PHYSICAL REVIEW D VOLUME 11, NUMBER 12 15 JUNE 1975

Comment on the modifications of Maxwell's equations proposed by Trimmer and Baierlein*

G. C. Duncan and L. P. Fulcher
Department of Physics, Bowling Green State University, Bowling Green, Ohio 43403
(Received 25 November 1974)

It is shown that the recent modifications of Maxwell's equations proposed by Trimmer and Baierlein do not allow one to discuss energy conservation of the electromagnetic field from the point of view of a generalization of Poynting's theorem.

Trimmer and Baierlein[1] have recently examined the plane-wave solutions of a modified set of Maxwell equations. This work was no doubt stimulated by their experimental search[2] for an anisotropy in the speed of light, and these equations were presumably invented in order to emphasize the kinds of new features that may arise if the speed of light were not isotropic. Of course, their equations have implications for the character of electromagnetic phenomena[3] in situations other than those encompassed by interferometer experiments.[2,4] Thus, it is appropriate to ask what kinds of experiments would set the most stringent upper limits on the small parameters in the theory.

On the other hand, one might study the equations further to see which of the conventional ideas about electromagnetism must be modified. As the beginning of such an effort, we present below the derivation of a theorem that forms the basis for a discussion of energy conservation. We illustrate some of the features of the new terms that arise with the example of plane waves.

First, we record the modified Maxwell equations utilized by Trimmer and Baierlein:

$$\vec{\nabla}\cdot\vec{E}=4\pi\rho, \tag{1}$$

$$\vec{\nabla}\cdot\vec{B}=0, \tag{2}$$

$$\vec{\nabla}\times(\vec{E}+h_1\hat{\mu}\times\vec{B})=-\frac{1}{c_0}\frac{\partial\vec{B}}{\partial t}, \tag{3}$$

$$\vec{\nabla}\times(\vec{B}+h_2\hat{\mu}\times\vec{E})=\frac{4\pi}{c_0}\vec{J}+\frac{1}{c_0}\frac{\partial\vec{E}}{\partial t}. \tag{4}$$

Here $\vec{E}$ and $\vec{B}$ have the usual interpretation as electric and magnetic fields in the limit that h_1 and h_2 vanish. The constant unit vector $\hat{\mu}$ characterizes the preferred direction in space that introduces the anisotropy in the speed of light, while c_0 denotes the vacuum speed of light in the absence of any preferred direction. We now derive an "energy integral" of Eqs. (1)–(4). First, take the dot product of Eq. (3) with $\vec{B}$. From the result we subract the dot product of Eq. (4) with $\vec{E}$. These operations yield, after rearrangement of terms,

$$\vec{B}\cdot(\vec{\nabla}\times\vec{E})-\vec{E}\cdot(\vec{\nabla}\times\vec{B})+h_1\vec{B}\cdot\vec{\nabla}\times(\hat{\mu}\times\vec{B})-h_2\vec{E}\cdot\vec{\nabla}\times(\hat{\mu}\times\vec{E})=-\frac{1}{c_0}\left(\vec{B}\cdot\frac{\partial\vec{B}}{\partial t}+\vec{E}\cdot\frac{\partial\vec{E}}{\partial t}\right)-\frac{4\pi}{c_0}\vec{E}\cdot\vec{J}. \tag{5}$$

Next, making use of the vector identities

$$\vec{\nabla}\cdot(\vec{E}\times\vec{B})=\vec{B}\cdot(\vec{\nabla}\times\vec{E})-\vec{E}\cdot(\vec{\nabla}\times\vec{B})$$

and

$$\frac{1}{2}\frac{\partial}{\partial t}(E^2+B^2)=\vec{B}\cdot\frac{\partial\vec{B}}{\partial t}+\vec{E}\cdot\frac{\partial\vec{E}}{\partial t},$$

we obtain the following intermediate result:

$$\vec{\nabla}\cdot(\vec{E}\times\vec{B})+h_1\vec{B}\cdot\vec{\nabla}\times(\hat{\mu}\times\vec{B})-h_2\vec{E}\cdot\vec{\nabla}\times(\hat{\mu}\times\vec{E})+\frac{1}{2c_0}\frac{\partial}{\partial t}(E^2+B^2)+\frac{4\pi}{c_0}\vec{E}\cdot\vec{J}=0. \tag{6}$$

Let us consider $\vec{B}\cdot\vec{\nabla}\times(\hat{\mu}\times\vec{B})$. The curl may be computed as follows:

$$\vec{\nabla}\times(\hat{\mu}\times\vec{B})=\hat{\mu}(\vec{\nabla}\cdot\vec{B})-\vec{B}(\vec{\nabla}\cdot\hat{\mu})+(\vec{B}\cdot\vec{\nabla})\hat{\mu}-(\hat{\mu}\cdot\vec{\nabla})\vec{B}.$$

But $\vec{B}$ is divergenceless and $\hat{\mu}$ is constant. Thus

$$\vec{\nabla}\times(\hat{\mu}\times\vec{B})=-(\hat{\mu}\cdot\vec{\nabla})\vec{B}.$$

Hence we have

$$\vec{B}\cdot\vec{\nabla}\times(\hat{\mu}\times\vec{B})=-\vec{B}\cdot(\hat{\mu}\cdot\vec{\nabla})\vec{B}=-\tfrac{1}{2}\vec{\nabla}\cdot(B^2\hat{\mu}). \tag{7}$$

The last step may readily be verified by using the Cartesian components of $\vec{B}$ and $\vec{\nabla}$. However, when the same procedure is applied to the analogous term involving $\vec{E}$ in Eq. (6), we do not obtain quite the same result. The difference is due to the nonvanishing of the divergence of the electric field. We state the result, namely,

$$\vec{E}\cdot\vec{\nabla}\times(\hat{\mu}\times\vec{E})=4\pi\rho\hat{\mu}\cdot\vec{E}-\tfrac{1}{2}\vec{\nabla}\cdot(E^2\hat{\mu}). \tag{8}$$

Thus, using Eqs. (7) and (8) in Eq. (6) and collecting terms, we find that

$$\frac{c_0}{4\pi}\vec{\nabla}\cdot[\vec{E}\times\vec{B}+\tfrac{1}{2}\hat{\mu}(h_2E^2-h_1B^2)]+\frac{1}{8\pi}\frac{\partial}{\partial t}(E^2+B^2)=-\vec{J}\cdot\vec{E}+h_2c_0\rho\hat{\mu}\cdot\vec{E}, \tag{9}$$

which embodies the theorem that we wished to establish. Taking the limits $h_1 \to 0$ and $h_2 \to 0$ we may recover Poynting's theorem.[5]

Equation (9) contains several features not encountered in the conventional theory of electromagnetism. The most troublesome of these arise from the last term. For example, the last term would seem to prevent one from considering the charges, currents, and fields as an isolated system. If one attempts to combine the two terms on the right-hand side of Eq. (9) by introducing an effective current $\vec{J}' = \vec{J} - h_2 c_0 \rho \hat{\mu}$, then this current is not conserved. If this is not done, then the last term seems to represent an energy gain or loss at those points in space where the charge density is not zero. It is interesting to note, however,[7,8] that the integral of the last term is equal to zero for any static charge distribution. Thus, the energy content of the fields surrounding a static charge distribution would not change although there would be local effects due to this term. Another implication of the last term may be a new means of communication between an electromagnetic wave and a charge. It would presumably have to be taken into account in the scattering of a beam of light from a charged particle in addition to the usual mechanism where the light accelerates the charged particle and the charged particle emits radiation. These features, of course, would not arise if $h_2 = 0$.

Tentatively, we identify the first term in Eq. (9) as the divergence of a modified Poynting vector $\vec{S}$ and the second term as the time rate of change of the usual energy density U. Now we calculate these two quantities for plane waves in free space. As Trimmer and Baierlein[1] show, Eq. (3) implies that

$$\vec{B} = \frac{\hat{k} \times \vec{E}}{(\omega / k c_0) + h_1 \hat{\mu} \cdot \hat{k}}\,, \tag{10}$$

where ω is the frequency and $\vec{k}$ is the wave vector. One may use their dispersion relation [Eq. (25) of Ref. 1] to solve for the phase velocities,

$$v^{\pm} = \omega_{\pm}/k = c_0[\alpha + \beta \pm (1+\beta^2)^{1/2}], \tag{11}$$

where $\alpha = -h_1 \hat{\mu} \cdot \hat{k}$ and $\beta = (h_1 + h_2)\hat{\mu} \cdot \hat{k}/2$. Using Eq. (10) for $\vec{B}$, the energy density takes the form

$$U_{\pm} = \frac{\langle E^2 \rangle}{8\pi} \left[1 + \frac{1}{[\beta \pm (1+\beta^2)^{1/2}]^2}\right], \tag{12}$$

where the angular brackets refer to an average of the square of the electric field over one complete oscillation. The calculation of the modified Poynting vector is also straightforward, and the ratio of its component in the direction of $\vec{k}$ to the energy density is given by

$$\frac{\vec{S} \cdot \hat{k}_{\pm}}{U_{\pm}} = \frac{c_0\{\beta \pm (1+\beta^2)^{1/2} - \alpha/2 + (\beta + \alpha/2)[\beta \pm (1+\beta^2)^{1/2}]^2\}}{\beta^2 + 1 \pm \beta(1+\beta^2)^{1/2}}\,. \tag{13}$$

We have found that this ratio does not reduce to the phase velocity unless the parameters are restricted. If $h_2 = 0$, then it is equal to $c_0[-\beta \pm (1+\beta^2)^{1/2}]$, the phase velocity. Similarly, if $h_1 = 0$, then it is equal to $c_0[\beta \pm (1+\beta^2)^{1/2}]$, the phase velocity in this case. If $h_1 = h_2 \neq 0$, then the ratio bears no simple relation to the phase velocity. Thus, if the conventional idea that the phase velocity is the velocity of energy transport in the direction of $\vec{k}$ is to be retained the parameters of the theory must be restricted.

In conclusion, we have found that two peculiar features of the equations studied by Trimmer and Baierlein can be eliminated if $h_2 = 0$.

The authors gratefully acknowledge stimulating conversations with Dr. Trimmer and useful correspondence with Dr. Trimmer and Dr. Baierlein.

*Work supported in part by two grants from the Faculty Research Committee at Bowling Green State University.

[1]W. S. N. Trimmer and R. F. Baierlein, Phys. Rev. D 8, 3326 (1973).

[2]W. S. N. Trimmer *et al.*, Phys. Rev. D 8, 3321 (1973); 9, 2489(E) (1974).

[3]W. S. N. Trimmer, private communication; L. M. Winer, Phys. Rev. D (to be published).

[4]See, for example, G. Joos, Ann. Phys. (Leipz.) 7, 385 (1930).

[5]J. D. Jackson, *Classical Electrodynamics* (Wiley, New York, 1962).

[6]Trimmer and Baierlein showed that their modifications preserved the continuity equation for $\vec{J}$ and ρ. See Eqs. (11)–(13) of Ref. 1.

[7]W. S. N. Trimmer, Ph.D. thesis, Wesleyan University (unpublished).

[8]W. S. N. Trimmer and R. Baierlein, private communication.