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The Relativistic Factor in the Orbital Dynamics of Point Masses

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Abstract There is a growing population of relativistically relevant minor bodies in the Solar System and a growing population of massive extrasolar planets with orbits very close to the central star where relativistic effects should have some signature. Our purpose is to review how general relativity affects the orbital dynamics of the planetary systems and to define a suitable relativistic correction for Solar System orbital studies when only point masses are considered. Using relativistic formulae for the N body problem suited for a planetary system given in the literature we present a series of numerical orbital integrations designed to test the relevance of the effects due to the general theory of relativity in the case of our Solar System. Comparison between different algorithms for accounting for the relativistic corrections are performed. Relativistic effects generated by the Sun or by the central star are the most relevant ones and produce evident modifications in the secular dynamics of the inner Solar System. The Kozai mechanism, for example, is modified due to the relativistic effects on the argument of the perihelion. Relativistic effects generated by planets instead are of very low relevance but detectable in numerical simulations.

Keywords celestial mechanics · relativistic effects · Solar System · small bodies

1 Introduction

General Relativity Theory (GRT), while widely used in some areas of astrophysics, has not yet been fully exploited in the study of planetary system dynamics, despite the fact that one of the most famous relativistic effects,

that of the precession of perihelia, shows that Relativity can play a role, albeit somewhat secondary, in the evolution of planetary systems.

The design of a detailed general relativistic reference frame for use in the description of our Solar System was achieved in a compelling though very technical way by Brumberg and Kopeikin and also by Damour et al. in a series of publications (Kopeikin, 1988; Brumberg & Kopeikin, 1989a,b; Brumberg, 1991; Damour et al., 1991, 1992, 1993). In those papers, each object in the system was equipped with a local reference frame (for measuring internal degrees of freedom) which was fitted within a global reference frame. This work culminated in resolutions of the IAU 2000 General Assembly (Soffel et al., 2003). The methods of Brumberg and Kopeikin, and Damour et al., allow for the development of relativistic equations of motion for extended objects, but for purposes of studying the general dynamics of planetary systems, the limit of point (monopole) masses can be expected to give good qualitative results.

Also, there has been exhaustive studies of the validity of GRT by means of the so called Parameterized Post Newtonian (PPN) theories (Will, 1981, 1993; Klioner & Soffel, 2000; Kopeikin & Vlasov, 2004) and comparison with observations made in the Solar System (for a recent review, see Will (2006)). Nevertheless, the effects measured for that purpose are in general not related to any dynamical study of the planetary system. There have also been a number of studies of Post Newtonian effects, such as the Lens-Thirring effect in the motion of satellites (see for example Iorio (2005) and also Ciufolini (2007)), but again these were not concerned with planetary dynamics studies.

Except for very precise simulations (for example Quinn et al. (1991), Varadi et al. (2003)) relativistic effects are in general not taken into account in the study of Solar System dynamics, and when they are, only the effects due to the Sun are considered, using a simplification of the above mentioned more complete PPN model found in Will (1981). This simplified approximation was also used in the study of some extrasolar planetary systems. For example, Barnes & Quinn (2001) studied the stability of the ν -Andromedae planetary system using that approximation and Nagasawa & Lin (2005) showed strong differences between classic and relativistic evolutions of planet b of that system. A more general study of the contribution of the relativistic terms to the secular evolution of extrasolar systems was done by Adams & Laughlin (2006). In the case of extrasolar planets orbiting Pulsars, a specific study of relativistic dynamics is still pending, as stated in (Gozdzwieski et al., 2005).

In the case of minor bodies, as explained by Shahid-Saless & Yeomans (1994) GRT should be used when studying low perihelia orbits. They also pointed out that inconsistencies arise when using newtonian dynamics with masses derived from relativistically derived ephemerides.

Apart from the aforementioned studies, relativistic effects were included in the pioneering work at high precision ephemerides at the JPL, and by now they have a measurable impact on observations, with today's high precision astronomical techniques (see e.g. Pitjeva (2005) for recent advances in the numerical integrations of planetary ephemerides).

It seems without doubt that future development of dynamical astronomy and experimental gravitation will fundamentally rely on the study of General

Relativity effects. This is a common point of view expressed in resolutions of the IAU 2000 General Assembly (Soffel et al., 2003)

The aim of this paper is to evaluate how GRT affects the orbital dynamics of the Solar System, in the limit of point masses, by means of a series of numerical tests. By extension we can extract some conclusions for the extrasolar systems. Hopefully, the numerical studies here presented will help to elucidate in which circumstances it is necessary to account for relativistic effects.

2 The Post-Newtonian Algorithm

We are interested in the first corrections to the classical Newtonian equations of motion of a system of point masses, as derived from the GRT. They can be found by means of an approximation of this last theory, known as the Post Newtonian approximation. This approximation can be accomplished by making an expansion of the GRT equations in terms of \mathbf{v}/c , where the \mathbf{v} are the velocities, and c is the speed of light. Equation of motion can thus be found for point masses, which include the Newtonian term, as well as a number of other terms which are suppressed by factors of $1/c$. The PN equation of motion used in this work is given by Newhall et al. (1983):

$$\begin{aligned} \ddot{\mathbf{r}}_i = & \sum_{j \neq i} \frac{\mu_j (\mathbf{r}_j - \mathbf{r}_i)}{r_{ij}^3} \left[1 - \frac{4}{c^2} \sum_{k \neq i} \frac{\mu_k}{r_{ik}} - \frac{1}{c^2} \sum_{k \neq j} \frac{\mu_k}{r_{jk}} + \frac{\mathbf{v}_i^2}{c^2} + \right. \\ & + 2 \frac{\mathbf{v}_j^2}{c^2} - \frac{4}{c^2} \mathbf{v}_i \cdot \mathbf{v}_j - \frac{3}{2c^2} \left(\frac{(\mathbf{r}_i - \mathbf{r}_j) \cdot \mathbf{v}_j}{r_{ij}} \right)^2 + \frac{1}{2c^2} (\mathbf{r}_j - \mathbf{r}_i) \cdot \ddot{\mathbf{r}}_j \left. \right] + \\ & + \frac{1}{c^2} \sum_{j \neq i} \frac{\mu_j}{r_{ij}^3} [(\mathbf{r}_i - \mathbf{r}_j) \cdot (4\mathbf{v}_i - 3\mathbf{v}_j)] (\mathbf{v}_i - \mathbf{v}_j) + \frac{7}{2c^2} \sum_{j \neq i} \frac{\mu_j \ddot{\mathbf{r}}_j}{r_{ij}} \quad (1) \end{aligned}$$

with μ_i standing for Gm_i , \mathbf{v}_i is the barycentric velocity of m_i and dots denoting usual derivation with respect to an inertial time, making $\ddot{\mathbf{r}}_j$ the acceleration of m_j . Here, “velocity”, and “acceleration” must be understood in a coordinate sense. These quantities do not in fact correspond exactly to the velocity and acceleration as measured by any physical observer - a subtle difference not found in Newtonian Mechanics. The summation is over all massive bodies including the Sun and the origin is the barycenter of the system. This expression is known as the Einstein-Infeld-Hoffman equation, and the first version of it is found in Einstein et al. (1938). For a modern deduction, see Brumberg (2007). The expression also coincides with the one given by Damour et al. (1991) in the case of point monopole masses. A pedagogical derivation of this equation, starting from GRT, is given in appendix A

As it stands, this algorithm would greatly slow down any numerical calculation, and it is only useful in situations where the relativistic contribution of *every* object in the system is to be taken into account. For the simplified case where only one object (e.g. the Sun) contributes with relevant corrections, the sums in (1) are changed into a single term, and in the reference frame of

this central, massive object with mass \mathcal{M} , the relativistic correction would read

$$\Delta\ddot{\mathbf{r}} = \frac{GM}{r^3 c^2} \left[\left(\frac{4GM}{r} - \mathbf{v}^2 \right) \mathbf{r} + 4(\mathbf{v} \cdot \mathbf{r})\mathbf{v} \right] \quad (2)$$

which is the correction proposed by Anderson et al. (1975) (in fact, the equation as written in Anderson et al. (1975) depends on two PPN parameters β and γ , which are equal to unity in the GRT case) and used by most papers that consider relativistic corrections (Quinn et al., 1991; Shahid-Saless & Yeomans, 1994; Varadi et al., 2003). This algorithm, although less exact than Eq. (1), is computationally much more affordable and, as we will show, allows for the most important effects. Anyway, in certain cases the complete algorithm (1) may become necessary, when there are more than just one relativistically relevant object in the system under consideration. Sometimes, in very simplified analysis, the second term in the right hand side of (2) is ignored but this can only be justified when considering near circular orbits.

In going from equations (1) to (2) one is not taking into account possible relativistic effects due to the changes in the velocity of the central object caused by other, large but not necessarily relativistically relevant, objects. In the case of the Solar System, the influence of Jupiter on the motion of the Sun could in principle introduce such an effect, but a simple order of magnitude analysis shows that this effect is highly suppressed and can be consistently taken out.

It is possible to show that the only secular contribution –that means non short period– from relativistic effect generated by the central star affects only the argument of the perihelion ω and the mean anomaly M which is related to an effect on the time of perihelion passage as measured from a far observer at rest. The effect on ω due to the relativistic effects of the Sun is given by $\Delta\omega = 0.0384/(a^{5/2}(1 - e^2))$ arcseconds per year where a is in AUs (Will, 1981; Sitarski, 1983; Shahid-Saless & Yeomans, 1994). For low eccentricity orbits the variation in M as measured by a far observer at rest is given approximately by $\Delta M \sim -0.115/(a^{5/2}\sqrt{1 - e^2})$ arcseconds per year and it is possible to show that the short period effects are negligible (Iorio, 2005). Both secular effects should be considered, for example, in the case of comets when obtaining their non gravitational forces from the observed temporal evolution of the orbital elements (Yeomans et al., 2005). These analytical predictions due to pure relativistic effects can be easily verified by means of a numerical integration of a massless particle orbiting the Sun under the effects of Eq. (2). We show as an example a numerical integration of a particle with orbital elements similar to 2P/Encke at Fig. 1. The short period effects are undetectable if any at all, and the effects on ω are exactly as predicted. The effects on mean anomaly M are not exactly as predicted because of the high value of e . The other orbital elements remain unchanged. In the figure the time is as measured from a distant observer at rest.

Some symplectic numerical integrators use an artificial separable Hamiltonian that can mimic the effects on ω and M that Eq. (2) generates (Saha & Tremaine, 1992). Using this artifact the relativistic effects can be included in the per-

turbing Hamiltonian and symplectic form is maintained but this approximation, usually used in studies of our planetary system, is only valid for nearly constant and not very high eccentricity orbits.

We are interested in exploring how relativistic effects generated by Eqs. (1) and (2) affect the dynamical evolution of our planetary system and the dynamics of small bodies. In order to analyze the relativistic effects in the Solar System we implemented three different models: one using classic newtonian accelerations (model C), another which considers only the relativistic effects generated by the Sun (R_S) and the last with the expression that takes into account the relativistic effects generated by all massive bodies (R_{all}).

3 Effects in Our Planetary System

Our planetary system is mainly composed by an inner set of terrestrial low mass planets and an outer set of giant very massive planets. As they evolve in different timescales we will analyze both two groups separately. Our model for planetary system includes the Sun and the eight mayor planets from Mercury to Neptune and we consider the Earth-Moon system as unique body located at its baricenter. This model is simpler than, for example, the one of Quinn et al. (1991) so our results will not be more precise. Our aim is to study the relevance of the relativistic corrections in order to have an idea if they are or not important in modeling the orbital dynamics of bodies evolving in the Solar System.

3.1 The inner planetary system

The inner planetary system is strongly perturbed by the giant planets and it is known that the relativistic terms due to the Sun have signatures in timescales of millions of years (Laskar, 1988; Quinn et al., 1991). Are the relativistic effects generated by the planets (model R_{all}) relevant or on the contrary are the Sun's relativistic effects (model R_S) enough to explain the dynamics of the inner planetary system?

In order to investigate this point we have integrated all the planetary system from Mercury to Neptune by 2 million years. We performed three different runs according to the three different models: 1) model C using MERCURY integrator (Chambers, 1999) with the Bulirsch–Stoer method including only classical newtonian gravitation, 2) model R_S using the same integrator with BS method including an user defined acceleration given by Sun's relativistic effects according to Eq. (2) and 3) model R_{all} with our own fully relativistic routine implemented using RA15 (Everhart, 1985) that includes relativistic effects from all massive bodies according to Eq. (1). Runs with models C and R_S were repeated for control using the classic and relativistic versions of the integrator EVORB (Fernandez et al. (2002), <http://www.fisica.edu.uy/~gallardo/evorb.html>).

Figures 2-5 show the evolutions of the planets from Mercury to Mars. It is possible to see that the main results from Laskar (1988) are recovered. Notable differences between classic and relativistic models are evident in the

case of Mercury (Fig. 2). In the case of Venus (Fig. 3) and the Earth (Fig. 4), the relativistic integrations present also some variations with respect to the classical one in the evolution of eccentricities and inclinations, in addition to a precession of the perihelion. For Mars, effects are of much lower importance and start to be appreciable at the end of the integration, also mainly in the evolution of eccentricity and inclination.

For the considered time intervals the integration results from both relativistic models R_S and R_{all} were basically undistinguishable. We want to stress that both relativistic models R_S and R_{all} were implemented in very different numerical integrators with very different algorithms (BS and RA15 respectively), so the coincidence of the results is a prove of their precision and a confirmation that the differences detected with model C are real and not a numerical artifact.

We tested the relevance of the relativistic effects by running an integration that included Earth plus the outer planetary system (that is without the less massive terrestrial planets) for a time span of 2 million years. By doing this, we eliminate the forced modes generated by the other terrestrial planets and we isolate the relativistic effects from the inner Solar System chaos (Laskar, 1996). The resulting evolution of Earth's eccentricity and specially the inclination are much less affected by the relativistic corrections. Then, the discrepancies between classic and relativistic evolutions found in the real Earth in Fig. 4 are mainly due to the complexity that characterize the dynamics of the complete inner planetary system, making the small relativistic effects on the inner planets to interact in an complex way.

3.2 The outer planetary system

We have shown that relativistic effects generated by the Sun are relevant but the ones generated by the planets are negligible in the inner planetary system. Can we conclude the same for the outer more massive planetary system? We are interested in relativistic effects due to the massive outer planetary system and to compare with the relativistic effects due exclusively to the Sun, that means model R_{all} versus R_S . In this experiment we integrated the outer planetary system from Jupiter to Neptune by 401 million years. We performed three different runs according to the three different models defined earlier.

In Figs. 6 and 7 we show the last years of the evolution of the eccentricity and inclination respectively for the four giant planets. There is a small shift in e and a very small shift in i between the different models which should affect very slightly the fundamental frequencies (Laskar, 1988) but the global behavior is not substantially changed.

Results for the more massive and near Jupiter and Saturn are somehow different than results for the less massive and remote planets Uranus and Neptune. Analyzing the eccentricity showed at Fig. 6 we can see that model R_{all} departs from R_S and both from the classic model C. In Uranus both relativistic models are almost coincident and they depart from C whereas for Neptune differences are something small. Analyzing the inclinations for Jupiter and Saturn in Fig. 7 we can see that model C and R_S coincide and

they depart from R_{all} , so mutual relativistic effects between Jupiter and Saturn seem to be more important than the relativistic effects of the Sun on Jupiter and Saturn. No differences in inclination are appreciated in Uranus and Neptune. We can interpret this as follows: the two most massive and near planets Jupiter and Saturn generate some additional perturbations in model R_{all} but these effects are not enough in the more distant and less massive planets Uranus and Neptune. Effects in the inclination are of lower magnitude than in the eccentricity and only in the more massive and near planets R_{all} show up.

It is known that quasi resonant angles like $2\lambda_J - 5\lambda_S$ or $2\lambda_N - \lambda_U$ where λ is the mean longitude can generate non negligible perturbations on minor bodies (Ferraz-Mello et al., 1998; Michtchenko & Ferraz-Mello, 2001; Marzari & Scholl, 2002). A modification of the time evolution of the quasi resonant angles due to relativistic effects could introduce dynamical effects in the time evolution of these minor bodies. Analyzing our integrations we conclude that the circulation periods of both critical angles remain practically unchanged by the relativistic terms, so that there do not exist relevant relativistic effects on these angles.

4 Effects in Solar System's Low (a, q) Orbits

4.1 Secular dynamics of fictitious particles

According to the expression for the relativistic effect $\Delta\omega$ we can infer that the low (a, q) orbits should be the most sensible to the relativistic effects, and according to the results of section 3.1, if we exclude planet crossing orbits, we can ignore the relativistic effects generated by the planets in the case of low (a, q) orbits. So, we have integrated by 10^5 years the system composed by the Sun, the four giant planets and a population of test particles with low perihelion ($0.05 < q < 0.5$ AU) and low semimajor axis ($0.2 < a < 2$ AU) with and without the relativistic terms due to the Sun, that is, models R_S and C respectively. We also performed a run with the model R_S including the terms corresponding to the solar quadrupole moment of the Sun $J_2 = 2 \times 10^{-7}$ (Pireaux & Rozelot, 2003) obtaining a completely negligible effect, so these effects can be ignored in comparison with the relativistic effects due to the Sun.

We considered the time evolution of the elements e, i, ω and computed the differences between both runs, classic and relativistic. The maximum differences $\Delta\omega$, Δe and Δi found during the integration are showed at Fig. 8 as a function of the initial semimajor axis and perihelion distance. That figure was constructed with 100 test particles all with initial $i = 20^\circ$, $\Omega = 0^\circ$, $\omega = 60^\circ$, so it is not a map valid for any set of orbital elements but give us an idea of how relativity modify the secular dynamics in that region.

This experiment designed excluding the terrestrial planets avoids the chaotic dynamics typical of objects with close encounters with them and the differences between both runs give us and indication of the differences generated by the relativistic terms in the secular, more regular, dynamics.

Besides some secular resonances present in this region (Michel & Thomas, 1996; Michel & Froeschlé, 1997) the secular evolution that dominates this region is the Kozai mechanism which produces strong variations in (e, i) linked to the evolution of ω maintaining a constant value of $H = \sqrt{1 - e^2} \cos i$, where the inclination is measured with respect to the invariable plane of the Solar System. The relativistic terms affect ω modifying the Kozai mechanism and in consequence the time evolution of (e, i) (Morbidelli, 2002).

At Fig. 9 we show an example of how relativity modifies the Kozai mechanism. It is a typical secular evolution with constant $a = 0.4$ AU in both models within all the time integration period. Strong differences between the other orbital elements appear conserving H nearly constant.

In the real Solar System these effects can be increased several times due to the secular forced modes generated by the terrestrial planets but in the case of orbits having close encounters with the planets the chaotic dynamics will overcome any improvement to the model introduced by the relativistic corrections.

4.2 The known relativistic asteroid population

After the successful explanation of the precession of perihelion of Mercury by Einstein's GRT, an unavoidable step was to study the evolution of the first very low perihelion asteroid known, 1566 Icarus (Shapiro et al., 1968), where relativistic effects were expected and observed in its dynamical evolution. Icarus, being maybe the most studied "relativistic" asteroid, is by no means the only one to deserve such treatment. Mercury is the Solar System's body with greatest relativistic $\Delta\omega$ (0.4304 "/yr) but there are at present around half hundred asteroids with $\Delta\omega$ greater than Icarus ($\Delta\omega = 0.1006$ "/yr). The objects with the greatest values for $\Delta\omega$ are shown at Table 1 which was constructed from the Near Earth Asteroids database of the Minor Planet Center and can be considered as an update of Table 1 of Shahid-Saless & Yeomans (1994). These asteroids are the ones with the most important relativistic effects on ω but not necessarily the ones with the most diverging evolutions from classic model C. Divergence depends also on the degree of chaos of their dynamics.

We have integrated the full planetary system plus all these asteroids and also some others for a time span of 10^5 years using both the classical and relativistic models C and R_S . All asteroids from the table exhibit chaotic evolutions with several close encounters with terrestrial planets that generate quick departures between classic and relativistic trajectories. Resonance sticking (Duncan & Levison, 1997; Lykawka & Mukai, 2006) is also present with switches between resonances that can be easily identified following Gallardo (2006). Asteroid 2004 XY60 is a good example (Fig. 10) of an evolution dominated by close encounters plus resonance sticking. It starts inside resonance 6:5 with Venus and after 13000 years the resonance's strength drops and high order resonances with Earth start to dominate following different dynamical trajectories the classic and the relativistic models.

As an extreme case we can mention 2000 LK, not included in Table 1, in which the relativistic particle hits the Sun after 83000 years, and the classical

one survive during the whole integration (Fig. 11). A more interesting case is 2003 CP20 (Fig. 12) neither included in the table, which did not have any close encounter with planets in any of the runs but it is strongly affected by exterior resonance 27:28 with Venus as we have found following Gallardo (2006). The critical angle $\sigma = -27\lambda_{Ven} + 28\lambda - \varpi$ is shown at Fig. 13 for the two models C and R_S . In this case the Kozai mechanism is also present generating important variations in e, i in analogy to other situations observed in the transneptunian region (Gomes et al., 2005).

We have also studied some real asteroids and fictitious particles in mean motion resonances with Jupiter obtaining very different evolutions between models C and R_S . This is an expected result because the chaotic nature of the resonant dynamics makes that small variation in the models to generate exponential divergences. For completeness we have studied the dynamical evolution of fictitious particles in the transneptunian region but no relativistic effects were found there.

5 Extrasolar Systems

The relevance of relativistic effects in extrasolar systems were already shown by, for example, Mardling & Lin (2004) and Nagasawa & Lin (2005). In particular Nagasawa & Lin (2005) showed that relativistic terms have dramatically affected the evolution of planet b in ν Andromedae. Adams & Laughlin (2006) have extended the secular interaction theory including GR terms and they found that relativistic corrections are relevant for systems with close planets (~ 4 days orbits). Then, it is not necessary to insist here on the importance of including relativistic corrections specially when semimajor axes are of the order of 10^{-1} AU or less, but we can analyze the relevance of the use of model R_{all} instead of R_S .

With this in mind we integrated a fictitious planetary system with low eccentricity and close orbits of massive planets (see Table 2) using the three models. We avoided experiments with very close orbits in order to avoid chaotic dynamics. The global dynamics obtained by the three models are very similar and regular due to the small eccentricity of the orbits. Differences are mainly in the frequencies of the secular evolution. We have Fourier analyzed one of the fundamental frequencies of the system obtaining $f = 1.63677 \times 10^{-3} \text{yr}^{-1}$ for model C, $f = 1.63682 \times 10^{-3} \text{yr}^{-1}$ for model R_S and $f = 1.63681 \times 10^{-3} \text{yr}^{-1}$ for model R_{all} . We have checked that the differences were not a numerical artifact repeating the simulation R_{all} with lower precision parameter for RA15 and verifying that the new frequency was exactly the same. Then, differences between models R_S and R_{all} seem to be real and not due to numerical artifacts but, at least in this example, negligibly small. We repeated the experiment with the same planetary system but in this case with smaller semimajor axes (0.2 and 0.8 AU) and an analogue behavior was observed regarding the differences between fundamental frequencies in the three models.

However, relativistic effects due to the planets (model R_{all}) could be important for planetary systems in mean motion resonances or under the mechanism of perihelion alignment or anti-alignment evolving very near the

separatrix in the phase space because a small difference in the evolution of ϖ could generate a very different behavior in the phase space with an associated very different evolution in e .

The problem of the relevance of model R_{all} deserves a more detailed study. In principle we can conclude that in planetary systems, even with low e , it is worthwhile to consider relativistic corrections at least with model R_S and a test with model R_{all} will be convenient.

6 Conclusions

The secular relativistic effects generated by the Sun are relevant for the argument of the perihelion and mean anomaly (related to the time of perihelion passage) whereas the short period effects are vanishingly small. In our planetary system relativistic effects generated by the Sun are appreciable specially in the inner Solar System, but those generated by the planets are very small and are only detectable in the long time scales in the giant planets, specially Jupiter and Saturn.

Due to the secular dynamics the departure between classic and relativistic trajectories is enhanced in the inner Solar System. In particular the Kozai mechanism is substantially modified driving notorious differences in the evolution of the eccentricity and inclination.

Nowadays there exist a noticeable population of relativistic minor bodies in the Solar System. Around a half hundred asteroids at present exhibit greater relativistic effects than 1566 Icarus. The relativistic corrections due to the Sun are enough for the correct modeling of the dynamics of low perihelion or low semimajor axis bodies. Minor bodies evolving having close approaches to the planets or being in resonances are the most affected by the relativistic corrections not because of the magnitude of the relativistic corrections but due to the chaotic nature of their dynamics. Relativistic corrections should also be applied to comets but in this case in order to be consistent a suitable model for the non gravitational forces is necessary.

In general, in extrasolar planets the relativistic corrections due to the star are enough and, in some cases, are indispensable as in the case of systems evolving near a separatrix of trajectories in the phase space. In some particular cases the relativistic effects generated by the planets could have some importance, but this point deserves future research.

In the case of exoplanets orbiting around pulsars, a more detailed study needs to be performed. Such a study could be done following the same lines as those presented in this paper, and it constitutes the obvious direction for future research. As for the case of binary pulsar systems, its study is more complex and the algorithms used in this paper would need to be improved.

Some years ago Sitarski (1983) concluded "...it seems that in all the modern investigations it is the very time to replace Newtonian equations of motions by those resulting from general relativity theory". Following the results here presented, it seems that relativistic corrections are not unavoidable for all dynamical studies, but they are necessary for the precise dynamical modeling of bodies evolving in the inner Solar System and for the dynamical modeling of close extrasolar planets.

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A Appendix: A Closer Look at the Post-Newtonian Algorithm

General Relativity is a theory written in a language quite unlike the one of newtonian gravitation. In GR, space–time is equipped with a *metric* $g_{\mu\nu}$, which measures distances and angles at every point. There does not exist any gravitational *force* in the theory, and the gravitational interactions are given by the form of the metric itself. Free particles in the theory follow *geodesics*, that is, trajectories which minimize distances as given by the metric.

Nevertheless, as any new theory, GR must, and of course does, recover the classical theory as a limit of weak gravitational fields and low velocities. This is what is called the *newtonian limit* of the theory, and in it the classical field ϕ is recovered as a term in the temporal part of the metric $g_{\mu\nu}$ of space–time (Weinberg (1972); Thorne et al. (1973))

$$g_{00} = -(1 + 2\phi)$$

In our case, we take this limit and improve it using an expansion in the typical velocities of the particles of the system $\mathbf{v}^2 \sim GM/r$, following Weinberg (1972). For velocities which are small compared with the speed of light c , that means $GM/rc^2 \ll 1$, this approximation is sufficient to describe the dynamics of point masses in the Solar System. Following the usual convention in GR, we work in units where $c = 1$. We can then write

$$\begin{aligned} g_{00} &= -1 + g_{00}^{(2)} + g_{00}^{(4)} + \dots \\ g_{ij} &= \delta_{ij} + g_{ij}^{(2)} + g_{ij}^{(4)} + \dots \\ g_{i0} &= g_{i0}^{(3)} + g_{i0}^{(5)} + \dots \end{aligned}$$

where numbers in parenthesis denote the order in \mathbf{v}/c of the terms, and the Latin indexes represent the spatial components. Notice that, in order to preserve properties of symmetry under time reversal at this order of the expansion (which is a valid symmetry, due to the fact that gravitational radiation effects are of a higher order in the expansion (Weinberg, 1972; Thorne et al., 1973)), only terms with a given parity are allowed in the expansion.

Similar expansions can be produced for the inverse metric tensor $g^{\mu\nu}$ and for the Ricci tensor of curvature $R_{\alpha\beta}$. In GR, curvature of space–time is given by second derivatives of the metric tensor, and measures how much geodesics are different to straight lines. The relation between these expansions can be readily deduced, giving expressions in the form of

$$R_{ij}^{(2)} = -\frac{1}{2} \frac{\partial^2 g_{00}^{(2)}}{\partial x^i \partial x^j} + \frac{1}{2} \frac{\partial^2 g_{kk}^{(2)}}{\partial x^i \partial x^j} - \frac{1}{2} \frac{\partial^2 g_{ik}^{(2)}}{\partial x^k \partial x^j} - \frac{1}{2} \frac{\partial^2 g_{kj}^{(2)}}{\partial x^k \partial x^i} + \nabla^2 g_{ij}^{(2)}$$

for the spatial components, making use of Einstein's summation convention.

These expressions can be simplified by the use of some arbitrary *gauge conditions*, which are permitted due to the invariance properties of GR theory under arbitrary coordinate transformations. This kind of freedom is analogous to those that exist in electromagnetic theory. In our case we followed Weinberg (1972) in choosing the so-called *Harmonic Coordinate Conditions*, which force the metric to satisfy

$$\Gamma^\lambda \equiv g^{\mu\nu} \Gamma_{\mu\nu}^\lambda = 0$$

where $\Gamma_{\mu\nu}^\lambda$ are the Christoffel symbols, related to the first derivatives of the metric, and also to the parallel transport of vectors along a curve in space–time. The use of these conditions impose restrictions to our expansion terms, e.g.

$$0 = \frac{1}{2} \frac{\partial g_{00}^{(2)}}{\partial t} - \frac{\partial g_{i0}^{(3)}}{\partial x^i} + \frac{1}{2} \frac{\partial g_{ii}^{(2)}}{\partial t}$$

these restriction help to simplify our equations.

Then we can proceed to make use of *Einstein's equation*, which relates in each point the curvature of space–time to the presence of matter and energy:

$$R_{\mu\nu} = -8\pi G(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T^\lambda_\lambda) \quad \text{with} \quad T^\lambda_\lambda = g^{\mu\lambda}T_{\mu\lambda}$$

where $T_{\mu\nu}$ is the *energy–momentum* tensor, including the density of energy in space–time. In the case of point particles, this density will reduce to a summation over Dirac's δ functions at the position of each particle. This tensor can also be expanded in terms of \mathbf{v}/c , and when we substitute the expansions in Einstein's equation we reach equations of the form

$$\begin{aligned} \nabla^2 g_{00}^{(2)} &= -8\pi G T^{(0)00} \\ \nabla^2 g_{00}^{(4)} &= \frac{\partial^2 g_{00}^{(2)}}{\partial t^2} + g_{ij}^{(2)} \frac{\partial^2 g_{00}^{(2)}}{\partial x^i \partial x^j} - \left(\frac{\partial g_{00}^{(2)}}{\partial x^i} \right) \left(\frac{\partial g_{00}^{(2)}}{\partial x^i} \right) \\ &\quad - 8\pi G \left[T^{(2)00} - 2g_{00}^{(2)} T^{(0)00} + T^{(2)ii} \right] \\ \nabla^2 g_{i0}^{(3)} &= 16\pi G T^{(1)i0} \\ \nabla^2 g_{ij}^{(2)} &= -8\pi G \delta_{ij} T^{(0)00} \end{aligned}$$

And we find, as was to be expected

$$g_{00}^{(2)} = -2\phi = 2G \int \frac{T^{(0)00}(\mathbf{r}', t)}{|\mathbf{r} - \mathbf{r}'|} d^3 r'$$

where $T^{(0)00}$ is just the mass density of the system, and ϕ the classical potential.

We can now define some new potentials, which will play the role of corrections to the classical theory.

$$g_{i0}^{(3)} = \zeta_i$$

where

$$\zeta_i(\mathbf{r}, t) = -4G \int \frac{T^{(1)i0}(\mathbf{r}', t)}{|\mathbf{r} - \mathbf{r}'|} d^3 r'$$

$$g_{00}^{(4)} = -2\phi^2 - 2\psi$$

where

$$\psi(\mathbf{r}, t) = - \int \frac{1}{|\mathbf{r} - \mathbf{r}'|} \left[\frac{1}{4\pi} \frac{\partial^2 \phi(\mathbf{r}', t)}{\partial t^2} + G T^{(2)00}(\mathbf{r}', t) + G T^{(2)ii}(\mathbf{r}', t) \right] d^3 r'$$

It is possible now to write a relativistic equation of motion in terms of these potentials. As we have already stated, in GR test particles move following geodesics,

and the equation for a geodesic is given, in terms of a generic curve length parameter τ by

$$\frac{d^2 x^\mu}{d\tau^2} + \Gamma_{\nu\lambda}^\mu \frac{dx^\nu}{d\tau} \frac{dx^\lambda}{d\tau} = 0$$

where the Christoffel symbols, which are needed for making covariant derivatives in a curved space–time, are also written in terms of the metric, and of its expansion.

Splitting up of temporal and spatial indexes leads to the expression

$$\frac{d^2 x^i}{dt^2} = -\Gamma_{00}^i - 2\Gamma_{0j}^i \frac{dx^j}{dt} - \Gamma_{jk}^i \frac{dx^j}{dt} \frac{dx^k}{dt} + \left[\Gamma_{00}^0 + 2\Gamma_{0j}^0 \frac{dx^j}{dt} + \Gamma_{jk}^0 \frac{dx^j}{dt} \frac{dx^k}{dt} \right] \frac{dx^i}{dt}$$

finally, after recovering the factors of c for a more intuitive expression, substitution of the three potentials give

$$\begin{aligned} \frac{d\mathbf{v}}{dt} = & -\nabla(\phi + 2\frac{\phi^2}{c^2} + \psi) - \frac{1}{c} \frac{\partial\zeta}{\partial t} + \frac{\mathbf{v}}{c} \times (\nabla \times \zeta) + \\ & + \frac{3}{c^2} \mathbf{v} \frac{\partial\phi}{\partial t} + \frac{4}{c^2} \mathbf{v}(\mathbf{v} \cdot \nabla)\phi - \frac{\mathbf{v}^2}{c^2} \nabla\phi \end{aligned}$$

Given this equation of motion, it becomes possible to understand the physical meaning of the three potentials, ϕ , ψ and ζ . The case of ϕ is easiest, as it is nothing but the classical gravitational potential from Newton's theory, thus recovering the classical limit when $\mathbf{v} \ll c$. The vector potential ζ is quite similar to the vector potential \mathbf{A} in electrodynamics. Just as the magnetic field can be seen as a (special) relativistic effect of the electric field, so ζ appears as a relativistic correction to the classical field, which exerts a velocity–dependent force normal to the distance vector and to the relative velocity of two massive objects. This field is sometimes known in the literature as the *gravitomagnetic* field, for its obvious analogies with the magnetic field in electrodynamics.

The ψ scalar field contains corrections to ϕ due to many relativistic effects. It can be shown that the time derivative term in the ψ defining integral accounts for the delay of the gravitational signal, which travels at the speed of light, and which moves following a geodesic instead of a straight line. The other two terms are corrections due to the equivalence between mass and energy, by means of which the kinetic and potential energies of the particles modify the classical field.

The equation of motion contains the classical equation of Newton as its first term, and many small corrections to it. The equation in itself can prove to be very insightful for a theorist, but it is not fit yet for use in a numerical algorithm. We would like to find a more straightforward algorithm, in which the acceleration of each point particle would be given explicitly by the positions and velocities of every particle in the system.

To accomplish this we must use the relativistic expression for the momentum–energy tensor of a system of particles, in order to find ϕ , ψ and ζ as functions of positions and velocities. In the case of ψ , the delay–of–signal term can prove very hard to simplify. Some methods and approximations from electrodynamics can be used, as the delay–of–signal term is almost identical to the Lenard–Jones electric potential (e.g Griffiths (1999)). After some algebra it is possible to arrive to the formula given by Newhall et al. (1983), which is more explicit – and numerically useful – albeit somewhat less physically meaningful:

$$\ddot{\mathbf{r}}_i = \sum_{j \neq i} \frac{\mu_j (\mathbf{r}_j - \mathbf{r}_i)}{r_{ij}^3} \left[1 - \frac{4}{c^2} \sum_{k \neq i} \frac{\mu_k}{r_{ik}} - \frac{1}{c^2} \sum_{k \neq j} \frac{\mu_k}{r_{jk}} + \frac{\mathbf{v}_i^2}{c^2} + \right.$$

$$\begin{aligned}
& +2\frac{\mathbf{v}_j^2}{c^2} - \frac{4}{c^2}\mathbf{v}_i \cdot \mathbf{v}_j - \frac{3}{2c^2} \left(\frac{(\mathbf{r}_i - \mathbf{r}_j) \cdot \mathbf{v}_j}{r_{ij}} \right)^2 + \frac{1}{2c^2}(\mathbf{r}_j - \mathbf{r}_i) \cdot \ddot{\mathbf{r}}_j \Big] + \\
& + \frac{1}{c^2} \sum_{j \neq i} \frac{\mu_j}{r_{ij}^3} [(\mathbf{r}_i - \mathbf{r}_j) \cdot (4\mathbf{v}_i - 3\mathbf{v}_j)] (\mathbf{v}_i - \mathbf{v}_j) + \frac{7}{2c^2} \sum_{j \neq i} \frac{\mu_j \ddot{\mathbf{r}}_j}{r_{ij}}
\end{aligned}$$

which is the same as eq. (1) in the main text.

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Table 1 Solar System's bodies most affected by relativistic effects generated by the Sun.

Designation	a (AU)	e	$i(^{\circ})$	$\Delta\omega$ ($''$ /yr)
Mercury	0.387	0.206	7.0	0.430
2004 XY60	0.640	0.797	23.7	0.321
2000 BD19	0.876	0.895	25.6	0.268
2006 CJ	0.676	0.755	10.2	0.238
(66391) 1999 KW4	0.642	0.688	38.8	0.221
1995 CR	0.907	0.869	4.0	0.200
1999 MN	0.674	0.665	2.0	0.185
2006 KZ39	0.616	0.525	9.3	0.178
2001 TD45	0.797	0.777	25.4	0.171
2004 JG6	0.635	0.531	18.9	0.166
1998 SO	0.731	0.699	30.3	0.164
(85953) 1999 FK21	0.739	0.703	12.5	0.162
2004 XZ130	0.618	0.455	2.9	0.161
2006 JF42	0.672	0.582	5.9	0.157
2005 WS3	0.672	0.576	23.0	0.155
2004 UL	1.266	0.927	23.7	0.151

Table 2 A fictitious planetary system adopted to evaluate the effects of models C, R_S and R_{all} . The mass of the central star was taken equal to M_{\odot} .

Object	a (AU)	e	$i(^{\circ})$	m/m_{Jup}
Planet b	0.6	0.04	1.04	1.0
Planet c	1.0	0.02	1.45	1.0

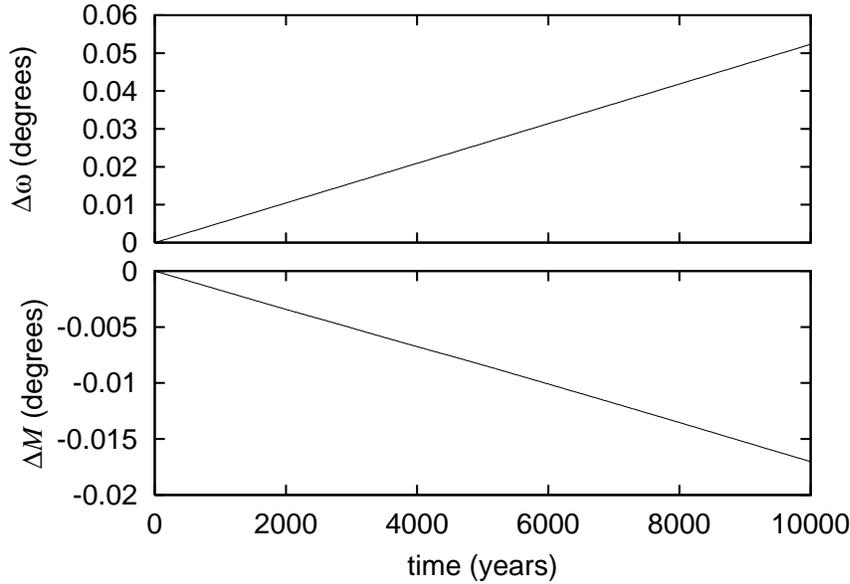


Fig. 1 A numerical integration considering only the Sun and a massless particle with orbital elements similar to 2P/Encke ($a = 2.22$, $e = 0.85$) designed to show the relativistic effects on ω and M due to Eq. (2). Variations are $\Delta\omega = \omega_{rel} - \omega_{cla}$ and $\Delta M = M_{rel} - M_{cla}$. The short period effects are undetectable.

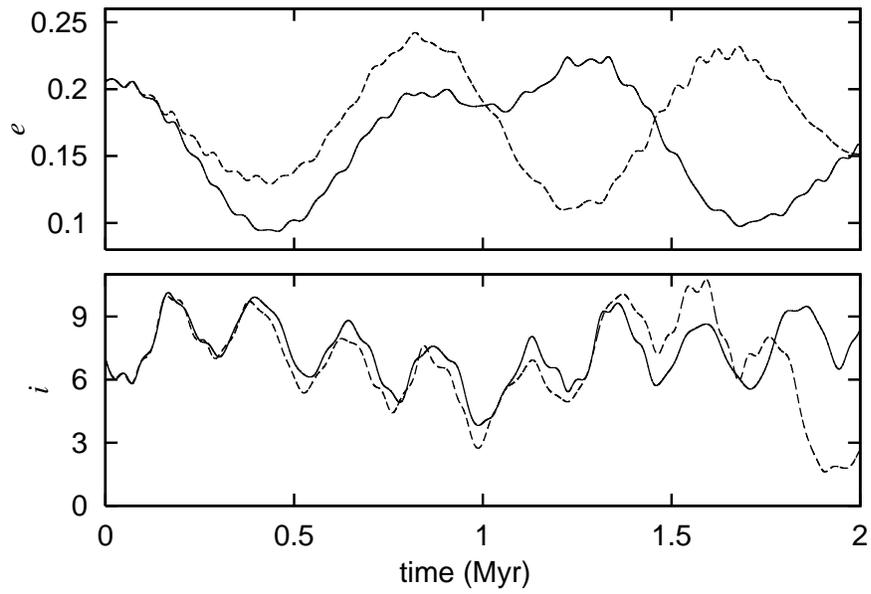


Fig. 2 Mercury's evolution. Top: eccentricity; bottom: inclination. Full line: classical model; dashed line: both relativistic models.

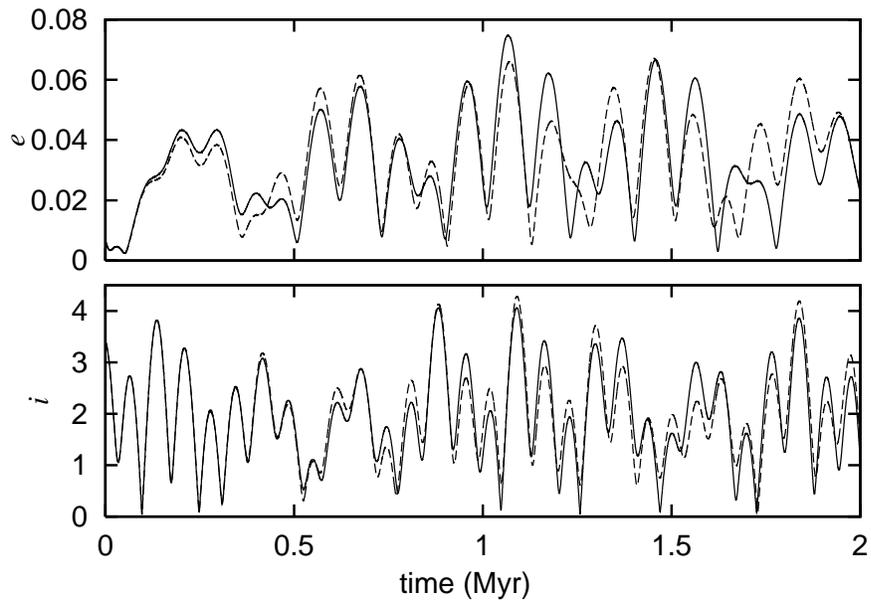


Fig. 3 Venus's evolution. Top: eccentricity; bottom: inclination. Full line: classical model; dashed line: both relativistic models.

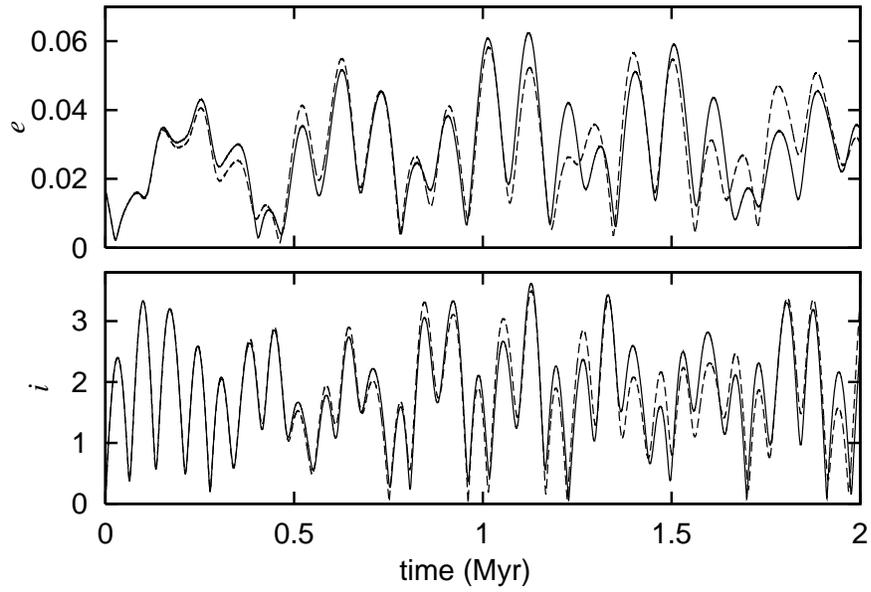


Fig. 4 Earth's evolution. Top: eccentricity; bottom: inclination. Full line: classical model; dashed line: both relativistic models.

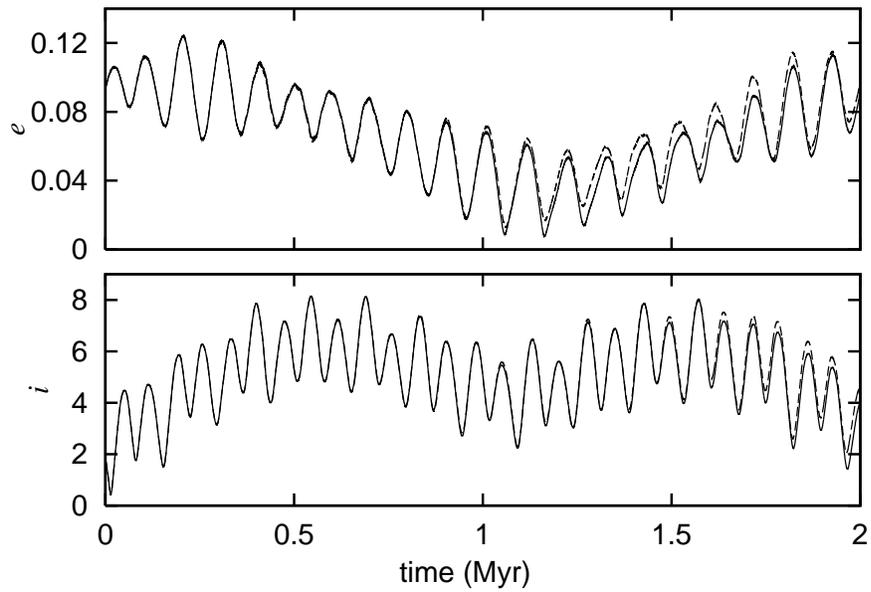


Fig. 5 Mars' evolution. Top: eccentricity; bottom: inclination. Full line: classical model; dashed line: both relativistic models.

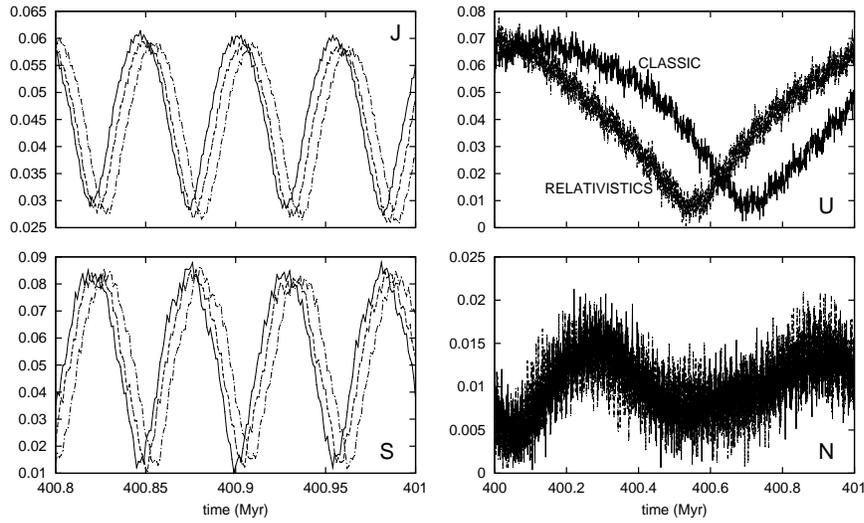


Fig. 6 Eccentricity evolution of the outer planetary system. Full line: classical model C; dashed line: relativistic model R_S ; dashed dotted line: relativistic model R_{all} .

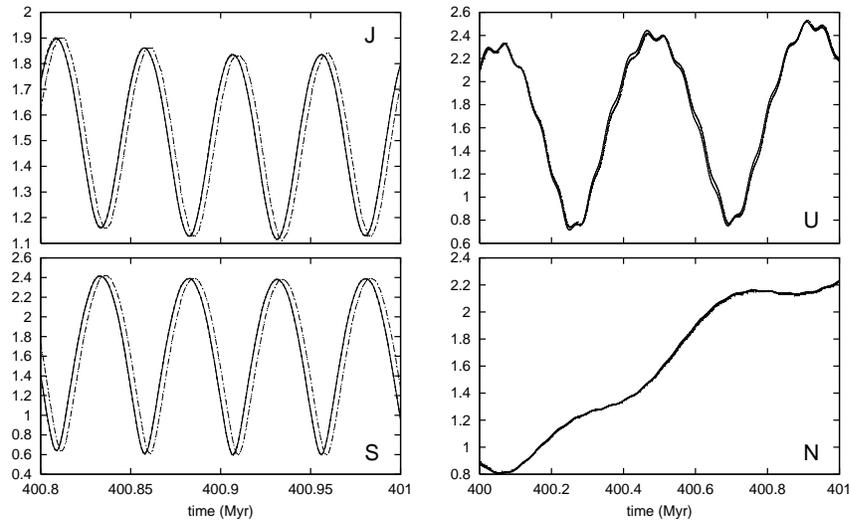


Fig. 7 Inclination evolution of the outer planetary system. Full line: classical model C; dashed line: relativistic model R_S ; dashed dotted line: relativistic model R_{all} .

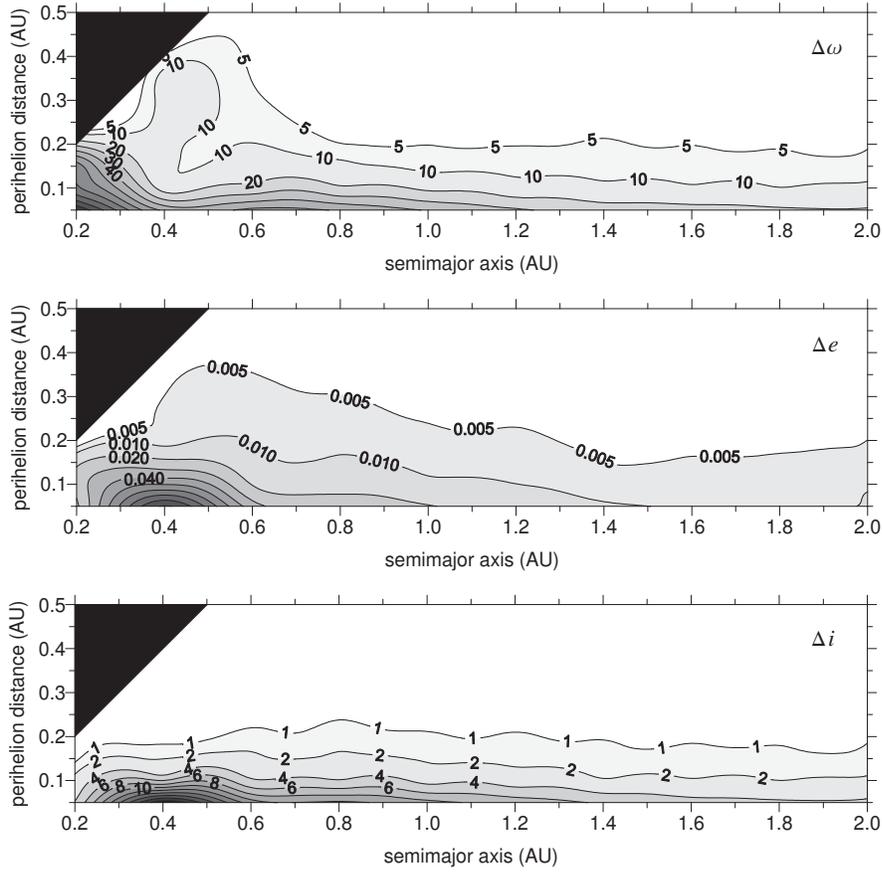


Fig. 8 Maximum differences between classic and relativistic secular dynamics generated by the giant planets as function of the initial (a, q) values in a numerical integration for 10^5 years. All test particles have initial $i = 20^\circ$, $\Omega = 0^\circ$, $\omega = 60^\circ$. The extreme changes observed in (e, i) at $a \sim 0.4$ are due to a strong Kozai mechanism (see Fig. 9). These differences increase when considering the secular effects of the entire planetary system.

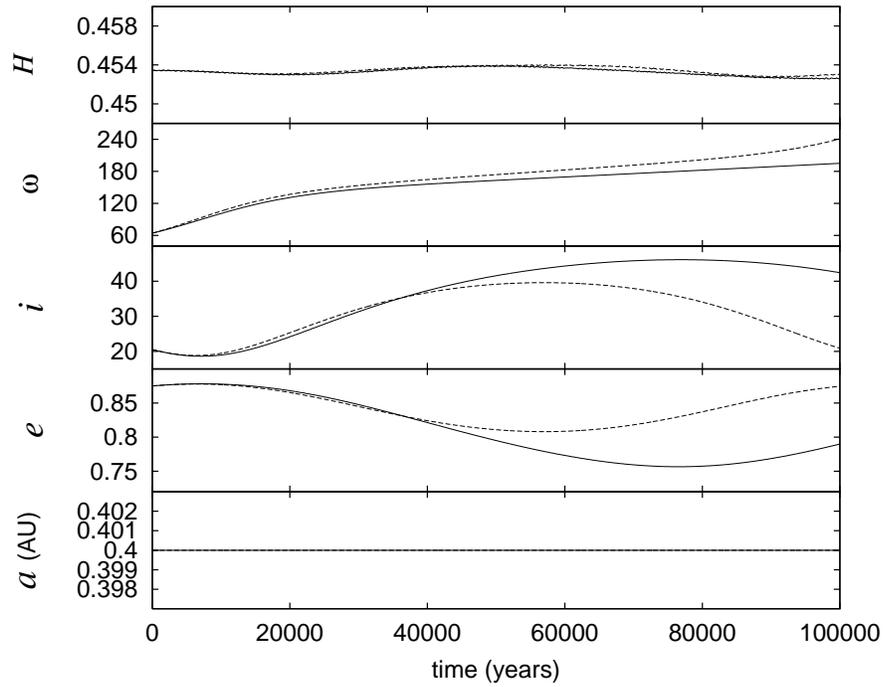


Fig. 9 Time evolution of a fictitious particle inside the secular Kozai mechanism generated by the giant planets. Full line: classical model C; dashed line: relativistic model R_S . In spite of considerable variations in e and i the parameter $H = \sqrt{1 - e^2} \cos i$ remains nearly constant in both models. Relativistic terms diminishes the amplitude of the oscillations in (e, i) .

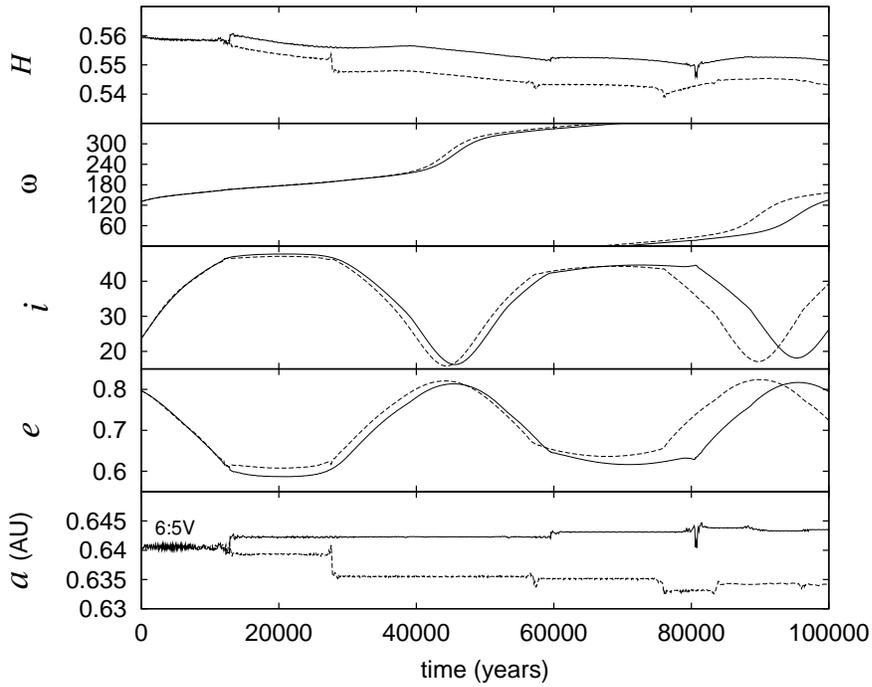


Fig. 10 Asteroid 2004 XY60. At the beginning the asteroid is in the 6:5 resonance with Venus. After 13000 years the strength of the resonance drops and both the classic and relativistic particles are captured in different high order mean motion resonances with Earth.

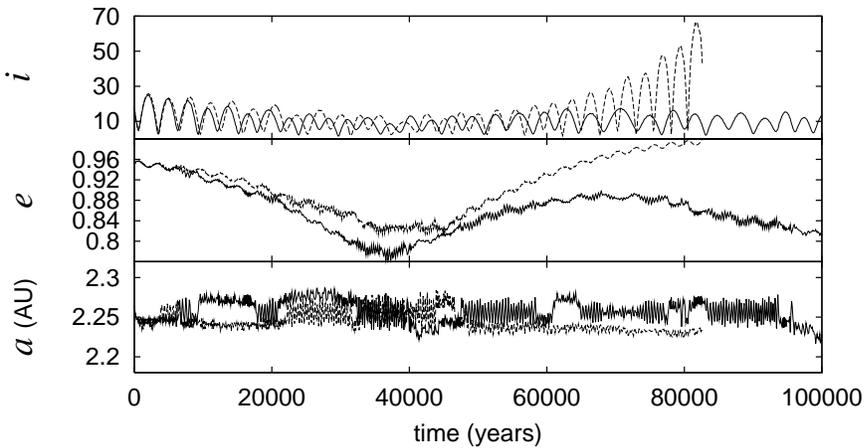


Fig. 11 Asteroid 2000 LK. An example where relativistic effects plus chaotic dynamics generate very different evolutions. Full line: classical model C; dashed line: relativistic model R_s . The relativistic simulation ends up colliding with the Sun. In this case the parameter H is not conserved.

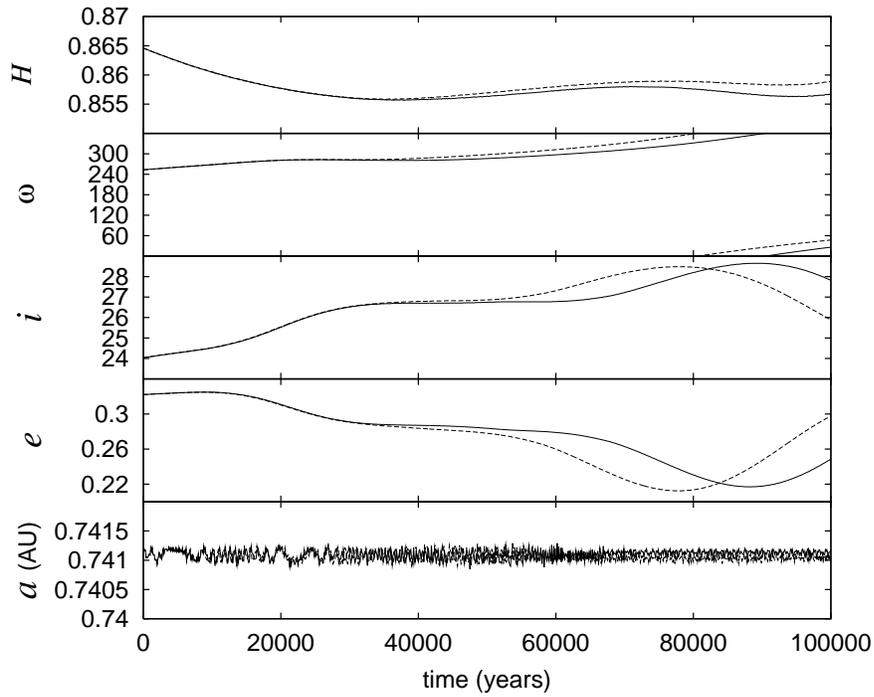


Fig. 12 Asteroid 2003 CP20. An example of evolution without encounters with the planets. Full line: classical model C; dashed line: relativistic model R_S . The asteroid is captured in resonance 27:28 with Venus and follows the effects of the Kozai mechanism.

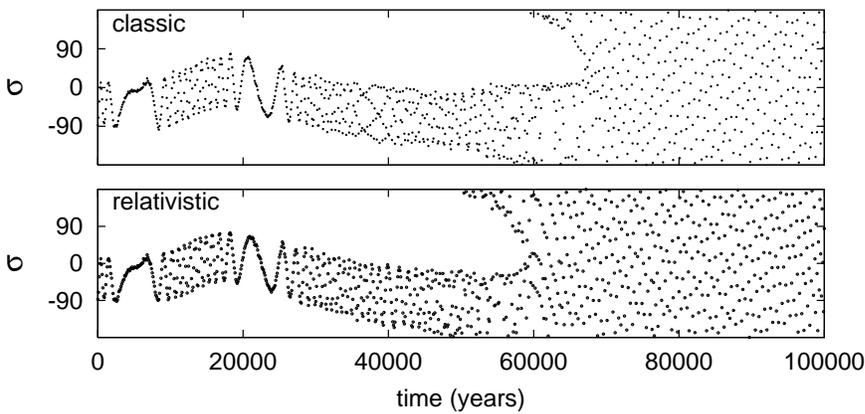


Fig. 13 Asteroid 2003 CP20. The critical angle of the resonant motion is $\sigma = -27\lambda_{Ven} + 28\lambda - \varpi$ according to the classic and relativistic models.