IV. "Note on the Free Vibrations of an infinitely long Cylindrical Shell." By LORD RAYLEIGH, Sec. R.S. Received February 26, 1889.

In a recent memoir* Mr. Love has considered this question among others; but he has not discussed his result [equation (95)], except in its application to a rather special case involving the existence of a free edge. When the cylinder is regarded as infinitely long, the problem is naturally of a simpler character; and I have thought that it might be worth while to express more fully the frequency equation, as applicable to all vibrations, independent of the thickness of the shell, which are periodic with respect both to the length and the circumference of the cylinder.

In order to prevent misunderstanding, it may be well to premise that the vibrations, whose frequency is to be determined, do not include the gravest of which a thin shell is capable. If the middle surface be simply bent, the potential energy of deformation is of a higher order of magnitude than in the contrary case, and according to the present method of treatment the frequency of vibration will appear to be zero. It is known, however, that the only possible modes of bending of a cylindrical shell are such as are not periodic along the length, or rather have the wave-length in this direction infinitely long.† When the middle surface is stretched, as well as bent, the potential energy of bending may be neglected, except in certain very special cases.

Taking cylindrical co-ordinates (r, ϕ, z), and denoting the displacements parallel to z, ϕ, r by u, v, w respectively, we have for the principal elongations and shear at any point (a, ϕ, z)‡—

$$\sigma_1 = \frac{du}{dz}, \qquad \sigma_2 = \frac{w}{a}+\frac{1}{a}\frac{dv}{d\phi}, \qquad \varpi = \frac{1}{a}\frac{du}{d\phi}+\frac{dv}{dz} \quad \ldots\ldots (1);$$

and the energy per unit of area is expressed by

$$2nh\left\{\sigma_1^2+\sigma_2^2+\tfrac{1}{2}\varpi^2+\frac{m-n}{m+n}(\sigma_1+\sigma_2)^2\right\} \quad \ldots\ldots\ldots (2),$$

where $2h$ denotes the thickness of the shell, and m, n are the elastic constants of Thomson and Tait's notation.

* "On the small Free Vibrations and Deformation of a thin Elastic Shell," 'Phil. Trans.,' A, vol. 179 (1888), p. 491.

† "On the Bending and Vibration of thin Elastic Shells, especially of Cylindrical Form," 'Roy. Soc. Proc.,' *supra*, p. 105.

‡ See a paper on the Infinitesimal Bending of Surfaces of Revolution ('London Math. Soc. Proc.,' vol. 13, p. 4, Nov. 1881), and those already cited.

The functions u, v, w are to be assumed proportional to the sines, or cosines, of μz and $s\phi$. These may be combined in various ways, but a sufficient example is

$$u = \mathrm{U}\cos s\phi\cos\mu z, \quad v = \mathrm{V}\sin s\phi\sin\mu z, \quad w = \mathrm{W}\cos s\phi\sin\mu z \;\ldots\; (3);$$

so that

$$\sigma_1 = -\mu\mathrm{U}\cos s\phi\sin\mu z \;\ldots\ldots\; (4),$$

$$\sigma_2 = (\mathrm{W}+s\mathrm{V})\cos s\phi\sin\mu z \;\ldots\ldots\; (5),$$

$$\varpi = (-s\mathrm{U}+\mu\mathrm{V})\sin s\phi\cos\mu z \;\ldots\ldots\; (6),$$

unity being written for convenience in place of a. The energy per unit area is thus

$$2nh\left[\cos^2 s\phi\sin^2\mu z\left\{\mu^2\mathrm{U}^2+(\mathrm{W}+s\mathrm{V})^2+\frac{m-n}{m+n}(\mathrm{W}+s\mathrm{V}-\mu\mathrm{U})^2\right\}\right.$$
$$\left.+\sin^2 s\phi\cos^2\mu z\,(-s\mathrm{U}+\mu\mathrm{V})^2\right]\;\ldots\ldots\; (7).$$

Again, the kinetic energy per unit area is, if ρ be the volume density,

$$\rho h\left[\left(\frac{d\mathrm{U}}{dt}\right)^2\cos^2 s\phi\cos^2\mu z+\left(\frac{d\mathrm{V}}{dt}\right)^2\sin^2 s\phi\sin^2\mu z+\left(\frac{d\mathrm{W}}{dt}\right)^2\cos^2 s\phi\sin^2\mu z\right]$$
$$\ldots\ldots\; (8).$$

In the integration of these expressions with respect to ϕ and z, the mean value of each $\sin^2$ or $\cos^2$ is $\frac{1}{2}$.* We may then apply Lagrange's method. If the type of vibration be $\cos pt$, and $p^2\rho/n = k^2$, the resulting equations may be written

$$\{2(\mathrm{M}+1)\mu^2+s^2-k^2\}\mathrm{U}-(2\mathrm{M}+1)\mu s\mathrm{V}-2\mathrm{M}\mu\mathrm{W} = 0 \;\ldots\; (9),$$

$$-(2\mathrm{M}+1)\mu s\mathrm{U}+\{\mu^2+2(\mathrm{M}+1)s^2-k^2\}\mathrm{V}+2(\mathrm{M}+1)s\mathrm{W} = 0 \;\ldots\; (10),$$

$$-2\mathrm{M}\mu\mathrm{U}+2(\mathrm{M}+1)s\mathrm{V}+\{2(\mathrm{M}+1)-k^2\}\mathrm{W} = 0 \;\ldots\; (11),$$

where

$$\mathrm{M} = \frac{m-n}{m+n} \;\ldots\ldots\; (12).$$

The frequency equation is that expressing the evanescence of the determinant of this triad of equations.

We will consider for a moment the simple case which arises when $\mu = 0$, that is, when the displacements are independent of z. The three equations reduce to

* In the physical problem the range of integration for ϕ is from 0 to 2π; but mathematically we are not confined to one revolution. We may conceive the shell to consist of several superposed convolutions, and then s is not limited to be a whole number.

$$(s^2-k^2)U = 0 \qquad (13),$$

$$\{2(M+1)s^2-k^2\}V+2(M+1)sW = 0 \qquad (14),$$

$$2(M+1)sV+\{2(M+1)-k^2\}W = 0 \qquad (15);$$

and they may be satisfied in two ways. First let $V = W = 0$; then U may be finite, provided

$$s^2-k^2 = 0 \qquad (16).$$

The corresponding type for U is

$$U = \cos s\phi \cos pt \qquad (17),$$

where

$$p^2 = \frac{ns^2}{\rho a^2} \qquad (18),$$

a being restored, as can be done at any moment by consideration of dimensions. In this motion the material is sheared without extension, every generating line of the cylinder moving along its own length. The frequency depends upon the circumferential wave-length, and not upon the curvature of the cylinder.

The second kind of vibrations are those in which $U = 0$, so that the motion is strictly in two dimensions. The elimination of the ratio V/W from (14), (15) gives

$$k^2\{k^2-2(M+1)(1+s^2)\} = 0 \qquad (19),$$

as the frequency equation. The first root is $k^2 = 0$, indicating infinitely slow motion. These are the flexural vibrations already referred to, and the corresponding relation between V and W is by (14)

$$sV+W = 0 \qquad (20),$$

giving by (4), (5), (6),

$$\sigma_1 = \sigma_2 = \varpi = 0.$$

The other root of (19) gives on restoration of a,

$$k^2a^2 = \frac{4m}{m+n}(1+s^2) \qquad (21),$$

or

$$p^2 = \frac{4mn}{m+n}\,\frac{1+s^2}{a^2\rho} \qquad (22);$$

while the relation between V and W is

$$-V+sW = 0 \qquad (23).$$

It will be observed that when s is very large, the flexural vibrations tend to become exclusively normal, and the extensional vibrations to become exclusively tangential, as might have been expected from the theory of plane plates.

Returning now to the general case, the determinant of (9), (10), (11) gives on reduction

$$[k^2-\mu^2-s^2]\{k^2[k^2-2(\mathrm{M}+1)(\mu^2+s^2+1)]+4(2\mathrm{M}+1)\mu^2\}$$
$$+4(2\mathrm{M}+1)\mu^2s^2=0 \quad\ldots\ldots\ldots\ldots\ldots\ldots \quad (24).$$

If $\mu=0$, we have the three solutions already considered,

$$k^2=0, \qquad k^2=s^2, \qquad k^2=2(\mathrm{M}+1)(s^2+1).$$

If $s=0$, that is, if the deformation be symmetrical about the axis, we have

$$k^2=\mu^2, \quad \text{or} \quad k^2[k^2-2(\mathrm{M}+1)(\mu^2+1)]+4(2\mathrm{M}+1)\mu^2=0 \quad .. \quad (25).$$

Corresponding to the first root we have $\mathrm{U}=0$, $\mathrm{W}=0$, as is readily proved on reference to the original equations with $s=0$. The vibrations are the purely torsional ones represented by

$$v=\sin\mu z\cos pt \quad\ldots\ldots\ldots\ldots\ldots\ldots \quad (26),$$

where

$$p^2=\frac{n\mu^2}{\rho} \quad\ldots\ldots\ldots\ldots\ldots\ldots \quad (27).$$

The frequency depends upon the wave-length parallel to the axis, and not upon the radius of the cylinder.

The remaining roots of (25) correspond to motions for which $\mathrm{V}=0$, or which take place in planes through the axis. The general character of these vibrations may be illustrated by the case where μ is small, or the wave-length a large multiple of the radius of the cylinder. We find approximately from the quadratic (on restoration of a)

$$\frac{k^2a^2}{\mathrm{M}+1}=2+\frac{2\mathrm{M}^2\mu^2a^2}{(\mathrm{M}+1)^2} \quad\ldots\ldots\ldots\ldots\ldots\ldots \quad (28),$$

or

$$k^2=\frac{2(2\mathrm{M}+1)\mu^2}{(\mathrm{M}+1)} \quad\ldots\ldots\ldots\ldots\ldots\ldots \quad (29).$$

The vibrations of (28) are nearly purely radial. If we suppose that μ vanishes, we fall back upon

$$k^2a^2=2(\mathrm{M}+1),$$

or

$$p^2 = \frac{4mn}{m+n}\,\frac{1}{a^2\rho} \quad\text{.................} \quad (30),*$$

as may be seen from (22), by putting $s = 0$.

On the other hand, the vibrations of (29) are nearly purely axial. In terms of m and n,

$$p^2 = \frac{n\mu^2}{\rho}\,\frac{3m-n}{m} \quad\text{..................} \quad (31).$$

Now, if q denote Young's modulus,

$$q = \frac{n(3m-n)}{m} \quad\text{..................} \quad (32);$$

so that

$$p^2 = \frac{q\mu^2}{\rho} \quad\text{.........................} \quad (33).$$

This is the ordinary formula for the longitudinal vibrations of a rod, the fact that the section is here a thin annulus not influencing the result to this order of approximation.

Another extreme case worthy of notice occurs when s is very great. Equation (24) then reduces to

$$k^2[k^2-\mu^2-s^2]\,[k^2-2(M+1)\,(\mu^2+s^2)] = 0 \;\text{......}\; (34);$$

so that k^2 becomes a function of μ and s only through (μ^2+s^2), as might have been expected from the theory of plane plates. The first root relates to flexural vibrations; the second to vibrations of shearing, as in (18); the third to vibrations involving extension of the middle surface, analogous to those in (22).

It is scarcely necessary to add, in conclusion, that the most general deformation of the middle surface can be expressed by means of a series of such as are periodic with respect to z and ϕ, so that the problem considered is really the most general small motion of an infinite cylindrical shell.

[Another particular case worth notice arises when $s = 1$, so that (24) assumes the form

$$k^2(k^2-\mu^2-1)[k^2-2(M+1)(\mu^2+2)]$$
$$+4\mu^2(k^2-\mu^2)(2M+1) = 0 \;\text{........}\; (35).$$

As we have already seen, if μ be zero, one of the values of k^2 vanishes. If μ be small, the corresponding value of k^2 is of the order μ^4. Equation (35) gives in this case

$$k^2 = \frac{2M+1}{M+1}\,\mu^4 \quad\text{..................} \quad (36);$$

* This equation is given, in a slightly different form, by Love (*loc. cit.*, p. 523).

or in terms of p, q, and with restoration of a,

$$p^2 = \frac{q\mu^4 a^2}{2\rho} \quad \text{.......................(37).}$$

This agrees with the usual formula* for the transverse vibrations of rods.—Added April 3.]

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* 'Theory of Sound,' § 181.