a straightforward and consistent working out of MAXWELL'S principles without any plainly-expressed statement on the question of the relative motion of matter and ether; for the part played in MAXWELL'S theory by matter is merely (and, of course, roughly) formularised by supposing that it causes the etherial constants to take different values, whilst introducing new properties, that of dissipating energy being the most prominent and important. We may, therefore, think of merely one medium, the most of which is uniform (the ether), whilst certain portions (matter as well) have different powers of supporting electric displacement and magnetic induction from the rest, as well as a host of additional properties; and of these we can include the power of supporting conduction current with dissipation of energy according to JOULE'S law, the change from isotropy to eolotropy in respect to the distribution of the several fluxes, the presence of intrinsic sources of energy, &c.*

§ **3.** We do not in any way form the equations of motion of such a medium, even as regards the uniform simple ether, away from gross matter; we have only to discuss it as regards the electric and magnetic fluxes it supports, and the stresses and fluxes of energy thereby necessitated. First, we suppose the medium to be stationary, and examine the flux of electromagnetic energy. This is the POYNTING flux of energy. Next we set the medium into motion of an unrestricted kind. We have now necessarily a convection of the electric and magnetic energy, as well as the POYNTING flux. Thirdly, there must be a similar convection of the kinetic energy, &c., of the translational motion; and fourthly, since the motion of the medium involves the working of ordinary (Newtonian) force, there is associated with the previous a flux of energy due to the activity of the corresponding stress. The question is therefore a complex one, for we have to properly fit together these various fluxes of energy in harmony with the electromagnetic equations. A side issue is the determination of the proper measure of the activity of intrinsic forces, when the medium moves; in another form, it is the determination of the proper meaning of "true current" in MAXWELL'S sense.

§ **4.** The only general principle that we can bring to our assistance in interpreting electromagnetic results relating to activity and flux of energy, is that of the per-

* Perhaps it is best to say as little as possible at present about the connection between matter and ether, but to take the electromagnetic equations in an abstract manner. This will leave us greater freedom for future modifications without contradiction. There are, also, cases in which it is obviously impossible to suppose that matter in bulk carries on with it the ether in bulk which permeates it. Either, then, the mathematical machinery must work between the molecules; or else, we must make such alterations in the equations referring to bulk as will be practically equivalent in effect. For example, the motional magnetic force $\mathbf{VDq}$ of equations (88), (92), (93) may be modified either in $\mathbf{q}$ or in $\mathbf{D}$, by use of a smaller effective velocity $\mathbf{q}$, or by the substitution in $\mathbf{D}$ or $c\mathbf{E}$ of a modified reckoning of c for the effective permittivity.

sistence of energy. But it would be quite inadequate in its older sense referring to integral amounts; the definite localisation by MAXWELL, of electric and magnetic energy, and of its waste, necessitates the similar localisation of sources of energy; and in the consideration of the supply of energy at certain places, combined with the continuous transmission of electrical disturbances, and therefore of the associated energy, the idea of a flux of energy through space, and therefore of the continuity of energy in space and in time, becomes forced upon us as a simple, useful, and necessary principle, which cannot be avoided.

When energy goes from place to place, it traverses the intermediate space. Only by the use of this principle can we safely derive the electromagnetic stress from the equations of the field expressing the two laws of circuitation of the electric and magnetic forces; and this again becomes permissible only by the postulation of the definite localisation of the electric and magnetic energies. But we need not go so far as to assume the objectivity of energy. This is an exceedingly difficult notion, and seems to be rendered inadmissible by the mere fact of the relativity of motion, on which kinetic energy depends. We cannot, therefore, definitely individualise energy in the same way as is done with matter.

If ρ be the density of a quantity whose total amount is invariable, and which can change its distribution continuously, by actual motion from place to place, its equation of continuity is

$$\text{conv } \mathbf{q}\rho = \dot{\rho}, \qquad (1)$$

where $\mathbf{q}$ is its velocity, and $\mathbf{q}\rho$ the flux of ρ. That is, the convergence of the flux of ρ equals the rate of increase of its density. Here ρ may be the density of matter. But it does not appear that we can apply the same method of representation to the flux of energy. We may, indeed, write

$$\text{conv } \mathbf{X} = \dot{\mathrm{T}}, \qquad (2)$$

if $\mathbf{X}$ be the flux of energy from all causes, and T the density of localisable energy. But the assumption $\mathbf{X} = \mathrm{T}\mathbf{q}$ would involve the assumption that T moved about like matter, with a definite velocity. A part of T may, indeed, do this, viz., when it is confined to, and is carried by matter (or ether); thus we may write

$$\text{conv } (\mathbf{q}\mathrm{T} + \mathbf{X}) = \dot{\mathrm{T}}, \qquad (3)$$

where T is energy which is simply carried, whilst $\mathbf{X}$ is the total flux of energy from other sources, and which we cannot symbolise in the form $\mathrm{T}\mathbf{q}$; the energy which comes to us from the Sun, for example, or radiated energy. It is, again, often impossible to carry out the principle in this form, from a want of knowledge of how energy gets to a certain place. This is, for example, particularly evident in the case of gravitational energy, the distribution of which, before it is communicated to matter, increasing its kinetic energy, is highly speculative. If it come from the ether (and where else *can* it come from ?), it should be possible to symbolise this in **X**, if not in **q**T; but in default of a knowledge of its distribution in the ether, we cannot do so, and must therefore turn the equation of continuity into

$$S + \text{conv}\,(\mathbf{q}T + \mathbf{X}) = \dot{T}, \qquad (4)$$

where S indicates the rate of supply of energy per unit volume from the gravitational source, whatever that may be. A similar form is convenient in the case of intrinsic stores of energy, which we have reason to believe are positioned within the element of volume concerned, as when heat gives rise to thermoelectric force. Then S is the activity of the intrinsic sources. Then again, in special applications, T is conveniently divisible into different kinds of energy, potential and kinetic. Energy which is dissipated or wasted comes under the same category, because it may either be regarded as stored, though irrecoverably, or passed out of existence, so far as any immediate useful purpose is performed. Thus we have as a standard practical form of the equation of continuity of energy referred to the unit volume,

$$S + \text{conv}\,\{\mathbf{X} + \mathbf{q}\,(U + T)\} = Q + \dot{U} + \dot{T}, \qquad (5)$$

where S is the energy supply from intrinsic sources, U potential energy and T kinetic energy of localisable kinds, **q** (U + T) its convective flux, Q the rate of waste of energy, and **X** the flux of energy other than convective, *e.g.*, that due to stresses in the medium and representing their activity. In the electromagnetic application we shall see that U and T must split into two kinds, and so must **X**, because there is a flux of energy even when the medium is at rest.

§ **5.** Sometimes we meet with cases in which the flux of energy is either wholly or partly of a circuital character. There is nothing essentially peculiar to electromagnetic problems in this strange and apparently useless result. The electromagnetic instances are paralleled by similar instances in ordinary mechanical science, when a body is in motion and is also strained, especially if it be in rotation. This result is a necessary consequence of our ways of reckoning the activity of forces and of stresses, and serves to still further cast doubt upon the "thinginess" of energy. At the same time, the flux of energy is going on all around us, just as certainly as the flux of matter, and it is impossible to avoid the idea; we should, therefore, make use of it and formularise it whenever and as long as it is found to be useful, in spite of the occasional failure to obtain readily understandable results.

The idea of the flux of energy, apart from the conservation of energy, is by no means a new one. Had gravitational energy been less obscure than it is, it might have found explicit statement long ago. Professor POYNTING* brought the principle

* POYNTING, 'Phil. Trans.,' 1884.

into prominence in 1884, by making use of it to determine the electromagnetic flux of energy. Professor LODGE* gave very distinct and emphatic expression of the principle generally, apart from its electromagnetic aspect, in 1885, and pointed out how much more simple and satisfactory it makes the principle of the conservation of energy become. So it would, indeed, could we only understand gravitational energy ; but in that, and similar respects, it is a matter of faith only. But Professor LODGE attached, I think, too much importance to the identity of energy, as well as to another principle he enunciated, that energy cannot be transferred without being transformed, and conversely ; the transformation being from potential to kinetic energy or conversely. This obviously cannot apply to the convection of energy, which is a true flux of energy ; nor does it seem to apply to cases of wave motion in which the energy, potential and kinetic, of the disturbance, is transferred through a medium unchanged in relative distribution, simply because the disturbance itself travels without change of type ; though it may be that in the unexpressed internal actions associated with the wave propagation there might be found a better application.

It is impossible that the ether can be fully represented, even merely in its transmissive functions, by the electromagnetic equations. Gravity is left out in the cold ; and although it is convenient to ignore this fact, it may be sometimes usefully remembered, even in special electromagnetic work ; for, if a medium have to contain and transmit gravitational energy as well as electromagnetic, the proper system of equations should show this, and, therefore, include the electromagnetic. It seems, therefore, not unlikely that in discussing purely electromagnetic speculations, one may be within a stone's throw of the explanation of gravitation all the time. The consummation would be a really substantial advance in scientific knowledge.

On the Algebra and Analysis of Vectors without Quaternions. Outline of Author's System.

§ **6.** The proper language of vectors is the algebra of vectors. It is, therefore, quite certain that an extensive use of vector-analysis in mathematical physics generally, and in electromagnetism, which is swarming with vectors, in particular, is coming and may be near at hand. It has, in my opinion, been retarded by the want of special treatises on vector analysis adapted for use in mathematical physics, Professor TAIT's well-known profound treatise being, as its name indicates, a treatise on Quaternions. I have not found the HAMILTON-TAIT notation of vector operations convenient, and have employed, for some years past, a simpler system. It is not, however, entirely a question of notation that is concerned. I reject the quaternionic basis of vector-analysis. The anti-quaternionic argument has been recently ably stated by Professor WILLARD GIBBS.† He distinctly separates this from the question

* LODGE, 'Phil. Mag.,' June, 1885, "On the Identity of Energy."

† Professor GIBBS's letters will be found in 'Nature,' vol. 43, p. 511, and vol. 44, p. 79 ; and Professor

of notation, and this may be considered fortunate, for whilst I can fully appreciate and (from practical experience) endorse the anti-quaternionic argument, I am unable to appreciate his notation, and think that of HAMILTON and TAIT is, in some respects, preferable, though very inconvenient in others.

In HAMILTON'S system the quaternion is the fundamental idea, and everything revolves round it. This is exceedingly unfortunate, as it renders the establishment of the algebra of vectors without metaphysics a very difficult matter, and in its application to mathematical analysis there is a tendency for the algebra to get more and more complex as the ideas concerned get simpler, and the quaternionic basis forms a real difficulty of a substantial kind in attempting to work in harmony with ordinary Cartesian methods.

Now, I can confidently recommend, as a really practical working system, the modification I have made. It has many advantages, and not the least amongst them is the fact that the quaternion does not appear in it at all (though it may, without much advantage, be brought in sometimes), and also that the notation is arranged so as to harmonise with Cartesian mathematics. It rests entirely upon a few definitions, and may be regarded (from one point of view) as a systematically abbreviated Cartesian method of investigation, and be understood and practically used by any one accustomed to Cartesians, without any study of the difficult science of Quaternions. It is simply the elements of Quaternions without the quaternions, with the notation simplified to the uttermost, and with the very inconvenient *minus* sign before scalar products done away with.*

TAIT'S in vol. 43, pp. 535, 608. This rather one-sided discussion arose out of Professor TAIT stigmatising Professor GIBBS as "a retarder of quaternionic progress." This may be very true; but Professor GIBBS is anything but a retarder of progress in vector analysis and its application to physics.

* §§ 7, 8, 9 contain an introduction to vector-analysis (without the quaternion), which is sufficient for the purposes of the present paper, and, I may add, for general use in mathematical physics. It is an expansion of that given in my paper "On the Electromagnetic Wave Surface," 'Phil. Mag.,' June, 1885. The algebra and notation are substantially those employed in all my papers, especially in "Electromagnetic Induction and its Propagation," 'The Electrician,' 1885.

Professor GIBBS'S vectorial work is scarcely known, and deserves to be well known. In June, 1888, I received from him a little book of 85 pages, bearing the singular imprint NOT PUBLISHED, Newhaven, 1881–4. It is indeed odd that the author should not have published what he had been at the trouble of having printed. His treatment of the linear vector operator is specially deserving of notice. Although "for the use of students in physics," I am bound to say that I think the work much too condensed for a first introduction to the subject.

In 'The Electrician' for Nov. 13, 1891, p. 27, I commenced a few articles on elementary vector-algebra and analysis, specially meant to explain to readers of my papers how to work vectors. I am given to understand that the earlier ones, on the algebra, were much appreciated; the later ones, however, are found difficult. But the vector-algebra is identically the same in both, and is of quite a rudimentary kind. The difference is, that the later ones are concerned with analysis, with varying vectors; it is the same as the difference between common algebra and differential calculus. The difficulty, whether real or not, does not indicate any difficulty in the vector-algebra. I mention this on account of the great prejudice which exists against vector-algebra.

§ **7.** Quantities being divided into scalars and vectors, I denote the scalars, as usual, by ordinary letters, and put the vectors in the plain black type, known, I believe, as Clarendon type, rejecting MAXWELL'S German letters on account of their being hard to read. A special type is certainly not essential, but it facilitates the reading of printed complex vector investigations to be able to see at a glance which quantities are scalars and which are vectors, and eases the strain on the memory. But in MS. work there is no occasion for specially formed letters.

Thus **A** stands for a vector. The tensor of a vector may be denoted by the same letter plain; thus A is the tensor of **A**. (In MS. the tensor is A_0.) Its rectangular scalar components are A_1, A_2, A_3. A unit vector parallel to **A** may be denoted by $\mathbf{A}_1$, so that $\mathbf{A} = A\mathbf{A}_1$. But little things of this sort are very much matters of taste. What is important is to avoid as far as possible the use of letter prefixes, which, when they come two (or even three) together, as in Quaternions, are very confusing.

The scalar product of a pair of vectors **A** and **B** is denoted by **AB**, and is defined to be

$$\mathbf{AB} = A_1B_1 + A_2B_2 + A_3B_3 = AB\cos\widehat{\mathbf{AB}} = \mathbf{BA}. \qquad (6)$$

The addition of vectors being as in the polygon of displacements, or velocities, or forces; *i.e.*, such that the vector length of any closed circuit is zero; either of the vectors **A** and **B** may be split into the sum of any number of others, and the multiplication of the two sums to form **AB** is done as in common algebra; thus

$$\left.\begin{aligned}(\mathbf{a}+\mathbf{b})(\mathbf{c}+\mathbf{d}) &= \mathbf{ac}+\mathbf{ad}+\mathbf{bc}+\mathbf{bd},\\ &= \mathbf{ca}+\mathbf{da}+\mathbf{cb}+\mathbf{db}.\end{aligned}\right\} \qquad (7)$$

If **N** be a unit vector, **NN** or $\mathbf{N}^2 = 1$; similarly $\mathbf{A}^2 = A^2$ for any vector.

The reciprocal of a vector **A** has the same direction; its tensor is the reciprocal of the tensor of **A**. Thus

$$\mathbf{AA}^{-1} = \frac{\mathbf{A}}{\mathbf{A}} = 1;$$

and

$$\mathbf{AB}^{-1} = \mathbf{B}^{-1}\mathbf{A} = \frac{\mathbf{A}}{\mathbf{B}} = \frac{A}{B}\cos\widehat{\mathbf{AB}}. \qquad (8)$$

The vector product of a pair of vectors is denoted by **VAB**, and is defined to be the vector whose tensor is $AB\sin\widehat{\mathbf{AB}}$, and whose direction is perpendicular to the plane of **A** and **B**. Or

$$\mathrm{V}\mathbf{AB} = \mathbf{i}(A_2B_3 - A_3B_2) + \mathbf{j}(A_3B_1 - A_1B_3) + \mathbf{k}(A_1B_2 - A_2B_1) = -\mathrm{V}\mathbf{BA}, \qquad (9)$$

where **i**, **j**, **k**, are any three mutually rectangular unit vectors. The tensor of **VAB** is $\mathrm{V}_0\mathbf{AB}$; or

$$\mathrm{V}_0\mathbf{AB} = AB\sin\widehat{\mathbf{AB}}. \qquad (10)$$

Its components are **iVAB, jVAB, kVAB.**

In accordance with the definitions of the scalar and vector products, we have

$$\left.\begin{array}{lll} \mathbf{i}^2 = 1, & \mathbf{j}^2 = 1, & \mathbf{k}^2 = 1\,; \\ \mathbf{ij} = 0, & \mathbf{jk} = 0, & \mathbf{ki} = 0\,; \\ \mathrm{V}\mathbf{ij} = \mathbf{k}, & \mathrm{V}\mathbf{jk} = \mathbf{i}, & \mathrm{V}\mathbf{ki} = \mathbf{j}\,; \end{array}\right\} \qquad (11)$$

and from these we prove at once that

$$\mathrm{V}\,(\mathbf{a} + \mathbf{b})\,(\mathbf{c} + \mathbf{d}) = \mathrm{V}\mathbf{ac} + \mathrm{V}\mathbf{ad} + \mathrm{V}\mathbf{bc} + \mathrm{V}\mathbf{bd},$$

and so on, for any number of component vectors. The order of the letters in each product has to be preserved, since $\mathrm{V}\mathbf{ab} = -\mathrm{V}\mathbf{ba}$.

Two very useful formulæ of transformation are

$$\begin{aligned} \mathbf{A}\mathrm{V}\mathbf{BC} &= \mathbf{B}\mathrm{V}\mathbf{CA} = \mathbf{C}\mathrm{V}\mathbf{AB} \\ &= \mathrm{A}_1\,(\mathrm{B}_2\mathrm{C}_3 - \mathrm{B}_3\mathrm{C}_2) + \mathrm{A}_2\,(\mathrm{B}_3\mathrm{C}_1 - \mathrm{B}_1\mathrm{C}_3) + \mathrm{A}_3\,(\mathrm{B}_1\mathrm{C}_2 - \mathrm{B}_2\mathrm{C}_1)\,; \end{aligned} \qquad (12)$$

and

$$\left.\begin{aligned} \mathrm{V}\mathbf{A}\mathrm{V}\mathbf{BC} &= \mathbf{B}.\mathbf{CA} - \mathbf{C}.\mathbf{AB}, \\ &= \mathbf{B}(\mathbf{CA}) - \mathbf{C}(\mathbf{AB}). \end{aligned}\right\} \qquad (13)$$

Here the dots, or the brackets in the alternative notation, merely act as separators, separating the scalar products **CA** and **AB** from the vectors they multiply. A space would be equivalent, but would be obviously unpractical.

As $\frac{\mathbf{A}}{\mathbf{B}}$ is a scalar product, so in harmony therewith, there is the vector product $\mathrm{V}\frac{\mathbf{A}}{\mathbf{B}}$. Since $\mathrm{V}\mathbf{AB} = -\mathrm{V}\mathbf{BA}$, it is now necessary to make a convention as to whether the denominator comes first or last in $\mathrm{V}\frac{\mathbf{A}}{\mathbf{B}}$. Say therefore, $\mathrm{V}\mathbf{AB}^{-1}$. Its tensor is

$$\mathrm{V}_0\frac{\mathbf{A}}{\mathbf{B}} = \frac{\mathrm{A}}{\mathrm{B}}\sin \widehat{\mathbf{AB}}. \qquad (14)$$

§ **8.** Differentiation of vectors, and of scalar and vector functions of vectors with respect to scalar variables is done as usual. Thus,

$$\left.\begin{aligned} \dot{\mathbf{A}} &= \mathbf{i}\dot{\mathrm{A}}_1 + \mathbf{j}\dot{\mathrm{A}}_2 + \mathbf{k}\dot{\mathrm{A}}_3. \\ \frac{d}{dt}\mathbf{AB} &= \mathbf{A}\dot{\mathbf{B}} + \mathbf{B}\dot{\mathbf{A}}. \\ \frac{d}{dt}\mathbf{A}\mathrm{V}\mathbf{BC} &= \dot{\mathbf{A}}\mathrm{V}\mathbf{BC} + \mathbf{A}\mathrm{V}\dot{\mathbf{B}}\mathbf{C} + \mathbf{A}\mathrm{V}\mathbf{B}\dot{\mathbf{C}}. \end{aligned}\right\} \qquad (15)$$

The same applies with complex scalar differentiators, *e.g.*, with the differentiator

$$\frac{\partial}{\partial t} = \frac{d}{dt} + \mathbf{q}\nabla,$$

used when a moving particle is followed, **q** being its velocity. Thus,

$$\frac{\partial}{\partial t}\mathbf{AB} = \mathbf{A}\frac{\partial \mathbf{B}}{\partial t} + \mathbf{B}\frac{\partial \mathbf{A}}{\partial t}$$
$$= \mathbf{A\dot{B}} + \mathbf{B\dot{A}} + \mathbf{A.q}\nabla\mathbf{.B} + \mathbf{B.q}\nabla\mathbf{.A}. \quad \ldots \quad (16)$$

Here $\mathbf{q}\nabla$ is a scalar differentiator given by

$$\mathbf{q}\nabla = q_1\frac{d}{dx} + q_2\frac{d}{dy} + q_3\frac{d}{dz}, \quad \ldots \quad (17)$$

so that $\mathbf{A.q}\nabla\mathbf{.B}$ is the scalar product of **A** and the vector $\mathbf{q}\nabla\mathbf{.B}$; the dots here again act essentially as separators. Otherwise, we may write it $\mathbf{A}\,(\mathbf{q}\nabla)\,\mathbf{B}$.

The fictitious vector ∇ given by

$$\nabla = \mathbf{i}\nabla_1 + \mathbf{j}\nabla_2 + \mathbf{k}\nabla_3 = \mathbf{i}\frac{d}{dx} + \mathbf{j}\frac{d}{dy} + \mathbf{k}\frac{d}{dz} \quad \ldots \quad (18)$$

is *very* important. Physical mathematics is very largely the mathematics of ∇. The name Nabla seems, therefore, ludicrously inefficient. In virtue of **i**, **j**, **k**, the operator ∇ behaves as a vector. It also, of course, differentiates what follows it.

Acting on a scalar P, the result is the vector

$$\nabla \mathrm{P} = \mathbf{i}\nabla_1\mathrm{P} + \mathbf{j}\nabla_2\mathrm{P} + \mathbf{k}\nabla_3\mathrm{P}, \quad \ldots \quad (19)$$

the vector rate of increase of P with length.

If it act on a vector **A**, there is first the scalar product

$$\nabla\mathbf{A} = \nabla_1\mathrm{A}_1 + \nabla_2\mathrm{A}_2 + \nabla_3\mathrm{A}_3 = \mathrm{div}\,\mathbf{A}, \quad \ldots \quad (20)$$

or the divergence of **A**. Regarding a vector as a flux, the divergence of a vector is the amount leaving the unit volume.

The vector product $\mathrm{V}\nabla\mathbf{A}$ is

$$\mathrm{V}\nabla\mathbf{A} = \mathbf{i}\,(\nabla_2\mathrm{A}_3 - \nabla_3\mathrm{A}_2) + \mathbf{j}\,(\nabla_3\mathrm{A}_1 - \nabla_1\mathrm{A}_3) + \mathbf{k}\,(\nabla_1\mathrm{A}_2 - \nabla_2\mathrm{A}_1),$$
$$= \mathrm{curl}\,\mathbf{A}. \quad \ldots \quad (21)$$

The line-integral of **A** round a unit area equals the component of the curl of **A** perpendicular to the area.

We may also have the scalar and vector products $\mathbf{N}\nabla$ and $\mathrm{V}\mathbf{N}\nabla$, where the vector **N** is not differentiated. These operators, of course, require a function to follow them on which to operate; the previous $\mathbf{q}\nabla\mathbf{.A}$ of (16) illustrates.

The Laplacean operator is the scalar product ∇^2 or $\nabla\nabla$; or

$$\nabla^2 = \nabla_1^2 + \nabla_2^2 + \nabla_3^2; \quad \ldots \quad (22)$$

and an example of (13) is

$$\mathrm{V}\nabla\mathrm{V}\nabla\mathbf{A} = \nabla.\nabla\mathbf{A} - \nabla^2\mathbf{A},$$

or

$$\text{curl}^2\mathbf{A} = \nabla \,\text{div}\, \mathbf{A} - \nabla^2\mathbf{A}, \quad \quad (23)$$

which is an important formula.

Other important formulæ are the next three.

$$\text{div}\, \mathrm{P}\mathbf{A} = \mathrm{P}\, \text{div}\, \mathbf{A} + \mathbf{A}\nabla.\mathrm{P}, \quad \quad (24)$$

P being scalar. Here note that $\mathbf{A}\nabla.\mathrm{P}$ and $\mathbf{A}\nabla\mathrm{P}$ (the latter being the scalar product of $\mathbf{A}$ and $\nabla\mathrm{P}$) are identical. This is not true when for P we substitute a vector. Also

$$\text{div}\, \mathrm{V}\mathbf{A}\mathbf{B} = \mathbf{B}\, \text{curl}\, \mathbf{A} - \mathbf{A}\, \text{curl}\, \mathbf{B}; \quad \quad (25)$$

which is an example of (12), noting that both $\mathbf{A}$ and $\mathbf{B}$ have to be differentiated. And

$$\text{curl}\, \mathrm{V}\mathbf{A}\mathbf{B} = \mathbf{B}\nabla.\mathbf{A} + \mathbf{A}\, \text{div}\, \mathbf{B} - \mathbf{A}\nabla.\mathbf{B} - \mathbf{B}\, \text{div}\, \mathbf{A}, \quad \quad (26)$$

This is an example of (13).

§ **9.** When one vector $\mathbf{D}$ is a *linear* function of another vector $\mathbf{E}$, that is, connected by equations of the form

$$\left.\begin{aligned} \mathrm{D}_1 &= c_{11}\mathrm{E}_1 + c_{12}\mathrm{E}_2 + c_{13}\mathrm{E}_3, \\ \mathrm{D}_2 &= c_{21}\mathrm{E}_1 + c_{22}\mathrm{E}_2 + c_{23}\mathrm{E}_3, \\ \mathrm{D}_3 &= c_{31}\mathrm{E}_1 + c_{32}\mathrm{E}_2 + c_{33}\mathrm{E}_3, \end{aligned}\right\}, \quad \quad (27)$$

in terms of the rectangular components, we denote this simply by

$$\mathbf{D} = c\mathbf{E}, \quad \quad (28)$$

where c is the linear operator. The conjugate function is given by

$$\mathbf{D}' = c'\mathbf{E}, \quad \quad (29)$$

where $\mathbf{D}'$ is got from $\mathbf{D}$ by exchanging c_{12} and c_{21}, &c. Should the nine coefficients reduce to six by $c_{12} = c_{21}$, &c., $\mathbf{D}$ and $\mathbf{D}'$ are identical, or $\mathbf{D}$ is a self-conjugate or symmetrical linear function of $\mathbf{E}$.

But, in general, it is the sum of $\mathbf{D}$ and $\mathbf{D}'$ which is a symmetrical function of $\mathbf{E}$, and the difference is a simple vector-product. Thus

$$\left.\begin{aligned} \mathbf{D} &= c_0\mathbf{E} + \mathrm{V}\boldsymbol{\epsilon}\mathbf{E}, \\ \mathbf{D}' &= c_0\mathbf{E} - \mathrm{V}\boldsymbol{\epsilon}\mathbf{E}, \end{aligned}\right\} \quad \quad (30)$$

where c_0 is a self-conjugate operator, and $\boldsymbol{\epsilon}$ is the vector given by

$$\boldsymbol{\epsilon} = \mathbf{i}\frac{c_{32} - c_{23}}{2} + \mathbf{j}\frac{c_{13} - c_{31}}{2} + \mathbf{k}\frac{c_{21} - c_{12}}{2}. \qquad (31)$$

The important characteristic of a self-conjugate operator is

$$\left.\begin{aligned} \mathbf{E}_1\mathbf{D}_2 &= \mathbf{E}_2\mathbf{D}_1, \\ \text{or} \quad \mathbf{E}_1 c_0 \mathbf{E}_2 &= \mathbf{E}_2 c_0 \mathbf{E}_1, \end{aligned}\right\} \qquad (32)$$

where $\mathbf{E}_1$ and $\mathbf{E}_2$ are any two **E**'s, and $\mathbf{D}_1$, $\mathbf{D}_2$ the corresponding **D**'s. But when there is not symmetry, the corresponding property is

$$\left.\begin{aligned} \mathbf{E}_1\mathbf{D}_2 &= \mathbf{E}_2\mathbf{D}'_1, \\ \text{or} \quad \mathbf{E}_1 c \mathbf{E}_2 &= \mathbf{E}_2 c' \mathbf{E}_1. \end{aligned}\right\} \qquad (33)$$

Of these operators we have three or four in electromagnetism connecting forces and fluxes, and three more connected with the stresses and strains concerned. As it seems impossible to avoid the consideration of rotational stresses in electromagnetism, and these are not usually considered in works on elasticity, it will be desirable to briefly note their peculiarities here, rather than later on.

On Stresses, irrotational and rotational, and their Activities.

§ **10.** Let $\mathbf{P}_N$ be the vector stress on the **N** plane, or the plane whose unit normal is **N**. It is a linear function of **N**. This will fully specify the stress on any plane. Thus, if $\mathbf{P}_1$, $\mathbf{P}_2$, $\mathbf{P}_3$ are the stresses on the **i**, **j**, **k** planes, we shall have

$$\left.\begin{aligned} \mathbf{P}_1 &= \mathbf{i}P_{11} + \mathbf{j}P_{12} + \mathbf{k}P_{13}, \\ \mathbf{P}_2 &= \mathbf{i}P_{21} + \mathbf{j}P_{22} + \mathbf{k}P_{23}, \\ \mathbf{P}_3 &= \mathbf{i}P_{31} + \mathbf{j}P_{32} + \mathbf{k}P_{33}. \end{aligned}\right\} \qquad (34)$$

Let, also, $\mathbf{Q}_N$ be the conjugate stress; then, similarly,

$$\left.\begin{aligned} \mathbf{Q}_1 &= \mathbf{i}P_{11} + \mathbf{j}P_{21} + \mathbf{k}P_{31}, \\ \mathbf{Q}_2 &= \mathbf{i}P_{12} + \mathbf{j}P_{22} + \mathbf{k}P_{32}, \\ \mathbf{Q}_3 &= \mathbf{i}P_{13} + \mathbf{j}P_{23} + \mathbf{k}P_{33}. \end{aligned}\right\} \qquad (35)$$

Half the sum of the stresses $\mathbf{P}_N$ and $\mathbf{Q}_N$ is an ordinary irrotational stress; so that

$$\left.\begin{aligned} \mathbf{P}_N &= \phi_0\mathbf{N} + \mathrm{V}\boldsymbol{\epsilon}\mathbf{N}, \\ \mathbf{Q}_N &= \phi_0\mathbf{N} - \mathrm{V}\boldsymbol{\epsilon}\mathbf{N}, \end{aligned}\right\} \qquad (36)$$

where ϕ_0 is self-conjugate, and

$$2\boldsymbol{\epsilon} = \mathbf{i}\,(P_{23} - P_{32}) + \mathbf{j}\,(P_{31} - P_{13}) + \mathbf{k}\,(P_{12} - P_{21}). \qquad (37)$$

Here $2\boldsymbol{\epsilon}$ is the torque per unit volume arising from the stress **P**.

The translational force, **F**, per unit volume is (by inspection of a unit cube)

$$\mathbf{F} = \nabla_1 \mathbf{P}_1 + \nabla_2 \mathbf{P}_2 + \nabla_3 \mathbf{P}_3 \qquad (38)$$

$$= \mathbf{i}\,\mathrm{div}\,\mathbf{Q}_1 + \mathbf{j}\,\mathrm{div}\,\mathbf{Q}_2 + \mathbf{k}\,\mathrm{div}\,\mathbf{Q}_3; \qquad (39)$$

or, in terms of the self-conjugate stress and the torque,

$$\mathbf{F} = (\mathbf{i}\,\mathrm{div}\,\phi_0\mathbf{i} + \mathbf{j}\,\mathrm{div}\,\phi_0\mathbf{j} + \mathbf{k}\,\mathrm{div}\,\phi_0\mathbf{k}) - \mathrm{curl}\,\boldsymbol{\epsilon}, \qquad (40)$$

where $-\mathrm{curl}\,\boldsymbol{\epsilon}$ is the translational force due to the rotational stress alone, as in Sir W. THOMSON's latest theory of the mechanics of an "ether."*

Next, let **N** be the unit normal drawn outward from any closed surface. Then

$$\Sigma \mathbf{P}_N = \Sigma \mathbf{F}, \qquad (41)$$

where the left summation extends over the surface and the right summation throughout the enclosed region. For

$$\mathbf{P}_N = N_1\mathbf{P}_1 + N_2\mathbf{P}_2 + N_3\mathbf{P}_3$$
$$= \mathbf{i}.\mathbf{N}\mathbf{Q}_1 + \mathbf{j}.\mathbf{N}\mathbf{Q}_2 + \mathbf{k}.\mathbf{N}\mathbf{Q}_3; \qquad (42)$$

so the well-known theorem of divergence gives immediately, by (39),

$$\Sigma \mathbf{P}_N = \Sigma\,(\mathbf{i}\,\mathrm{div}\,\mathbf{Q}_1 + \mathbf{j}\,\mathrm{div}\,\mathbf{Q}_2 + \mathbf{k}\,\mathrm{div}\,\mathbf{Q}_3) = \Sigma \mathbf{F}. \qquad (43)$$

Next, as regards the equivalence of rotational effect of the surface-stress to that of the internal forces and torques. Let **r** be the vector distance from any fixed origin. Then **VrF** is the vector moment of a force, **F**, at the end of the arm **r**. Another (not so immediate) application of the divergence theorem gives

$$\Sigma \mathrm{V}\mathbf{r}\mathbf{P}_N = \Sigma \mathrm{V}\mathbf{r}\mathbf{F} + \Sigma 2\boldsymbol{\epsilon}, \qquad (44)$$

Thus, any distribution of stress, whether rotational or irrotational, may be regarded as in equilibrium. Given any stress in a body, terminating at its boundary, the body will be in equilibrium both as regards translation and rotation. Of course, the boundary discontinuity in the stress has to be reckoned as the equivalent of internal divergence in the appropriate manner. Or, more simply, let the stress fall off continuously from the finite internal stress to zero through a thin surface-layer. We

* 'Mathematical and Physical Papers,' vol. 3, Art. 99, p. 436.

then have a distribution of forces and torques in the surface-layer which equilibrate the internal forces and torques.

To illustrate; we know that MAXWELL arrived at a peculiar stress, compounded of a tension parallel to a certain direction, and an equal lateral pressure, which would account for the mechanical actions apparent between electrified bodies, and endeavoured similarly to determine the stress in the interior of a magnetised body to harmonise with the similar external magnetic stress of the simple type mentioned. This stress in a magnetised body I believe to be thoroughly erroneous; nevertheless, so far as accounting for the forcive on a magnetised body is concerned, it will, when properly carried out with due attention to surface-discontinuity, answer perfectly well, not because it *is* the stress, but because *any* stress would do the same, the only essential feature concerned being the external stress in the air.

Here we may also note the very powerful nature of the stress-function, considered merely as a mathematical engine, apart from physical reality. For example, we may account for the forcive on a magnet in many ways, of which the two most prominent are by means of forces on imaginary magnetic matter, and by forces on imaginary electric currents, in the magnet and on its surface. To prove the equivalence of these two methods (and the many others) involves very complex surface- and volume-integrations and transformations in the general case, which may be all avoided by the use of the stress-function instead of the forces.

§ **11.** Next as regards the activity of the stress $\mathbf{P}_N$ and the equivalent translational, distortional, and rotational activities. The activity of $\mathbf{P}_N$ is $\mathbf{P}_N\mathbf{q}$ per unit area, if $\mathbf{q}$ be the velocity. Here

$$\mathbf{P}_N\mathbf{q} = q_1.\mathbf{N}\mathbf{Q}_1 + q_2.\mathbf{N}\mathbf{Q}_2 + q_3.\mathbf{N}\mathbf{Q}_3, \qquad (45)$$

by (42); or, re-arranging,

$$\mathbf{P}_N\mathbf{q} = \mathbf{N}(q_1\mathbf{Q}_1 + q_2\mathbf{Q}_2 + q_3\mathbf{Q}_3) = \mathbf{N}\Sigma\mathbf{Q}q,$$

$$= \mathbf{N}q\mathbf{Q}_q, \qquad (46)$$

where $\mathbf{Q}_q$ is the conjugate stress on the $\mathbf{q}$ plane. That is, $q\mathbf{Q}_q$ or $\Sigma\mathbf{Q}q$ is the negative of the vector flux of energy expressing the stress-activity. For we choose P_{NN} so as to mean a pull when it is positive, and when the stress $\mathbf{P}_N$ works in the same sense with $\mathbf{q}$ energy is transferred against the motion, to the matter which is pulled.

The convergence of the energy-flux, or the divergence of $q\mathbf{Q}_q$, is therefore the activity per unit volume. Thus

$$\text{div}(\mathbf{Q}_1q_1 + \mathbf{Q}_2q_2 + \mathbf{Q}_3q_3) = \mathbf{q}(\mathbf{i}\,\text{div}\,\mathbf{Q}_1 + \mathbf{j}\,\text{div}\,\mathbf{Q}_2 + \mathbf{k}\,\text{div}\,\mathbf{Q}_3) + (\mathbf{Q}_1\nabla q_1 + \mathbf{Q}_2\nabla q_2 + \mathbf{Q}_3\nabla q_3)\,. \qquad (47)$$

$$= \mathbf{q}(\nabla_1\mathbf{P}_1 + \nabla_2\mathbf{P}_2 + \nabla_3\mathbf{P}_3) + \mathbf{P}_1\nabla_1\mathbf{q} + \mathbf{P}_2\nabla_2\mathbf{q} + \mathbf{P}_3\nabla_3\mathbf{q} \qquad (48)$$

where the first form (47) is generally most useful. Or

$$\text{div}\,\Sigma \mathbf{Q}q = \mathbf{F}\mathbf{q} + \Sigma \mathbf{Q}\nabla q\,; \quad \ldots \quad (49)$$

where the first term on the right is the translational activity, and the rest is the sum of the distortional and rotational activities. To separate the latter introduce the strain velocity vectors (analogous to $\mathbf{P}_1$, $\mathbf{P}_2$, $\mathbf{P}_3$)

$$\mathbf{p}_1 = \tfrac{1}{2}(\nabla q_1 + \nabla_1 \mathbf{q}), \qquad \mathbf{p}_2 = \tfrac{1}{2}(\nabla q_2 + \nabla_2 \mathbf{q}), \qquad \mathbf{p}_3 = \tfrac{1}{2}(\nabla q_3 + \nabla_3 \mathbf{q})\,; \quad \ldots \quad (50)$$

and generally

$$\mathbf{p}_N = \tfrac{1}{2}(\nabla.\mathbf{q}\mathbf{N} + \mathbf{N}\nabla.\mathbf{q}). \quad \ldots \quad (51)$$

Using these we obtain

$$\begin{aligned} \Sigma \mathbf{Q}\nabla \mathbf{q} &= \mathbf{Q}_1\mathbf{p}_1 + \mathbf{Q}_2\mathbf{p}_2 + \mathbf{Q}_3\mathbf{p}_3 + \mathbf{Q}_1 \frac{\nabla q_1 - \nabla_1 \mathbf{q}}{2} + \mathbf{Q}_2 \frac{\nabla q_2 - \nabla_2 \mathbf{q}}{2} + \mathbf{Q}_3 \frac{\nabla q_3 - \nabla_3 \mathbf{q}}{2} \\ &= \Sigma \mathbf{Q}\mathbf{p} + \tfrac{1}{2}\mathbf{Q}_1 \mathrm{V}\mathbf{i}\,\text{curl}\,\mathbf{q} + \tfrac{1}{2}\mathbf{Q}_2 \mathrm{V}\mathbf{j}\,\text{curl}\,\mathbf{q} + \tfrac{1}{2}\mathbf{Q}_3 \mathrm{V}\mathbf{k}\,\text{curl}\,\mathbf{q} \\ &= \Sigma \mathbf{Q}\mathbf{p} + \boldsymbol{\epsilon}\,\text{curl}\,\mathbf{q}. \quad \ldots \quad (52) \end{aligned}$$

Thus $\Sigma \mathbf{Q}\mathbf{p}$ is the distortional activity and $\boldsymbol{\epsilon}$ curl $\mathbf{q}$ the rotational activity. But since the distortion and the rotation are quite independent, we may put $\Sigma \mathbf{P}\mathbf{p}$ for the distortional activity; or else use the self-conjugate stress, and write it $\frac{1}{2}\Sigma(\mathbf{P} + \mathbf{Q})\,\mathbf{p}$.

§ **12.** In an ordinary "elastic solid," when isotropic, there is elastic resistance to compression and to distortion. We may also imaginably have elastic resistance to translation and to rotation; nor is there, so far as the mathematics is concerned, any reason for excluding dissipative resistance to translation, distortion, and rotation; and kinetic energy may be associated with all three as well, instead of with the translation alone, as in the ordinary elastic solid.

Considering only three elastic moduli, we have the old k and n of THOMSON and TAIT (resistance to compression and rigidity), and a new coefficient, say n_1, such that

$$\boldsymbol{\epsilon} = n_1\,\text{curl}\,\mathbf{D}, \quad \ldots \quad (53)$$

if $\mathbf{D}$ be the displacement and $2\boldsymbol{\epsilon}$ the torque, as before.

The stress on the $\mathbf{i}$ plane (any plane) is

$$\begin{aligned} \mathbf{P}_1 &= n(\nabla \mathrm{D}_1 + \nabla_1 \mathbf{D}) + \mathbf{i}(k - \tfrac{2}{3}n)\,\text{div}\,\mathbf{D} + n_1 \mathrm{V}\,\text{curl}\,\mathbf{D}.\mathbf{i} \\ &= (n + n_1)\nabla_1 \mathbf{D} + (n - n_1)\nabla \mathrm{D}_1 + (k - \tfrac{2}{3}n)\,\mathbf{i}\,\text{div}\,\mathbf{D}\,; \quad \ldots \quad (54) \end{aligned}$$

and its conjugate is

$$\begin{aligned} \mathbf{Q}_1 &= n(\nabla \mathrm{D}_1 + \nabla_1 \mathbf{D}) + \mathbf{i}(k - \tfrac{2}{3}n)\,\text{div}\,\mathbf{D} - n_1(\nabla_1 \mathbf{D} - \nabla \mathrm{D}_1) \\ &= (n - n_1)\nabla_1 \mathbf{D} + (n + n_1)\nabla \mathrm{D}_1 + \mathbf{i}(k - \tfrac{2}{3}n)\,\text{div}\,\mathbf{D}\,; \quad \ldots \quad (55) \end{aligned}$$

from which

$$\mathrm{F}_1 = \text{div}\,\mathbf{Q}_1 = (n - n_1 + k - \tfrac{2}{3}n)\nabla_1\,\text{div}\,\mathbf{D} + (n + n_1)\nabla^2 \mathrm{D}_1 \quad \ldots \quad (56)$$

is the $\mathbf{i}$ component of the translational force; the complete force $\mathbf{F}$ is therefore

$$\mathbf{F} = (n + n_1)\, \nabla^2\mathbf{D} + (k + \tfrac{1}{3}n - n_1)\, \nabla \operatorname{div} \mathbf{D}; \quad . \quad . \quad . \quad (57)$$

or, in another form, if

$$\mathrm{P} = -k \operatorname{div} \mathbf{D},$$

P being the isotropic pressure,

$$\mathbf{F} = -\nabla\mathrm{P} + n\,(\nabla^2\mathbf{D} + \tfrac{1}{3}\nabla \operatorname{div} \mathbf{D}) - n_1 \operatorname{curl}^2 \mathbf{D}, \quad . \quad . \quad . \quad (58)$$

remembering (23) and (53).

We see that in (57) the term involving div **D** may vanish in a compressible solid by the relation $n_1 = k + \frac{1}{3}n$; this makes

$$n + n_1 = k + \tfrac{4}{3}n, \qquad n_1 - n = k - \tfrac{2}{3}n, \quad . \quad . \quad . \quad (59)$$

which are the moduli, longitudinal and lateral, of a simple longitudinal strain; that is, multiplied by the extension, they give the longitudinal traction, and the lateral traction required to prevent lateral contraction.

The activity per unit volume, other than translational, is

$$\begin{aligned}\Sigma\mathbf{Q}\nabla q &= (n - n_1)\big(\nabla_1\mathbf{D}.\nabla q_1 + \nabla_2\mathbf{D}.\nabla q_2 + \nabla_3\mathbf{D}.\nabla q_3\big)\\ &\quad + (n + n_1)\big(\nabla\mathrm{D}_1.\nabla q_1 + \nabla\mathrm{D}_2.\nabla q_2 + \nabla\mathrm{D}_3.\nabla q_3\big)\\ &\quad + (k - \tfrac{2}{3}n) \operatorname{div} \mathbf{D} \operatorname{div} \mathbf{q}\\ &= n\big(\nabla_1\mathbf{D}.\nabla q_1 + \nabla_2\mathbf{D}.\nabla q_2 + \nabla_3\mathbf{D}.\nabla q_3 + \nabla\mathrm{D}_1\nabla q_1 + \nabla\mathrm{D}_2\nabla q_2 + \nabla\mathrm{D}_3\nabla q_3\big)\\ &\quad + (k - \tfrac{2}{3}n) \operatorname{div} \mathbf{D} \operatorname{div} \mathbf{q} + n_1 \operatorname{curl} \mathbf{D} \operatorname{curl} \mathbf{q}; \quad . \quad . \quad . \quad (60)\end{aligned}$$

or, which is the same,

$$\begin{aligned}\Sigma\mathbf{Q}\nabla q &= \frac{d}{dt}\Big[\tfrac{1}{2}k\,(\operatorname{div} \mathbf{D})^2 + \tfrac{1}{2}n_1\,(\operatorname{curl} \mathbf{D})^2\\ &\quad + \tfrac{1}{2}n\{(\nabla\mathrm{D}_1)^2 + (\nabla\mathrm{D}_2)^2 + (\nabla\mathrm{D}_3)^2 + \nabla\mathrm{D}_1.\nabla_1\mathbf{D} + \nabla\mathrm{D}_2.\nabla_2\mathbf{D} + \nabla\mathrm{D}_3.\nabla_3\mathbf{D}\} - \tfrac{1}{3}n\,(\operatorname{div} \mathbf{D})^2\Big], \quad . \quad (61)\end{aligned}$$

where the quantity in square brackets is the potential energy of an *infinitesimal* distortion and rotation. The italicised reservation appears to be necessary, as we shall see from the equation of activity later, that the convection of the potential energy destroys the completeness of the statement

$$\Sigma\mathbf{Q}\nabla q = \dot{\mathrm{U}},$$

if U be the potential energy.

In an elastic solid of the ordinary kind, with $n_1 = 0$, we have

$$\left.\begin{aligned}\mathbf{P}_{\mathrm{N}} &= n\big(2 \operatorname{curl} \mathrm{V}\mathbf{D}\mathbf{N} + \mathrm{V}\mathbf{N} \operatorname{curl} \mathbf{D}\big),\\ \mathbf{F} &= -n \operatorname{curl}^2 \mathbf{D}.\end{aligned}\right\} \quad . \quad . \quad . \quad (62)$$

In the case of a medium in which n is zero but n_1 finite (Sir W. THOMSON'S rotational ether),

$$\left.\begin{aligned}\mathbf{P}_{\mathrm{N}} &= n_1 \mathrm{V}\,\mathrm{curl}\,\mathbf{D}.\mathbf{N},\\ \mathbf{F} &= -\,n_1\,\mathrm{curl}^2\,\mathbf{D}.\end{aligned}\right\} \qquad (63)$$

Thirdly, if we have both $k = -\frac{4}{3}n$ and $n = n_1$, then

$$\left.\begin{aligned}\mathbf{P}_{\mathrm{N}} &= 2n\,\mathrm{curl}\,\mathrm{V}\mathbf{D}\mathbf{N},\\ \mathbf{F} &= -\,2n\,\mathrm{curl}^2\,\mathbf{D},\end{aligned}\right\} (\mathbf{\epsilon} = n\,\mathrm{curl}\,\mathbf{D}), \qquad (64)$$

i.e., the sums of the previous two stresses and forces.

§ **13.** As already observed, the vector flux of energy, due to the stress, is

$$-\Sigma \mathbf{Q}q = -\,\mathbf{Q}_q q = -\,(\mathbf{Q}_1 q_1 + \mathbf{Q}_2 q_2 + \mathbf{Q}_3 q_3). \qquad (65)$$

Besides this, there is the flux of energy

$$\mathbf{q}\,(\mathrm{U} + \mathrm{T})$$

by convection, where U is potential and T kinetic energy. Therefore,

$$\mathbf{W} = \mathbf{q}\,(\mathrm{U} + \mathrm{T}) - \Sigma \mathbf{Q}q \qquad (66)$$

represents the complete energy flux, so far as the stress and motion are concerned. Its convergence increases the potential energy, the kinetic energy, or is dissipated. But if there be an impressed translational force $\mathbf{f}$ its activity is $\mathbf{fq}$. This supply of energy is independent of the convergence of $\mathbf{W}$. Hence

$$\mathbf{fq} + \mathrm{Q} + \dot{\mathrm{U}} = \dot{\mathrm{T}} + \mathrm{div}\left[\mathbf{q}\,(\mathrm{U} + \mathrm{T}) - \Sigma \mathbf{Q}q\right] \qquad (67)$$

is the equation of activity.

But this splits into two parts at least. For (67) is the same as

$$(\mathbf{f} + \mathbf{F})\,\mathbf{q} + \Sigma \mathbf{Q}\nabla q = \mathrm{Q} + \dot{\mathrm{U}} + \dot{\mathrm{T}} + \mathrm{div}\,\mathbf{q}\,(\mathrm{U} + \mathrm{T}), \qquad (68)$$

and the translational portion may be removed altogether. That is,

$$(\mathbf{f} + \mathbf{F})\,\mathbf{q} = \mathrm{Q}_0 + \dot{\mathrm{U}}_0 + \dot{\mathrm{T}}_0 + \mathrm{div}\,\mathbf{q}\,(\mathrm{U}_0 + \mathrm{T}_0), \qquad (69)$$

if the quantities with the zero suffix are only translationally involved. For example, if

$$\mathbf{f} + \mathbf{F} = \rho\frac{\partial \mathbf{q}}{\partial t}, \qquad (70)$$

as in fluid motion, without frictional or elastic forces associated with the translation, then

$$(\mathbf{f} + \mathbf{F})\,\mathbf{q} = \rho\mathbf{q}\frac{\partial\mathbf{q}}{\partial t} = \dot{\mathrm{T}} + \operatorname{div}\mathbf{q}\mathrm{T}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (71)$$

if $\mathrm{T} = \frac{1}{2}\rho q^2$, the kinetic energy per unit volume. The complete form (69) comes in by the addition of elastic and frictional resisting forces. So deducting (69) from (68) there is left

$$\Sigma\mathbf{Q}\nabla q = \mathrm{Q}_1 + \dot{\mathrm{U}}_1 + \dot{\mathrm{T}}_1 + \operatorname{div}\mathbf{q}\,(\mathrm{U}_1 + \mathrm{T}_1), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (72)$$

where the quantities with suffix unity are connected with the distortion and the rotation, and there may plainly be two sets of dissipative terms, and of energy (stored) terms. Thus the relation

$$\boldsymbol{\epsilon} = \left(n_1 + n_2\frac{d}{dt} + n_3\frac{d^2}{dt^2}\right)\operatorname{curl}\mathbf{D} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (73)$$

will bring in dissipation and kinetic energy, as well as the former potential energy of rotation associated with n_1.

That there can be dissipative terms associated with the distortion is also clear enough, remembering STOKES's theory of a viscous fluid. Thus, for simplicity, do away with the rotating stress, by putting $\boldsymbol{\epsilon} = 0$, making $\mathbf{P}_{\mathrm{N}}$ and $\mathbf{Q}_{\mathrm{N}}$ identical. Then take the stress on the $\mathbf{i}$ plane to be given by

$$\mathbf{P}_1 = \left(n + \mu\frac{d}{dt} + \nu\frac{d^2}{dt^2}\right)(\nabla\mathrm{D}_1 + \nabla_1\mathbf{D}) - \mathbf{i}\left\{\mathrm{P} + \tfrac{2}{3}\left(n + \mu\frac{d}{dt} + \nu\frac{d^2}{dt^2}\right)\operatorname{div}\mathbf{D}\right\}, \quad . \quad . \quad (74)$$

and similarly for any other plane; where $\mathrm{P} = -k\operatorname{div}\mathbf{D}$.

When $\mu = 0$, $\nu = 0$, we have the elastic solid with rigidity and compressibility. When $n = 0$, $\nu = 0$, we have the viscous fluid of STOKES. When $\nu = 0$ only, we have a viscous elastic solid, the viscous resistance being purely distortional, and proportional to the speed of distortion. But with n, μ, ν, all finite, we still further associate kinetic energy with the potential energy and dissipation introduced by n and μ.

We have

$$\Sigma\mathbf{P}\nabla q = \mathrm{Q}_2 + \dot{\mathrm{U}}_2 + \dot{\mathrm{T}}_2$$

for infinitesimal strains, omitting the effect of convection of energy; where

$$\mathrm{T}_2 = \tfrac{1}{2}\nu\left[-\tfrac{2}{3}(\operatorname{div}\mathbf{q})^2 + \nabla q_1(\nabla q_1 + \nabla_1\mathbf{q}) + \nabla q_2(\nabla q_2 + \nabla_2\mathbf{q}) + \nabla q_3(\nabla q_3 + \nabla_3\mathbf{q})\right], \quad . \quad . \quad . \quad . \quad (75)$$

$$\mathrm{Q}_2 = \mu\left[-\tfrac{2}{3}(\operatorname{div}\mathbf{q})^2 + \nabla q_1(\nabla q_1 + \nabla_1\mathbf{q}) + \nabla q_2(\nabla q_2 + \nabla_2\mathbf{q}) + \nabla q_3(\nabla q_3 + \nabla_3\mathbf{q})\right], \quad . \quad . \quad . \quad . \quad (76)$$

$$\mathrm{U}_2 = \tfrac{1}{2}n\left[\left(\frac{k}{n} - \tfrac{2}{3}\right)(\operatorname{div}\mathbf{D})^2 + \nabla\mathrm{D}_1(\nabla\mathrm{D}_1 + \nabla_1\mathbf{D}) + \nabla\mathrm{D}_2(\nabla\mathrm{D}_2 + \nabla_2\mathbf{D}) + \nabla\mathrm{D}_3(\nabla\mathrm{D}_3 + \nabla_3\mathbf{D})\right]. \quad (77)$$

Observe that T_2 and Q_2 only differ in the exchange of μ to $\frac{1}{2}\nu$; but U_2, the potential energy, is not the same function of n and $\mathbf{D}$ that T_2 is of ν and $\mathbf{q}$. But if we take $k = 0$, we produce similarity. An elastic solid having no resistance to compression is also one of Sir W. THOMSON's ethers.

When $n = 0$, $\mu = 0$, $\nu = 0$, we come down to the frictionless fluid, in which

$$\mathbf{f} - \nabla \mathrm{P} = \rho \frac{\partial \mathbf{q}}{\partial t}, \quad \text{(78)}$$

and

$$\Sigma \mathbf{P} \nabla q = - \mathrm{P} \operatorname{div} \mathbf{q}, \quad \text{(79)}$$

with the equation of activity

$$\mathbf{fq} = \dot{\mathrm{U}} + \dot{\mathrm{T}} + \operatorname{div} (\mathrm{U} + \mathrm{T} + \mathrm{P})\, \mathbf{q}, \quad \text{(80)}$$

the only parts of which are not always easy to interpret are the $\mathrm{P}\mathbf{q}$ term, and the proper measure of U. By analogy, and conformably with more general cases, we should take

$$\mathrm{P} = - k \operatorname{div} \mathbf{D}, \qquad \text{and} \qquad \mathrm{U} = \tfrac{1}{2} k (\operatorname{div} \mathbf{D})^2,$$

reckoning the expansion or compression from some mean condition.

The Electromagnetic Equations in a Moving Medium.

§ **14.** The study of the forms of the equation of activity in purely mechanical cases, and the interpretation of the same is useful, because in the electromagnetic problem of a moving medium we have still greater generality, and difficulty of safe and sure interpretation. To bring it as near to abstract dynamics as possible, all we need say regarding the two fluxes, electric displacement $\mathbf{D}$ and magnetic induction $\mathbf{B}$, is that they are linear functions of the electric force $\mathbf{E}$ and magnetic force $\mathbf{H}$, say

$$\mathbf{B} = \mu \mathbf{H}, \qquad \mathbf{D} = c \mathbf{E}, \quad \text{(81)}$$

where c and μ are linear operators of the symmetrical kind, and that associated with them are the stored energies U and T, electric and magnetic respectively (per unit volume), given by

$$\mathrm{U} = \tfrac{1}{2} \mathbf{ED}, \qquad \mathrm{T} = \tfrac{1}{2} \mathbf{HB}, \quad \text{(82)}$$

In isotropic media c is the permittivity, μ the inductivity. It is unnecessary to say more regarding the well-known variability of μ and hysteresis than that a magnet is here an ideal magnet of constant inductivity.

As there may be impressed forces, $\mathbf{E}$ is divisible into the force of the field and an impressed part; for distinctness, then, the complete $\mathbf{E}$ may be called the "force of the flux" $\mathbf{D}$. Similarly as regards $\mathbf{H}$ and $\mathbf{B}$.

There is also waste of energy (in conductors, namely) at the rates

$$Q_1 = \mathbf{EC}, \qquad Q_2 = \mathbf{HK}, \quad \ldots \ldots \quad (83)$$

where the fluxes **C** and **K** are also linear functions of **E** and **H** respectively; thus

$$\mathbf{C} = k\mathbf{E}, \qquad \mathbf{K} = g\mathbf{H}, \quad \ldots \ldots \quad (84)$$

where, when the force is parallel to the flux, and k is scalar, it is the electric conductivity. Its magnetic analogue is g, the magnetic conductivity. That is, a magnetic conductor is a (fictitious) body which cannot support magnetic force without continuously dissipating energy.

Electrification is the divergence of the displacement, and its analogue, magnetification, is the divergence of the induction; thus

$$\rho = \operatorname{div} \mathbf{D}, \qquad \sigma = \operatorname{div} \mathbf{B} \quad \ldots \ldots \quad (85)$$

are their volume densities. The quantity σ is probably quite fictitious, like **K**.

According to MAXWELL's doctrine, the true electric current is always circuital, and is the sum of the conduction current and the current of displacement, which is the time rate of increase of the displacement. But, to preserve circuitality, we must add the convection current when electrification is moving, so that the true current becomes

$$\mathbf{J} = \mathbf{C} + \dot{\mathbf{D}} + \mathbf{q}\rho, \quad \ldots \ldots \quad (86)$$

where **q** is the velocity of the electrification ρ. Similarly

$$\mathbf{G} = \mathbf{K} + \dot{\mathbf{B}} + \mathbf{q}\sigma \quad \ldots \ldots \quad (87)$$

should be the corresponding magnetic current.

§ **15.** MAXWELL's equation of electric current in terms of magnetic force in a medium at rest, say,

$$\operatorname{curl} \mathbf{H}_1 = \mathbf{C} + \dot{\mathbf{D}},$$

where $\mathbf{H}_1$ is the force of the field, should be made, using **H** instead,

$$\operatorname{curl} (\mathbf{H} - \mathbf{h}_0) = \mathbf{C} + \dot{\mathbf{D}} + \mathbf{q}\rho,$$

and here $\mathbf{h}_0$ will be the force of intrinsic magnetisation, such that $\mu\mathbf{h}_0$ is the intensity of intrinsic magnetisation. But I have shown that when there is motion, another impressed term is required, viz., the motional magnetic force

$$\mathbf{h} = \mathrm{V}\mathbf{Dq}, \quad \ldots \ldots \quad (88)$$

making the first circuital law become

$$\text{curl}\,(\mathbf{H} - \mathbf{h}_0 - \mathbf{h}) = \mathbf{J} = \mathbf{C} + \dot{\mathbf{D}} + \mathbf{q}\rho. \quad \ldots \quad (89)$$

MAXWELL'S other connection to form the equations of propagation is made through his vector-potential **A** and scalar potential Ψ. Finding this method not practically workable, and also not sufficiently general, I have introduced instead a companion equation to (89) in the form

$$-\text{curl}\,(\mathbf{E} - \mathbf{e}_0 - \mathbf{e}) = \mathbf{G} = \mathbf{K} + \dot{\mathbf{B}} + \mathbf{q}\sigma, \quad \ldots \quad (90)$$

where $\mathbf{e}_0$ expresses intrinsic force, and $\mathbf{e}$ is the motional electric force given by

$$\mathbf{e} = \text{V}\mathbf{qB}, \quad \ldots \quad (91)$$

which is one of the terms in MAXWELL'S equation of electromotive force. As for $\mathbf{e}_0$, it includes not merely the force of intrinsic electrification, the analogue of intrinsic magnetisation, but also the sources of energy, voltaic force, thermoelectric force, &c.

(89) and (90) are thus the working equations, with (88) and (91) in case the medium moves; along with the linear relations before mentioned, and the definitions of energy and waste of energy per unit volume. The fictitious **K** and σ are useful in symmetrizing the equations, if for no other purpose.

Another way of writing the two equations of curl is by removing the **e** and **h** terms to the right side. Let

$$\left.\begin{array}{ll} \text{curl}\,\mathbf{h} = \mathbf{j}, & \mathbf{J} + \mathbf{j} = \mathbf{J}_0, \\ -\text{curl}\,\mathbf{e} = \mathbf{g}, & \mathbf{G} + \mathbf{g} = \mathbf{G}_0. \end{array}\right\} \quad \ldots \quad (92)$$

Then (89) and (90) may be written

$$\left.\begin{array}{r} \text{curl}\,(\mathbf{H} - \mathbf{h}_0) = \mathbf{J}_0 = \mathbf{C} + \dot{\mathbf{D}} + \mathbf{q}\rho + \mathbf{j}, \\ -\text{curl}\,(\mathbf{E} - \mathbf{e}_0) = \mathbf{G}_0 = \mathbf{K} + \dot{\mathbf{B}} + \mathbf{q}\sigma + \mathbf{g}. \end{array}\right\} \quad \ldots \quad (93)$$

So far as circuitality of the current goes, the change is needless, and still further complicates the make-up of the true current, supposed now to be $\mathbf{J}_0$. On the other hand, it is a simplification on the left side, deriving the current from the force of the flux or of the field more simply.

A question to be settled is whether **J** or $\mathbf{J}_0$ should be the true current. There seems only one crucial test, viz., to find whether $\mathbf{e}_0\mathbf{J}$ or $\mathbf{e}_0\mathbf{J}_0$ is the rate of supply of energy to the electromagnetic system by an intrinsic force $\mathbf{e}_0$. This requires, however, a full and rigorous examination of all the fluxes of energy concerned.

The Electromagnetic Flux of Energy in a stationary Medium.

§ **16.** First let the medium be at rest, giving us the equations

$$\text{curl}\ (\mathbf{H} - \mathbf{h}_0) = \mathbf{J} = \mathbf{C} + \dot{\mathbf{D}}, \qquad (94)$$

$$-\ \text{curl}\ (\mathbf{E} - \mathbf{e}_0) = \mathbf{G} = \mathbf{H} + \dot{\mathbf{B}}. \qquad (95)$$

Multiply (94) by $(\mathbf{E} - \mathbf{e}_0)$, and (95) by $(\mathbf{H} - \mathbf{h}_0)$, and add the results. Thus,

$$(\mathbf{E} - \mathbf{e}_0)\ \mathbf{J} + (\mathbf{H} - \mathbf{h}_0)\ \mathbf{G} = (\mathbf{E} - \mathbf{e}_0)\ \text{curl}\ (\mathbf{H} - \mathbf{h}_0) - (\mathbf{H} - \mathbf{h}_0)\ \text{curl}\ (\mathbf{E} - \mathbf{e}_0),$$

which, by the formula (25), becomes

$$\mathbf{e}_0\mathbf{J} + \mathbf{h}_0\mathbf{G} = \mathbf{EJ} + \mathbf{HG} + \text{div}\ \mathrm{V}\ (\mathbf{E} - \mathbf{e}_0)\ (\mathbf{H} - \mathbf{h}_0)\ ;$$

or, by the use of (82), (83),

$$\mathbf{e}_0\mathbf{J} + \mathbf{h}_0\mathbf{G} = \mathrm{Q} + \dot{\mathrm{U}} + \dot{\mathrm{T}} + \text{div}\ \mathbf{W}, \qquad (96)$$

where the new vector **W** is given by

$$\mathbf{W} = \mathrm{V}\ (\mathbf{E} - \mathbf{e}_0)\ (\mathbf{H} - \mathbf{h}_0). \qquad (97)$$

The form of (96) is quite explicit, and the interpretation sufficiently clear. The left side indicates the rate of supply of energy from intrinsic sources. These $(\mathrm{Q} + \dot{\mathrm{U}} + \dot{\mathrm{T}})$ shows the rate of waste and of storage of energy in this unit volume. The remainder, therefore, indicates the rate at which energy is passed out from the unit volume; and the flux **W** represents the flux of energy necessitated by the postulated localisation of energy and its waste, when **E** and **H** are connected in the manner shown by (94) and (95).

There might also be an independent circuital flux of energy, but, being useless, it would be superfluous to bring it in.

The very important formula (97) was first discovered and interpreted by Professor POYNTING, and independently discovered and interpreted a little later by myself in an extended form. It will be observed that in my mode of proof above there is no limitation as to homogeneity or isotropy as regards the permittivity, inductivity, and conductivity. But c and μ should be symmetrical. On the other hand, k and g do not require this limitation in deducing (97).*

* The method of treating MAXWELL's electromagnetic scheme employed in the text (first introduced in "Electromagnetic Induction and its Propagation," 'The Electrician,' January 3, 1885, and later)

It is important to recognize that this flux of energy is not dependent upon the translational motion of the medium, for it is assumed explicitly to be at rest. The vector $\mathbf{W}$ cannot, therefore, be a flux of the kind $\mathbf{Q}_q q$ before discussed, unless possibly it be merely a rotating stress that is concerned.

The only dynamical analogy with which I am acquainted which seems at all satisfactory is that furnished by Sir W. THOMSON's theory of a rotational ether. Take the case of $\mathbf{e}_0 = 0$, $\mathbf{h}_0 = 0$, $k = 0$, $g = 0$, and c and μ constants, that is, pure ether uncontaminated by ordinary matter. Then

$$\text{curl}\,\mathbf{H} = c\dot{\mathbf{E}}, \quad \ldots \quad (98)$$

$$-\text{curl}\,\mathbf{E} = \mu\dot{\mathbf{H}}. \quad \ldots \quad (99)$$

Now, let $\mathbf{H}$ be velocity, μ density; then, by (99), $-$ curl $\mathbf{E}$ is the translational force due to the stress, which is, therefore, a rotating stress; thus,

$$\mathbf{P}_N = \mathbf{VEN}, \qquad \mathbf{Q}_N = \mathbf{VNE}; \quad \ldots \quad (100)$$

and $2\mathbf{E}$ is the torque. The coefficient c represents the compliancy or reciprocal of the quasi-rigidity. The kinetic energy $\frac{1}{2}\mu\mathrm{H}^2$ represents the magnetic energy, and the potential energy of the rotation represents the electric energy; whilst the flux of energy is $\mathbf{VEH}$. For the activity of the torque is

$$2\mathbf{E}.\frac{\text{curl}\,\mathbf{H}}{2} = \mathbf{E}\,\text{curl}\,\mathbf{H},$$

and the translational activity is

$$-\mathbf{H}\,\text{curl}\,\mathbf{E}.$$

Their sum is

may, perhaps, be appropriately termed the Duplex method, since its characteristics are the exhibition of the electric, magnetic, and electromagnetic relations in a duplex form, symmetrical with respect to the electric and magnetic sides. But it is not merely a method of exhibiting the relations in a manner suitable to the subject, bringing to light useful relations which were formerly hidden from view by the intervention of the vector-potential and its parasites, but constitutes a method of working as well. There are considerable difficulties in the way of the practical employment of MAXWELL's equations of propagation, even as they stand in his treatise. These difficulties are greatly magnified when we proceed to more general cases, involving heterogeneity and eolotropy and motion of the medium supporting the fluxes. The duplex method supplies what is wanted. Potentials do not appear, at least initially. They are regarded strictly as auxiliary functions which do not represent any physical state of the medium. In special problems they may be of great service for calculating purposes; but in general investigations their avoidance simplifies matters greatly. The state of the field is settled by $\mathbf{E}$ and $\mathbf{H}$, and these are the primary objects of attention in the duplex system.

As the papers to which I have referred are not readily accessible, I may take this opportunity of mentioning that a Reprint of my 'Electrical Papers' is in the press (MACMILLAN and Co.), and that the first volume is nearly ready.

$$- \text{div}\, \mathbf{VEH},$$

making **VEH** the flux of energy.*

All attempts to construct an elastic solid analogy with a distortional stress fail to give satisfactory results, because the energy is wrongly localised, and the flux of energy incorrect. Bearing this in mind, the above analogy is at first sight very enticing. But when we come to remember that the d/dt in (98) and (99) should be $\partial/\partial t$, and find extraordinary difficulty in extending the analogy to include the conduction current, and also remember that the electromagnetic stress has to be accounted for (in other words, the known mechanical forces), the perfection of the analogy, as far as it goes, becomes disheartening. It would further seem, from the explicit assumption that $\mathbf{q} = 0$ in obtaining **W** above, that no analogy of this kind can be sufficiently comprehensive to form the basis of a physical theory. We must go altogether beyond the elastic solid with the additional property of rotational elasticity. I should mention, to avoid misconception, that Sir W. THOMSON does not push the analogy even so far as is done above, or give to μ and c the same interpretation. The particular meaning here given to μ is that assumed by Professor LODGE in his "Modern Views of Electricity," on the ordinary elastic solid theory, however. I have found it very convenient from its making the curl of the electric force be a Newtonian force (per unit volume). When impressed electric force $\mathbf{e}_0$ produces disturbances, their real source is, as I have shown, not the seat of $\mathbf{e}_0$, but of curl $\mathbf{e}_0$. So we may with facility translate problems in electromagnetic waves into elastic solid problems by taking the electromagnetic source to represent the mechanical source of motion, impressed Newtonian force.

Examination of the Flux of Energy in a moving Medium, and Establishment of the Measure of "True" Current.

§ **17.** Now pass to the more general case of a moving medium with the equations

$$\text{curl}\, \mathbf{H}_1 = \quad \text{curl}\, (\mathbf{H} - \mathbf{h}_0 - \mathbf{h}) = \mathbf{J} = \mathbf{C} + \dot{\mathbf{D}} + \mathbf{q}\rho, \quad \ldots \ldots \quad (101)$$

$$- \text{curl}\, \mathbf{E}_1 = - \text{curl}\, (\mathbf{E} - \mathbf{e}_0 - \mathbf{e}) = \mathbf{G} = \mathbf{K} + \dot{\mathbf{B}} + \mathbf{q}\sigma, \quad \ldots \ldots \quad (102)$$

where $\mathbf{E}_1$ is, for brevity, what the force **E** of the flux becomes after deducting the intrinsic and motional forces; and similarly for $\mathbf{H}_1$.

From these, in the same way as before, we deduce

$$(\mathbf{e}_0 + \mathbf{e})\, \mathbf{J} + (\mathbf{h}_0 + \mathbf{h})\, \mathbf{G} = \mathbf{EJ} + \mathbf{HG} + \text{div}\, \mathbf{VE}_1\mathbf{H}_1; \quad \ldots \ldots \quad (103)$$

and it would seem at first sight to be the same case again, but with impressed forces

* This form of application of the rotating ether I gave in 'The Electrician,' January 23, 1891, p. 360.

$(\mathbf{e} + \mathbf{e}_0)$ and $(\mathbf{h} + \mathbf{h}_0)$ instead of $\mathbf{e}_0$ and $\mathbf{h}_0$, whilst the POYNTING flux requires us to reckon only $\mathbf{E}_1$ and $\mathbf{H}_1$ as the effective electric and magnetic forces concerned in it.*

But we must develop $(Q + \dot{U} + \dot{T})$ plainly first. We have, by (86), (87), used in (103),

$$\mathbf{e}_0\mathbf{J} + \mathbf{h}_0\mathbf{G} = \mathbf{E}\,(\mathbf{C} + \dot{\mathbf{D}} + \mathbf{q}\rho) + \mathbf{H}\,(\mathbf{K} + \dot{\mathbf{B}} + \mathbf{q}\sigma) - (\mathbf{eJ} + \mathbf{hG}) + \text{div}\,\mathbf{V}\mathbf{E}_1\mathbf{H}_1. \quad . \quad . \quad (104)$$

Now here we have

$$\left.\begin{aligned} \dot{U} = \frac{d}{dt}\tfrac{1}{2}\mathbf{ED} &= \tfrac{1}{2}\mathbf{E}\dot{\mathbf{D}} + \tfrac{1}{2}\mathbf{D}\dot{\mathbf{E}} \\ &= \mathbf{E}\dot{\mathbf{D}} + \tfrac{1}{2}(\mathbf{D}\dot{\mathbf{E}} - \mathbf{E}\dot{\mathbf{D}}) \\ &= \mathbf{E}\dot{\mathbf{D}} - \tfrac{1}{2}\mathbf{E}\dot{c}\mathbf{E} \\ &= \mathbf{E}\dot{\mathbf{D}} - \dot{U}_c. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (105)$$

* It will be observed that the constant 4π, which usually appears in the electrical equations, is absent from the above investigations. This demands a few words of explanation. The units employed in the text are rational units, founded upon the principle of continuity in space of vector functions, and the corresponding appropriate measure of discontinuity, viz., by the amount of divergence. In popular language, the *unit* pole sends out *one* line of force, in the rational system, instead of 4π lines, as in the irrational system. The effect of the rationalisation is to introduce 4π into the formulæ of central forces and potentials, and to abolish the swarm of 4π's that appear in the practical formulæ of the practice of theory on FARADAY-MAXWELL lines, which receives its fullest and most appropriate expression in the rational method. The rational system was explained by me in 'The Electrician,' in 1882, and applied to the general theory of potentials and connected functions in 1883. (Reprint, vol. 1, p. 199, and later, especially p. 262.) I then returned to irrational formulæ because I did not think, then, that a reform of the units was practicable, partly on account of the labours of the B. A. Committee on Electrical Units, and partly on account of the ignorance of, and indifference to, theoretical matters which prevailed at that time. But the circumstances have greatly changed, and I do think a change is now practicable. There has been great advance in the knowledge of the meaning of MAXWELL'S theory, and a diffusion of this knowledge, not merely amongst scientific men, but amongst a large body of practicians called into existence by the extension of the practical applications of electricity. Electricity is becoming, not only a master science, but also a very practical one. It is fitting, therefore, that learned traditions should not be allowed to control matters too greatly, and that the units should be rationalised. To make a beginning, I am employing rational units throughout in my work on "Electromagnetic Theory," commenced in 'The Electrician,' in January, 1891, and continued as fast as circumstances will permit; to be republished in book form. In Section XVII. (October 16, 1891, p. 655), will be found stated more fully the nature of the change proposed, and the reasons for it. I point out, in conclusion, that as regards theoretical treatises and investigations, there is no difficulty in the way, since the connection of the rational and irrational units may be explained separately; and I express the belief that when the merits of the rational system are fully recognised, there will arise a demand for the rationalisation of the practical units. We are, in the opinion of men qualified to judge, within a measurable distance of adopting the metric system in England. Surely the smaller reform I advocate should precede this. To put the matter plainly, the present system of units contains an absurdity running all through it of the same nature as would exist in the metric system of common units were we to define the unit area to be the area of a circle of unit diameter. The absurdity is only different in being less obvious in the electrical case. It would not matter much if it were not that electricity is a practical science.

Comparison of the third with the second form of (105) defines the generalised meaning of $\dot{c}$ when c is not a mere scalar. Or thus,

$$\dot{U}_c = \mathbf{E}^2\dot{c} = \tfrac{1}{2}\frac{d}{dt}(\mathbf{ED})_c$$
$$= \tfrac{1}{2}\dot{c}_{11}E_1^2 + \tfrac{1}{2}\dot{c}_{22}E_2^2 + \tfrac{1}{2}\dot{c}_{33}E_3^2 + \dot{c}_{12}E_1E_2 + \dot{c}_{23}E_2E_3 + \dot{c}_{31}E_1E_3, \quad (106)$$

representing the time-variation of U due to variation in the c's only.

Similarly

$$\dot{T} = \mathbf{H}\dot{\mathbf{B}} - \tfrac{1}{2}\mathbf{H}\dot{\mu}\mathbf{H} = \mathbf{H}\dot{\mathbf{B}} - \dot{T}_\mu, \quad (107)$$

with the equivalent meaning for $\dot{\mu}$ generalised.

Using these in (104) we have the result

$$\mathbf{e}_0\mathbf{J} + \mathbf{h}_0\mathbf{G} = (Q + \dot{U} + \dot{T}) + \mathbf{q}(\mathbf{E}\rho + \mathbf{H}\sigma) + (\tfrac{1}{2}\mathbf{E}\dot{c}\mathbf{E} + \tfrac{1}{2}\mathbf{H}\dot{\mu}\mathbf{H}) - (\mathbf{eJ} + \mathbf{hG}) + \operatorname{div} V\mathbf{E}_1\mathbf{H}_1. \quad (108)$$

Here we have, besides $(Q + \dot{U} + \dot{T})$, terms indicating the activity of a translational force. Thus $\mathbf{E}\rho$ is the force on electrification ρ, and $\mathbf{Eq}\rho$ its activity. Again,

$$\frac{\partial c}{\partial t} = \dot{c} + \mathbf{q}\nabla.c;$$

so that we have

$$\dot{c} = \frac{\partial c}{\partial t} - \mathbf{q}\nabla.c,$$

and, similarly,

$$\dot{\mu} = \frac{\partial \mu}{\partial t} - \mathbf{q}\nabla.\mu, \quad (109)$$

the generalised meaning of which is indicated by

$$-\frac{\partial U_c}{\partial t} + \tfrac{1}{2}\mathbf{E}\dot{c}\mathbf{E} = -\tfrac{1}{2}\mathbf{E}(\mathbf{q}\nabla.c)\mathbf{E} = -\mathbf{q}\nabla U_c; \quad (110)$$

where, in terms of scalar products involving **E** and **D**,

$$-\mathbf{q}\nabla U_c = -\tfrac{1}{2}(\mathbf{E.q}\nabla.\mathbf{D} - \mathbf{D.q}\nabla.\mathbf{E}). \quad (111)$$

This is also the activity of a translational force. Similarly,

$$-\frac{\partial T_\mu}{\partial t} + \tfrac{1}{2}\mathbf{H}\dot{\mu}\mathbf{H} = -\mathbf{q}\nabla T_\mu \quad (112)$$

is the activity of a translational force. Then again

$$-(\mathbf{eJ} + \mathbf{hG}) = -\mathbf{J}V\mathbf{qB} - \mathbf{G}V\mathbf{Dq} = \mathbf{q}(V\mathbf{JB} + V\mathbf{DG}) \quad (113)$$

expresses a translational activity. Using them all in (108) it becomes

$$\mathbf{e}_0\mathbf{J} + \mathbf{h}_0\mathbf{G} = (\mathrm{Q} + \dot{\mathrm{U}} + \dot{\mathrm{T}}) + \mathbf{q}\left(\mathbf{E}\rho + \mathbf{H}\sigma - \nabla \mathrm{U}_c - \nabla \mathrm{T}_\mu^{\,s} + \mathrm{V}\mathbf{JB} + \mathrm{V}\mathbf{DG}\right)$$
$$+ \operatorname{div} \mathrm{V}\mathbf{E}_1\mathbf{H}_1 + \frac{\partial}{\partial t}(\mathrm{U}_c + \mathrm{T}_\mu). \quad . \quad . \quad . \quad . \quad (114)$$

It is clear that we should make the factor of $\mathbf{q}$ be the complete translational force. But that has to be found; and it is equally clear that, although we appear to have exhausted all the terms at disposal, the factor of $\mathbf{q}$ in (114) is not the complete force, because there is no term by which the force on intrinsically magnetised or electrized matter could be exhibited. These involve $\mathbf{e}_0$ and $\mathbf{h}_0$. But as we have

$$\mathbf{q}\,(\mathrm{V}\mathbf{j}_0\mathbf{B} + \mathrm{V}\mathbf{D}\mathbf{g}_0) = -(\mathbf{e}\mathbf{j}_0 + \mathbf{h}\mathbf{g}_0), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (115)$$

a possible way of bringing them in is to add the left member and subtract the right member of (115) from the right member of (114); bringing the translational force to $\mathbf{f}$, say, where

$$\mathbf{f} = \mathbf{E}\rho + \mathbf{H}\sigma - \nabla \mathrm{U}_c - \nabla \mathrm{T} \;\; + \mathrm{V}(\mathbf{J} + \mathbf{j}_0)\,\mathbf{B} + \mathrm{V}\,(\mathbf{G} + \mathbf{g}_0)\,\mathbf{D}. \quad . \quad . \quad . \quad . \quad . \quad (116)$$

But there is still the right number of (115) to be accounted for. We have

$$-\operatorname{div}(\mathrm{V}\mathbf{e}\mathbf{h}_0 + \mathrm{V}\mathbf{e}_0\mathbf{h}) = \mathbf{e}\mathbf{j}_0 + \mathbf{h}\mathbf{g}_0 + \mathbf{e}_0\mathbf{j} + \mathbf{h}_0\mathbf{g}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (117)$$

and, by using this in (114), through (115), (116), (117), we bring it to

$$\mathbf{e}_0\mathbf{J} + \mathbf{h}_0\mathbf{G} = (\mathrm{Q} + \dot{\mathrm{U}} + \dot{\mathrm{T}}) + \mathbf{f}\mathbf{q} - (\mathbf{e}_0\mathbf{j} + \mathbf{h}_0\mathbf{g}) + \operatorname{div}\left(\mathrm{V}\mathbf{E}_1\mathbf{H}_1 - \mathrm{V}\mathbf{e}\mathbf{h}_0 - \mathrm{V}\mathbf{e}_0\mathbf{h}\right) + \frac{\partial}{\partial t}(\mathrm{U}_c + \mathrm{T}_\mu); \quad (118)$$

or, transferring the $\mathbf{e}_0$, $\mathbf{h}_0$ terms from the right to the left side,

$$\mathbf{e}_0\mathbf{J}_0 + \mathbf{h}_0\mathbf{G}_0 = \mathrm{Q} + \dot{\mathrm{U}} + \dot{\mathrm{T}} + \mathbf{f}\mathbf{q} + \operatorname{div}\left(\mathrm{V}\mathbf{E}_1\mathbf{H}_1 - \mathrm{V}\mathbf{e}\mathbf{h}_0 - \mathrm{V}\mathbf{e}_0\mathbf{h}\right) + \frac{\partial}{\partial t}(\mathrm{U}_c + \mathrm{T}_\mu) \quad . \quad . \quad (119)$$

Here we see that we have *a* correct form of activity equation, though it may not be *the* correct form. Another form, equally probable, is to be obtained by bringing in $\mathrm{V}\mathbf{e}\mathbf{h}$; thus

$$\operatorname{div} \mathrm{V}\mathbf{e}\mathbf{h} = \mathbf{h} \operatorname{curl} \mathbf{e} - \mathbf{e} \operatorname{curl} \mathbf{h} = -(\mathbf{e}\mathbf{j} + \mathbf{h}\mathbf{g}) = \mathbf{q}\,(\mathrm{V}\mathbf{j}\mathbf{B} + \mathrm{V}\mathbf{D}\mathbf{g}), \quad . \quad . \quad . \quad . \quad (120)$$

which converts (119) to

$$\mathbf{e}_0\mathbf{J}_0 + \mathbf{h}_0\mathbf{G}_0 = \mathrm{Q} + \dot{\mathrm{U}} + \dot{\mathrm{T}} + \mathbf{F}\mathbf{q} + \operatorname{div}\left(\mathrm{V}\mathbf{E}_1\mathbf{H}_1 - \mathrm{V}\mathbf{e}\mathbf{h} - \mathrm{V}\mathbf{e}\mathbf{h}_0 - \mathrm{V}\mathbf{e}_0\mathbf{h}\right) + \frac{\partial}{\partial t}(\mathrm{U}_c + \mathrm{T}_\mu) \;. \quad (121)$$

where $\mathbf{F}$ is the translational force

$$\mathbf{F} = \mathbf{E}\rho + \mathbf{H}\sigma - \nabla \mathrm{U}_c - \nabla \mathrm{T}_\mu + \mathrm{V} \operatorname{curl} \mathbf{H}.\mathbf{B} + \mathrm{V} \operatorname{curl} \mathbf{E}.\mathbf{D}, \quad . \quad . \quad . \quad . \quad (122)$$

which is perfectly symmetrical as regards **E** and **H**, and in the vector products utilises the fluxes and their complete forces, whereas former forms did this only partially. Observe, too, that we have only been able to bring the activity equation to a correct form (either (119) or (122)) by making $\mathbf{e}_0\mathbf{J}_0$ be the activity of intrinsic force $\mathbf{e}_0$, which requires that $\mathbf{J}_0$ should be the true electric current, according to the energy criterion, not **J**.

§ **18.** Now, to test (119) and (121), we must interpret the flux in (121), or say

$$\mathbf{Y} = \mathrm{V}\mathbf{E}_1\mathbf{H}_1 - \mathrm{V}\mathbf{eh} - \mathrm{V}\mathbf{eh}_0 - \mathrm{V}\mathbf{e}_0\mathbf{h}, \quad \ldots \ldots \quad (123)$$

which has replaced the Poynting flux $\mathrm{V}\mathbf{E}_1\mathbf{H}_1$ when $\mathbf{q} = 0$, along with the other changes. Since **Y** reduces to $\mathrm{V}\mathbf{E}_1\mathbf{H}_1$ when $\mathbf{q} = 0$, there must still be a Poynting flux when **q** is finite, though we do not know its precise form of expression. There is also the stress flux of energy and the flux of energy by convection, making a total flux

$$\mathbf{X} = \mathbf{W} + \mathbf{q}\,(\mathrm{U} + \mathrm{T}) - \Sigma\mathbf{Q}q + \mathbf{q}\,(\mathrm{U}_0 + \mathrm{T}_0), \quad \ldots \ldots \quad (124)$$

where **W** is the Poynting flux, and $-\Sigma\mathbf{Q}q$ that of the stress, whilst $\mathbf{q}\,(\mathrm{U}_0 + \mathrm{T}_0)$ means convection of energy connected with the translational force. We should therefore have

$$\mathbf{e}_0\mathbf{J}_0 + \mathbf{h}_0\mathbf{G}_0 = (\mathrm{Q} + \dot{\mathrm{U}} + \dot{\mathrm{T}}) + (\mathrm{Q}_0 + \dot{\mathrm{U}}_0 + \dot{\mathrm{T}}_0) + \mathrm{div}\,\mathbf{X} \quad \ldots \ldots \quad (125)$$

to express the continuity of energy. More explicitly

$$\begin{aligned}\mathbf{e}_0\mathbf{J}_0 + \mathbf{h}_0\mathbf{G}_0 = \mathrm{Q} + \dot{\mathrm{U}} + \dot{\mathrm{T}} + \mathrm{div}\left[\mathbf{W} + \mathbf{q}\,(\mathrm{U} + \mathrm{T})\right] \\ + \mathrm{Q}_0 + \dot{\mathrm{U}}_0 + \dot{\mathrm{T}}_0 + \mathrm{div}\left[-\Sigma\mathbf{Q}q + \mathbf{q}\,(\mathrm{U}_0 + \mathrm{T}_0)\right] \quad \ldots \ldots \quad (126)\end{aligned}$$

But here we may simplify by using the result (69) (with, however, **f** put $= 0$), making (126) become

$$\mathbf{e}_0\mathbf{J}_0 + \mathbf{h}_0\mathbf{G}_0 = (\mathrm{Q} + \dot{\mathrm{U}} + \dot{\mathrm{T}}) + \mathbf{Fq} + \mathbf{Sa} + \mathrm{div}\left[\mathbf{W} + \mathbf{q}\,(\mathrm{U} + \mathrm{T}) - \Sigma\mathbf{Q}q\right], \quad \ldots \quad (127)$$

where **S** is the torque, and **a** the spin.

Comparing this with (121), we see that we require

$$\mathbf{W} + \mathbf{q}\,(\mathrm{U} + \mathrm{T}) - \Sigma\mathbf{Q}q = \mathrm{V}\mathbf{E}_1\mathbf{H}_1 - \mathrm{V}\mathbf{eh} - \mathrm{V}\mathbf{e}_0\mathbf{h} - \mathrm{V}\mathbf{eh}_0, \quad \ldots \ldots \quad (128)$$

with a similar equation when (119) is used instead; and we have now to separate the right member into two parts, one for the Poynting flux, the other for the stress flux, in such a way that the force due to the stress is the force **F** in (121), (122), or the force **f** in (119), (116); or similarly in other cases. It is unnecessary to give the failures; the only one that stands the test is (121), which satisfies it completely.

I argued that

$$\mathbf{W} = \mathrm{V}(\mathbf{E} - \mathbf{e}_0)(\mathbf{H} - \mathbf{h}_0) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (129)$$

was the probable form of the POYNTING flux in the case of a moving medium, not $\mathrm{V}\mathbf{E}_1\mathbf{H}_1$, because when a medium is endowed with a *uniform* translational motion, the transmission of disturbances through it takes place just as if it were at rest. With this expression (129) for $\mathbf{W}$, we have, identically,

$$\mathrm{V}\mathbf{E}_1\mathbf{H}_1 - \mathrm{V}\mathbf{eh} - \mathrm{V}\mathbf{e}_0\mathbf{h} - \mathrm{V}\mathbf{eh}_0 = \mathbf{W} - \mathrm{V}\mathbf{eH} - \mathrm{V}\mathbf{Eh}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (130)$$

Therefore, by (128) and (130), we get

$$\Sigma \mathbf{Q}q = \mathrm{V}\mathbf{eH} + \mathrm{V}\mathbf{Eh} + \mathbf{q}(\mathrm{U} + \mathrm{T}), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (131)$$

to represent the negative of the stress flux of energy, so that, finally, the fully significant equation of activity is

$$\begin{aligned}\mathbf{e}_0\mathbf{J}_0 + \mathbf{h}_0\mathbf{G}_0 = \mathrm{Q} + \dot{\mathrm{U}} + \dot{\mathrm{T}} + \mathbf{Fq} + \mathbf{Sa} + \mathrm{div}\left[\mathrm{V}(\mathbf{E} - \mathbf{e}_0)(\mathbf{H} - \mathbf{h}_0) + \mathbf{q}(\mathrm{U} + \mathrm{T})\right] \\ - \mathrm{div}\left[\mathrm{V}\mathbf{eH} + \mathrm{V}\mathbf{Eh} + \mathbf{q}(\mathrm{U} + \mathrm{T})\right]. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (132)\end{aligned}$$

This is, of course, an identity, subject to the electromagnetic equations we started from, and is only one of the multitude of forms which may be given to it, many being far simpler. But the particular importance of this form arises from its being the only form apparently possible which shall exhibit the principle of continuity of energy without outstanding terms, and without loss of generality; and this is only possible by taking $\mathbf{J}_0$ as the proper flux for $\mathbf{e}_0$ to work upon.*

* In the original an erroneous estimate of the value of $(\partial/\partial t)(\mathrm{U}_c + \mathrm{T}_\mu)$ was used in some of the above equations. This is corrected. The following contains full details of the calculation. We require the value of $(\partial/\partial t)\mathrm{U}_c$, or of $\frac{1}{2}\mathbf{E}(\partial c/\partial t)\mathbf{E}$, where $\partial c/\partial t$ is the linear operator whose components are the time-variations (for the same matter), of those of c. The calculation is very lengthy in terms of these six components. But vectorially it is not difficult. In (27), (28), we have

$$\left.\begin{aligned}\mathbf{D} = c\mathbf{E} &= \mathbf{i}.\mathbf{c}_1\mathbf{E} + \mathbf{j}.\mathbf{c}_2\mathbf{E} + \mathbf{k}.\mathbf{c}_3\mathbf{E} \\ &= (\mathbf{i}.\mathbf{c}_1 + \mathbf{j}.\mathbf{c}_2 + \mathbf{k}.\mathbf{c}_3)\mathbf{E},\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (132a)$$

if the vectors $\mathbf{c}_1$, $\mathbf{c}_2$, $\mathbf{c}_3$, are given by

$$\mathbf{c}_1 = \mathbf{i}c_{11} + \mathbf{j}c_{12} + \mathbf{k}c_{13}, \qquad \mathbf{c}_2 = \mathbf{i}c_{21} + \mathbf{j}c_{22} + \mathbf{k}c_{23}, \qquad \mathbf{c}_3 = \mathbf{i}c_{31} + \mathbf{j}c_{32} + \mathbf{k}c_{33}.$$

We, therefore, have

$$\mathbf{E}\frac{\partial c}{\partial t}\mathbf{E} = \mathbf{E}\left(\frac{\partial \mathbf{i}}{\partial t}.\mathbf{c}_1 + \frac{\partial \mathbf{j}}{\partial t}.\mathbf{c}_2 + \frac{\partial \mathbf{k}}{\partial t}.\mathbf{c}_3\right)\mathbf{E} + \mathbf{E}\left(\mathbf{i}.\frac{\partial \mathbf{c}_1}{\partial t} + \mathbf{j}.\frac{\partial \mathbf{c}_2}{\partial t} + \mathbf{k}.\frac{\partial \mathbf{c}_3}{\partial t}\right)\mathbf{E} \quad . \quad . \quad . \quad . \quad (132b)$$

The part played by the dots is to clearly separate the scalar products.

Now suppose that the eolotropic property symbolised by c is intrinsically unchanged by the shift of the matter. The mere translation does not, therefore, affect it, nor does the distortion; but the rotation