$$= -1 + \frac{x}{b} + \sum_{s=1}^{s=\infty} \frac{(2)(-1)^s}{s\pi} \cos(s\pi) \sin\left(\frac{s\pi x}{b}\right) \epsilon^{-(s\pi/b)^2(t/LG)}. \tag{72}$$

Substituting (71) and (72) in (56),

$$u(x) = \frac{4E}{\pi L} \frac{1}{p} \left[\sum_{s=1}^{s=\infty} \frac{\sin [(2s-1)(\pi x/b)]}{(2s-1)} \epsilon^{-((2s-1)^2\pi^2 t)/(b^2 LG)} \right]. \tag{73}$$

But operationally

$$\frac{1}{p} = \int_{t=0}^{t=t} dt$$

so

$$u(x) = \frac{4E}{\pi L} \cdot \left(-\frac{b^2 LG}{\pi^2} \right) \sum_{s=1}^{s=\infty} \frac{\sin [(2s-1)(\pi x/b)]}{(2s-1)^3} \epsilon^{-((2s-1)^2\pi^2 t)/(b^2 LG)} \Bigg|_{t=0}^{t=t} \tag{74}$$

or

$$u(x) = \frac{4EGb^2}{\pi^3} \sum_{s=1}^{s=\infty} \frac{\sin [(2s-1)(\pi x/b)]}{(2s-1)^3} \left[1 - \epsilon^{-((2s-1)^2\pi^2 t)/(b^2 LG)} \right]. \tag{75}$$

which is (47), a solution of (46).

The Distribution of Current Along a Symmetrical Center-Driven Antenna*

RONOLD KING†, ASSOCIATE, I.R.E., AND CHARLES W. HARRISON, JR.†, ASSOCIATE, I.R.E.

***Summary*—The cylindrical, center-driven antenna is analyzed as a boundary-value problem of electromagnetic theory. An integral equation in the current (originally obtained in a different way by Hallén) is derived. Its solution is outlined briefly and the general formula is given. Complete curves for the distribution of current for a wide range of lengths and ratios of length to radius are given. These include curves showing the components of current in phase with the driving potential difference and in quadrature with this, and curves giving the magnitude of the current and its phase angle referred to the driving potential difference. The conventionally assumed sinusoidal distribution of current is shown to be a fair approximation for extremely thin antennas and for thicker antennas which do not greatly exceed $\lambda/2$ in length.**

INTRODUCTION

THE distribution of current along a center-driven symmetrical antenna of small circular cross section and half-length h is not the same as the distribution along the same conductors when these are folded together to form a closely spaced parallel-wire line of length h. Because the conductors are actually and identically the same in the two arrangements one might legitimately assume that the distributions of current would be similar. On the other hand the geometrical configuration of the two wires differs in such a fundamental way from the point of view of general electromagnetic theory that great differences in the distribution of current might also be expected. The fact is, that two parallel wires which carry equal and opposite currents sufficiently close together in terms of the wavelengths may be analyzed to a good approximation in terms of ordinary electric-circuit theory, whereas the same two wires placed end to end may not. The criterion is this: If the resultant force acting at any instant on the charges in any small element of a conductor due to charges moving at appropriate earlier times in the rest of an extended circuit includes *significant* contributions *only* from neighboring parts of the circuit (that are not more than a very small fraction of a wavelength away) then ordinary electric-circuit theory is a good approximation. In this case radiation is neglected *because it is negligible.* If the spacing b of a parallel-wire line is sufficiently small ($b <<< \lambda$) then the forces on the charges in a given element ds of one of the two wires due to equal and opposite currents and charges in parallel elements which are more distant than ten times the spacing b practically cancel. All significant forces are due to charges moving in immediately adjacent parts of the two wires. In the case of the antenna no such cancellation of forces due to moving charges which are separated more than a small fraction of a wavelength occurs, and ordinary electric-circuit theory is not applicable. This is equivalent to stating that radiation is not negligible.

From the point of view of electromagnetic theory the parallel-wire line with an open end is a special case of the center-driven antenna, and it may be analyzed rigorously as such. On the other hand the antenna is in no fundamental sense a folded-open section of transmission line.

Two methods of attacking the problem of the distribution of current along the center-driven antenna suggest themselves. In the first of these one depends upon the similarity between the antenna and the open-end parallel-wire line, and assumes that by suitably correcting transmission-line theory a satisfactory approximation for the antenna may be devised. One might, for example, measure the input impedance of the

* Decimal classification: R121. Original manuscript received by the Institute, January 20, 1943.

† Cruft Laboratory and Research Laboratory of Physics, Harvard University, Cambridge, Massachusetts.

antenna for a given value of h, and then equate this to the general formula for the input impedance of a terminated section of transmission line of length h. By suitably adjusting two or more of the parameters involved, viz., the attenuation constant α, the phase constant β, the characteristic impedance Z_c, or the terminal impedance $Z=R+jX$, an "equivalent" line can be determined. If the value of h were varied over a wide range the "equivalence" which was established at one value would be only roughly maintained but the order of magnitude would be correctly given. One might then assume that the distribution of current along the "adjusted" parallel-wire line should be a rough approximation of that along the antenna. Correction factors to be applied to line theory in order to approximate the antenna may be devised in many ways, both experimental and theoretical.[1] All such methods are, however, essentially makeshifts which may lead to results which are adequate for many engineering purposes, but which do not actually solve the problem. By skillfully devising enough correction factors any theory can always be made to fit any problem. But such methods are justified only while a more rigorous approach has not been carried out.

The second method of attacking the problem of the distribution of current along a center-driven antenna does not attempt arbitrarily to correct a theory which does not actually apply. It proceeds rather, from the point of view that the antenna is a boundary-value problem in its own right which can certainly be formulated in general terms. If it cannot be solved in closed form, it can at least be evaluated approximately in terms of parameters which characterize the antenna itself, parameters such as the length and radius of the wire, rather than in terms of a characteristic impedance, an attenuation constant, or a terminal impedance which are essentially foreign to the antenna. The antenna was investigated from this general point of view by L. V. King[2] and by Hallén[3] using different but comparable methods. Both are analytically complicated. Actually the problem can be set up formally much more directly than was done by either of these two investigators, and this will be done below. The formulation leads directly to the integral equation obtained by Hallén, rather than to that derived by King. The solution of this equation will be described only briefly because it differs in no essential way from that carried out by Hallén. Since Hallén's paper is not readily available it seems desirable to provide at least an outline of the analysis.

[1] See, for example, S. A. Schelkunoff and C. B. Feldman, "On radiation from antennas," Proc. I.R.E., vol. 30, pp. 511–516; November, 1942.

[2] L. V. King, "On the radiation field of a perfectly conducting base-insulated cylindrical antenna over a perfectly conducting plane earth, and the calculation of radiation resistance and reactance" *Phil. Trans. Royal Soc.* (London), vol. 236, pp. 381–422; November 2, 1937.

[3] E. Hallén, "Theoretical investigations into the transmitting and receiving qualities of antennas," *Nova Acta, Royal Soc. Sciences*, (Uppsala) vol. 11, pp. 1–44, November, 1938.

The Differential Equation

The analytical problem for the determination of the distribution of current in a cylindrical antenna of half-length h and radius a may be formulated in terms of the general boundary condition which requires continuity of the tangential component of the electric field across any boundary surface between two media. If the axis of the antenna is made to fall along the z axis of a system of cylindrical co-ordinates, r, θ, z, the following boundary conditions obtain:

$$(E_z^0)_{r=a} = (E_z^i)_{r=a} \quad \text{on the cylindrical surface} \qquad (1a)$$

$$(E_r^0)_{z=\pm h} = (E_r^i)_{z=\pm h} \text{ on the end faces.} \qquad (1b)$$

The superscript i refers to the interior of the conductor, the superscript 0 to space outside the conductor. The electric field in the conductor everywhere satisfies the relation

$$i = \sigma E. \qquad (2)$$

Here σ is the conductivity and i is the volume density of current. In the idealized case of a perfect conductor the tangential components of E would vanish on the surface. If the end-faces are required to be small so that the following conditions are fulfilled:

$$a \ll h \qquad (3a)$$

$$\beta a = (2\pi a/\lambda) \ll 1, \qquad (3b)$$

then the average electric field $(E_r)_{z=\pm h}$ at the end faces must be less than the average field $(E_z)_{r=a}$ along the cylindrical surface. This follows because i_r near the end faces must vanish at $r=a$, and with (3*b*) it cannot reach a large amplitude between $r=0$ and $r=a$. Accordingly nothing of significance is neglected in so far as the antenna as a whole is concerned if no account is taken of the end faces and hence of $(E_r)_{z=\pm h}$. Thus one may assume the current to vanish at $z=\pm h$ without flowing radially inward on the end faces. (*Note added in proof:* The significance of the end faces and of the approximations involved in neglecting them has been considered by L. Brillouin in a paper which formulates the antenna problem in a mathematically more precise but very much more intricate way. For very thick cylindrical antennas this is important; for moderately thin ones as required by (3). The end faces can certainly have no greater effect than that due to an increase in h by a.)

In carrying out the analysis it will be taken for granted that the cross-sectional and axial distributions of current are mutually independent. This is always true to a very high degree of approximation in a good conductor provided (3) is fulfilled. It is commonly assumed in the derivation of the interval impedance per unit length z^i due to skin effect; it is also assumed in the derivation of the transmission-line equations. Accordingly, one may write

$$(E_z^i)_{r=a} = z^i I_z. \qquad (4a)$$

Here I_z is the total current in the conductor at the cross section z, z^i is the internal impedance in ohms per meter. At high frequencies it is

$$z^i = \frac{1}{2\pi a}\sqrt{\frac{\omega\mu\Pi}{2\sigma}}. \tag{4b}$$

Here $\Pi = 4\pi\times 10^{-7}$ henry per meter; σ is the conductivity in mhos per meter; a the radius in meters, μ the relative permeability of the antenna.

The electric field at outside points is conveniently calculated from the vector potential defined by

$$\operatorname{curl} A = B \tag{5a}$$

$$\operatorname{div} A = -j\,(\omega/c^2)\,\phi \tag{5b}$$

using the relation defining the scalar potential ϕ. It is

$$-\operatorname{grad}\phi = E + j\omega A. \tag{6}$$

This may be written in the form,

$$E = -\frac{jc^2}{\omega}\left(\operatorname{grad}\operatorname{div} A + \frac{\omega^2}{c^2}A\right) \tag{7}$$

if ϕ is eliminated from (6) using (5b).

Except at points very near the end faces one can write

$$A_r \ll A_z. \tag{8a}$$

One also has at all points,

$$A_\theta = 0. \tag{8b}$$

Accordingly the z component of the electric field has the following value except very near the end faces:

$$E_z = -j\frac{\omega}{\beta^2}\left(\frac{d^2A_z}{dz^2} + \beta^2 A_z\right). \tag{9}$$

Here
$$\beta^2 \equiv \frac{\omega^2}{c^2}. \tag{10}$$

Upon substituting (9) and (4) in (1) the following differential equation in the vector potential is obtained:

$$\frac{d^2A_z^0}{dz^2} + \beta^2 A_z^0 = j\frac{\beta^2}{\omega}z^i I_z. \tag{11}$$

The vector potential is thus seen to satisfy a one-dimensional wave equation which is homogeneous in the idealized case of an antenna which is a perfect conductor so that $z^i = 0$. It is readily verified using (5b) that the scalar potential satisfies an entirely similar equation. The total current does not satisfy such a simple equation as will be shown directly.

The Formal Solution of the Equation

The differential equation (11) is a nonhomogeneous equation which has a general solution involving the sum of a complementary function A_c^0 and a particular integral A_p^0. The former may be written in the form

$$A_c^0 = \frac{-j}{c}\left[C_1\cos\beta z + C_2\sin\beta z\right] \tag{12a}$$

with C_1 and C_2 arbitrary constants of integration. A particular integral is

$$A_p^0 = \frac{jz^i}{c}\int_0^z I(s)\sin\beta(z-s)ds. \tag{12b}$$

It is readily verified by substituting (12b) in (11) that it satisfies the equation. Thus the general solution of (11) is

$$A_z^0 = \frac{-j}{c}\left[C_1\cos\beta z + C_2\sin\beta z - z^i\int_0^z I(s)\sin\beta(z-s)ds\right]. \tag{13}$$

Let it be required that the antenna under consideration be symmetrical with respect to a pair of closely spaced driving points 0 and 0′ at its center in such a way that the following symmetry conditions obtain:

$$I(z) = I(-z); \qquad A^0(z) = A^0(-z). \tag{14}$$

The relation (13) is easily specialized to satisfy (14) by writing $|z|$ for z in $\sin\beta z$. Thus

$$A_z^0 = \frac{-j}{c}\left[C_1\cos\beta z + C_2\sin\beta|z| - z^i\int_0^z I(s)\sin\beta(z-s)ds\right]. \tag{15}$$

It is readily verified that (15) is unchanged if $-z$ is everywhere written for z. (In the integral the variable is changed by writing $s = -u$ after writing $-z$ for z.)

The Driving-Potential Difference

Let it be assumed that a driving-potential difference V_0^e is maintained between the two terminals 0 and 0′ which are assumed to be separated an infinitesimal distance. In practice, terminals are always separated a finite distance but it is here postulated that it is in any case a negligible fraction of a wavelength. The actual case is readily reduced to the assumed one as shown in Fig. 1. The actual terminals are A and B and a transmission line is connected to them, as shown on the left. By filling the gap between A and B in the manner shown on the right, the equal and opposite currents in the indefinitely close parallel conductors from B' to 0′ and from A' to 0 completely cancel in so far as could be determined at outside points. Thus the antenna may be assumed to extend without break across AB; it includes a point generator maintaining the potential difference V_0^e across its terminals. In the same way the transmission line may be taken to extend from A' to B' without break with a point load concentrated midway between A' and B'.

The boundary condition on the scalar potential is

$$V_0^e = \lim_{z\to 0}\{\phi^0(+z) - \phi^0(-z)\}. \tag{16}$$

From (5b) one has, since $A_r \ll A_z$, $A_\theta = 0$,

$$\frac{\partial A_z^0}{\partial z} = -j\frac{\omega}{c^2}\phi^0. \tag{17a}$$

Also,
$$\frac{\partial A_z^0(+z)}{\partial z} = -j\frac{\omega}{c^2}\phi^0(+z) \tag{17b}$$

$$\frac{\partial A_z^0(-z)}{-\partial z} = -j\frac{\omega}{c^2}\phi^0(-z) \tag{17c}$$

so that
$$\phi^0(-z) = -\phi^0(z) = j\frac{c^2}{\omega}\frac{\partial A_z^0(-z)}{\partial z} \tag{18}$$

and
$$V_0^e = 2\lim_{z\to 0}\phi^0(+z) = \frac{2jc^2}{\omega}\lim_{z\to 0}\frac{\partial A_z^0(+z)}{\partial z}. \tag{19}$$

Upon differentiating (15) with respect to z and allowing z to approach zero one has

$$\lim_{z\to 0}\left(\frac{\partial A_z^0}{\partial z}\right) = \frac{-j\beta}{c} C_2 \tag{20}$$

so that with $\beta=\omega/c$, one obtains

$$C_2 = \tfrac{1}{2}V_0^e. \tag{21}$$

The Integral Equation

It is shown in Appendix I that the vector potential at all points outside a cylindrical conductor (including its surface) except those within distances of an end face comparable with its radius is given to a good approximation by

$$A_z^0 = \frac{\Pi}{4\pi}\int_{-h}^{+h} I_z' \frac{e^{-j\beta R}}{R}\,dz'. \tag{22}$$

Here R is the distance from the point (r,θ,z) outside the conductor where A_z^0 is calculated to the center of the element dz' at z' on the axis. That is,

$$R = \sqrt{(z-z')^2 + r^2}. \tag{23}$$

The universal magnetic constant is

$$\Pi = 4\pi\times 10^{-7} \text{ henry per meter.} \tag{24}$$

If the integral (15) is specialized to the surface of the antenna, i.e., to $r=a$, and is then substituted in (13) one obtains

$$j\frac{c\Pi}{4\pi}\int_{-h}^{+h} I_z' \frac{e^{-j\beta R}}{R}\,dz' = C_1\cos\beta z + \tfrac{1}{2}V_0^e \sin\beta|z| - z^i\int_0^z I(s)\sin\beta(z-s)\,ds. \tag{25}$$

In terms of the fundamental electric constant

$$\Delta = 8.85\times 10^{-12} \text{ farad per meter} \tag{26}$$

and the magnetic constant defined in (24) one has

$$c = \frac{1}{\sqrt{\Pi\Delta}} = 3\times 10^8 \text{ meters per second} \tag{27a}$$

and,

$$R_c = \sqrt{\frac{\Pi}{\Delta}} = c\Pi = 376.7 \text{ ohms.} \tag{27b}$$

Then

$$c\Pi/4\pi = R_c/4\pi = 30 \text{ ohms.} \tag{28}$$

This may be substituted in (25). The notation in terms of R_c (ohms) will be retained so that simple dimensional relations are at all times in view.

As a first step in the solution, the integral on the left in (25) may be expanded in the following way:

$$\int_{-h}^{+h} I_z' \frac{e^{-j\beta R}}{R}\,dz' = I_z\int_{-h}^{+h}\frac{dz'}{R} + \int_{-h}^{+h}\frac{I_z' e^{-j\beta R} - I_z}{R}\,dz'. \tag{29}$$

The first integral on the right can now be evaluated directly. It is

$$\int_{-h}^{+h}\frac{dz'}{R} = \ln\left[\frac{\sqrt{(h-z)^2+a^2}+(h-z)}{\sqrt{(h+z)^2+a^2}-(h+z)}\right]. \tag{30}$$

With the notation,

$$\Omega \equiv 2\ln\left(\frac{2h}{a}\right) \tag{31}$$

$$\delta \equiv \ln\left\{\frac{1}{4}\left[\sqrt{1+\left(\frac{a}{h-z}\right)^2}+1\right]\cdot\left[\sqrt{1+\left(\frac{a}{h+z}\right)^2}+1\right]\right\}. \tag{32}$$

Equation (30) may be written as follows:

$$\int_{-h}^{+h}\frac{dz'}{R} = \Omega + \ln(1 - z^2/h^2) + \delta. \tag{33}$$

It is to be noted that δ is negligible except very near the ends of the antenna and that (30), (which includes

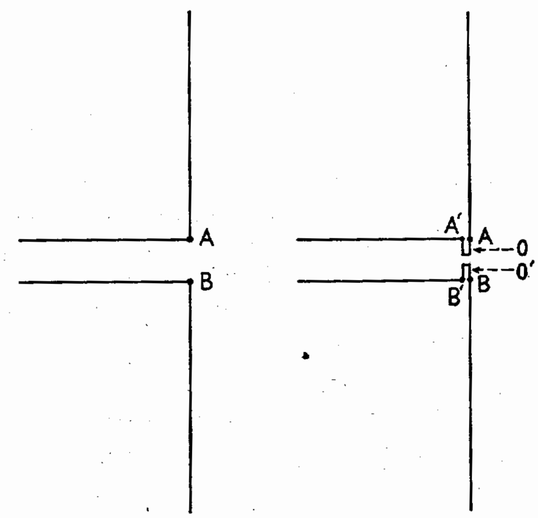

Fig. 1—Actual method of feeding and analytically equivalent method.

δ) is everywhere finite reducing to the following value at the ends:

$$\left[\int_{-h}^{+h}\frac{dz'}{R}\right]_{z=\pm h} = \tfrac{1}{2}\Omega + \ln 2. \tag{34}$$

Upon substituting (33) in (29) and then inserting (29) in (25) one readily obtains

$$\begin{aligned} I_z = \frac{-j4\pi}{\Omega R_c}&\left\{C_1\cos\beta z + \tfrac{1}{2}V_0^e\sin\beta|z| - z^i\int_0^z I(s)\sin\beta(z-s)\,ds\right\} \\ &- \frac{1}{\Omega}\left\{I_z\ln(1-z^2/h^2) + I_z\delta + \int_{-h}^{+h}\left(\frac{I_z' e^{-j\beta R} - I_z}{R}\right)dz'\right\}. \end{aligned} \tag{35}$$

Since the current vanishes at the ends and $\ln(1-z^2/h^2)+\delta$ remains finite according to (33) and (34), one has, with $z=h$,

$$\begin{aligned} 0 = \frac{-j4\pi}{\Omega R_c}&\left\{C_1\cos\beta h + \tfrac{1}{2}V_0^e\sin\beta h\right\} \\ &- \frac{1}{\Omega}\left\{\frac{j4\pi z^i}{R_c}\int_0^h I(s)\sin\beta(h-s)\,ds - \int_{-h}^{+h} I_z'\frac{e^{-j\beta R_h}}{R_h}\,dz'\right\}. \end{aligned} \tag{36}$$

Here

$$R_h = \sqrt{(h-z')^2 + a^2}. \tag{37}$$

If (36) is subtracted from (35) one has

$$I_z = \frac{-j4\pi}{\Omega R_c}\{C_1(\cos\beta z - \cos\beta h) + \tfrac{1}{2}V_0^e(\sin\beta|z| - \sin\beta h)\} - \frac{1}{\Omega}\left\{I_z \ln(1 - z^2/h^2) + I_z\delta + \int_{-h}^{+h}\left(\frac{I_z' e^{-j\beta R} - I_z}{R}\right)dz' - \frac{j4\pi z^i}{R_c}\int_0^z I(s)\sin\beta(z-s)ds\right\} + \frac{1}{\Omega}\left\{\int_{-h}^{+h} I_z' \frac{e^{-j\beta R_h}}{R_h}dz' - \frac{j4\pi z^i}{R_c}\int_0^h I(s)\sin\beta(h-s)ds\right\}. \quad (38)$$

This expression is the same as that originally derived by Hallén in a somewhat different way. Since its evaluation from this point on follows in all essential respects the method used by Hallén it will not be reproduced in detail. A brief outline is given in Appendix II because Hallén's paper is not generally available at the present time.

The First-Order Solution

By the method of successive approximations outlined in Appendix II, (38) may be expressed in the form of a series in the small quantity $1/\Omega$. By substituting this series in (36) the constant of integration C_1 may be evaluated. The zeroth and first-order terms in the solution are

$$I_z = \frac{j2\pi V_0^e}{\Omega R_c}\left\{\frac{\sin\beta(h-|z|) + (1/\Omega)\,[M_1^I + jM_1^{II}]}{\cos\beta h + (1/\Omega)\,[A_1^I + jA_1^{II}]}\right\}. \quad (39)$$

Terms involving factors of order $1/\Omega^2$, $1/\Omega^3$, etc., are neglected in (39). The real functions M_1^I, M_1^{II}, A_1^I, and A_1^{II}, which are functions of h and z only and not at all of the radius a, are defined as follows in terms of the complex F and G functions given in Appendix II.

$$M_1^I + jM_1^{II} \equiv F_1(z)\sin\beta h - F_1(h)\sin\beta|z| + G_1(h)\cos\beta z - G_1(z)\cos\beta h \quad (40)$$

$$A_1^I + jA_1^{II} \equiv F_1(h). \quad (41)$$

These have been computed for several values of h as shown graphically in Figs. 2 to 6.

Let the numerator in the brace of (39) be denoted by

$$N^I + jN^{II} = Ne^{j\psi_n} \quad (42a)$$

with

$$N^I = \sin\beta(h - |z|) + M_1^I/\Omega; \quad N^{II} = M_1^{II}/\Omega. \quad (42b)$$

Similarly let the denominator in (39) be

$$D^I + jD^{II} = De^{j\psi_D} \quad (42c)$$

with $D^I = \cos\beta h + A_1^I/\Omega$; $D^{II} = A_1^{II}/\Omega$. (42d)

Also let

$$f' = N\cos(\psi_D - \psi_N) \quad (42e)$$

$$f'' = N\sin(\psi_D - \psi_N). \quad (42f)$$

Both f' and f'' are functions of z, while D is not. With this notation, (39) reduces to

$$I_z = \frac{2\pi V_0^e}{\Omega R_c D}(f'' + jf') = \frac{V_0^e}{60\Omega D}(f'' + jf'). \quad (43)$$

In amplitude-phase-angle form one has

$$I_z = \frac{V_0^e}{60\Omega D}\sqrt{(f'')^2 + (f')^2}\,e^{j\theta} \quad (44)$$

with

$$\theta = \tan^{-1}\left(\frac{f'}{f''}\right). \quad (45)$$

If the applied voltage varies according to

$$v_0^e = V_0^e \sin\omega t \quad (46)$$

then the instantaneous current at a distance z from the center of the symmetrical center-driven antenna is

$$i_z = \frac{V_0^e}{60\Omega D}\sqrt{(f'')^2 + (f')^2}\sin(\omega t + \theta). \quad (47)$$

The distribution of current along a cylindrical antenna which satisfies the condition

$$\Omega^2 \gg 1 \quad (48)$$

has thus been obtained.

If (48) is interpreted to mean

$$\Omega \geqq 10, \quad (49)$$

one can also write as an equivalent

$$\frac{h}{a} \geqq 75. \quad (50)$$

Curves showing the functions f'' and f' for use with the formula (43) are reproduced in Figs. 7 to 12 for several lengths and three different thicknesses covering most of the practical range. Actually it is merely necessary to multiply the values of f'' or f' in the curves by $1/60\Omega D$ in order to obtain the corresponding components of current in amperes per input volt. Numerical values of this factor for the several cases plotted in the figures are given in Table I. Curves giving $\sqrt{(f'')^2+(f')^2}$ and $\theta = \tan^{-1}(f'/f'')$ for use with (44) are shown in Figs. 13 to 18. Thus Figs. 7 to 18 together with (43) and (44) completely characterize the distribution of current along a typical center-fed antenna of circular cross section with radius a and of length h. Before discussing these general results it is well to consider first the input impedance and then two special cases.

The Input Impedance

In considering the significance of the distribution curves for current it is instructive to examine simultaneously the input impedance of the antenna. This is defined simply as the potential difference V_0^e maintained at the input terminals divided by the input current. It may be obtained directly from (39) by writing $z=0$. Thus

$$Z_{00} = \frac{V_0^e}{I_0} = \frac{-j\Omega R_c}{2\pi}\left\{\frac{\cos\beta h + (1/\Omega)(A_1^I + jA_1^{II})}{\sin\beta h + (1/\Omega)(B_1^I + jB_1^{II})}\right\} \quad (51)$$

with

$$B_1^I + iB_1^{II} = F_1(0)\sin\beta h + G_1(h) - G_1(0)\cos\beta h. \quad (52)$$

TABLE I

$\frac{h}{\lambda}$	H (radians)	$\frac{1}{60\Omega D}$ (multiply all numbers by 10^{-3})			$Y_{00}=G_{00}-jB_{00}$ (in mhos $\times 10^{-3}$)		
		$\Omega=10$	$\Omega=20$	$\Omega=30$	$\Omega=10$	$\Omega=20$	$\Omega=30$
0.2425	1.538	13.396	14.094	13.805	15.54 $-j3.67$	15.36 $-j0.0$	13.8 $+j3.67$
0.25	1.571	11.819	11.815	11.810	12.74 $-j5.9$	11.39 $-j5.90$	11.04 $-j5.9$
0.375	2.356	2.320	1.176	0.785	0.992 $-j1.443$	0.255 $-j0.805$	0.115 $-j0.552$
0.50	3.141	1.820	0.870	0.589	0.574 $+j0.423$	0.140 $+j0.095$	0.066 $+j0.041$
0.625	3.927	2.806	1.284	0.830	0.834 $+j2.806$	0.174 $+j1.11$	0.073 $+j0.675$
0.75	4.712	8.670			10.2 $-j3.54$		

This is exactly the expression from which curves for the input impedance have been computed.[4]

The Distribution of Current for an Indefinitely Thin Antenna

The distribution of current along an indefinitely thin antenna is obtained from (39) by allowing the radius a to approach zero. This is equivalent to allowing the parameter Ω to approach infinity. Let the applied potential difference V_0^e be increased with Ω so that the ratio V_0^e/Ω remains finite. One then has

$$I_z = \frac{j2\pi V_0^e}{\Omega R_c} \frac{\sin\beta(h-|z|)}{\cos\beta h}. \tag{53}$$

The input impedance is formally expressed by

$$Z_{00} = jX_{00} = -j\Omega\,(R_c/2\pi)\cot\beta h. \tag{54}$$

(It is to be noted that (53) and (54) are actually good approximations for an antenna of very small but non-vanishing radius over those limited parts of the ranges of βh and $\beta(h-|z|)$ for which the trigonometric factors in both numerator and denominator in (39) and in (51) are large compared with the magnitudes of the factors involving $1/\Omega$.)

The distribution of current along an indefinitely thin antenna is seen to be very simple in form. Referred to the input current I_0, defined by

$$I_0 = j\frac{2\pi V_0^e}{\Omega R_c}\tan\beta h, \tag{55}$$

it is

$$I_z = I_0 \frac{\sin\beta(h-|z|)}{\sin\beta h}. \tag{56}$$

Or in terms of the maximum value defined by

$$I_{\max} = I_0/\sin\beta h \tag{57}$$

it is

$$I_z = I_{\max}\sin\beta(h-|z|). \tag{58}$$

The current I_{max} is fictitious in all antennas for which h is shorter than $\lambda/4$.

The distribution (56), or its equivalent (58), is the one usually assumed for all straight antennas regardless of radius. It is here shown to be strictly correct for an antenna of indefinitely small radius. Distribution

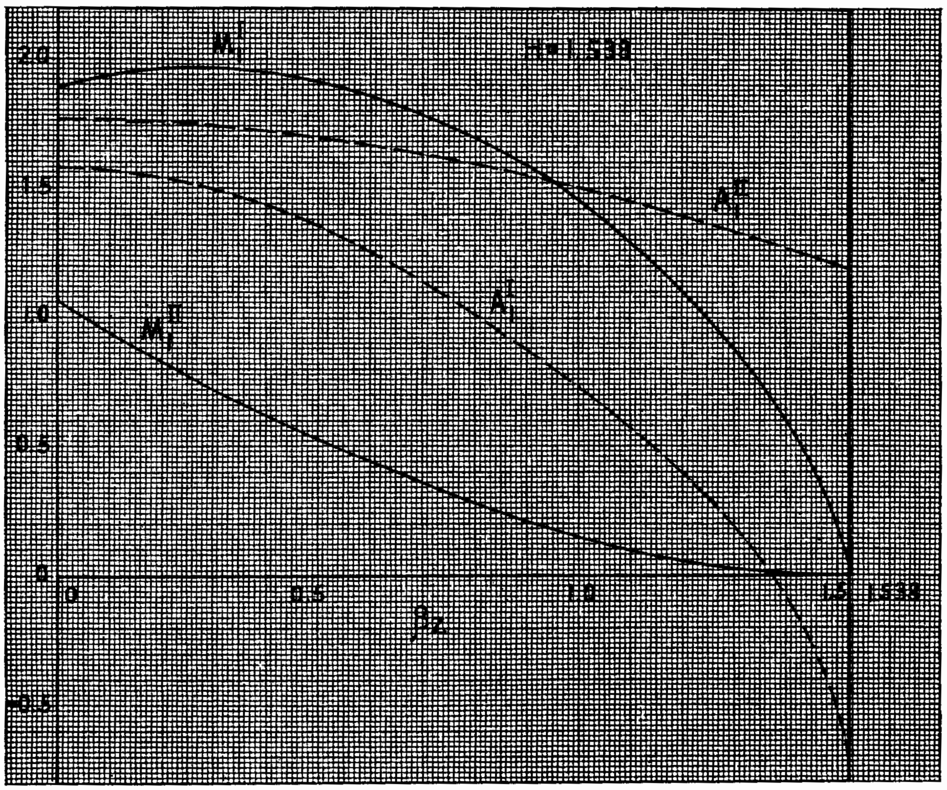

Fig. 2—The function A_1^I, A_1^{II}, M_1^I, and M_1^{II} for $H=1.538$.

[4] Ronold King and F. G. Blake, "The self-impedance of a symmetrical antenna," Proc. I.R.E., vol. 30, pp. 335–349; July, 1942.

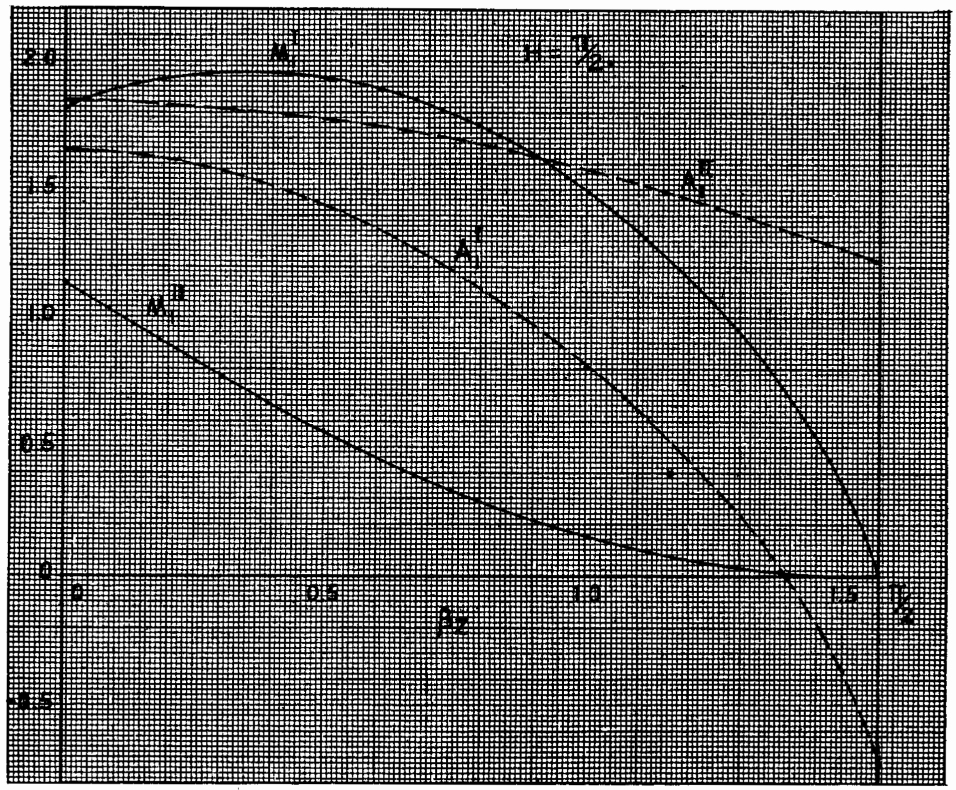

Fig. 3—The functions $A_1{}^I$, $A_1{}^{II}$, $M_1{}^I$, and $M_1{}^{II}$ for $H=\pi/2$.

curves computed from (58) are well known. A few are shown in Figs. 13 to 18 marked sine curve.

The input impedance of an infinitely thin antenna as given by (54) with Ω increasing without limit requires X_{00} to be negatively infinite for βh between 0 and $\pi/2$, π and $3\pi/2$, etc., and positively infinite for βh between $\pi/2$ and π, $3\pi/2$ and 2π, etc. The values at $\beta h=\pi/2$, π, $3\pi/2$, etc., are indeterminate. The formula (54) would be a good approximation for an extremely thin antenna except near the values of βh listed above. It is not a good approximation for thick antennas.

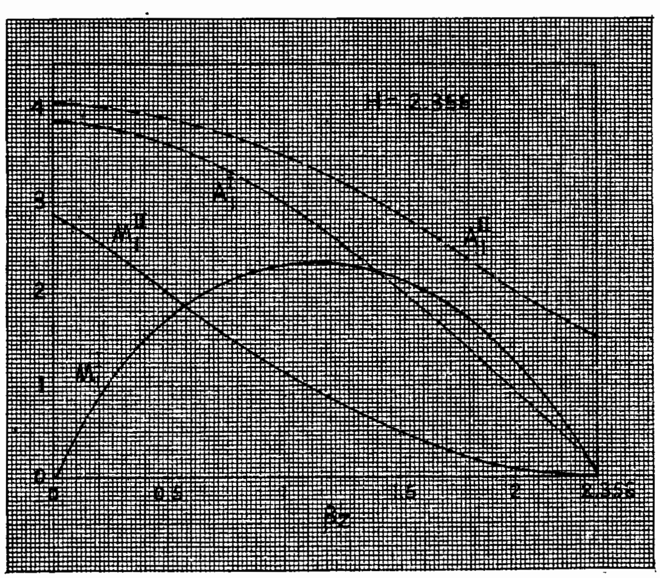

Fig. 4—The functions $A_1{}^I$, $A_1{}^{II}$, $M_1{}^I$, and $M_1{}^{II}$ for $H=3\pi/4$.

The Distribution of Current for an Antenna Approximately a Half Wavelength Long

The simple sinusoidal form (53) for the distribution of current and the equally simple expression (54) for the input impedance are not useful at $\beta h=n\pi/2$ with n any integer even for indefinitely thin antennas. At $\beta h=n\pi/2$, one has from (39) and (51)

$$I_z = \frac{j2\pi V_0{}^e}{R_c}\left\{\frac{\cos \beta z + (1/\Omega)\ [M_1{}^I + jM_1{}^{II}]}{A_1{}^I + jA_1{}^{II}}\right\} \tag{59}$$

$$Z_{00} = \frac{-jR_c}{2\pi}\left\{\frac{A_1{}^I + jA_1{}^{II}}{1 + (1/\Omega)\ (B_1{}^I + jB_1{}^{II})}\right\}. \tag{60}$$

These formulas are limited only by (48).

Since the functions A, B, and M appearing in (59) and (60) depend upon the radius a only through terms involving the ohmic resistance (which are negligible in good conductors), it follows that (59) and (60) depend upon the radius only through the one term in which $1/\Omega$ appears as a factor. That means that the distribution of current (59) and the input impedance (60) of antennas for which $h=\lambda/4$ will vary only slightly with radius as compared with antennas of other lengths which depend on the general expressions (39) and (51) that have Ω as a factor in all terms. Thus, one might expect that a reasonably satisfactory approximation for a moderately thin antenna would be given by neglecting the terms in $1/\Omega$ in (59) and (60). If this is done one obtains formulas which are *independent* of the radius, and which are strictly accurate in the limit as the radius is made increasingly small

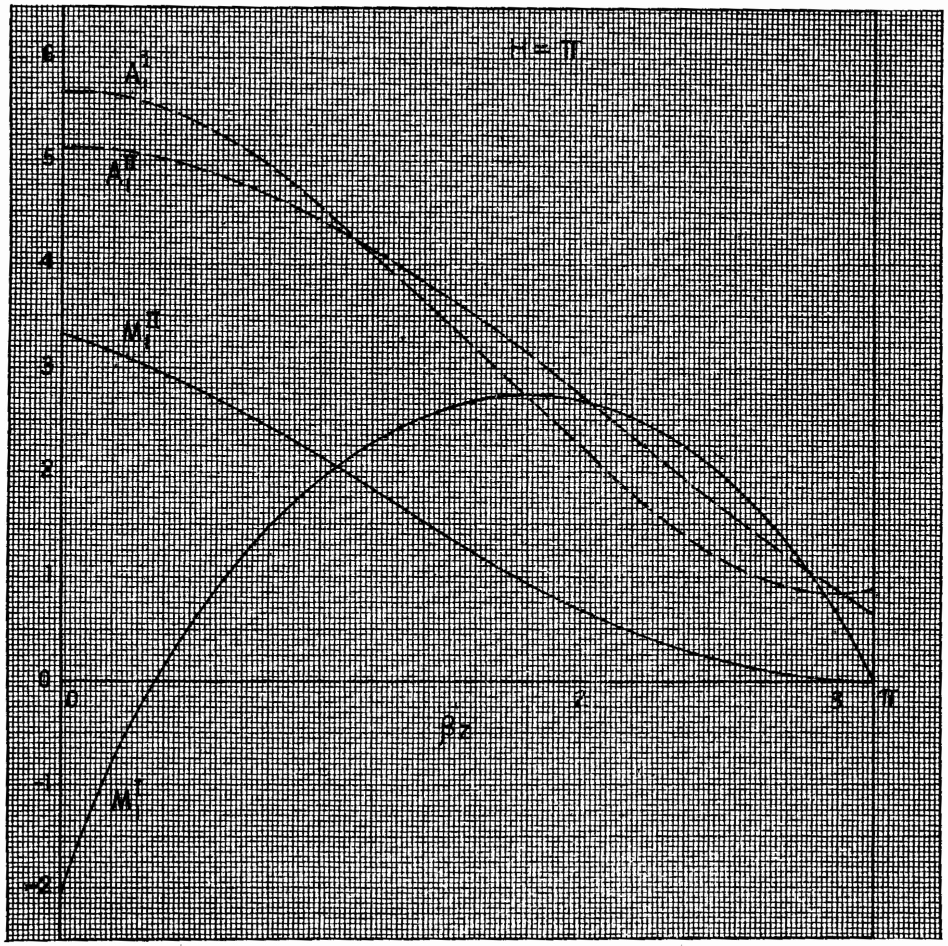

Fig. 5—The functions $A_1{}^I$, $A_1{}^{II}$, $M_1{}^I$, and $M_1{}^{II}$ for $H=\pi$.

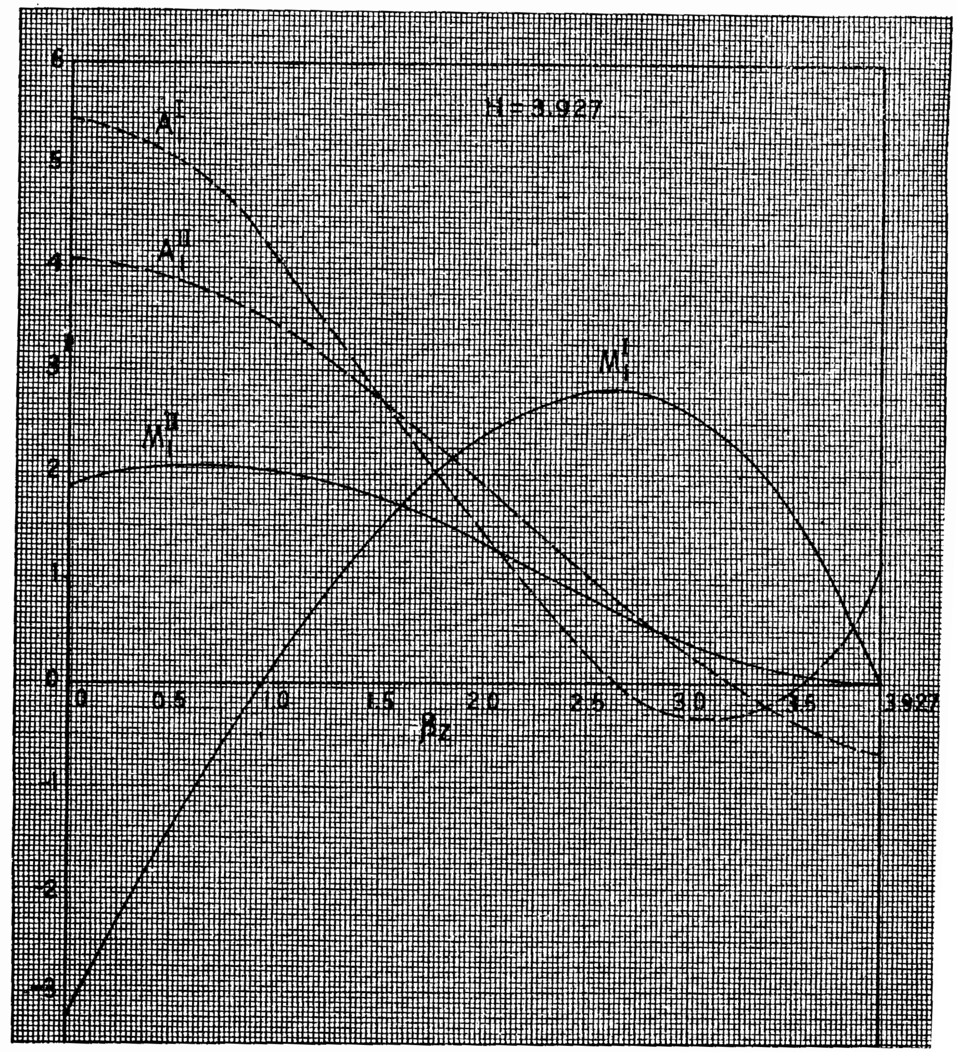

Fig. 6—The functions $A_1{}^I$, $A_1{}^{II}$, $M_1{}^I$, and $M_1{}^{II}$ for $H=5\pi/4$.

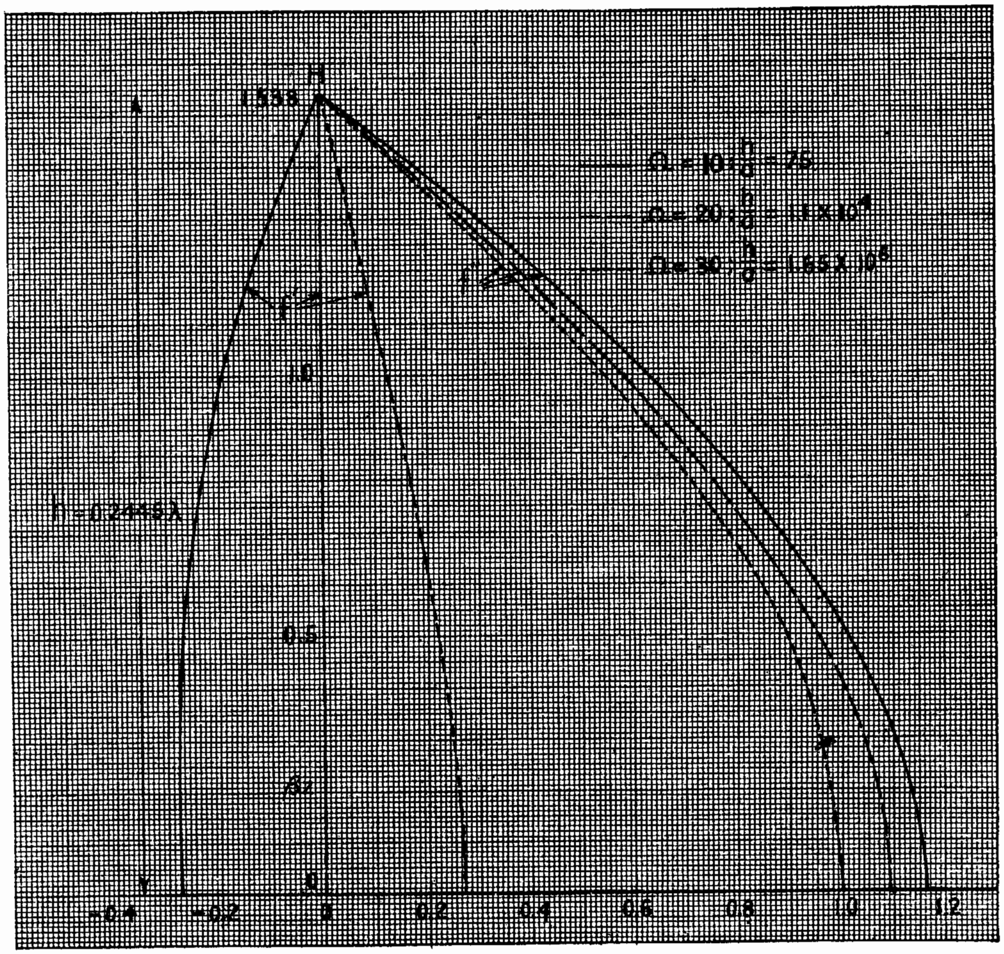

Fig. 7—The distribution functions f' and f'' for $H=1.538$.

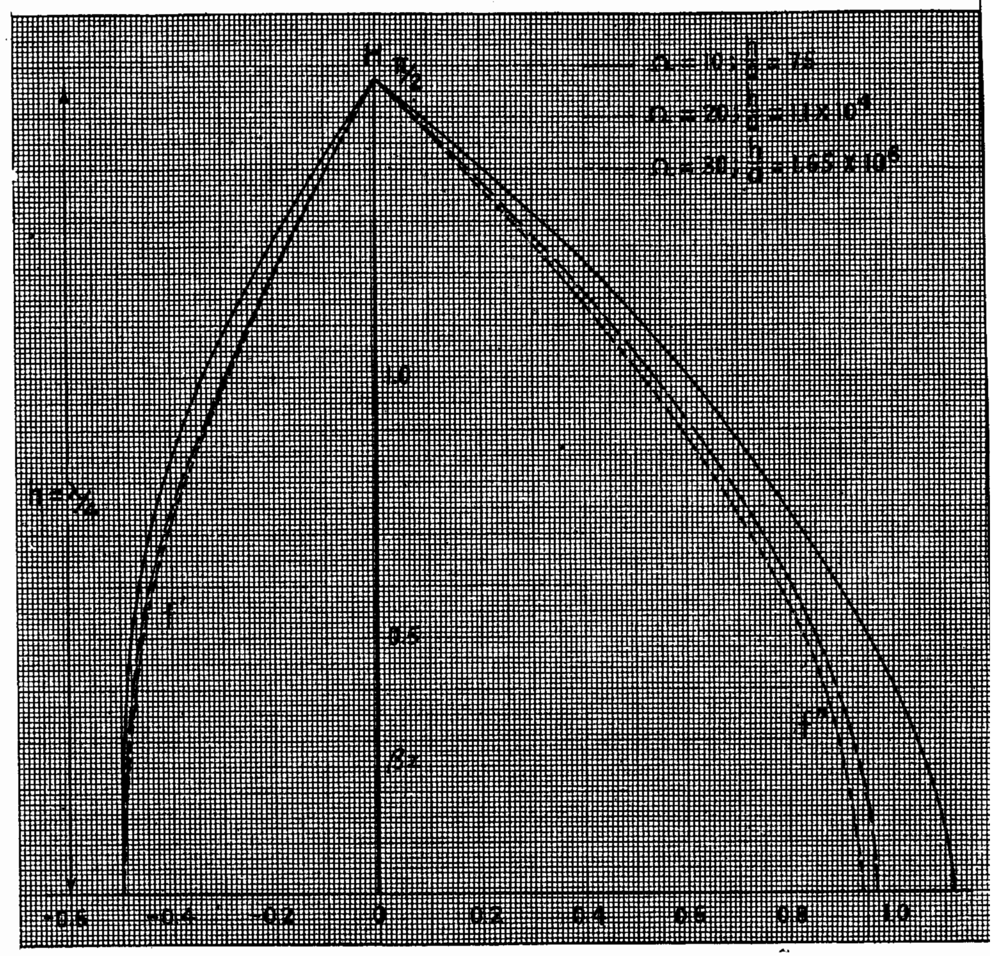

Fig. 8—The distribution functions f' and f'' for $H=\pi/2$.

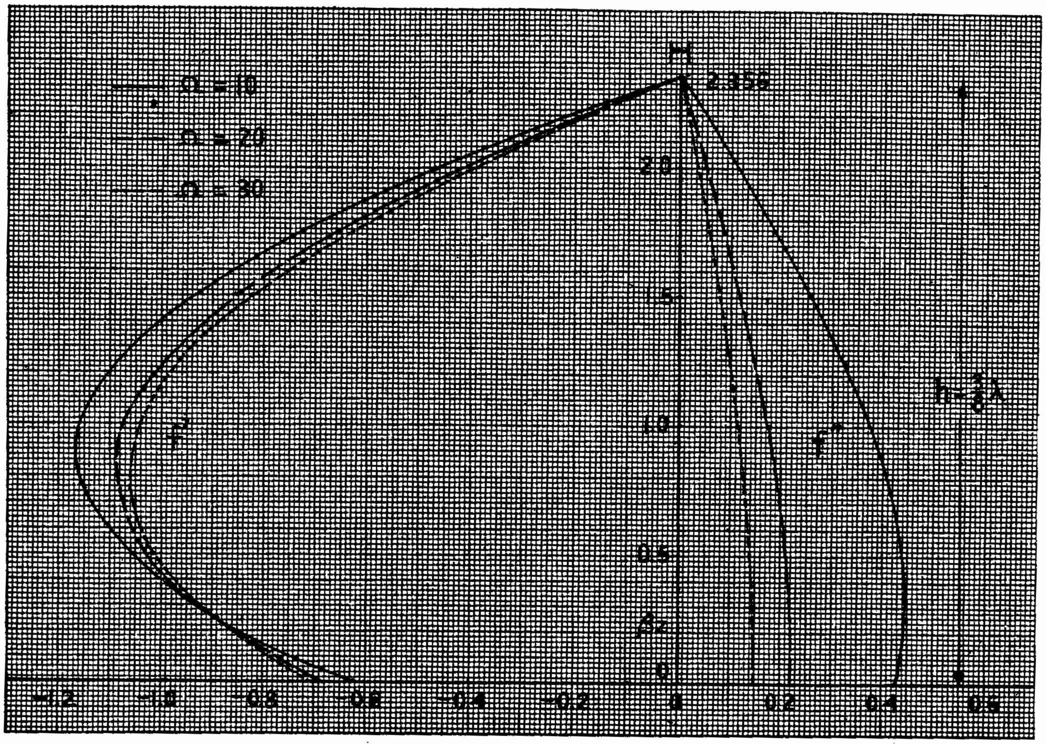

Fig. 9—The distribution functions f' and f'' for $H=3\pi/4$.

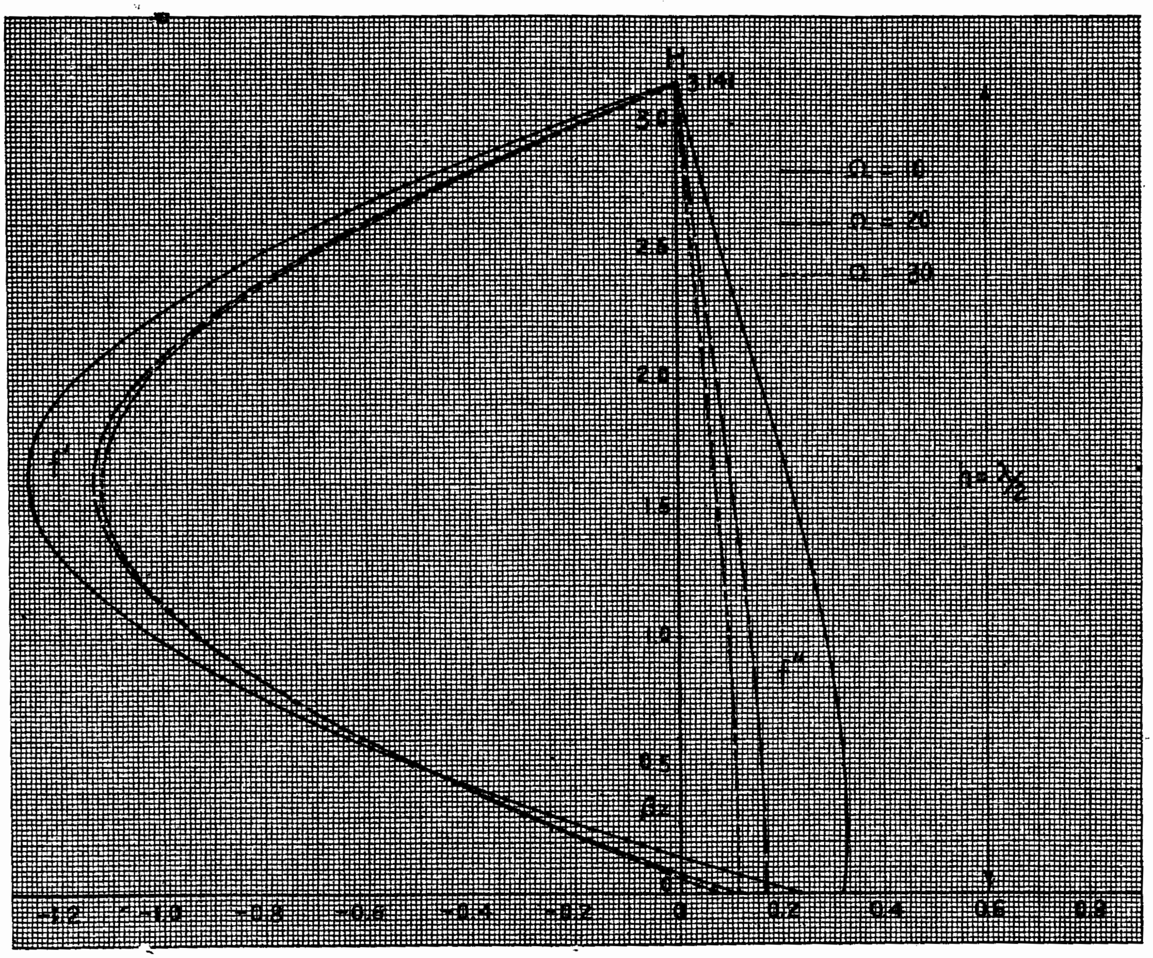

Fig. 10—The distribution functions f' and f'' for $H=\pi$.

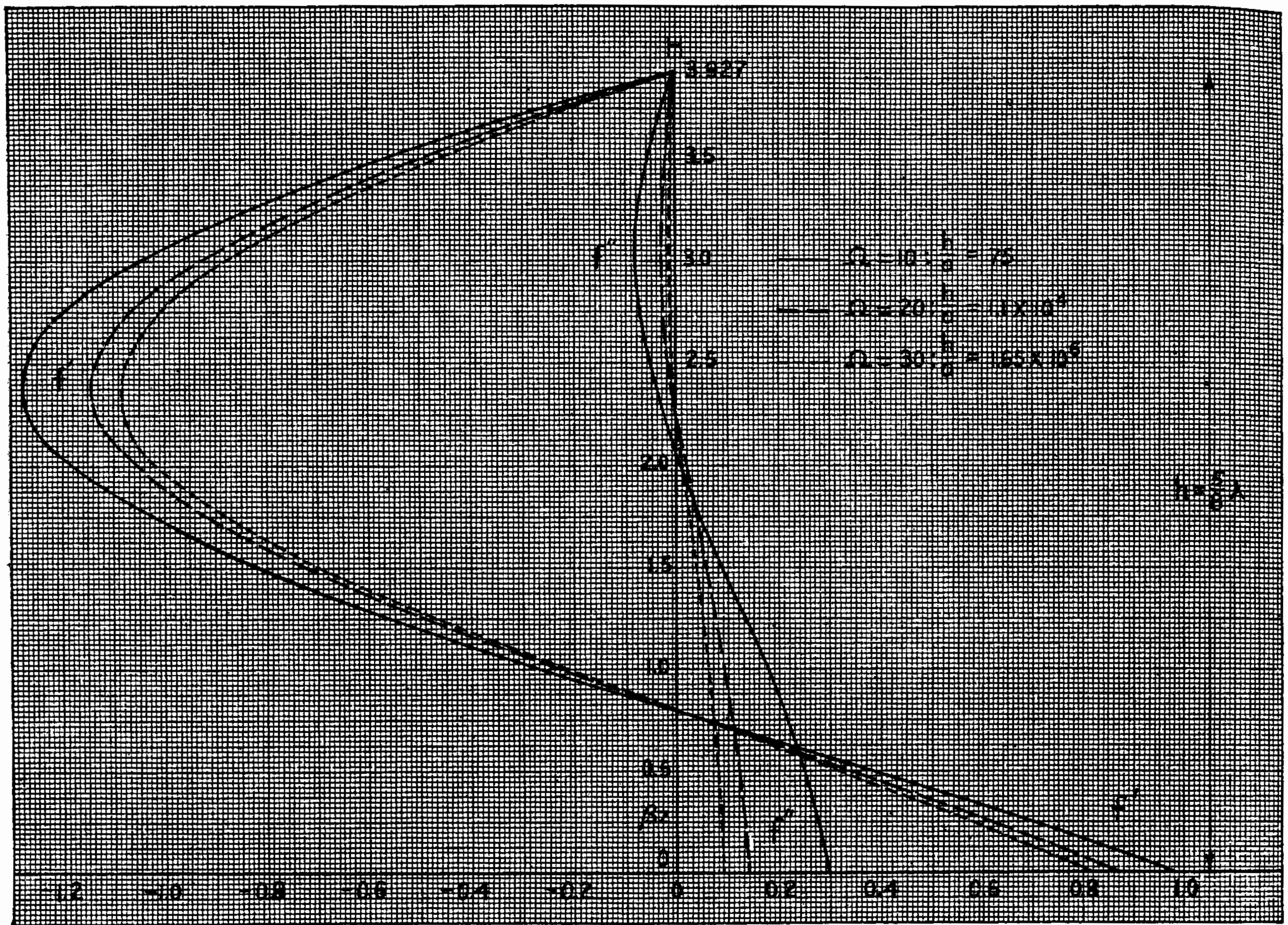

Fig. 11—The distribution functions f' and f'' for $H=5\pi/4$.

after βh has been fixed at $n\pi/2$. It can be seen from the left half of Fig. 8 in footnote reference 4 that as the ratio a/λ becomes smaller and smaller the input-reactance curve approaches the vertical and its intersection with the $X_{00}=0$ axis moves to the right. Just before one passes to the actual limit, $a=0$, the reactance curve intersects the $X_{00}=0$ axis an infinitesimal distance to the left of $\beta h=\pi/2$, while its value at $\beta h=\pi/2$ is just a trifle below $X_{00}=42.5$. In the limit, the reactance curve is a vertical line from $-\infty$ to $+\infty$ at $\beta h=\pi/2$, so that values of $X_{00}=0$ or $X_{00}=42.5$ are equally correct but actually meaningless. The formulas which apply to $\beta h=\pi/2$ with the radius a approaching, but not quite reaching, zero are given below. They are not correct for the condition of resonance $X_{00}=0$, which occurs indefinitely near $\beta h=\pi/2$, as $a\rightarrow 0$ but considerably below this value even for small radii.

$$I_z = (V_0^e/Z_{00}) \cos \beta z \qquad (61)$$

with $Z_{00} = R_{00} + jX_{00} = (R_c/2\pi)(A_1^{II} - jA_1^{I}).$ (62)

For $\beta h=\pi/2$ the numerical value is

$$Z_{00} = 73.13 + j42.5 = 84.5 \angle 30°.2. \qquad (63)$$

The formula (61) is as simple in form as (53) and permits writing

$$I_z = I_0 \cos \beta z \qquad (64)$$

with I_0 now complex and given by

$$I_0 = V_0^e/Z_{00} \qquad (65)$$

instead of the pure imaginary defined by (55). Thus (56) and (58) apply to an almost infinitely thin antenna with $\beta h=\pi/2$ and with $I_0=I_{max}$ defined above.

The simple formula (61) is not strictly applicable to antennas of practical thickness any more than is (53). However, because $1/\Omega$ appears in the more correct formula (59) only in one small term, (61) is a better approximation for the case $\beta h=\pi/2$ than is (53) for other values of βh. Indeed, an examination of Figs. 8 and 14, (which give the distribution of current along antennas for which $\beta h=\pi/2$ and Ω has the values 10, 20, and 30) reveals that even for the thickest antenna ($\Omega=10$), I_z does not differ greatly from a simple cosine as given by (61), but with Z_{00} standing for the actual input impedance calculated from (60). These are

$$\Omega = 10, \quad Z_{00} = 64.8 + j29.7 = 71.2 \angle 24°.6 \qquad (66a)$$

$$\Omega = 20, \quad Z_{00} = 69.6 + j35.7 = 78.6 \angle 27°.2 \qquad (66b)$$

$$\Omega = 30, \quad Z_{00} = 70.3 + j37.6 = 79.8 \angle 28°.2 \qquad (66c)$$

Although the phase angle θ of the current as shown in Fig. 14 is not perfectly constant over the length of the antenna it varies only a few degrees from the value at the input terminals.

Antennas in General

If an antenna differs even slightly in half length from quarter wavelength for which $\beta h=\pi/2$ it is not at all lear from (39) that an approximate formula of the ype (61) or (53) may be used unless the antenna is nfinitely thin. Because the term in $1/\Omega$ in the deominator of (39) is small, the term $\cos \beta h$ will be sigificant even though βh differs very little from $\pi/2$. he impedance formula (51) also changes very rapidly n the vicinity of $\beta h=\pi/2$. Nevertheless the actual omputation of I_z for βh sufficiently below $\pi/2$ so that n antenna for which $\Omega=20$ is self-resonant with $Y_{00}=0$ (Figs. 7 and 13) shows that $|I_z|$ differs but little rom a sine curve measured from the upper ends with $-|z|$ as variable, and that the phase angle θ stays ery nearly constant at a value near that for I_0 even or thick antennas. On the other hand, antennas which re sufficiently long so that βh appreciably exceeds /2, as in Figs. 15 to 18, $|I_z|$ cannot be represented ery satisfactorily by a simple sine curve nor does the hase angle θ remain constant at anywhere near its alue for I_0. One must conclude, therefore, that the istribution of current along antennas only of such ngths that βh does not exceed appreciably the value /2 may be represented with fair accuracy by

$$I_z = I_0 \frac{\sin \beta(h-|z|)}{\sin \beta h} \tag{67}$$

ith

$$I_0 = V_0^e/Z_{00} \tag{68}$$

ere Z_{00} is the input impedance computed from the ccurate formula (51) or obtained from the curves iven in footnote reference 4. If the half-length of the ntenna is much greater than $\lambda/4$, in particular, if it pproaches or exceeds $\lambda/2$, a representation in terms of 67) and (68) is not satisfactory.

Since a single distribution function with a constant hase angle is not in general adequate, one is faced ith the necessity of complicating the representation. learly a much better approximation at the expense of nly a small increase in complexity would result if each omponent I_z' and I_z'' in

$$I_z = I_z'' + jI_z' = \frac{V_0^e}{60\Omega D}(f''+jf') \tag{69}$$

ere separately represented by a simple trigonometric nction. This is actually possible to a very satisfactory egree of approximation. The representation is the ollowing:

$$I_z = V_0^e\left[G_{00}\left(\frac{\cos\beta z - \cos H}{1-\cos H}\right) - jB_{00}\left(\frac{\sin(H-\beta|z|)}{\sin H}\right)\right]. \tag{70}$$

ere $H=\beta h$, G_{00} is the input conductance, and B_{00} the nput susceptance of the antenna in question. The adittance, $1/Z_{00}=Y_{00}=G_{00}-j\,B_{00}$, for each of the sevral lengths considered above is given in Table I. In rder to show that (70) is a good representation of the actual distribution one notes in the first place that the input current I_0 is exactly right for all lengths. The distribution along the antenna given by (70) may be compared with the actual distributions using Figs. 19 and 20. These show the true distribution functions, f' and f'', as obtained from Figs. 8 to 12 each plotted with the appropriate trigonometric function which is supposed to represent it approximately in (70). (In Figs. 19 and 20, f' and f'' have been adjusted in scale so that the values at $z=0$ coincide. Actually the admittance factors G_{00} and $-B_{00}$ in (70) serve to change the scales of the trigonometric functions respectively to give the correct values at $z=0$. For purposes of plotting and comparison it was more convenient to adjust f' and f'' to the trigonometric functions rather than vice versa.) It is seen that the representation of the analytically extremely complex functions f' and f'' in terms of the simple trigonometric functions is surprisingly good over practically the entire range of lengths shown. The poorest approximation is near $h=\frac{3}{8}$ for $\Omega=10$. Some difficulty in representing f' is encountered at $h=\lambda/2$, as shown in Fig. 19. Because no antenna of physically realizable radius is antiresonant at $h=\lambda/2$, B_{00} does not vanish and the imaginary term in (70) becomes infinite. This difficulty can be avoided by introducing a fictitious

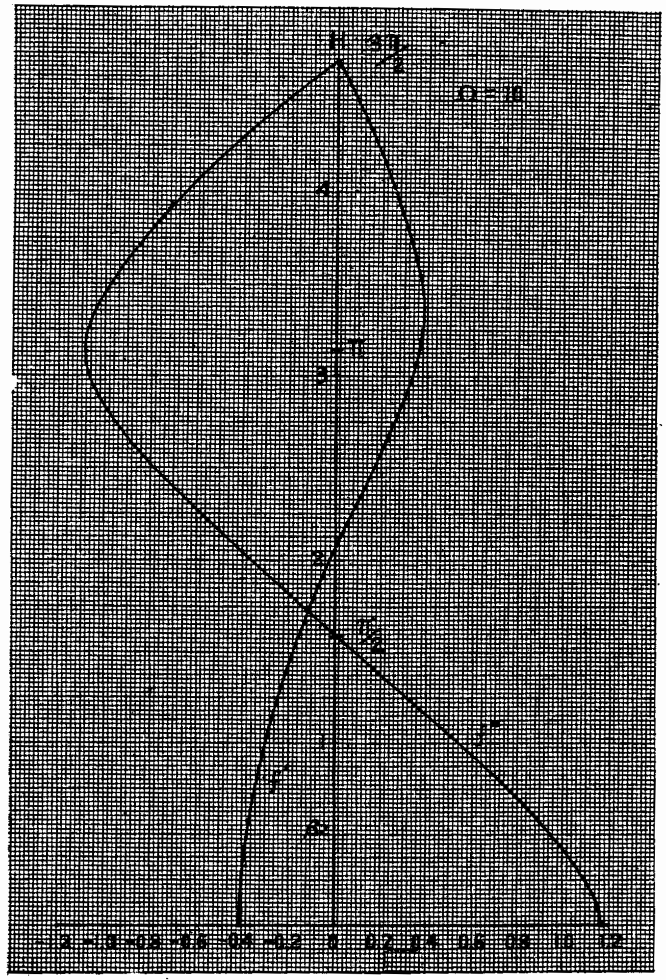

Fig. 12—The distribution functions f' and f'' for $H=3\pi/2$.

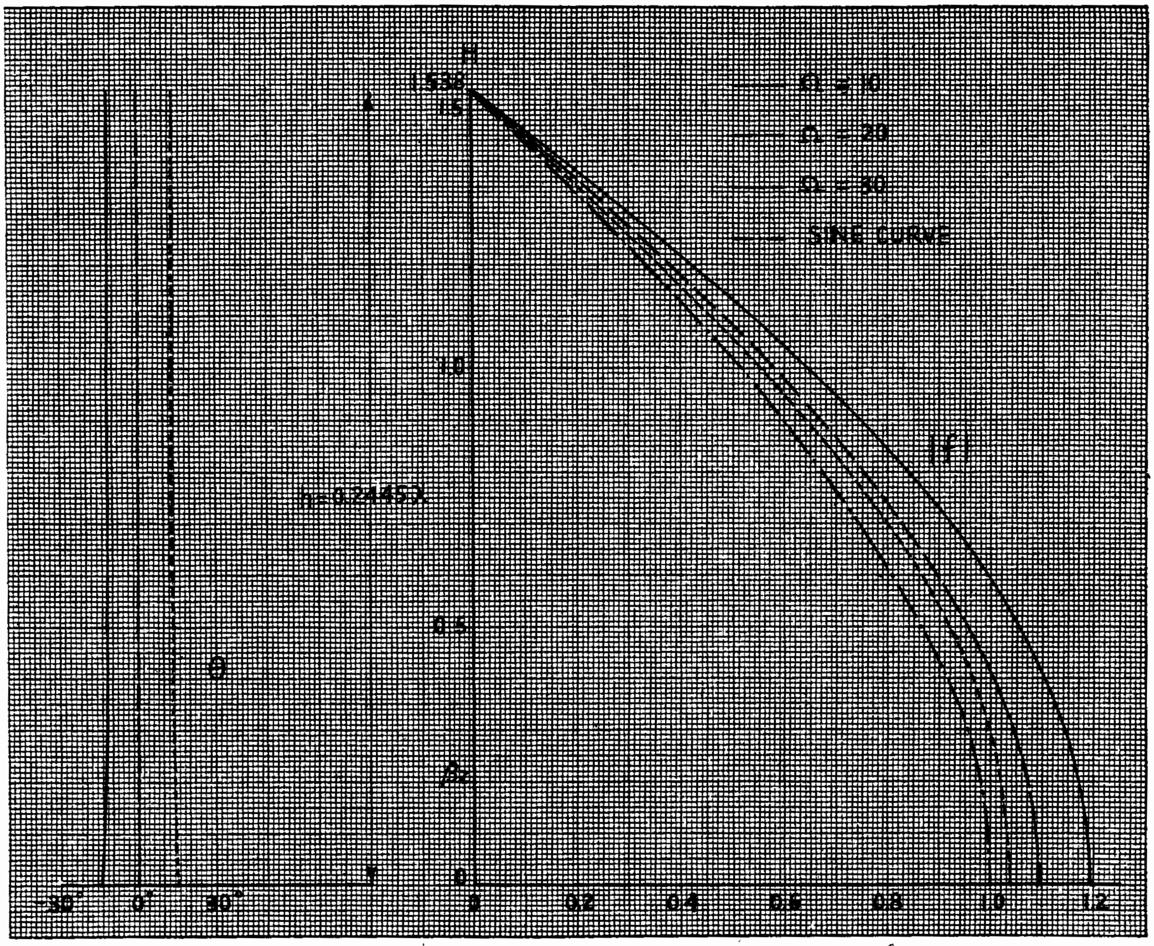

Fig. 13—The amplitude function $|f|$ and the phase function θ for $H=1.538$.

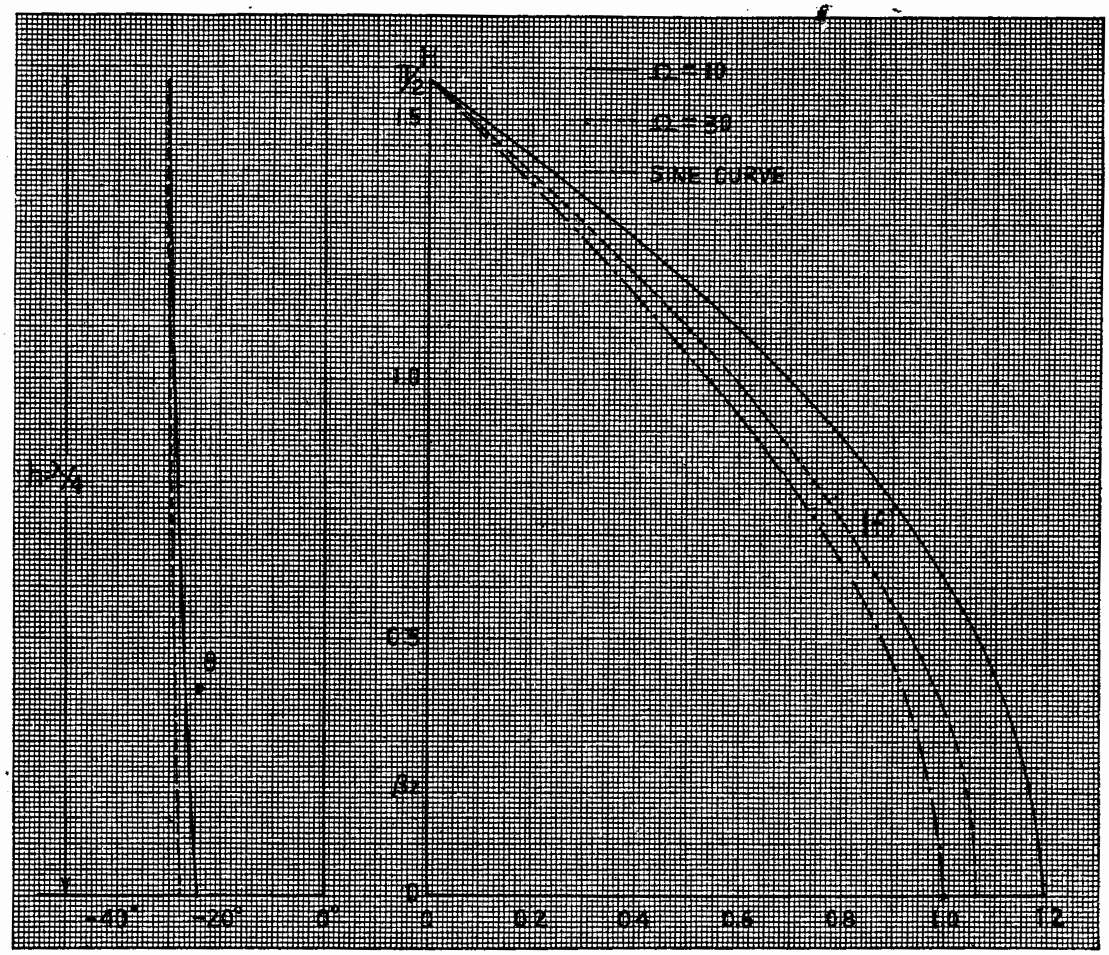

Fig. 14—The amplitude function $|f|$ and the phase function θ for $H=\pi/2$.

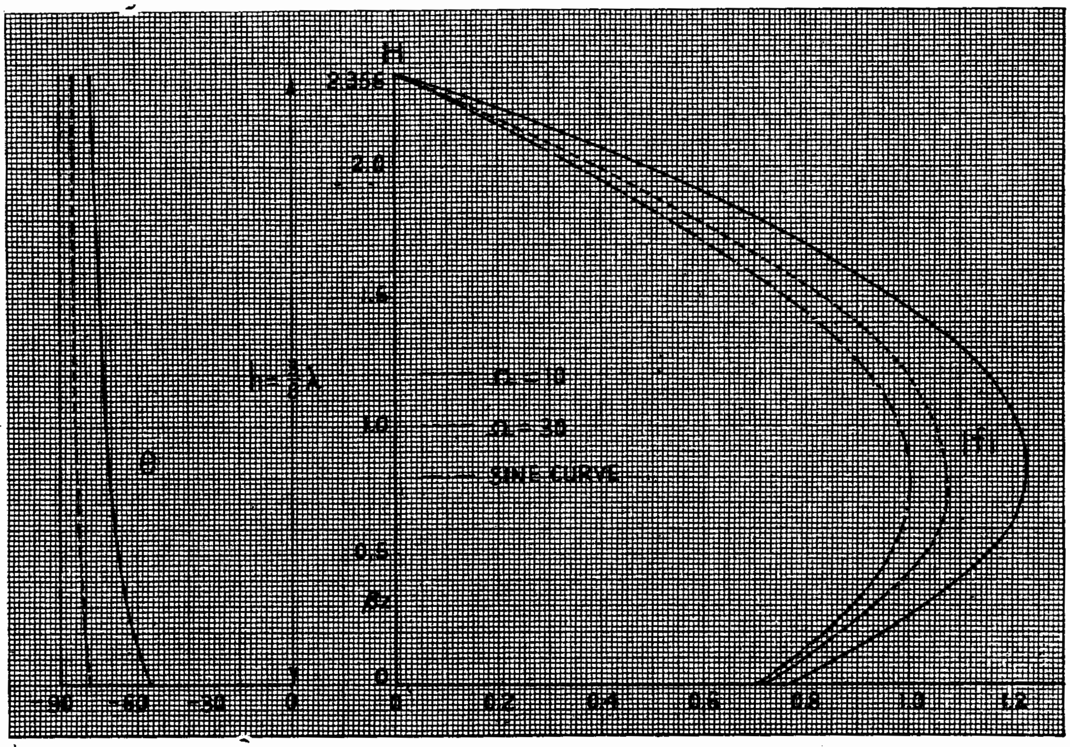

Fig. 15—The amplitude function $|f|$ and the phase function θ for $H=3\pi/4$.

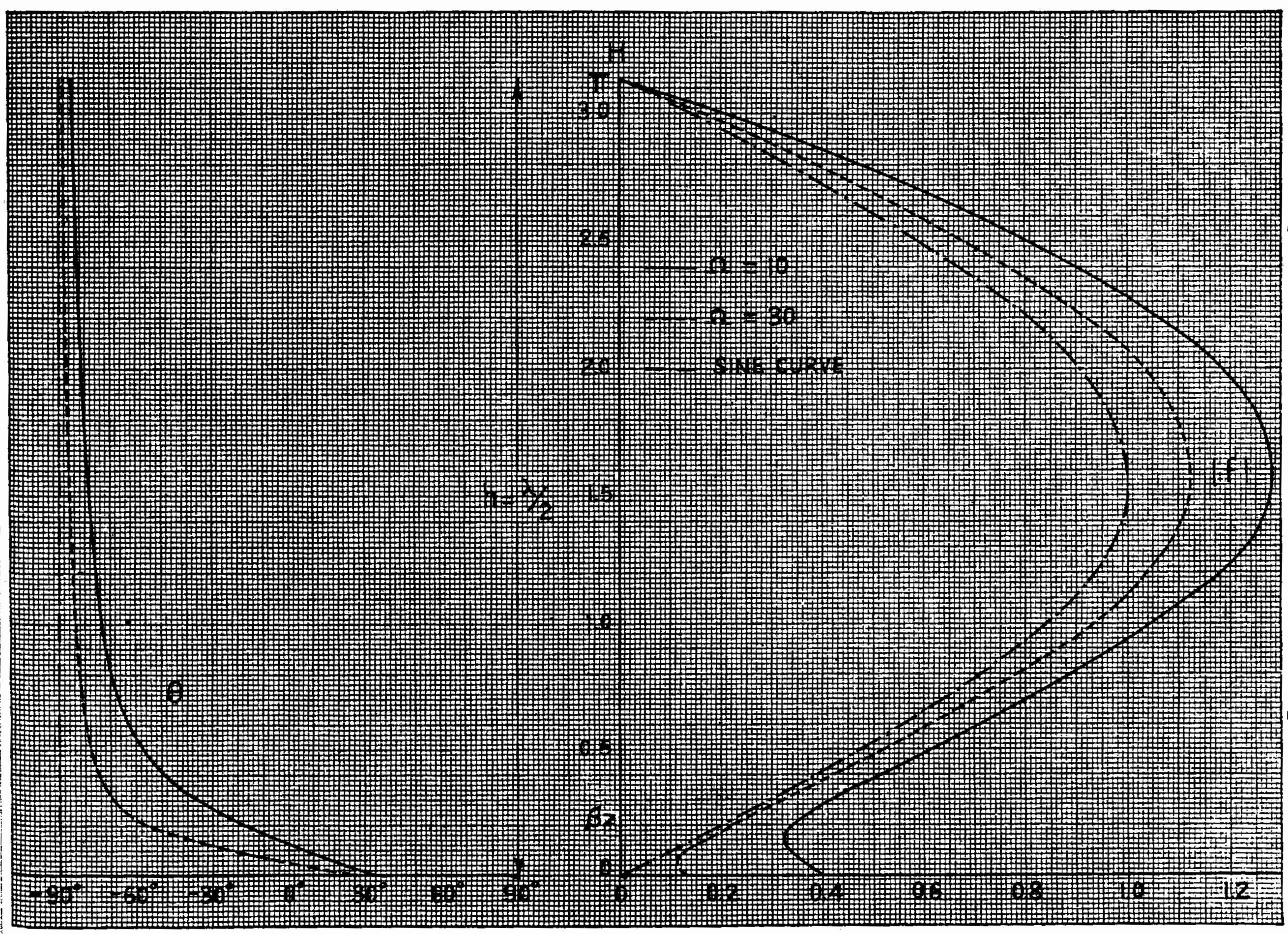

Fig. 16—The amplitude function $|f|$ and the phase function θ for $H=\pi$.

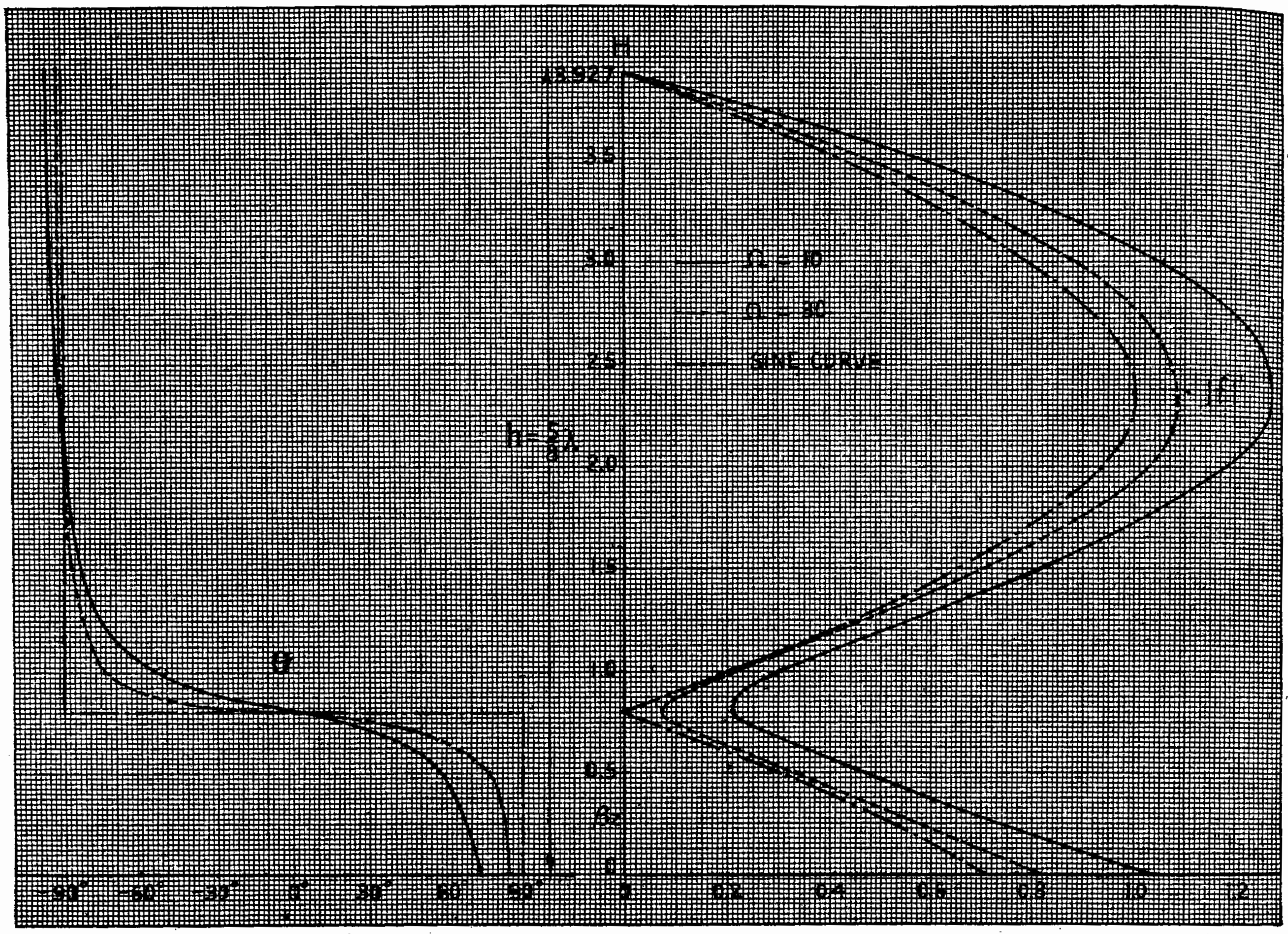

Fig. 17—The amplitude function $|f|$ and the phase function θ for $H=5\pi/4$.

and slightly greater length h' in the form $\beta h'=H'$ for H' in (70), and adjusting h' so that $\sin(H'-\beta|z|)/(\sin H')$ crosses the axis at or near the point where f' crosses it. ($H'-H$ should be approximately equal to $\pi/2-H_r$ where H_r is the value of H producing antiresonance for the particular choice of Ω. It may be obtained from Fig. 12 or equation (29b) in footnote reference 4.) At antiresonance, both B_{00} and $\sin H'$ must vanish. The indeterminate form $(V_0^e B_{00})/(\sin H')$ must then be replaced by the maximum value of I_z'. This always occurs near $z=h-(\lambda/4)$.

For purposes of calculating electromagnetic fields due to antennas of practically encountered thicknesses the distribution function (70) is a very much better approximation than the form (67) with (68) which is unsatisfactory over most of the range. (It is to be noted that at $H=n\pi/2$ with n odd (70) reduces exactly to (67).) The application of (70) in computing electromagnetic fields is reserved for a later paper.

Radiation Resistance Referred to Maximum Current

A common method of estimating the total power radiated from an antenna is to integrate the Poynting vector over a spherical surface in the far zone of the antenna. The calculation of the Poynting vector from the electric and magnetic fields is based on the assumption that the distribution of current has the simple sinusoidal form which is strictly accurate only for an indefinitely thin antenna. The total power so computed is then divided by the square of the maximum current (at $\lambda/4$ from the end of the antenna) to obtain the radiation resistance referred to maximum current. The curve marked $\Omega=\infty$ in Fig. 21 is that obtained and commonly reproduced for the sinusoidal distribution of current. Because the distribution of current in practical antennas is never exactly sinusoidal, even for $h=\lambda/4$ where the approximation is best, the radiation resistance so obtained is not accurate for them. Its correct value (neglecting power consumed in heating the antenna which is less than 3 per cent for copper antennas) can be determined as follows: Let the radiation resistance referred to maximum current be defined by

$$R_m^e = \frac{P_0}{|I_m|^2}. \tag{71}$$

Here P_0 is the power supplied to the antenna at its input terminals and, neglecting power consumed in heating the conductor, also the power radiated.

$$|I_m|^2 = \left[\sqrt{I_z'^2 + I_z''^2}\right]_{\max}. \tag{72}$$

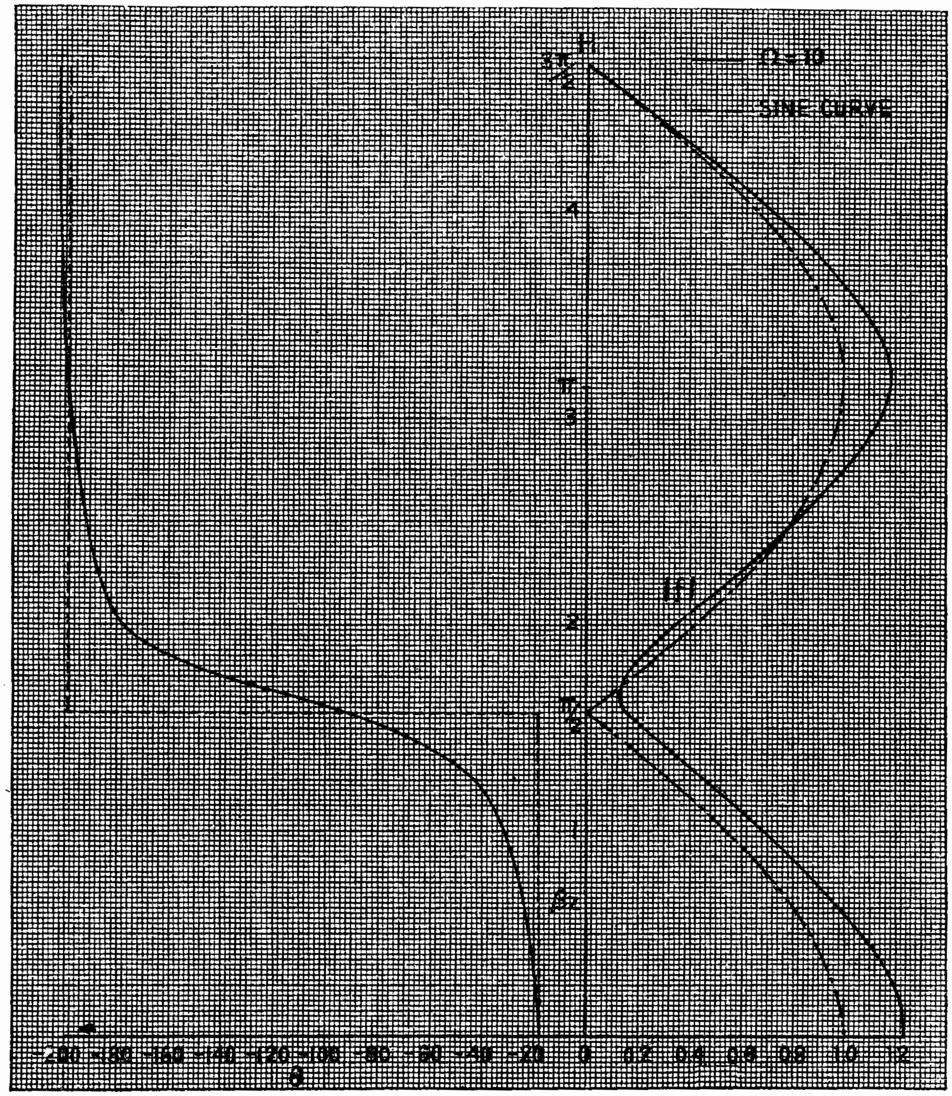

Fig. 18—The amplitude function $|f|$ and the phase function θ for $H=3\pi/2$.

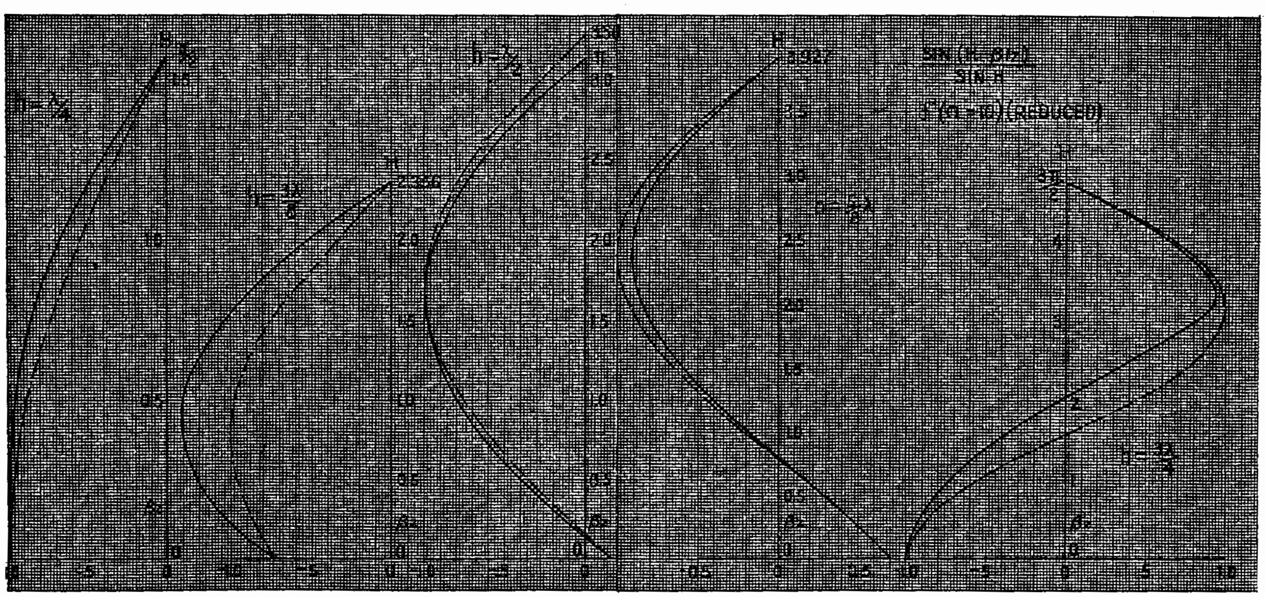

Fig. 19—The function $\sin (H-\beta|z|)/\sin H$ compared with the distribution function f' reduced to the same value at $z=0$.

Fig. 20—The function $(\cos \beta z - \cos H)/(1-\cos H)$ compared with the distribution function f'' reduced to the same value at $z=0$.

Since
$$P_0 = |I_0|^2 R_{00} \tag{73}$$
one has
$$R_m{}^e = \left|\frac{I_0}{I_m}\right|^2 R_{00}. \tag{74}$$

Since $|I_0|$ and $|I_m|$ are readily determined from Figs. 13 to 18 and R_{00} may be computed directly from (11a), using Table I in footnote reference 4 for the particular value of Ω, $R_m{}^e$ may be determined. The values so computed for the several lengths for which current data are available are shown plotted in small circles in Fig. 21 for a relatively thin antenna ($\Omega=30$) and a moderately thick antenna ($\Omega=10$). Although the number of available points is not sufficient to determine accurately the resulting curves near their maxima and minima, the general shape and position relative to the familiar curve for the simple sine distribution ($\Omega=\infty$) are correctly given. It is clear that the curve based on the sine distribution is not at all a good approximation for even moderately thick antennas except for lengths with H near $H=n\pi/2$ with n odd where the sinusoidal distribution is least in error. Even here it may be in error by as much as 50 per cent for very thick antennas. Actually $R_m{}^e$ is completely unnecessary if the input impedance of an antenna is known.

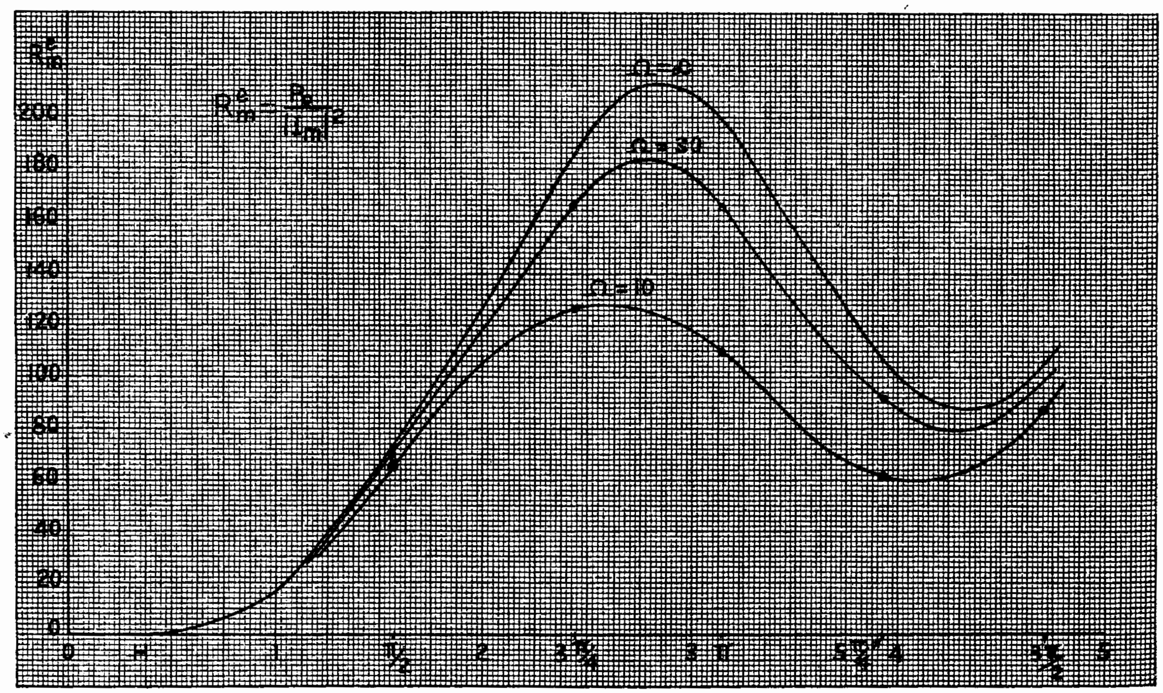

Fig. 21—The radiation resistance $R_m{}^e$ referred to maximum current I_m. The curve marked $\Omega=\infty$ is accurately computed throughout. The curves marked $\Omega=10$ and $\Omega=30$ are estimated using only the insufficient number of accurately computed points shown by small circles.

Appendix I

The complex amplitude of the vector potential defined by (5) satisfies the Helmholtz equation

$$\nabla^2 A + \beta^2 A = i\Pi. \tag{75}$$

Here i is the volume density of current flowing, in this case, in the antenna. The Helmholtz integral which satisfies (75) is

$$A = \frac{\Pi}{4\pi}\int_\tau \frac{i'}{R_1} e^{-i\beta R_1} d\tau'. \tag{76}$$

Here τ is the volume of the cylindrical antenna, $d\tau'$ is an element of τ, i' is the current density at $d\tau'$, and R_1 is the distance between the point (r, θ, z) where A is computed and the element $d\tau'$ at (r', θ', z'). Since the radial component A_r can be due to i_r only, the inequality (8a) is readily verified. The z component is

$$A_z = \frac{\Pi}{4\pi}\int_\tau \frac{i_z'}{R_1} e^{-i\beta R_1} d\tau'. \tag{77}$$

It is to be proved that A_z evaluated from (22) on the cylindrical surface of the antenna differs by a negligible amount from A_z computed from the exact formula (75). I_z' in (22) is defined by

$$I_z' = \int_0^a \int_0^{2\pi} i_z' r dr d\theta = 2\pi \int_0^a i_z' r dr \tag{78}$$

since rotational symmetry may be assumed. Thus it must be shown that the following difference is vanishingly small.

$$D = \int_{-h}^{+h}\int_0^a\int_0^{2\pi} \frac{i_z'}{R_1} e^{-i\beta R_1} r d\theta' dr' dz' - \int_{-h}^{+h} \frac{I_z'}{R} e^{-i\beta R} dz'. \tag{79}$$

Here R is the distance from any point outside the antenna and not near the end faces where A is to be calculated, in particular a point on its cylindrical surface, to the element dz' on the *axis* of the antenna. R_1 is the distance from the same outside point to an element $r d\theta' dr' dz'$ in a cross section of the antenna at the center of which dz' is defined. This is illustrated in Fig. 22.

Because of rotational symmetry i_z' is independent of θ' and may be removed outside the sign of integration with respect to θ'. Thus with (78)

$$D = \int_{-h}^{+h} dz' \int_0^a i_z' r dr' \int_0^{2\pi} \left[\frac{e^{-i\beta R_1}}{R_1} - \frac{e^{-i\beta R}}{R}\right] d\theta'. \tag{80a}$$

It follows directly from (3) or from Fig. 22 that R_1 can differ from R at most by magnitudes of the order of the radius a of the cylinder. If R_1 and R are large compared with a, they will differ from each other by a negligible amount and the difference in the brackets in (6) will be vanishingly small. Accordingly, significant contributions to D for points on the surface of the antenna where $r = a$ can come only from that part of the integration with respect to z' for which $(z - z')$ is not large compared with a, i.e., from sections of the surface which are very close to the circumference at z'. It follows that the distribution of current at more distant points and even the length of the antenna can make no difference in the integral except at points very near the ends. Thus one may integrate from $-\infty$ to $+\infty$ and assume i_z' independent of the integration with respect to z'. Furthermore, since significant contributions

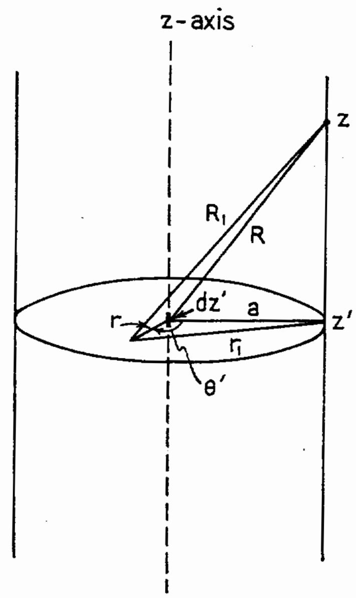

Fig. 22—A section through the antenna.

are obtained only for distances comparable in magnitude with a, and since with (3) $e^{-i\beta a} \doteq 1$, the two exponentials may be set equal to unity in the range of significant contributions. This leaves

$$D = \int_0^a i_z' r dr' \int_0^{2\pi} d\theta' \int_{-\infty}^{+\infty} \left(\frac{1}{R_1} - \frac{1}{R}\right) dz'. \tag{80b}$$

The integral with respect to z' can be integrated directly into $\ln (a/r_1)^2$ with r_1 indicated in Fig. 22. If this is integrated with respect to θ' the integral vanishes. Thus D is entirely negligible except at points within distances of the ends of the antenna comparable with a. The contributions of these short sections were made negligible by imposing (3a).

(It is interesting to note that the rigorous derivation of the transmission-line equations for parallel wires from fundamental electromagnetic theory depends upon exactly the same demonstration in that (22) is assumed valid on the surface of each parallel wire.)

Appendix II

In order to derive the solution (39) from the integral equation (38) it is convenient to introduce the following shorthand notation:

$$F_0(z) \equiv \cos \beta z; \qquad G_0(z) \equiv \sin \beta |z| \tag{81}$$

$$F_{0z} \equiv F_0(z) - F_0(h); \qquad G_{0z} \equiv G_0(z) - G_0(h). \tag{82}$$

The first brace in (38) will be denoted by $(I_z)_0$. Using the above notation it is

$$(I_z)_0 = \frac{-j4\pi}{\Omega R_c}\left\{C_1 F_{0z} - \tfrac{1}{2} V_0^e G_{0z}\right\}. \tag{83}$$

It may be regarded as a zeroth order approximation for I_z. If it is substituted in the rest of the terms in (38), all of which include the current as a factor, terms will be obtained which are multiplied by $1/\Omega^2$. They will be denoted by $(I_z')_0$. They are

$$(I_z')_0 = \frac{-j4\pi}{\Omega^2 R_c}\{C_1F_{1z} - \tfrac{1}{2}V_0^e G_{1z}\} \tag{84}$$

with

$$F_{1z} \equiv F_1(z) - F_1(h); \quad G_{1z} \equiv G_1(z) - G_1(h) \tag{85}$$

and

$$F_1(z) = -F_{0z}\ln\left(1 - \frac{z^2}{h^2}\right) + F_{0z}\delta - \int_{-h}^{+h}\frac{F_{0z}'e^{-j\beta R} - F_{0z}}{R}dz' \left\{-\frac{j4\pi z^i}{R_c}\int_0^z F_{0s}\sin\beta(z-s)ds\right\} \tag{86}$$

$$F_1(h) = -\left\{\int_{-h}^{+h}F_{0z}'\frac{e^{-j\beta R_h}}{R_h}dz' - \frac{j4\pi z^i}{R_c}\int_0^h F_{0s}\sin\beta(h-s)ds\right\}. \tag{87}$$

$G_1(z)$ is exactly like $F_1(z)$ with G written for F throughout. (88)

$G_1(h)$ is exactly like $F_1(h)$ with G written for F throughout. (89)

The first-order approximation for I_z is now given by

$$(I_z)_1 \equiv (I_z)_0 + (I_z')_0 = \frac{-j4\pi}{\Omega R_c}\left\{C_1\left[F_{0z} + \frac{F_{1z}}{\Omega}\right] + \tfrac{1}{2}V_0^e\left[G_{0z} + \frac{G_{1z}}{\Omega}\right]\right\}. \tag{90}$$

If this expression is substituted for I_z on the right in (38) a second-order approximation may be obtained. This may be substituted back in (38) to obtain a third-order solution. The process may be continued indefinitely to obtain a series solution of the form

$$I_z = \frac{-j4\pi}{\Omega R_c}\left\{C_1\left[F_{0z} - \frac{F_{1z}}{\Omega} + \frac{F_{2z}}{\Omega^2} + \cdots\right] + \tfrac{1}{2}V_0^e\left[G_{0z} + \frac{G_{1z}}{\Omega} + \frac{G_{2z}}{\Omega^2} + \cdots\right]\right\}. \tag{91}$$

The constant of integration C_1, can now be evaluated directly by substituting the solution (91) for I_z in (36). If this is done one has

$$0 = \frac{-j4\pi}{\Omega R_c}\left\{C_1\left[F_0(h) + \frac{F_1(h)}{\Omega} + \cdots\right] + \tfrac{1}{2}V_0^e\left[G_0(h) + \frac{G_1(h)}{\Omega} + \cdots\right]\right\}. \tag{92}$$

This can be solved for C_1 as follows:

$$C_1 = -\tfrac{1}{2}V_0^e\left\{\frac{G_0(h) + (1/\Omega)G_1(h) + \cdots}{F_0(h) + (1/\Omega)F_1(h) + \cdots}\right\}. \tag{93}$$

If this is inserted in (91) one has the solution for I_z. It is

$$I_z = \frac{j2\pi V_0^e}{\Omega R_c}\left\{\frac{[F_{0z}+F_{1z}/\Omega+\cdots][G_0(h)+G_1(h)/\Omega+\cdots] - [G_{0z}+G_{1z}/\Omega+\cdots][F_0(h)+F_1(h)/\Omega+\cdots]}{F_0(h)+F_1(h)/\Omega+\cdots}\right\}. \tag{94}$$

After rearranging using (82) and (85) one has precisely (39).

In carrying out the evaluation of the function $F_1(z)$, $F_1(h)$, $G_1(z)$, and $G_1(h)$ as defined in (86) and (87), advantage can be taken of the fact that a^2 is negligible compared with $(h-z)^2$ and $(h+z)^2$ except for points very near the ends of the antenna. At the ends errors as large as 50 per cent are involved. However, since the current necessarily vanishes at the ends, the distribution of current is actually not significantly affected. At most the current within distances of the ends comparable with the radius a may be in error by an appreciable amount, but this error becomes negligible at distances of three or four times the radius from the end. Actually in computing the current, points need not be taken within distances of the ends comparable with the radius a and a curve connecting points at distances of $5a$ or more from the ends to zero values at the ends must give the correct distribution. Accordingly the term in δ may be neglected and one may write

$$R \doteq |z' - z| \tag{95}$$

in the integrals. It is especially important to note that the approximations here introduced are extremely good for the current at all points except near the ends where it is known to vanish. In particular, the input current, and hence the input impedance, is in no way affected if (3) is fulfilled.

If one makes use of (95) all integrals in (86) to (89) are readily evaluated without further approximations in terms of trigonometric functions or the integral functions defined below.

$$\int_a^b \frac{1-\cos u}{u}du = \overline{\mathrm{Ci}}\,(b) - \overline{\mathrm{Ci}}\,(a) \tag{96}$$

$$\int_a^b \frac{\sin u}{u}du = \mathrm{Si}\,(b) - \mathrm{Si}\,(a). \tag{97}$$

The final forms are

$$\begin{aligned} F_1(z) = &-(\cos\beta z - \cos\beta h)\ln(1 - z^2/h^2) \\ &+\tfrac{1}{2}\cos\beta z[\overline{\mathrm{Ci}}\,2\beta(h+z) + \overline{\mathrm{Ci}}\,2\beta(h-z) \\ &+j\,\mathrm{Si}\,2\beta(h+z) + j\,\mathrm{Si}\,2\beta(h-z)] \\ &-\tfrac{1}{2}\sin\beta z[\mathrm{Si}\,2\beta(h+z) - \mathrm{Si}\,2\beta(h-z) \\ &-j\,\overline{\mathrm{Ci}}\,2\beta(h+z) + j\,\overline{\mathrm{Ci}}\,2\beta(h-z)] \\ &-\cos\beta h[\overline{\mathrm{Ci}}\,\beta(h+z) + \overline{\mathrm{Ci}}\,\beta(h-z) \\ &+j\,\mathrm{Si}\,\beta(h+z) + j\,\mathrm{Si}\,\beta(h-z)] \\ &+j\frac{4\pi z^i h}{R_c}\left[\frac{z}{2h}\sin\beta z - \frac{\cos\beta h}{\beta h}(1-\cos\beta z)\right] \end{aligned} \tag{98}$$

$$\begin{aligned} G_1(z) = &-(\sin\beta|z| - \sin\beta h)\ln(1 - z^2/h^2) \\ &-\tfrac{1}{2}\cos\beta z[\mathrm{Si}\,2\beta(h+z) + \mathrm{Si}\,2\beta(h-z) \\ &-2\,\mathrm{Si}\,(2\beta|z|) - j\,\overline{\mathrm{Ci}}\,2\beta(h+z) \\ &-j\,\overline{\mathrm{Ci}}\,2\beta(h-z) + 2j\,\overline{\mathrm{Ci}}\,(2\beta z)] \\ &-\tfrac{1}{2}\sin\beta z[\overline{\mathrm{Ci}}\,2\beta(h+z) - \overline{\mathrm{Ci}}\,2\beta(h-z) \\ &+j\,\mathrm{Si}\,2\beta(h+z) - j\,\mathrm{Si}\,2\beta(h-z) - 2j\,\mathrm{Si}\,(2\beta z)] \end{aligned}$$

$$
\begin{aligned}
&-\tfrac{1}{2}\sin\beta|z|\left[4\ln\frac{|z|}{h+|z|}-2\,\overline{\mathrm{Ci}}\,(2\beta z)\right]\\
&-\sin\beta h[\overline{\mathrm{Ci}}\,\beta(h+z)+\overline{\mathrm{Ci}}\,\beta(h-z)\\
&+j\,\mathrm{Si}\,\beta(h+z)+j\,\mathrm{Si}\,\beta(h-z)]\\
&+j\frac{4\pi z^i h}{R_c}\left[\frac{\sin\beta|z|}{2\beta h}-\frac{|z|\cos\beta z}{2h}\right.\\
&\left.-\frac{\sin\beta h}{\beta h}(1-\cos\beta z)\right]
\end{aligned}
\tag{99}
$$

$$
\begin{aligned}
F_1(h)=&\tfrac{1}{2}\cos\beta h[\overline{\mathrm{Ci}}\,(4\beta h)+j\,\mathrm{Si}\,(4\beta h)]\\
&-\tfrac{1}{2}\sin\beta h[\mathrm{Si}\,(4\beta h)-j\,\overline{\mathrm{Ci}}\,(4\beta h)]\\
&-\cos\beta h[\overline{\mathrm{Ci}}\,(2\beta h)+j\,\mathrm{Si}\,(2\beta h)]\\
&+j\frac{4\pi z^i h}{R_c}\left[\frac{\sin\beta h}{2}-\frac{\cos\beta h}{\beta h}(1-\cos\beta h)\right]
\end{aligned}
\tag{100}
$$

$$
\begin{aligned}
G_1(h)=&-\tfrac{1}{2}\cos\beta h[\mathrm{Si}\,(4\beta h)-2\,\mathrm{Si}\,(2\beta h)\\
&-j\,\overline{\mathrm{Ci}}\,(4\beta h)+2j\,\overline{\mathrm{Ci}}\,(2\beta h)]\\
&-\tfrac{1}{2}\sin\beta h[\overline{\mathrm{Ci}}\,(4\beta h)+j\,\mathrm{Si}\,(4\beta h)\\
&-2j\,\mathrm{Si}\,(2\beta h)-2\,\overline{\mathrm{Ci}}\,(2\beta h)-4\ln 2]\\
&-\sin\beta h[\overline{\mathrm{Ci}}\,(2\beta h)+j\,\mathrm{Si}\,(2\beta h)]\\
&-j\frac{4\pi z^i h}{R_c}\left[\frac{\sin\beta h}{2\beta h}+\tfrac{1}{2}\cos\beta h-\frac{\sin\beta h\cos\beta h}{\beta h}\right]
\end{aligned}
\tag{101}
$$

The terms in z^i were neglected in computing the curves of Figs. 2 to 6. They are entirely negligible if the antenna is a good conductor.

It has been planned to present in the PROCEEDINGS of the I.R.E. instructional material of timely interest. This procedure was instituted some time ago, and here continues by the publication, in successive issues of the PROCEEDINGS, of a series of coordinated parts, together entitled "Some Aspects of Radio Reception at Ultra-High Frequency" by Messrs. E. W. Herold and L. Malter. Parts IV and V of the five parts are here presented. Each Part is preceded by its own related summary. *The Editor*

Some Aspects of Radio Reception at Ultra-High Frequency*

E. W. HEROLD†, MEMBER, I.R.E., AND L. MALTER‡, ASSOCIATE, I.R.E.

PART IV. GENERAL SUPERHETERODYNE CONSIDERATIONS AT ULTRA-HIGH FREQUENCIES

L. MALTER‡

Summary —This paper presents a general survey of the problems encountered in the mixer or converter stage of superheterodyne receivers, particularly at ultra-high frequencies. The application of a strong local-oscillator voltage causes a periodic variation of the signal-electrode transconductance as a consequence of which intermediate-frequency-current components appear in the output circuit when a signal is also impressed upon the signal electrode. It is demonstrated that intermediate-frequency-current components are present in the output, which differ from the signal frequency by integral multiples of the local-oscillator frequency, if the Fourier analysis of the signal-electrode transconductance contains components which are integral multiples of the local-oscillator frequency. Methods of determining the conversion transconductance for so-called fundamental and harmonic conversion are given.

It is shown that the noise output and input loading of a mixer stage are given by averaging these quantities over a local-oscillator cycle. A discussion of mixer gain is included, with a demonstration that the gain of a mixer stage is given approximately by the product of the conversion transconductance and the impedance of the output circuit (for high-output-impedance tubes).

Considerations regarding image rejection and the undesirability of radiation of oscillator power lead to the conclusion that high intermediate frequencies are desirable.

An extended discussion of whether to use an amplifier or mixer stage in the first stage of a superheterodyne receiver is included. If the received signal is strong, one should convert immediately, unless image rejection or the prevention of oscillator radiation necessitate the use of radio-frequency stages. If the received signal is weak, an amplifier stage should be used below a certain frequency and a mixer above, the transition frequency depending upon the characteristics of the tubes available and the bandwidth required. In general the transition frequency occurs at the point where available tubes will no longer give appreciable radio-frequency gain for the bandwidth required.

* Decimal classification: R361.2. Original manuscript received by the Institute, August 2, 1943.

† RCA Laboratories, Princeton, N. J.
‡ Formerly, RCA Laboratories, Princeton, N. J.; now, RCA Victor Division, Radio Corporation of America, Lancaster, Pa.

I. INTRODUCTION

IN PART II of this series we concerned ourselves primarily with the case wherein the signal voltages applied to the circuits and tubes of a receiver are so low in amplitude that the tubes can be considered as linear devices, wherein the output voltage or current is proportional to the signal-electrode voltage. This case will be recognized as being precisely that of the linear amplifier.

It is frequently convenient, however, to make use of the superheterodyne principle in receivers. In receivers of this type, the incoming signal is combined with a locally produced oscillation of different frequency to produce a third signal at a frequency referred to as the intermediate frequency, which is related to both the frequencies of the incoming signal and the locally produced oscillation. It is an essential characteristic of any