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Anisotropic Modification of Maxwell's Equations

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This paper examines a modified form of Maxwell's equations, one designed to produce anisotropic light propagation in a vacuum. The equations predict anisotropies in the speed of light that behave as $\cos\phi$ and $\cos 2\phi$, where ϕ is the angle between the actual direction of propagation and some single preferred direction in space. The predicted index of refraction also has terms that behave as $\cos\phi$ and $\cos 2\phi$.

I. INTRODUCTION

In the preceding paper,[1] we discussed an experiment designed to measure anisotropies in the speed of light. It is difficult to use Maxwell's equations to justify the assumptions made in the earlier paper because Maxwell's equations do not predict an anisotropy in the speed of light in a vacuum. This paper will examine a modified form of Maxwell's equations which predict an anisotropy of interest. The index of refraction, phase velocity, and optical path length will be discussed.

II. THE OPTICAL PATH LENGTH

In Ref. 1 the extra optical path length due to a piece of glass was used to search for an anisotropy in the speed of light. This anisotropy has the form $c(\phi)=c_0(1+b_1\cos\phi)^{-1}$, where $c(\phi)$ is the vacuum phase velocity of light, c_0 is a constant, b_1 is a small number which determines the size of the anisotropy, and ϕ is the angle between the actual direction of light propagation and some single preferred direction $\hat{\mu}$. The interferometer used always passes the light through exactly the same portion of the glass. The entire interferometer is periodically rotated, however, and it is possible that the index of refraction, $n(\phi)$, might change due to the rotation of the glass. If the index changes in a perverse manner, the change could just cancel the effect due to the $\cos\phi$ anisotropy. To examine what this perverse form is, consider the optical path length, $L_{\text{o.p.}}$,

$$L_{\text{o.p.}} = \int_A^D \lambda^{-1} dl. \tag{1}$$

$L_{\text{o.p.}}$ is the number of wavelengths contained in the optical path between points A and D. When one uses the fundamental relationship

$$\lambda\nu = \frac{c(\phi)}{n(\phi)} = \text{phase velocity}, \tag{2}$$

Eq. (1) becomes

$$L_{\text{o.p.}} = \int_A^D \nu n(\phi)[c(\phi)]^{-1} dl. \tag{3}$$

In these equations λ is the wavelength, and ν is the frequency of the monochromatic light under consideration. Now if glass is present, the index of refraction can be written as its vacuum value, 1, plus a term due to the glass, $g(\phi)$:

$$n(\phi) = 1 + g(\phi). \tag{4}$$

Equation (3) now becomes

$$L_{\text{o.p.}} = \int_A^D \nu[c(\phi)]^{-1}\, 1\, dl + \int_A^D \nu[c(\phi)]^{-1} g(\phi)\, dl. \tag{5}$$

The anisotropy in the first term integrates to zero (to first order in b_1) when the integral is over a closed path, as pertains to most interferometers (see Sec. II of Ref. 1). The second integral is nonzero only over the portions of the path containing glass. Unless $g(\phi)$ has exactly the same form as $c(\ )$, the second integral can be arranged so it has a nonvanishing dependence. As long as the second term has a ϕ dependence, the interferometer described in the companion paper can detect the $\cos\phi$ anisotropy.

The next section will solve a modified form of Maxwell's equations for a traveling plane wave. Once the phase velocity in a vacuum, $c(\phi)$, and the phase velocity in the glass, $c_g(\phi)$, are known, the index of refraction is easily calculated:

$$n(\phi) = c(\phi)/c_g(\phi). \tag{6}$$

The index calculated below is not of the perverse form that cancels the effect of the $\cos\phi$ anisotropy.

III. AN ANISOTROPIC MODIFICATION OF MAXWELL'S EQUATIONS

The modified form of Maxwell's equations is

$$\vec{\nabla}\cdot\vec{E} = 4\pi\rho\ , \tag{7}$$

$$\vec{\nabla}\cdot\vec{B} = 0, \tag{8}$$

$$\vec{\nabla}\times(\vec{E} + h_1\hat{\mu}\times\vec{B}) = -\frac{1}{c_0}\frac{\partial\vec{B}}{\partial t}, \tag{9}$$

$$\vec{\nabla}\times(\vec{B} + h_2\hat{\mu}\times\vec{E}) = \frac{4\pi}{c_0}\vec{J} + \frac{1}{c_0}\frac{\partial\vec{E}}{\partial t}. \tag{10}$$

$\vec{E}$ and $\vec{B}$ are the electric and magnetic fields, and ρ and $\vec{J}$ are the charge density and current density. The constants h_1 and h_2 are small dimensionless numbers that are a measure of the size of the anisotropy, and c_0 is a constant which turns out to be the average of $c(\phi)$. The vector $\hat{\mu}$ is a unit vector along some preferred direction in space.

Though there are a number of ways to modify Maxwell's equations to give an anisotropy in the speed of light, the above form was used for several reasons. First, these equations predict a $\cos\phi$ anisotropy, the form that interests us. Second, it was desired to have a single preferred direction, $\hat{\mu}$, to keep the equations as simple as possible. Third, these equations predict conservation of charge. This is easily shown by taking the time derivative of Eq. (7) and the divergence of Eq. (10):

$$\frac{\partial}{\partial t}\vec{\nabla}\cdot\vec{E} = 4\pi\frac{\partial}{\partial t}\rho, \tag{11}$$

$$\frac{4\pi}{c_0}\vec{\nabla}\cdot\vec{J} + \frac{1}{c_0}\frac{\partial}{\partial t}\vec{\nabla}\cdot\vec{E} = 0, \tag{12}$$

which yield the statement of conservation of charge,

$$\vec{\nabla}\cdot\vec{J} + \frac{\partial}{\partial t}\rho = 0. \tag{13}$$

Fourth, the equations preserve much of the symmetry between $\vec{E}$ and $\vec{B}$. And fifth, the equations are linear in $\vec{E}$ and $\vec{B}$ and also maintain the invariance of Maxwell's equations under the parity operation.

Finally, the equations also predict a null result for the Cavendish experiment. Initially, consider a solid conducting body, which may have a nonzero charge. There is no $\vec{E}$ field in the interior since a steady state $\vec{E}$ field cannot exist in a conductor. Equation (7) then implies that there is zero charge density inside the body, and the central region can be hollowed out without changing the boundary conditions. The resulting hollow conducting body has no interior steady state $\vec{E}$ field. This is equivalent to a null Cavendish experiment.

To examine the index of refraction in glass, one must also assume specific force laws for the electrons in the glass. The force law assumed in this paper is the standard Lorentz force, $\vec{F} = q(\vec{E} + c_0^{-1}\vec{v}\times\vec{B})$. In glass, the electrons behave as linear harmonic oscillators, oscillating below their natural frequency. Hence, the dipole moments are proportional to the instantaneous electric field, and the current is proportional to the derivative of $\vec{E}$:

$$\vec{J} = \alpha\frac{\partial\vec{E}}{\partial t}, \tag{14}$$

where α is a real proportionality constant.

IV. THE INDEX OF REFRACTION

First, the traveling-wave solution to these equations will be found. For a plane wave in a region containing no net charge, assume the solution

$$\vec{E} = \vec{A}\,e^{i(\vec{k}\cdot\vec{r} - \omega t)}, \tag{15}$$

$$\vec{B} = \vec{C}\,e^{i(\vec{k}\cdot\vec{r} - \omega t + \tilde{\varphi})}. \tag{16}$$

If a relationship between $\vec{k}$ and ω can be found which satisfies the modified equations, then a traveling wave solution exists. Equations (7) and (8) become

$$i\vec{k}\cdot\vec{E} = 0, \tag{17}$$

$$i\vec{k}\cdot\vec{B} = 0. \tag{18}$$

Hence, $\vec{E}$ is perpendicular to $\vec{k}$ and $\vec{B}$ is perpendicular to $\vec{k}$. Equations (9) and (10) become

$$i\vec{k}\times(\vec{E} + h_1\hat{\mu}\times\vec{B}) = i\omega c_0^{-1}\vec{B}, \tag{19}$$

$$i\vec{k}\times(\vec{B} + h_2\hat{\mu}\times\vec{E}) = 4\pi c_0^{-1}\vec{J} - i\omega c_0^{-1}\vec{E}. \tag{20}$$

When one uses Eq. (14) and the triple cross product relationship

$$\vec{A}\times(\vec{B}\times\vec{C}) = \vec{B}(\vec{A}\cdot\vec{C}) - \vec{C}(\vec{A}\cdot\vec{B}),$$

these equations become

$$\vec{k}\times\vec{E}=[\omega c_0^{-1}+h_1(\vec{k}\cdot\hat{\mu})]\vec{B}\,, \quad (21)$$

$$\vec{k}\times\vec{B}=[-(4\pi\alpha+1)\omega c_0^{-1}+h_2(\vec{k}\cdot\hat{\mu})]\vec{E}\,. \quad (22)$$

For convenience set

$$\beta=4\pi\alpha+1\,. \quad (23)$$

Crossing $\vec{k}$ into Eq. (21) and substituting $\vec{k}\times\vec{B}$ from Eq. (22), one has

$$-k^2\vec{E}=[\omega c_0^{-1}+h_1(\vec{k}\cdot\hat{\mu})][-\beta\omega c_0^{-1}+h_2(\vec{k}\cdot\hat{\mu})]\vec{E}\,, \quad (24)$$

$$\left[\beta\left(\frac{\omega}{kc_0}\right)^2+(h_1\beta-h_2)\cos\phi\left(\frac{\omega}{kc_0}\right)-h_1h_2\cos^2\phi-1\right]\vec{E}=0. \quad (25)$$

The $\cos\phi$ is the cosine of the angle between $\vec{k}$ and $\hat{\mu}$. Solving this quadratic equation in (ω/kc_0), one can find the expression for the phase velocity $c_g(\phi)$.

$$c_g(\phi)=\omega k^{-1}\,,$$

$$c_g(\phi)\simeq c_0\beta^{-1/2}+c_0(h_2-\beta h_1)(2\beta)^{-1}\cos\phi+c_0(8\beta^{3/2})^{-1}(h_1\beta+h_2)^2\cos^2\phi \quad (26)$$

to quadratic order in the h's. The phase velocity has an anisotropy that behaves both as $\cos\phi$ and as $\cos2\phi$ (because $\cos^2\phi=\frac{1}{2}\cos2\phi+\frac{1}{2}$). These remain if β is set to unity in order to represent the situation in vacuum.

Since h_1 and h_2 can reasonably be expected to be small numbers, terms of order h^2 will be neglected:

$$c_g(\phi)\simeq c_0\beta^{-1/2}[1+\tfrac{1}{2}\beta^{-1/2}(h_2-\beta h_1)\cos\phi]\,. \quad (27)$$

The index of refraction n can now be calculated from Eq. (6):

$$n=c(\phi)/c_g(\phi)$$
$$=\beta^{1/2}+\tfrac{1}{2}\beta^{1/2}[(\beta^{1/2}-1)h_1+(1-\beta^{-1/2})h_2]\cos\phi. \quad (28)$$

As a result, this theory predicts a refractive index that has a ϕ dependence. The second integral in Eq. (5) becomes

$$\int\nu g(\phi)[c(\phi)]^{-1}dl=\int\nu(\beta^{1/2}-1)c_0^{-1}[1+\tfrac{1}{2}(1+\beta^{1/2})h_1\cos\phi]\,dl\,. \quad (29)$$

So long as $h_1\neq0$, the integration through the glass gives a nonzero integral, and hence the interferometer described in Ref. 1 would detect the anisotropy induced by the h_1 term in the modified Maxwell's equations.

V. CONCLUSION

The modified Maxwell's equations discussed in this paper predict two anisotropies in the speed of light. One that behaves as $\cos\phi$ and is first order in the h's. The other behaves as $\cos^2\phi$ or equivalently $\cos2\phi$, and is second order in the h's. Here ϕ is the angle between a preferred direction in space and the direction of light propagation. The refractive index has an anisotropy term that behaves as $\cos\phi$. Such anisotropy could be detected by the interferometer described in Ref. 1.

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