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On the Theory of Resonance

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V. *On the Theory of Resonance. By the Hon.* J. W. STRUTT, *M.A., Fellow of Trinity College, Cambridge. Communicated by* W. SPOTTISWOODE, *F.R.S.*

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Introduction.

ALTHOUGH the theory of aërial vibrations has been treated by more than one generation of mathematicians and experimenters, comparatively little has been done towards obtaining a clear view of what goes on in any but the more simple cases. The extreme difficulty of any thing like a general deductive investigation of the question is no doubt one reason. On the other hand, experimenters on this, as on other subjects, have too often observed and measured blindly without taking sufficient care to simplify the conditions of their experiments, so as to attack as few difficulties as possible at a time. The result has been vast accumulations of isolated facts and measurements which lie as a sort of dead weight on the scientific stomach, and which must remain undigested until theory supplies a more powerful solvent than any now at our command. The motion of the air in cylindical organ-pipes was successfully investigated by BERNOULLI and EULER, at least in its main features; but their treatment of the question of the open pipe was incomplete, or even erroneous, on account of the assumption that at the open end the air remains of invariable density during the vibration. Although attacked by many others, this difficulty was not finally overcome until HELMHOLTZ †, in a paper which I shall have repeated occasion to refer to, gave a solution of the problem under certain restrictions, free from any arbitrary assumptions as to what takes place at the open end. POISSON and STOKES‡ have solved the problem of the vibrations communicated to an infinite mass of air from the surface of a sphere or circular cylinder. The solution for the sphere is very instructive, because the vibrations outside any imaginary sphere enclosing vibrating bodies of any kind may be supposed to take their rise in the surface of the sphere itself.

More important in its relation to the subject of the present paper is an investigation by HELMHOLTZ of the air-vibrations in cavernous spaces (*Hohlraüme*), whose three dimensions are very small compared to the wave-length, and which communicate with the external atmosphere by small holes in their surfaces. If the opening be circular of area σ, and if S denote the volume, n the number of vibrations per second in the fundamental

* Additions made since the paper was first sent to the Royal Society are inclosed in square brackets [ ].

† Theorie der Luftschwingungen in Röhren mit offenen Enden. Crelle, 1860.

‡ Phil. Trans. 1868, or Phil. Mag. Dec. 1868.

note, and a the velocity of sound,

$$n=\frac{a\sigma^{\frac{1}{4}}}{2^{\frac{1}{2}}\pi^{\frac{5}{4}}S^{\frac{1}{2}}}.$$

HELMHOLTZ's theory is also applicable when there are more openings than one in the side of the vessel.

In the present paper I have attempted to give the theory of vibrations of this sort in a more general form. The extension to the case where the communication with the external air is no longer by a mere hole in the side, but by a neck of greater or less length, is important, not only because resonators with necks are frequently used in practice, but also by reason of the fact that the theory itself is applicable within wider limits. The mathematical reasoning is very different from that of HELMHOLTZ, at least in form, and will I hope be found easier. In order to assist those who may wish only for clear general ideas on the subject, I have broken up the investigation as much as possible into distinct problems, the results of which may in many cases be taken for granted without the rest becoming unintelligible. In Part I. my object has been to put what may be called the dynamical part of the subject in a clear light, deferring as much as possible special mathematical calculations. In the first place, I have considered the general theory of resonance for air-spaces confined nearly all round by rigid walls, and communicating with the external air by any number of passages which may be of the nature of necks or merely holes, under the limitation that both the length of the necks and the dimensions of the vessel are very small compared to the wave-length. To prevent misapprehension, I ought to say that the theory applies only to the fundamental note of the resonators, for the vibrations corresponding to the overtones are of an altogether different character. There are, however, cases of multiple resonance to which our theory is applicable. These occur when two or more vessels communicate with each other and with the external air by necks or otherwise; and are easily treated by LAGRANGE's general dynamical method, subject to a restriction as to the relative magnitudes of the wave-lengths and the dimensions of the system corresponding to that stated above for a single vessel. I am not aware whether this kind of resonance has been investigated before, either mathematically or experimentally. Lastly, I have sketched a solution of the problem of the open organ-pipe on the same general plan, which may be acceptable to those who are not acquainted with HELMHOLTZ's most valuable paper. The method here adopted, though it leads to results essentially the same as his, is I think more calculated to give an insight into the real nature of the question, and at the same time presents fewer mathematical difficulties. For a discussion of the solution, however, I must refer to HELMHOLTZ.

In Part II. the calculation of a certain quantity depending on the form of the necks of common resonators, and involved in the results of Part I., is entered upon. This quantity, denoted by c, is of the nature of a length, and is identical with what would be called in the theory of electricity the *electric conductivity* of the passage, supposed to be occupied by uniformly conducting matter. The question is accordingly similar to that of determining the electrical resistance of variously shaped conductors—an analogy of which I have not hesitated to avail myself freely both in investigation and statement. Much circumlocution is in this way avoided on account of the greater completeness of electrical phraseology. Passing over the case of mere holes, which has been already considered by HELMHOLTZ, and need not be dwelt upon here, we come to the value of the resistance for necks in the form of circular cylinders. For the sake of simplicity each end is supposed to be in an infinite plane. In this form the mathematical problem is definite, but has not been solved rigorously. Two limits, however (a higher and a lower), are investigated, between which it is proved that the true resistance must lie. The lower corresponds to a correction to the length of the tube equal to $\frac{\pi}{4}\times$(radius) for each end. It is a remarkable coincidence that HELMHOLTZ also finds the same quantity as an approximate correction to the length of an organ-pipe, although the two methods are entirely different and neither of them rigorous. His consists of an exact solution of the problem for an approximate cylinder, and mine of an approximate solution for a true cylinder; while both indicate on which side the truth must lie. The final result for a cylinder infinitely long is that the correction lies between ·785 R and ·828 R. When the cylinder is finite, the upper limit is rather smaller. In a somewhat similar manner I have investigated limits for the resistance of a tube of revolution, which is shown to lie between

$$\int\frac{dx}{\pi y^2}$$

and

$$\int\frac{dx}{\pi y^2}\left\{1+\frac{1}{2}\left(\frac{dy}{dx}\right)^2\right\},$$

where y denotes the radius of the tube at any point x along the axis. These formulæ apply whatever may be in other respects the form of the tube, but are especially valuable when it is so nearly cylindrical that $\frac{dy}{dx}$ is everywhere small. The two limits are then very near each other, and either of them gives very approximately the true value. The resistance of tubes, which are either not of revolution or are not nearly straight, is afterwards approximately determined. The only experimental results bearing on the subject of this paper, and available for comparison with theory, that I have met with are some arrived at by SONDHAUSS* and WERTHEIM†. Besides those quoted by HELMHOLTZ, I have only to mention a series of observations by SONDHAUSS‡ on the pitch of flasks with long necks which led him to the empirical formula

$$n=46705\frac{\sigma^{\frac{1}{2}}}{L^{\frac{1}{2}}S^{\frac{1}{2}}},$$

σ, L being the area and length of the neck, and S the volume of the flask. The corresponding equation derived from the theory of the present paper is

$$n=54470\frac{\sigma^{\frac{1}{4}}}{L^{\frac{1}{2}}S^{\frac{1}{2}}},$$

* POGG. Ann. vol. lxxxi.

† Annales de Chimie, vol. xxxi.

‡ POGG. Ann. vol. lxxix.

which is only applicable, however, when the necks are so long that the corrections at the ends may be neglected—a condition not likely to be fulfilled. This consideration sufficiently explains the discordance. Being anxious to give the formulæ of Parts I. and II. a fair trial, I investigated experimentally the resonance of a considerable number of vessels which were of such a form that the theoretical pitch could be calculated with tolerable accuracy. The result of the comparison is detailed in Part III., and appears on the whole very satisfactory; but it is not necessary that I should describe it more minutely here. I will only mention, as perhaps a novelty, that the experimental determination of the pitch was not made by causing the resonators to speak by a stream of air blown over their mouths. The grounds of my dissatisfaction with this method are explained in the proper place.

[Since this paper was written there has appeared another memoir by Dr. SONDHAUSS* on the subject of resonance. An empirical formula is obtained bearing resemblance to the results of Parts I. and II., and agreeing fairly well with observation. No attempt is made to connect it with the fundamental principles of mechanics. In the Philosophical Magazine for September 1870, I have discussed the differences between Dr. SONDHAUSS's formula and my own from the experimental side, and shall not therefore go any further into the matter on the present occasion.]

PART I.

The class of resonators to which attention will chiefly be given in this paper are those where a mass of air confined almost all round by rigid walls communicates with the external atmosphere by one or more narrow passages. For the present it may be supposed that the boundary of the principal mass of air is part of an oval surface, nowhere contracted into any thing like a narrow neck, although some cases not coming under this description will be considered later. In its general character the fundamental vibration of such an air-space is sufficiently simple, consisting of a periodical rush of air through the narrow channel (if there is only one) into and out of the confined space, which acts the part of a reservoir. The channel spoken of may be either a mere hole of any shape in the side of the vessel, or may consist of a more or less elongated tube-like passage.

If the linear dimension of the reservoir be small as compared to the wave-length of the vibration considered, or, as perhaps it ought rather to be said, the quarter wave-length, the motion is remarkably amenable to deductive treatment. Vibration in general may be considered as a periodic transformation of energy from the potential to the kinetic, and from the kinetic to the potential forms. In our case the kinetic energy is that of the air in the neighbourhood of the opening as it rushes backwards or forwards. It may be easily seen that relatively to this the energy of the motion inside the reservoir is, under the restriction specified, very small. A formal proof would require the assistance of the general equations to the motion of an elastic fluid, whose use I wish to avoid in

* POGG. Ann. 1870.

this paper. Moreover the motion in the passage and its neighbourhood will not differ sensibly from that of an incompressible fluid, and its energy will depend only on the rate of total flow through the opening. A quarter of a period later this energy of motion will be completely converted into the potential energy of the compressed or rarefied air inside the reservoir. So soon as the mathematical expressions for the potential and kinetic energies are known, the determination of the period of vibration or resonant note of the air-space presents no difficulty.

The motion of an incompressible frictionless fluid which has been once at rest is subject to the same formal laws as those which regulate the flow of heat or electricity through uniform conductors, and depends on the properties of the potential function, to which so much attention has of late years been given. In consequence of this analogy many of the results obtained in this paper are of as much interest in the theory of electricity as in acoustics, while, on the other hand, known modes of expression in the former subject will save circumlocution in stating some of the results of the present problem.

Let h_0 be the density, and ϕ the velocity-potential of the fluid motion through an opening. The kinetic energy or *vis viva*

$$=\frac{1}{2}h_0\iiint\left[\left(\frac{d\phi}{dx}\right)^2+\left(\frac{d\phi}{dy}\right)^2+\left(\frac{d\phi}{dz}\right)^2\right]dxdydz,$$

the integration extending over the volume of the fluid considered

$$=\frac{1}{2}h_0\iint\phi\frac{d\phi}{dn}dS,$$

by GREEN'S theorem.

Over the rigid boundary of the opening or passage, $\frac{d\phi}{dn}=0$, so that if the portion of fluid considered be bounded by two equipotential surfaces, ϕ_1 and ϕ_2, one on each side of the opening,

$$vis\ viva=\frac{1}{2}h_0(\phi_1-\phi_2)\iint\frac{d\phi}{dn}dS=\frac{1}{2}h_0(\phi_1-\phi_2)\dot{X},$$

if $\dot{X}$ denote the rate of total flow through the opening.

At a sufficient distance on either side ϕ becomes constant, and the rate of total flow is proportional to the difference of its values on the two sides. We may therefore put

$$\phi_1-\phi_2=\frac{1}{c}\iint\frac{d\phi}{dn}dS=\frac{\dot{X}}{c},$$

where c is a linear quantity depending on the size and shape of the opening, and representing in the electrical interpretation the reciprocal of the *resistance* to the passage of electricity through the space in question, the specific resistance of the conducting matter being taken for unity. The same thing may be otherwise expressed by saying that c is the side of a cube, whose resistance between opposite faces is the same as that of the opening.

The expression for the *vis viva* in terms of the rate of total flow is accordingly

$$vis\ viva=\frac{h_0}{2}\frac{\dot{X}^2}{c}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

If S be the capacity of the reservoir, the condensation at any time inside it is given by

$\frac{X}{S}$, of which the mechanical value is

$$\frac{1}{2}h_0a^2\frac{X^2}{S}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

a denoting, as throughout the paper, the velocity of sound.

The whole energy at any time, both actual and potential, is therefore

$$\frac{h_0}{2}\frac{\dot{X}^2}{c}+\frac{h_0}{2}a^2\frac{X^2}{S}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

and is constant. Differentiating with respect to time, we arrive at

$$\ddot{X}+\frac{a^2c}{S}X=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

as the equation to the motion, which indicates simple oscillations performed in a time

$$2\pi\div\sqrt{\frac{a^2c}{S}}.$$

Hence if n denote the number of vibrations per second in the resonant note,

$$n=\frac{a}{2\pi}\sqrt{\frac{c}{S}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

The wave-length λ, which is the quantity most immediately connected with the dimensions of the resonant space, is given by

$$\lambda=\frac{a}{n}=2\pi\sqrt{\frac{S}{c}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

A law of SAVART, not nearly so well known as it ought to be, is in agreement with equations (5) and (6). It is an immediate consequence of the principle of dynamical similarity, of extreme generality, to the effect that *similar* vibrating bodies, whether they be gaseous, such as the air in organ-pipes or in the resonators here considered, or solid, such as tuning-forks, vibrate in a time which is directly as their linear dimensions. Of course the material must be the same in two cases that are to be compared, and the geometrical similarity must be complete, extending to the shape of the opening as well as to the other parts of the resonant vessel. Although the wave-length λ is a function of the size and shape of the resonator only, n or the position of the note in the musical scale depends on the nature of the gas with which the resonator is filled. And it is important to notice that it is on the nature of the gas in and near the opening that the note depends, and *not* on the gas in the interior of the reservoir, whose inertia does not come into play during vibrations corresponding to the fundamental note. In fact we may say that the mass to be moved is the air in the neighbourhood of the opening, and that the air in the interior acts merely as a spring in virtue of its resistance to compression. Of course this is only true under the limitation specified, that the diameter of the reservoir is small compared to the quarter wave-length. Whether this condition is fulfilled in the case of any particular resonator is easily seen, *à posteriori*, by calculating the value of λ from (6), or by determining it experimentally.

Several Openings.

When there are two or more passages connecting the interior of the resonator with the external air, we may proceed in much the same way, except that the equation of energy by itself is no longer sufficient. For simplicity of expression the case of two passages will be convenient, but the same method is applicable to any number. Let X_1, X_2 be the total flow through the two necks, c_1, c_2 constants depending on the form of the necks corresponding to the constant c in formula (6); then T, the *vis viva*, is given by

$$T=\frac{h_0}{2}\left(\frac{\dot{X}_1^2}{c_1}+\frac{\dot{X}_2^2}{c_2}\right),$$

the necks being supposed to be sufficiently far removed from one another not to *interfere* (in a sense that will be obvious). Further,

$$V=\text{Potential Energy}=\frac{1}{2}h_0a^2\frac{(X_1+X_2)^2}{S}.$$

Applying LAGRANGE's general dynamical equation, $\frac{d}{dt}\left(\frac{dT}{d\dot{\psi}}\right)-\frac{dT}{d\psi}=-\frac{dV}{d\psi}$,

we obtain

$$\left.\begin{aligned}\frac{\ddot{X}_1}{c_1}+\frac{a^2}{S}(X_1+X_2)=0,\\ \frac{\ddot{X}_2}{c_2}+\frac{a^2}{S}(X_1+X_2)=0\end{aligned}\right\} \quad \ldots \ldots \quad (7)$$

as the equations to the motion.

By subtraction,

$$\frac{\ddot{X}_1}{c_1}-\frac{\ddot{X}_2}{c_2}=0,$$

or, on integration,

$$\frac{X_1}{c_1}=\frac{X_2}{c_2} \quad \ldots \ldots \quad (8)$$

Equation (8) shows that the motions of the air in the two necks have the same period and are at any moment in the same phase of vibration. Indeed there is no essential distinction between the case of one neck and that of several, as the passage from one to the other may be made continuously without the failure of the investigation.

When, however, the separate passages are sufficiently far apart, the constant c for the system, considered as a single communication between the interior of the resonator and the external air, is the simple sum of the values belonging to them when taken separately, which would not otherwise be the case. This is a point to which we shall return later, but in the mean time, by addition of equations (7), we find

$$\ddot{X}_1+\ddot{X}_2+\frac{a^2}{S}(c_1+c_2)(X_1+X_2)=0,$$

so that

$$n=\frac{a}{2\pi}\sqrt{\frac{c_1+c_2}{S}} \quad \ldots \ldots \ldots \quad (9)$$

If there be any number of necks for which the values of c are c_1, c_2, c_3, ..., and no two of which are near enough to interfere, the same method is applicable, and gives

$$n=\frac{a}{2\pi}\sqrt{\frac{c_1+c_2+c_3+\ldots}{S}}; \quad \ldots \ldots \ldots \quad (9')$$

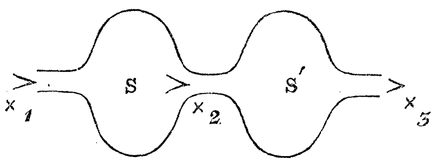
when there are two similar necks $c_2=c_1$, and

$$n=\sqrt{2}\times\frac{a}{2\pi}\sqrt{\frac{c}{S}}.$$

The note is accordingly higher than if there were only one neck in the ratio of $\surd 2:1$, a fact observed by SONDHAUSS and proved theoretically by HELMHOLTZ for the case of openings which are mere holes in the sides of the reservoir.

Double Resonance.

Fig. 1.



Suppose that there are two reservoirs, S, S′, communicating with each other and with the external air by narrow passages or necks. If we were to consider S S′ as a single reservoir and to apply equation (9), we should be led to an erroneous result; for the reasoning on which (9) is founded proceeds on the assumption that, within the reservoir, the inertia of the air may be left out of account, whereas it is evident that the *vis viva* of the motion through the connecting passage may be as great as through the two others. However, an investigation on the same general plan as before meets the case perfectly. Denoting by X_1, X_2, X_3 the total flows through the three necks, we have for the *vis viva* the expression

$$T=\frac{1}{2}h_0\left\{\frac{\dot{X}_1^2}{c_1}+\frac{\dot{X}_2^2}{c_2}+\frac{\dot{X}_3^2}{c_3}\right\},$$

and for the potential energy

$$V=\frac{1}{2}h_0a^2\left\{\frac{(X_2-X_1)^2}{S}+\frac{(X_3-X_2)^2}{S'}\right\}.$$

An application of LAGRANGE's method gives as the differential equations to the motion,

$$\left.\begin{aligned}&\frac{\ddot{X}_1}{c_1}+a^2\frac{X_1-X_2}{S}=0,\\&\frac{\ddot{X}_2}{c_2}+a^2\left\{\frac{X_2-X_1}{S}+\frac{X_2-X_3}{S'}\right\}=0,\\&\frac{\ddot{X}_3}{c_3}+a^2\frac{X_3-X_2}{S'}=0.\end{aligned}\right\} \quad \ldots \ldots \ldots \quad (10)$$

By addition and integration

$$\frac{X_1}{c_1}+\frac{X_2}{c_2}+\frac{X_3}{c_3}=0.$$

Hence, on elimination of X_2,

$$\left.\begin{aligned}&\ddot{X}_1+\frac{a^2}{S}\left\{(c_1+c_2)X_1+\frac{c_1c_2}{c_3}X_3\right\}=0,\\&\ddot{X}_3+\frac{a^2}{S'}\left\{(c_3+c_2)X_3+\frac{c_3c_2}{c_1}X_1\right\}=0.\end{aligned}\right\}$$

Assuming $X_1=A\varepsilon^{pt}$, $X_3=B\varepsilon^{pt}$, we obtain, on substitution and elimination of $A:B$,

$$p^4+p^2a^2\left\{\frac{c_1+c_2}{S}+\frac{c_3+c_2}{S'}\right\}+\frac{a^4}{SS'}\left\{c_1c_3+c_2(c_1+c_3)\right\}=0 \quad \ldots \ldots \ldots \quad (11)$$

as the equation to determine the resonant notes. If n be the number of vibrations per second, $n^2=-\frac{p^2}{4\pi^2}$, the values of p^2 given by (11) being of course both real and negative. The formula simplifies considerably if $c_3=c_1$, $S'=S$; but it will be more instructive to work this case from the beginning. Let $c_1=c_3=mc_2=mc$.

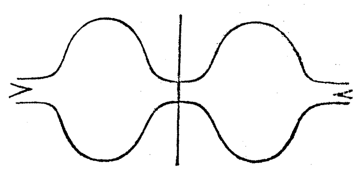
The differential equations take the form

$$\left.\begin{aligned}&\ddot{X}_1+\frac{a^2c}{S}\left\{(1+m)X_1+X_3\right\}=0,\\&\ddot{X}_3+\frac{a^2c}{S}\left\{(1+m)X_3+X_1\right\}=0,\end{aligned}\right\}\text{while } X_2=-\frac{X_1+X_3}{m}.$$

Hence

$$\left.\begin{aligned}&(X_1+X_3)^{\cdot\cdot}+\frac{a^2c}{S}(m+2)(X_1+X_3)=0,\\&(X_1-X_3)^{\cdot\cdot}+\frac{a^2c}{S}m(X_1-X_3)=0.\end{aligned}\right\}$$

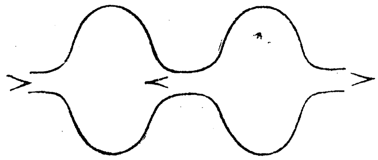
Fig. 2.



The whole motion may be regarded as made up of two parts, for the first of which $X_1+X_3=0$; which requires $X_2=0$. This motion is therefore the same as might take place were the communication between S and S′ cut off, and has its period given by

$$n^2=\frac{a^2c_1}{4\pi^2S}=\frac{a^2mc}{4\pi^2S}.$$

Fig. 3.



For the other component part, $X_1-X_3=0$, so that

$$X_2=-\frac{2X_1}{m}, \qquad n'^2=\frac{a^2(m+2)c}{4\pi^2 S} \quad . \quad . \quad . \quad (12)$$

Thus $\frac{n'^2}{n^2}=\frac{m+2}{m}$, which shows that the second note is the higher. It consists of vibrations in the two reservoirs opposed in phase and modified by the connecting passage, which acts in part as a second opening to both, and so raises the pitch. If the passage is small, so also is the difference of pitch between the two notes. A particular case worth notice is obtained by putting in the general equation $c_3=0$, which amounts to suppressing one of the communications with the external air. We thus obtain

$$p^4+a^2p^2\left(\frac{c_1+c_2}{S}+\frac{c_2}{S'}\right)+\frac{a^4}{SS'}c_1c_2=0\,;$$

or if $S=S'$, $c_1=mc_2=mc$,

$$p^4+a^2p^2\frac{c}{S}(m+2)+\frac{a^4c^2}{S^2}m=0,$$

$$n^2=\frac{a^2c}{8\pi^2S}\{m+2\pm\sqrt{m^2+4}\}.$$

If we further suppose $m=1$ or $c_2=c_1$,

$$n^2=\frac{a^2c}{8\pi^2S}(3\pm\sqrt{5}).$$

If N be the number of vibrations for a simple resonator (S, c),

$$N^2=\frac{a^2c}{4\pi^2S}\,;$$

$$\therefore\ n_1^2\div N^2=\frac{3+\sqrt{5}}{2}=2\cdot618,$$

$$N^2\div n_2^2=\frac{2}{3-\sqrt{5}}=2\cdot618.$$

It appears therefore that the interval from n_1 to N is the same as from N to n_2, namely, $\sqrt{2\cdot618}=1\cdot618$, or rather more than a fifth. It will be found that whatever the value of m may be, the interval between the resonant notes cannot be less than 2·414, which is about an octave and a minor third. The corresponding value of m is 2.

A similar method is applicable to any combination of reservoirs and connecting passages, no matter how complicated, under the single restriction as to the comparative magnitudes of the reservoirs and wave-lengths; but the example just given is sufficient to illustrate the theory of multiple resonance. In Part III. a resonator of this sort will be described, which was constructed for the sake of a comparison between the theory and experiment. In applying the formulæ (6) or (12) to an actual measurement, the question will arise whether the volume of the necks, especially when they are rather large, is to be included or not in S. At the moment of rest the air in the neck is compressed or rarefied as well as that inside the reservoir, though not to the same degree; in fact the condensation must vary continuously between the interior of the resonator and the external air. This consideration shows that, at least in the case of necks which are tolerably symmetrical, about half the volume of the neck should be included in S.

[In consequence of a suggestion made by Mr. CLERK MAXWELL, who reported on this paper, I have been led to examine what kind of effect would be produced by a deficient rigidity in the envelope which contains the alternately compressed and rarefied air. Taking for simplicity the case of a sphere, let us suppose that the radius, instead of remaining constant at its normal value R, assumes the variable magnitude $R+\varrho$. We have

$$\text{kinetic energy}=\frac{h_0\dot{X}^2}{2c}+\frac{m}{2}\dot{\varrho}^2,$$

$$\text{potential energy}=\frac{h_0a^2}{2S}\{X+4\pi R^2\varrho\}^2+\tfrac{1}{2}\beta\varrho^2,$$

where m and β are constants expressing the inertia and rigidity of the spherical shell. Hence, by LAGRANGE's method,

$$\left.\begin{aligned}&\ddot{X}+\frac{ca^2}{S}(X+4\pi R^2\varrho)=0,\\ &m\ddot{\varrho}+4\pi R^2\frac{h_0a^2}{S}(X+4\pi R^2\varrho)+\beta\varrho=0,\end{aligned}\right\}$$

equations determining the periods of the two vibrations of which the system is capable. It might be imagined at first sight that a yielding of the sides of the vessel would necessarily lower the pitch of the resonant note; but this depends on a tacit assumption that the capacity of the vessel is largest when the air inside is most compressed. But it may just as well happen that the opposite is true. Everything depends on the relative magnitudes of the periods of the two vibrations supposed for the moment independent of one another. If the note of the shell be very high compared to that of the air, the inertia of the shell may be neglected, and this part of the question treated statically. Putting in the equations $m=0$, we see that the phases of X and ϱ are opposed, and then X goes through its changes more slowly than before. On the other hand, if it be the note of the air-vibration, which is much the higher, we must put $\beta=0$, which leads to

$$4\pi R^2h_0\ddot{X}-cm\ddot{\varrho}=0,$$

showing that the phases of X and ϱ agree. Here the period of X is diminished by the yielding of the sides of the vessel, which indeed acts just in the same way as a second aperture would do. A determination of the actual note in any case of a spherical shell of given dimensions and material would probably be best obtained deductively.

But in order to see what probability there might be that the results of Part III. on glass flasks were sensibly modified by a want of rigidity, I thought it best to make a direct experiment. To the neck of a flask was fitted a glass tube of rather small bore, and the whole filled with water so as to make a kind of water-thermometer. On removing by means of an air-pump the pressure of the atmosphere on the outside of the bulb, the liquid fell in the tube, but only to an extent which indicated an increase in the capacity of the flask of about a ten-thousandth part. This corresponds in the ordinary arrangement to a doubled density of the contained air. It is clear that so small a yielding could produce no sensible effect on the pitch of the air-vibration.]

Open Organ-pipes.

Although the problem of open organ-pipes, whose diameter is very small compared to their length and to the wave-length, has been fully considered by HELMHOLTZ, it may not be superfluous to show how the question may be attacked from the point of view of the present paper, more especially as some important results may be obtained by a comparatively simple analysis. The principal difficulty consists in finding the connexion between the spherical waves which diverge from the open end of the tube into free space, and the waves in the tube itself, which at a distance from the mouth, amounting to several diameters, are approximately plane. The transition occupies a space which is large compared to the diameter, and in order that the present treatment may be applicable must be small compared to the wave-length. This condition being fulfilled, the compressibility of the air in the space mentioned may be left out of account and the difficulty is turned. Imagine a piston (of infinitely small thickness) in the tube at the place where the waves cease to be plane. The motion of the air on the free side is entirely determined by the motion of the piston, and the *vis viva* within the space considered may be expressed by

$$\frac{1}{2}h_0\frac{\dot{X}^2}{c},$$

where X denotes the rate of total flow at the place of the piston, and c is, as before, a linear quantity depending on the form of the mouth. If Q is the section of the tube and ψ the velocity potential,

$$\dot{X}=Q\frac{d\psi}{dx}.$$

The most general expression for the velocity-potential of plane waves is

$$\psi=\left(\frac{A}{k}\sin kx+B\cos kx\right)\cos 2\pi nt+\beta\cos kx\sin 2\pi nt, \quad \ldots \ldots \quad (13)$$

$$\frac{d\psi}{dx}=(A\cos kx-Bk\sin kx)\cos 2\pi nt-\beta k\sin kx\sin 2\pi nt,$$

where

$$k=\frac{2\pi}{\lambda}=\frac{2\pi n}{a}.$$

When $x=0$,

$$\left.\begin{aligned}\psi&=B\cos 2\pi nt+\beta\sin 2\pi nt,\\ \frac{d\psi}{dx}&=A\cos 2\pi nt.\end{aligned}\right\}$$

The variable part of the pressure on the tube side of the piston

$$=-h_0\frac{d\psi}{dt}.$$

The equation to the motion of the air in the mouth is therefore

$$\frac{Q}{c}\frac{d}{dt}\frac{d\psi}{dx}+\frac{d\psi}{dt}=0,$$

or, on integration,

$$\frac{Q}{c}\frac{d\psi}{dx}+\psi=0. \quad \ldots \ldots \ldots \quad (14)$$

This is the condition to be satisfied when $x=0$.

Substituting the values of ψ and $\frac{d\psi}{dx}$, we obtain

$$\cos 2\pi nt\left(A\frac{Q}{c}+B\right)+\beta\sin 2\pi nt=0,$$

which requires

$$A\frac{Q}{c}+B=0, \qquad \beta=0.$$

If there is a node at $x=-l$

$$A\cos kl+Bk\sin kl=0;$$

$$\therefore k\tan kl=-\frac{A}{B}=-\frac{c}{Q}. \quad \ldots \ldots \ldots \quad (15)$$

This equation gives the fundamental note of the tube closed at $x=-l$; but it must be observed that l is not the length of the tube, because the origin $x=0$ is not in the mouth. There is, however, nothing indeterminate in the equation, although the origin is to a certain extent arbitrary, for the values of c and l will change together so as to make the result for k approximately constant. This will appear more clearly when we come, in Part II., to calculate the actual value of c for different kinds of mouths. In the formation of (14) the pressure of the air on the positive side at a distance from the origin small against λ has been taken absolutely constant. Across such a loop surface no energy could be transmitted. In reality, of course, the pressure is variable on account of the spherical waves, and energy continually escapes from the tube and its vicinity. Although the pitch of the resonant note is not affected, it may be worth while to see what correction this involves.

We must, as before, consider the space in which the transition from plane to spherical waves is effected as small compared with λ. The potential in free space may be taken

$$\psi=\frac{A'}{r}\cos(kr+g-2\pi nt), \quad \ldots \ldots \ldots \quad (16)$$

expressing spherical waves diverging from the mouth of the pipe, which is the origin of r. The origin of x is still supposed to lie in the region of plane waves.

$*4\pi r^2\frac{d\psi}{dr}$= rate of total flow across the surface of the sphere whose radius is r

$$=-4\pi A'[\cos 2\pi nt\{\cos(kr+g)+kr\sin(kr+g)\}+\sin 2\pi nt\{\sin(kr+g)-kr\cos(kr+g)\}].$$

If the compression in the neighbourhood of the mouth is neglected, this must be the same as

$$Q\frac{d\psi}{dx=0}=QA\cos 2\pi nt.$$

Accordingly

$$AQ=-4\pi A'\{\cos(kr+g)+kr\sin(kr+g)\},$$

$$0=\quad\sin(kr+g)-kr(\cos kr+g).$$

These equations express the connexion between the plane and spherical waves. From the second, $\tan(kr+g)=kr$, which shows that g is a small quantity of the order $(kr)^2$. From the first

$$A'=-\frac{AQ}{4\pi},$$

so that

$$\psi_r=-\frac{AQ}{4\pi r}\cos 2\pi nt-\frac{AQk}{4\pi}\sin 2\pi nt,$$

the terms of higher order being omitted.

Now within the space under consideration the air moves according to the same laws as electricity, and so

$$\frac{Q}{c}\frac{d\psi}{dx=0}=-\psi_{x=0}+\psi_r,$$

$$\frac{d\psi}{dx=0}=A\cos 2\pi nt,$$

$$\psi_{x=0}=B\cos 2\pi nt+\beta\sin 2\pi nt.$$

Therefore on substitution and equation of the coefficients of $\sin 2\pi nt$, $\cos 2\pi nt$, we obtain

$$\left.\begin{aligned} AQ\left(\frac{1}{c}+\frac{1}{4\pi r}\right)&=-B,\\ \beta&=-\frac{AQk}{4\pi}.\end{aligned}\right\}$$

When the mouth is not much contracted c is of the order of the radius of the mouth, and when there is contraction it is smaller still. In all cases therefore the term $\frac{1}{4\pi r}$ is very small compared to $\frac{1}{c}$; and we may put

$$\frac{AQ}{c}=-B,\quad \beta=-\frac{AQk}{4\pi}, \qquad (17)$$

* Throughout HELMHOLTZ's paper the mouth of the pipe is supposed to lie in an infinite plane, so that the diverging waves are hemispherical. The calculation of the value of c is thereby simplified. Except for this reason it seems better to consider the diverging waves completely spherical as a nearer approximation to the actual circumstances of organ-pipes, although the sphere could never be quite complete.

which agree nearly with the results of HELMHOLTZ. In his notation a quantity α is used defined by the equation

$$-\frac{A}{Bk}=\cot k\alpha,$$

so that

$$\cot k\alpha=\tan kl \text{ by (15)},$$

or

$$k(l+\alpha)=(2m+1)\frac{\pi}{2};$$

α may accordingly be considered as the correction to the length of the tube (measured, however, in our method only on the negative side of the origin), and will be given by

$$\cot k\alpha=-\frac{c}{kQ}.$$

The value of c will be investigated in Part II.

The original theory of open pipes makes the pressure absolutely constant at the mouth, which amounts to neglecting the inertia of the air outside. Thus, if the tube itself were full of air, and the external space of hydrogen, the correction to the length of the pipe might be neglected. The first investigation, in which no escape of energy is admitted, would apply if the pipe and a space round its mouth, large compared to the diameter, but small compared to the wave-length, were occupied by air in an atmosphere otherwise composed of incomparably lighter gas. These remarks are made by way of explanation, but for a complete discussion of the motion as determined by (13) and (17), I must refer to the paper of HELMHOLTZ.

Long Tube in connexion with a Reservoir.

It may sometimes happen that the length of a neck is too large compared to the quarter wave-length to allow the neglect of the compressibility of the air inside. A cylindrical neck may then be treated in the same way as the organ-pipe. The potential of plane waves inside the neck may, by what has been proved, be put into the form

$$\psi=A'\sin k(x-\alpha)\cos 2\pi nt;$$

if we neglect the escape of energy, which will not affect the pitch of the resonant note,

$$\frac{d\psi}{dt}=-2\pi nA'\sin k(x-\alpha)\sin 2\pi nt,$$

$$\frac{d\psi}{dx}=kA'\cos k(x-\alpha)\cos 2\pi nt,$$

where α is the correction for the outside end.

The rate of flow out of $S=Q\frac{d\psi}{dx}$.

Total flow $=Q\int\frac{d\psi}{dx}dt=kA'Q\cos kL\frac{\sin 2\pi nt}{2\pi n}$, the reduced length of the tube, including the corrections for both ends, being denoted by L. Thus rarifaction in S

$$=k\frac{A'Q\cos kL}{S}\frac{\sin 2\pi nt}{2\pi n}=\frac{1}{a^2}\frac{d\psi}{dt}=\frac{2\pi nA'\sin kL}{a^2}\sin 2\pi nt.$$

This is the condition to be satisfied at the inner end. It gives

$$\tan kL=\frac{a^2}{4\pi^2n^2}\frac{kQ}{S}=\frac{Q}{kS}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

When kL is small,

$$\tan kL=kL+\frac{1}{3}(kL)^3=\frac{Q}{kS};$$

$$\therefore\ k^2=\frac{Q}{LS}\left(1-\frac{1}{3}\frac{LQ}{S}\right),$$

$$n=\frac{a}{2\pi}\sqrt{\frac{Q}{LS}}\left(1-\frac{1}{6}\frac{LQ}{S}\right)=\frac{a}{2\pi}\sqrt{\frac{Q}{L(S+\frac{1}{3}LQ)}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

In comparing this with (5), it is necessary to introduce the value of c, which is $\frac{Q}{L}$. (5) will accordingly give the same result as (19) if *one-third* of the contents of the neck be included in S. The first overtone, which is often produced by blowing in preference to the fundamental note, corresponds approximately to the length L of a tube open at both ends, modified to an extent which may be inferred from (18) by the finiteness of S.

The number of vibrations is given by

$$n=\frac{a}{2}\left(\frac{1}{L}+\frac{Q}{\pi^2S}\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

[The application of (20) is rather limited, because, in order that the condensation within S may be uniform as has been supposed, the linear dimension of S must be considerably less than the quarter wave-length; while, on the other hand, the method of approximation by which (20) is obtained from (18) requires that S should be large in comparison with QL.

A slight modification of (18) is useful in finding the pitch of pipes which are cylindrical through most of their length, but at the closed end expand into a bulb S of no great capacity. The only change required is to understand by L the length of the pipe down to the place where the enlargement begins with a correction for the *outer* end. Or if L denote the length of the tube simply, we have

$$\tan k(L+\alpha)=\frac{Q}{kS}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20\,a)$$

and $\alpha=\frac{\pi}{4}$ R approximately.

If S be very small we may derive from (20 *a*)

$$n=\frac{a}{4\left(L+\alpha+\frac{S}{Q}\right)}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20\,b)$$

In this form the interpretation is very simple, namely, that at the closed end the shape is of no consequence, and only the volume need be attended to. The air in this part of the pipe acts merely as a spring, its inertia not coming into play. A few measurements of this kind will be given in Part III.

The overtones of resonators which have not long necks are usually very high. Within the body of the reservoir a nodal surface must be formed, and the air on the further side vibrates as if it was contained in a completely closed vessel. We may form an idea of the character of these vibrations from the case of a sphere, which may be easily worked out from the equations given by Professor STOKES in his paper "On the Communication of Motion from a vibrating Sphere to a Gas"*. The most important vibration within a sphere is that which is expressed by the term of the first order in LAPLACE'S series, and consists of a swaying of the air from side to side like that which takes place in a doubly closed pipe. I find that for this vibration

$$\text{radius : wave-length} = \cdot 3313,$$

so that the note is higher than that belonging to a doubly closed (or open) pipe of the length of the diameter of the sphere by about a musical fourth. We might realize this vibration experimentally by attaching to the sphere a neck of such length that it would by itself, when closed at one end, have the same resonant note as the sphere.

Lateral Openings.

In most wind instruments the gradations of pitch are attained by means of lateral openings, which may be closed at pleasure by the fingers or otherwise. The common crude theory supposes that a hole in the side of, say, a flute establishes so complete a communication between the interior and the surrounding atmosphere, that a loop or point of no condensation is produced immediately under it. It has long been known that this theory is inadequate, for it stands on the same level as the first approximation to the motion in an open pipe in which the inertia of the air outside the mouth is virtually neglected. Without going at length into this question, I will merely indicate how an improvement in the treatment of it may be made.

Let ψ_1, ψ_2 denote the velocity-potentials of the systems of plane waves on the two sides of the aperture, which we may suppose to be situated at the point $x=0$. Then with our previous notation the conditions evidently are that when $x=0$,

$$\left.\begin{aligned} &\psi_1=\psi_2 \\ &\frac{Q}{c}\left(\frac{d\psi_1}{dx}-\frac{d\psi_2}{dx}\right)+\psi=0, \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots \quad (20\,c)$$

the escape of energy from the tube being neglected. These equations determine the connexion between the two systems of waves in any case that may arise, and the working out is simple. The results are of no particular interest, unless it be for a comparison with experimental measurements, which, so far as I am aware, have not hitherto been made.]

* Professor STOKES informs me that he had himself done this at the request of the Astronomer Royal.