VI. "On the Capillary Phenomena of Jets." By Lord RAYLEIGH, F.R.S. Received May 5, 1879.

When water issues under high pressure from a circular orifice in a thin plate, a jet is formed whose section, though diminished in area, retains the circular form. But if the orifice be not circular, the section of the jet undergoes remarkable transformations, which were elaborately investigated by Bidone,* many years ago. The peculiarities of the orifice are exaggerated in the jet, but in an inverted manner. The following examples are taken from Bidone's memoir.

FIG. 1.

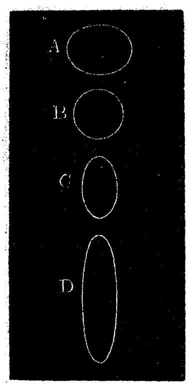


Fig. 1, orifice in the form of an ellipse (A), of which the major axis is horizontal, and 24 lines long; the minor axis is vertical, and 17 lines long. The head of water is 6 feet.

Near the orifice the sections of the vein are elliptical with major axis horizontal. The ellipticity gradually diminishes until at a distance of 30 lines from the orifice the section is circular. Beyond this point the vertical axis of the section increases, and the horizontal axis decreases, so that the vein reduces itself to a flat vertical sheet, very broad and thin. This sheet preserves its continuity to a distance of 6 feet from the orifice, where the vein is penetrated by air.

B represents the section at a distance of 30 lines from the orifice. It is a circle of 16 or 17 lines diameter.

C is the section at a distance of 6 inches from the orifice. It is an

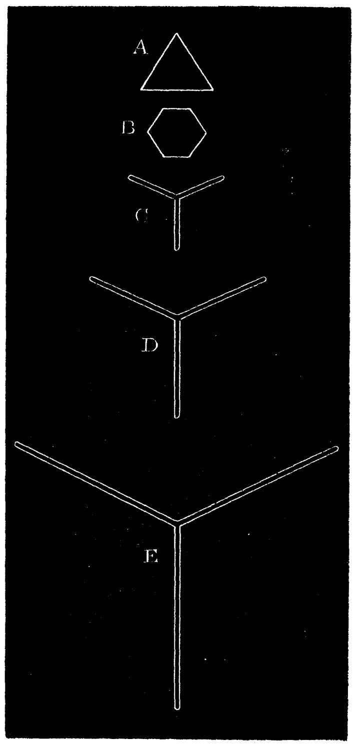
* "Expériences sur la Forme et sur la Direction des Veines et des Courans d'Eau lancés par diverses Ouvertures." Par George Bidone.

elliptical figure, whose major axis is 22 lines long and minor axis 14 lines long.

D is the section at 24 inches from the orifice. It also is an elliptical figure, whose vertical axis is 45 lines long and horizontal axis about 12 lines long.

In fig. 2, the orifice (A) is an equilateral triangle, with sides 2 inches long. The head of water is 6 feet. The vein resolves itself into three flat sheets disposed symmetrically round the axis, the planes

FIG. 2.



of the sheets being perpendicular to the sides of the orifice. These sheets are very thin, and retain their transparence and continuity to a distance of 42 inches, reckoned from the orifice. The sections represented by B, C, D, E are taken at distances from the orifice equal respectively to 1 inch, 6 inches, 12 inches, and 24 inches.

Similarly, a vein issuing from an orifice in the form of a regular polygon, of any number of sides, resolves itself into an equal number of thin sheets, whose planes are perpendicular to the sides of the polygon.

Bidone explains the formation of these sheets, in the main (as it appears to me), satisfactorily, by reference to simpler cases of meeting streams. Thus equal jets, moving in the same straight line, with equal and opposite velocities, flatten themselves into a disk, situated in the perpendicular plane. If the axes of the jets intersect obliquely, a sheet is formed symmetrically in the plane perpendicular to that of the impinging jets. Those portions of a jet which proceed from the outlying parts of an unsymmetrical orifice are considered to behave, in some degree, like independent meeting streams.

In many cases, more especially when the orifices are small and the heads of water low, the extension of the sheets in directions perpendicular to the jet reaches a limit. Sections taken at greater distances from the orifice show a gradual shortening of the sheets, until a compact form is attained, similar to that at the first contraction. Beyond this point, if the jet retains its coherence, sheets are gradually thrown out again, but in directions bisecting the angles between the directions of the former sheets. These sheets may, in their turn, reach a limit of development, again contract, and so on. The forms assumed in the case of orifices of various shapes, including the rectangle, the equilateral triangle, and the square, have been carefully investigated and figured by Magnus.* Phenomena of this kind are of every-day occurrence, and may generally be observed whenever liquid falls from the lip of a moderately elevated vessel.

Admitting the substantial accuracy of Bidone's explanation of the formation and primary expansion of the sheets or excrescences, we have to inquire into the cause of the subsequent contraction. Bidone attributes it to the viscosity of the fluid, which may certainly be put out of the question. In Magnus's view the cause is "cohesion;" but he does not explain what is to be understood under this designation, and it is doubtful whether he had a clear idea upon the subject. The true explanation appears to have been first given by Buff,† who refers the phenomenon distinctly to the capillary force. Under the operation of this force the fluid behaves as if enclosed in an envelope of constant tension, and the recurrent form of the jet is due to vibrations of the fluid column about the circular figure of equilibrium, superposed upon the general progressive motion. Since the phase of vibration depends upon the time elapsed, it is always the same at the same point in space, and thus the motion is *steady* in the hydro-dynamical sense, and the boundary of the jet is a fixed surface.

In so far as the vibrations may be considered to be isochronous, the

* "Hydraulische Untersuchungen." "Pogg. Ann.," xcv, 1855.

† "Pogg. Ann.," Bd. c, 1857.

distance between consecutive corresponding points of the recurrent figure, or, as it may be called, the *wave-length* of the figure, is directly proportional to the velocity of the jet, *i.e.*, to the *square root* of the head of water. This elongation of wave-length with increasing pressure was observed by Bidone and by Magnus, but no definite law was arrived at. As a jet falls under the action of gravity, its velocity increases, and thus an augmentation of wave-length might be expected; but, as will appear later, most of this augmentation is compensated by a change in the frequency of vibration due to the attenuation which is the necessary concomitant of the increased velocity. Consequently but little variation in the magnitudes of successive wave-lengths is to be noticed, even in the case of jets falling vertically with small initial velocity. In the following experiments the jets issued horizontally from orifices in thin plates, usually adapted to a large stoneware bottle, which served as reservoir or cistern. The plates were of tin, soldered to the ends of short brass tubes rather more than an inch in diameter, by the aid of which they could be conveniently fitted to a tubulure in the lower part of the bottle. The pressure at any moment of the outflow could be measured by a water manometer read with a scale of millimetres. Some little uncertainty necessarily attended the determination of the zero point; it was usually taken to be the reading of the scale at which the jet ceased to clear itself from the plate on the running out of the water. At the beginning of an experiment, the orifice was plugged with a small roll of clean paper, and the bottle was filled from an india-rubber tube in connexion with a tap. After a sufficient time had elapsed for the water in the bottle to come sensibly to rest, the plug was withdrawn, and the observations were commenced. The jet is exceedingly sensitive to disturbances in the reservoir, and no arrangement hitherto tried for maintaining the level of the water has been successful. The measurements of wave-length (λ) were made with the aid of a pair of dividers adjusted so as to include one or more wave-lengths; and as nearly as possible at the same moment the manometer was read. The distance between the points of the dividers was afterwards taken from a scale of millimetres. The facility, and in some cases the success, of the operation of observing the wave-length depends very much upon the suitability of the illumination.

The first set of observations here given refers to a somewhat elongated orifice of rectangular form. The pressures and wave-lengths are measured in millimetres. The third column contains numbers proportional to the square roots of the pressures.

Table I.—November 11, 1878.

Pressure.	Wave-length.	$\sqrt{}$(Pressure).
253	104	91
216	91	84
178	81	76
144	70	69
113*	61*	61*
83	51	52
58	42	43
39	33	36
21	24	26

The agreement of the second and third columns is pretty good on the whole. Small discrepancies at the bottom of the table may be due to the uncertainty attaching to the zero point of pressure, and also to another cause, which will be referred to later. At the higher pressures the observed wave-lengths have a marked tendency to increase more rapidly than the velocity of the jet. This result, which was confirmed by other observations, points to a departure from the law of isochronous vibration. Strict isochronism is only to be expected when vibrations are infinitely small, that is, in the present application when the section of the jet never deviates more than infinitesimally from the circular form. During the vibrations with which Table I is concerned, however, the departures from circularity are very considerable, and there is no reason for supposing that such vibrations will be executed in exactly the same time as vibrations of infinitely small amplitude. Nevertheless, this consideration would not lead to an explanation of the discrepancies in Table I, unless it were the fact that the amplitude of vibration increased with the pressure under which the jet issues.

As a matter of observation the increase of amplitude is very apparent, and was noticed by Magnus. It is also a direct consequence of theory, inasmuch as the *lateral* velocities to which the vibrations are due vary in direct proportion to the longitudinal velocity of the jet. Consequently the amplitude varies approximately as the square root of the pressure, or as the wave-length. The amplitude here spoken of is measured, of course, by the *departure* from circularity, and not by the value of the maximum radius itself.

The law of the square root of the pressure thus applies only to small amplitudes, and unfortunately it is precisely these small amplitudes which it is difficult to experiment upon. Still it is possible to approach theoretical requirements more nearly than in the experiments of Table I.

The next set of measurements (Table II) refer to an aperture in the form of an ellipse of moderate eccentricity. Two wave-lengths were included in the measurements; in other respects the arrangements were as before.

Table II.—November 12.

Pressure.	Wave-length.	√(Pressure.)
262	40½	40
208	36½	35½
182	34	33½
158	31	31
129	28½	28
107*	25½*	25½*
86	22½	23
69	20	20½
56	18	18½
42	15	16
34	13½	14½
27	12½	13
21	10	11½

In this case the law is fully verified, the discrepancies being decidedly within the limits of experimental error.

On the other hand, the discrepancies may be exaggerated by the use of higher pressures. Table III relates to the same orifice† as Table I. Instead of the stoneware bottle, a tall wooden box was used as reservoir.

Table III.—December 20.

Pressure.	Wave-length.	√(Pressure).
757	200	155
672	184	145
587	171	136
497	152	125
442	141	118
365	123	107
289	106	95½
234	93	86
189	79	77
154	70	70
123	62	62
107	58	58
89*	53*	53*
74	48	48
61	44½	44

The wave-lengths at the high pressures very greatly exceed those calculated from the lower pressures according to the law applicable to small vibrations.

† Its condition may have changed a little in the interval.

It is possible, however, to observe in cases where the amplitude is so small, that the discrepancies are moderate even at higher pressures than those recorded in Table III. The measurements in Table IV are of a jet from an elliptical aperture of small eccentricity. The ratio of axes is about 5 : 6. The wooden box was used. Two wave-lengths were measured.

Table IV.—December 18.

Pressure.	Wave-length.	√(Pressure).
1287	79†	83½
1195	82	80
1117	79½	77½
1023	73	74½
947	70½	71½
852	66¾	68
770	64½	64½
695	61¼	61
620	58	58
532	54½	53½
451	48¼	49¼
371	45½	44¾
290*	39½*	39½*
248	36½	36½
192	31½	32
158	28¾	29¼
133	26½	26¾
111	24¼	24½
94	21¾	22¼
85	21	21½

The following experiments relate to an orifice in the form of an equilateral triangle, with slightly rounded corners. The side measures about 3 millims. In this case the peculiarities of the contour are repeated *three* times in passing round the circumference. Two wave-lengths were measured.

Table V.—November 16.

Pressure.	Wave-length.	√(Pressure).
215	36	35
166	31½	31
127	27	27
92*	23*	23*
66	19	19½
43	14½	15½
27	11½	12½

† This is, doubtless, an error. At these high pressures the observation is difficult.

Here again we observe the tendency of the wave-length to increase more rapidly than the square root of the pressure.

At higher pressures the difference is naturally still more marked. With the same aperture, and the wooden box as reservoir, the results were:—

Table VI.—December 17.

Pressure.	Wave-length.	$\sqrt{}$(Pressure).
1072	102	80·4
992	94	77·5
888	89	73·2
827	86	70·7
762	81	67·8
702	77	65·0
619	70	61·1
539	66	57·0
468	$59\frac{1}{2}$	53·1
415	$54\frac{1}{2}$	50·0
337	47	44·6
292	42	42·0
251	$38\frac{1}{2}$	38·9
213	$34\frac{1}{2}$	35·8
189	33	33·7
163	31	31·3
140	$28\frac{1}{4}$	29·1
111	$24\frac{3}{4}$	25·9
90	$22\frac{3}{4}$	23·3
70	$19\frac{3}{4}$	20·6
57	$17\frac{1}{4}$	18·5
45	$16\frac{1}{2}$	16·5

The wave-lengths down to $34\frac{1}{2}$ are immediate measurements; those below are deduced from measurements of two wave-lengths.

Similar experiments were made with jets from a *square* hole (side = 2 millims.), the peculiarities of which are repeated *four* times in passing round the circumference. Two wave-lengths were measured.

Table VII.—December 14.

Pressure.	Wave-length.	$\sqrt{}$(Pressure).	Corrected.
447	32	30·2	29·9
377	$29\frac{1}{2}$	27·7	27·4
312	27	25·2	24·9
269	$24\frac{1}{2}$	23·4	23·1
247	23	22·5	22·1
218	$21\frac{1}{2}$	21·1	20·7

Pressure.	Wave-length.	√(Pressure).	Corrected.
192	20	19·8	19·3
167*	$18\frac{1}{2}$*	18·5*	18·0
136	$16\frac{1}{2}$	16·6	16·1
107	14	14·8	14·2
87	13	13·3	12·7
65	$10\frac{3}{4}$	11·5	10·8
47	$8\frac{1}{2}$	9·8	8·9

The third column contains numbers proportional to the square roots of the pressures. In the fourth column a correction is introduced, the significance of which will be explained later.

The value of λ, other things being the same, depends upon the nature of the fluid. Thus methylated alcohol gave a wave-length about twice as great as tap water. This is a consequence of the smaller capillarity.

If a water jet be touched by a fragment of wood moistened with oil, the waves in front of the place of contact are considerably drawn out; but no sensible effect appears to be propagated up the stream.

If a jet of mercury discharging into dilute sulphuric acid be polarized by an electric current, the change in the capillary constant discovered by Lipmann shows itself by alterations in the length of the wave.

When the wave-length is considerable in comparison with the diameter of the jet, the vibrations about the circular form take place practically in two dimensions, and are easily calculated mathematically. The more general case, in which there is no limitation upon the magnitude of the diameter, involves the use of Bessel's functions. The investigation will be found in Appendix I. For the present we will confine ourselves to a statement of the results for vibrations in two dimensions.

Let us suppose that the polar equation of the section is

$$r = a_0 + a_n \cos n\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad (1),$$

so that the curve is an undulating one, repeating itself n times over the circumference. The mean radius is a_0; and, since the deviation from the circular form is small, a_n is a small quantity in comparison with a_0. The vibration is expressed by the variation of a_n as a harmonic function of the time. Thus if $a_n \propto \cos(pt-\epsilon)$, it may be proved that

$$p = \pi^{\frac{1}{4}} T^{\frac{1}{2}} \rho^{-\frac{1}{2}} A^{-\frac{3}{4}} \sqrt{(n^3-n)} \quad . \quad . \quad . \quad . \quad . \quad (2).$$

In this equation T is the superficial tension, ρ the density, A the area of the section (equal to πa_0^2), and the frequency of vibration is $p \div 2\pi$.

For a jet of given fluid and of given area, the frequency of vibration varies as $\sqrt{(n^3-n)}$ or $\sqrt{(n-1)n(n+1)}$. The case of $n=1$ corresponds to a displacement of the jet as a whole, without alteration in the *form* of the boundary. Accordingly there is no potential energy, and the frequency of vibration is zero. For $n=2$ the boundary is elliptical, for $n=3$ triangular with rounded corners, and so on. With most forms of orifice the jet is subject to more than one kind of vibration at the same time. Thus with a square orifice vibrations would occur corresponding to $n=4$, $n=8$, $n=12$, &c. However, the higher modes of vibrations are quite subordinate, and may usually be neglected. The values of $\sqrt{(n^3-n)}$ for various values of n are shown below.

$n=2$,	$p=\sqrt{6} = \sqrt{6}\times 1$
$n=3$,	$p=\sqrt{6}\times\sqrt{4}=\sqrt{6}\times 2$
$n=4$,	$p=\sqrt{6}\times\sqrt{10}=\sqrt{6}\times 3{\cdot}16$
$n=5$,	$p=\sqrt{6}\times\sqrt{20}=\sqrt{6}\times 4{\cdot}47$
$n=6$,	$p=\sqrt{6}\times\sqrt{35}=\sqrt{6}\times 5{\cdot}92$
$n=7$,	$p=\sqrt{6}\times\sqrt{56}=\sqrt{6}\times 7{\cdot}48$
$n=8$,	$p=\sqrt{6}\times\sqrt{84}=\sqrt{6}\times 9{\cdot}17$
$n=9$,	$p=\sqrt{6}\times\sqrt{120}=\sqrt{6}\times 10{\cdot}95$
$n=12$,	$p=\sqrt{6}\times\sqrt{286}=\sqrt{6}\times 16{\cdot}95$

It appears that the frequency for $n=3$ is just double that for $n=2$; so that the wave-length for a triangular jet should be the half of that of an elliptical jet of equal area, the other circumstances being the same.

For a given fluid and mode of vibration (n), the frequency varies as $A^{-\frac{3}{4}}$, the thicker jet having the longer time of vibration. If v be the velocity of the jet, $\lambda=2\pi v p^{-1}$. If the jet convey a given volume of fluid, $v \propto A^{-1}$, and thus $\lambda \propto A^{-\frac{1}{4}}$. Accordingly in the case of a jet falling vertically, the increase of λ due to velocity is in great measure compensated by the decrease due to diminishing area of section.

The law of variation of p for a given mode of vibration with the nature of the fluid, and the area of the section, may be found by considerations of *dimensions*. T is a force divided by a line, so that its dimensions are 1 in mass, 0 in length, and -2 in time. The volume density ρ is of 1 dimension in mass, -3 in length, and 0 in time. A is of course of 2 dimensions in length, and 0 in mass and time. Thus the only combination of T, ρ, A, capable of representing a frequency, is $T^{\frac{1}{2}}\rho^{-\frac{1}{2}}A^{-\frac{3}{4}}$.

The above reasoning proceeds upon the assumption of the applicability of the law of isochronism. In the case of large vibrations, for which the law would not be true, we may still obtain a good deal of information by the method of dimensions. The *shape* of the orifice being given, let us inquire into the nature of the dependence of λ upon T, ρ, A, and P, the pressure under which the jet escapes. The dimensions of P, a force divided by an area, are 1 in mass, −1 in length, and −2 in time. Assume

$$\lambda \propto T^x \rho^y A^z P^u;$$

then by the method of dimensions we have the following relations among the exponents—

$$x+y+u=0, \qquad -3y+2z-u=0, \qquad -2x-2u=0,$$

whence $\qquad u=-x, \qquad y=0, \qquad z=\tfrac{1}{2}(1-x).$

Thus $$\lambda \propto T^x A^{\frac{1}{2}-\frac{1}{2}x} P^{-x} \propto A^{\frac{1}{2}} \left(\frac{T}{PA^{\frac{1}{2}}}\right)^x.$$

The exponent x is undetermined; and since any number of terms with different values of x may occur together, all that we can infer is that λ is of the form

$$\lambda = A^{\frac{1}{2}} f\left(\frac{T}{PA^{\frac{1}{2}}}\right),$$

where f is an arbitrary function, or if we prefer it

$$\lambda = T^{-\frac{1}{2}} P^{\frac{1}{2}} A^{\frac{3}{4}} \, F\left(\frac{PA^{\frac{1}{2}}}{T}\right),$$

where F is equally arbitrary. Thus for a given liquid and shape of orifice, there is complete dynamical similarity if the pressure be taken inversely proportional to the linear dimension, and this whether the deviation from the circular form be great or small.

In the case of water Quincke found T=81 on the C.G.S. system of units. On the same system $\rho=1$; and thus we get for the frequency of the gravest vibration ($n=2$),

$$\frac{p}{2\pi} = 3{\cdot}51\, a^{-\frac{3}{2}} = 8{\cdot}28\, A^{-\frac{3}{4}} \quad . \quad . \quad . \quad . \quad (3).$$

For a sectional area of one square centimetre, there are thus 8·28 vibrations per second. To obtain the pitch of middle C ($c'=256$) we should require a diameter

$$2a = \left(\frac{3{\cdot}51}{2{\cdot}56}\right)^{\frac{2}{3}} = {\cdot}115,$$

or rather more than a millimetre.

For the general value of n, we have

$$\frac{p}{2\pi} = 1{\cdot}43\, a^{-\frac{3}{2}} \sqrt{(n^3-n)} = 3{\cdot}38\, A^{-\frac{3}{4}} \sqrt{(n^3-n)} \quad . \quad . \quad (4).$$

If h be the head of water to which the velocity of the jet is due,

$$\lambda = \frac{\sqrt{(2gh)} \, . \, A^{\frac{3}{4}}}{3{\cdot}38 \sqrt{(n^3-n)}} \quad . \quad . \quad . \quad . \quad . \quad (5).$$

In applying this formula it must be remembered that A is the area of the section of the jet, and not the area of the aperture. We might indeed deduce the value of A from the area of the aperture by introduction of a coefficient of contraction (about ·62); but the area of the aperture itself is not very easily measured. It is much better to calculate A from an observation of the quantity of fluid (V), discharged under a measured head (h'), comparable in magnitude with that prevailing when λ is measured. Thus $A=V\,(2gh')^{-\frac{1}{2}}$. In the following calculations the C.G.S. system of units is employed.

In the case of the elliptical aperture of Table II, the value of A was found in this way to be ·0695. Hence at a head of 10·7 the wave-length should be

$$\lambda=\frac{\sqrt{(2g\times 10{\cdot}7)}\times(\cdot 0695)^{\frac{3}{4}}}{3{\cdot}38\sqrt{6}}=2{\cdot}37,$$

the value of g being taken at 981. The corresponding observed value of λ is 2·55.

Again, in the case of the experiments recorded in Table IV, it was found that A=·0660. Hence for $h=29{\cdot}0$ the value of the wave-length should be given by

$$\lambda=\frac{(2g\times 29{\cdot}0)\times(\cdot 0660)^{\frac{3}{4}}}{3{\cdot}38\times\sqrt{6}}=3{\cdot}76.$$

The corresponding observed value is 3·95.

We will next take the triangular orifice of Table V. The value of A was found to be ·154. Hence for a head of 9·2 the value of λ, calculated *à priori*, is

$$\lambda=\frac{\sqrt{(2g\times 9{\cdot}2)}\times(\cdot 154)^{\frac{3}{4}}}{3{\cdot}38\times\sqrt{24}}=1{\cdot}99,$$

as compared with 2·3 found by direct observation.

For the square orifice of Table VII, we have A=·153. Hence, if $h=16{\cdot}7$,

$$\lambda=\frac{\sqrt{(2g\times 16{\cdot}7)}\times(\cdot 153)^{\frac{3}{4}}}{3{\cdot}38\times\sqrt{60}}=1{\cdot}70,$$

as compared with 1·85 by observation.

It will be remarked that in every case the observed value of λ somewhat exceeds the calculated value. The discrepancies are to be attributed, not so much, I imagine, to errors of observation as to excessive amplitude of vibration, involving a departure from the frequency proper to infinitely small amplitudes. The closest agreement is in the case of Table IV, where the amplitude of vibration was smallest. It is also possible that the capillary tension actually operative in these experiments was somewhat less than that determined by Quincke for distilled water.

When the pressures are small, the wave-lengths are no longer considerable in comparison with the diameter of the jet, and the vibrations cannot be supposed to take place sensibly in two dimensions. The frequency of vibration then becomes itself a function of the wave-length. This question is investigated mathematically in Appendix I. For the case of $n=4$, it is proved that approximately

$$p^2=\frac{60T}{\rho a^3}\left(1+\frac{11\pi^2 a^2}{30\lambda^2}\right).$$

Hence for the aperture of Table VII,

$$\lambda \propto \sqrt{h}\,(1-\cdot 088\,\lambda^{-2}),$$

λ being expressed in centimetres. The numbers in the fourth column of the table are calculated according to this formula.

On the other hand at high pressures the frequency becomes a function of the pressure. Since frequency is always an *even* function of amplitude, and in the present application, the square of the amplitude varies as h, the wave-length is given approximately by an expression of the form $\sqrt{h}$ $(M+Nh)$, where M and N are constants. It appears from experiment, and might, I think, have been expected, that frequency *diminishes* as amplitude increases, so that N is *positive.*

When the aperture has the form of an exact circle, and when the flow of fluid in its neighbourhood is unimpeded by obstacles, there is a perfect balance of lateral motions and pressures, and consequently nothing to render the jet in its future course unsymmetrical. Even in this case, however, the phenomena are profoundly modified by the operation of the capillary force. Far from retaining the cylindrical form unimpaired, the jet rapidly resolves itself in a more or less regular manner into detached masses. It has, in fact, been shown by Plateau,* both from theory and experiment, that in consequence of surface-tension the cylinder is an unstable form of equilibrium, when its length exceeds its circumference.

The circumstances attending the resolution of a cylindrical jet into drops have been admirably examined and described by Savart,† and for the most part explained with great sagacity by Plateau. There are, however, a few points which appear not to have been adequately treated hitherto; and in order to explain myself more effectually

* "Statique Expérimentale et Théorique des Liquides soumis aux seules Forces Moléculaires." Paris, 1873.

† "Mémoire sur la Constitution des Veines Liquides lancées par des Orifices Circulaires en mince paroi." Ann. d. Chim., t. liii, 1833.

I propose to pass in review the leading features of Plateau's theory, imparting, where I am able, additional precision.

Let us conceive, then, an infinitely long circular cylinder of liquid, at rest,* and inquire under what circumstances it is stable, or unstable, for small displacements, symmetrical about the axis of figure.

Whatever the deformation of the originally straight boundary of the axial section may be, it can be resolved by Fourier's theorem into deformations of the harmonic type. These component deformations are in general infinite in number, of every wave-length, and of arbitrary phase; but in the first stages of the motion, with which alone we are at present concerned, each produces its effect independently of every other, and may be considered by itself. Suppose, therefore, that the equation of the boundary is

$$r=a+\alpha \cos kz, \quad . \quad . \quad . \quad . \quad . \quad (6),$$

where α is a small quantity, the axis of z being that of symmetry. The wave-length of the disturbance may be called λ, and is connected with k by the equation $k=2\pi\lambda^{-1}$. The capillary tension endeavours to contract the surface of the fluid; so that the stability, or instability, of the cylindrical form of equilibrium depends upon whether the surface (enclosing a given volume) be greater or less respectively after the displacement than before. It has been proved by Plateau (see also Appendix I) that the surface is greater than before displacement if $ka>1$, that is, if $\lambda<2\pi a$; but less, if $ka<1$, or $\lambda>2\pi a$. Accordingly, the equilibrium is stable, if λ be less than the circumference; but unstable, if λ be greater than the circumference of the cylinder. Disturbances of the former kind, like those considered in the earlier part of this paper, lead to *vibrations* of harmonic type, whose amplitudes always remain small; but disturbances, whose wave-length exceeds the circumference, result in a greater and greater departure from the cylindrical figure. The analytical expression for the motion in the latter case involves exponential terms, one of which (except in case of a particular relation between the initial displacements and velocities) increases rapidly, being equally multiplied in equal times. The coefficient (q) of the time in the exponential term (e^{qt}) may be considered to measure the degree of dynamical instability; its reciprocal q^{-1} is the time in which the disturbance is multiplied in the ratio $1:e$.

The degree of instability, as measured by q, is not to be determined from statical considerations only; otherwise there would be no limit to the increasing efficiency of the longer wave-lengths. The joint operation of superficial tension and *inertia* in fixing the wave-

* A motion common to every part of the fluid is necessarily without influence upon the stability, and may therefore be left out of account for convenience of conception and expression.

length of maximum instability was, I believe, first considered in a communication to the Mathematical Society,* on the "Instability of Jets." It appears that the value of q may be expressed in the form

$$q=\sqrt{\left(\frac{T}{\rho a^3}\right)} \cdot F(ka) \quad . \quad . \quad . \quad . \quad . \quad (7),$$

where, as before, T is the superficial tension, ρ the density, and F is given by the following table:—

k^2a^2.	$F(ka)$.	k^2a^2.	$F(ka)$.
·05	·1536	·4	·3382
·1	·2108	·5	·3432
·2	·2794	·6	·3344
·3	·3182	·8	·2701
		·9	·2015

The greatest value of F thus corresponds, not to a zero value of k^2a^2, but approximately to $k^2a^2=·4858$, or to $\lambda=4·508\times 2a$. Hence the maximum instability occurs when the wave-length of disturbance is about half as great again as that at which instability first commences.

Taking for water, in C.G.S. units, $T=81$, $\rho=1$, we get for the case of maximum instability,

$$q^{-1}=\frac{a^{\frac{3}{2}}}{81\times ·343}=·115\, d^{\frac{3}{2}} \quad . \quad . \quad . \quad . \quad (8),$$

if d be the diameter of the cylinder. Thus, if $d=1$, $q^{-1}=·115$; or for a diameter of one centimetre the disturbance is multiplied 2·7 times in about one-ninth of a second. If the disturbance be multiplied 1000 fold in time t, $qt=3\ log_e\ 10=6·9$, so that $t=·79d^{\frac{3}{2}}$. For example, if the diameter be one millimetre, the disturbance is multiplied 1000 fold in about one-fortieth of a second. In view of these estimates the rapid disintegration of a fine jet of water will not cause surprise.

The relative importance of two harmonic disturbances depends upon their initial magnitudes, and upon the rate at which they grow. When the initial values are very small, the latter consideration is much the more important; for, if the disturbances be represented by $\alpha_1 e^{q_1 t}$, $\alpha_2 e^{q_2 t}$, in which q_1 exceeds q_2, their ratio is $\frac{\alpha_2}{\alpha_1}e^{-(q_1-q_2)t}$; and this ratio decreases without limit with the time, whatever be the initial (finite) ratio $\alpha_2 : \alpha_1$. If the initial disturbances are small enough, that one is ultimately preponderant, for which the measure of instability is

* "Math. Soc. Proc.," November, 1878. See also Appendix I.

greatest. The smaller the causes by which the original equilibrium is upset, the more will the cylindrical mass tend to divide itself regularly into portions whose length is equal to 4·5 times the diameter. But a disturbance of less favourable wave-length may gain the preponderance in case its magnitude be sufficient to produce disintegration in a less time than that required by the other disturbances present.

The application of these results to actual jets presents no great difficulty. The disturbances, by which equilibrium is upset, are impressed upon the fluid as it leaves the aperture, and the continuous portion of the jet represents the distance travelled during the time necessary to produce disintegration. Thus the length of the continuous portion necessarily depends upon the character of the disturbances in respect of amplitude and wave-length. It may be increased considerably, as Savart showed, by a suitable isolation of the reservoir from tremors, whether due to external sources or to the impact of the jet itself in the vessel placed to receive it. Nevertheless it does not appear to be possible to carry the prolongation very far. Whether the residuary disturbances are of external origin, or are due to friction, or to some peculiarity of the fluid motion within the reservoir, has not been satisfactorily determined. On this point Plateau's explanations are not very clear, and he sometimes expresses himself as if the time of disintegration depended only upon the capillary tension, without reference to initial disturbances at all.

Two laws were formulated by Savart with respect to the length of the continuous portion of a jet, and have been to a certain extent explained by Plateau. For a given fluid and a given orifice the length is approximately proportional to the square root of the head. This follows at once from theory, if it can be assumed that the disturbances remain always of the same character, so that the *time* of disintegration is constant.* When the head is given, Savart found the length to be proportional to the diameter of the orifice. From (8) it appears that the time in which a disturbance is multiplied in a given ratio varies, not as d, but as $d^{\frac{3}{2}}$. Again, when the fluid is changed, the time varies as $\rho^{\frac{1}{2}}\,T^{-\frac{1}{2}}$. But it may be doubted, I think, whether the length of the continuous portion obeys any very simple laws, even when external disturbances are avoided as far as possible.

When the circumstances of the experiment are such that the reservoir is influenced by the shocks due to the impact of the jet, the disintegration usually establishes itself with complete regularity, and is attended by a musical note (Savart). The impact of the regular series of drops which is at any moment striking the sink (or vessel receiving the water), determines the rupture into similar drops of the portion of the jet at the same moment passing the orifice. The pitch

* For the sake of simplicity, I neglect the action of gravity upon the jet when formed. The question has been further discussed by Plateau.

of the note, though not absolutely definite, cannot differ much from that which corresponds to the division of the jet into wave-lengths of maximum instability; and, in fact, Savart found that the frequency was directly as the square root of the head, inversely as the diameter of the orifice, and independent of the nature of the fluid—laws which follow immediately from Plateau's theory.

From the pitch of the note due to a jet of given diameter, and issuing under a given head, the wave-length of the nascent divisions can be at once deduced. Reasoning from some observations of Savart, Plateau finds in this way 4·38 as the ratio of the length of a division to the diameter of the jet. The diameter of the orifice was 3 millims., from which that of the jet is deduced by the introduction of the coefficient ·8. Now that the length of a division has been estimated *à priori*, it is perhaps preferable to reverse Plateau's calculation, and to exhibit the frequency of vibration in terms of the other data of the problem. Thus

$$\text{frequency} = \frac{\sqrt{(2gh)}}{4{\cdot}508\, d} \quad . \quad . \quad . \quad . \quad . \quad (9).$$

But the most certain method of obtaining complete regularity of resolution is to bring the reservoir under the influence of an external vibrator, whose pitch is approximately the same as that proper to the jet. Magnus* employed a Neef's hammer, attached to the wooden frame which supported the reservoir. Perhaps an electrically maintained tuning-fork is still better. Magnus showed that the most important part of the effect is due to the forced vibration of that side of the vessel which contains the orifice, and that but little of it is propagated through the air. With respect to the limits of pitch, Savart found that the note might be a fifth above, and more than an octave below, that proper to the jet. According to theory, there would be no well-defined lower limit; on the other side, the external vibration cannot be efficient if it tends to produce divisions whose length is less than the circumference of the jet. This would give for the interval defining the upper limit π : 4·508, which is very nearly a fifth. In the case of Plateau's numbers (π : 4·38) the discrepancy is a little greater.

The detached masses into which a jet is resolved do not at once assume and retain a spherical form, but execute a series of vibrations, being alternately compressed and elongated in the direction of the axis of symmetry. When the resolution is effected in a perfectly periodic manner, each drop is in the same phase of its vibration as it passes through a given point of space; and thence arises the remarkable appearance of alternate swellings and contractions described by Savart. The interval from one swelling to the next is the space described by

* "Pogg. Ann.," bd. cvi, 1859.

the drop during one complete vibration, and is therefore (as Plateau shows) proportional *cæteris paribus* to the square root of the head.

The time of vibration is of course itself a function of the nature of the fluid and of the size of the drop. By the method of dimensions alone it may be seen that the time of infinitely small vibrations varies directly as the square root of the mass of the sphere and inversely as the square root of the capillary tension; and in Appendix II it is proved that its expression is

$$\tau = \sqrt{\left(\frac{3\pi\rho V}{8T}\right)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10),$$

V being the volume of the vibrating mass.

In an experiment arranged to determine the time of vibration, a stream of 19·7 cub. centims. per second was broken up under the action of a fork making 128 vibrations per second. Neglecting the mass of the small spherules (of which more will be said presently), we get for the mass of each sphere 19·7÷128, or ·154 grm.; and thence by (10), taking as before $T=81$,

$$\tau = \cdot 0473 \text{ second.}$$

The distance between the first and second swellings was by measurement 16·5 centims. The level of the contraction midway between the two swellings was 36·8 centims. below the surface of the liquid in the reservoir, corresponding to a velocity of 175 centims. per second. These data give for the time of vibration,

$$\tau = 16{\cdot}5 \div 36{\cdot}8 = \cdot 0612 \text{ second.}$$

The discrepancy between the two values of τ, which is greater than I had expected, is doubtless due in part to excessive amplitude, rendering the vibration slower than that calculated for infinitely small amplitudes.

A rough estimate of the degree of flattening to be expected at the first swelling may be arrived at by calculating the eccentricity of the *oblatum*, which has the same volume and *surface* as those appertaining to the portion of fluid in question when forming part of the undisturbed cylinder. In the case of the most natural mode of resolution, the volume of a drop is $9\pi a^3$, and its surface is $18\pi a^2$. The eccentricity of the *oblatum* which has this volume and this surface is ·944, corresponding to a ratio of principal axes equal to about 1 : 3.

In consequence of the rapidity of the motion some optical device is necessary to render apparent the phenomena attending the disintegration of a jet. Magnus employed a rotating mirror, and also a rotating disk from which a fine slit was cut out. The readiest method of obtaining instantaneous illumination is the electric spark, but with this Magnus was not successful. "The rounded masses of which the swellings consist reflect the light emanating from a point in such a manner that the eye sees only the single point of each, which is principally illuminated. Hence, when the stream is illuminated by the electric spark, the swellings appear like a string of pearls ; but their form cannot be recognised, because the intensity of the light reflected from the remaining portions of the masses is too small to allow this, on account of the velocity with which the impression is lost."* The electric spark had, however, been used successfully for this purpose some years before by Buff,† who observed the *shadow* of the jet on a white screen. Preferable to an opaque screen in my experience is a piece of ground glass, which allows the shadow to be examined from the further side. I have found also that the jet may be very well observed directly, if the illumination is properly managed. For this purpose it is necessary to place the jet between the source of light and the eye. The best effect is obtained when the light of the spark is somewhat diffused by being passed (for example) through a piece of ground glass.

The spark may be obtained from the secondary of an induction coil, whose terminals are in connexion with the coatings of a Leyden jar. By adjustment of the contact breaker the series of sparks may be made to fit more or less perfectly with the formation of the drops. A still greater improvement may be effected by using an electrically maintained fork, which performs the double office of controlling the resolution of the jet and of interrupting the primary current of the induction coil. In this form the experiment is one of remarkable beauty. The jet, illuminated only in one phase of transformation, appears almost perfectly steady, and may be examined at leisure. The fork that I used had a frequency of 128, and communicated its vibration to the reservoir through the table on which both were placed without any special provision for the purpose. The only weak point in the arrangement was the rather feeble character of the sparks, depending probably upon the use of an induction coil too large for the rate of intermittence. A change in the phase under observation could be effected by pressing slightly upon the reservoir, whereby the vibration communicated was rendered more or less intense.

The jet issued horizontally from an orifice of about half a centimetre in diameter, and almost immediately assumed a rippled outline. The gradually increasing amplitude of the disturbance, the formation of the elongated ligament, and the subsequent transformation of the ligament into a spherule, could be examined with ease. In consequence of the transformation being in a more advanced stage at the forward than at the hinder end, the ligament remains for a moment connected with the mass behind, when

* "Phil. Mag.," xviii, 1859, p. 172.

† "Liebig's Ann.," lxxviii, 1851.

it has freed itself from the mass in front, and thus the resulting spherule acquires a backwards relative velocity, which of necessity leads to a collision. Under ordinary circumstances the spherule rebounds, and may be thus reflected backwards and forwards several times between the adjacent masses. But if the jet be subject to moderate electrical influence, the spherule amalgamates with a larger mass at the first opportunity.* Magnus showed that the stream of spherules may be diverted into another path by the attraction of a powerfully electrified rod, held a little below the place of resolution.

Very interesting modifications of these phenomena are observed when a jet from an orifice in a thin plate† is directed obliquely upwards. In this case drops which break away with different velocities are carried under the action of gravity into different paths; and thus under ordinary circumstances a jet is apparently resolved into a "sheaf," or bundle of jets all lying in one vertical plane. Under the action of a vibrator of suitable periodic time the resolution is regularised, and then each drop, breaking away under like conditions, is projected with the same velocity, and therefore follows the same path. The apparent gathering together of the sheaf into a fine and well-defined stream is an effect of singular beauty.

In certain cases where the tremor to which the jet is subjected is compound, the single path is replaced by two, three, or even more paths, which the drops follow in a regular cycle. The explanation has been given with remarkable insight by Plateau. If for example besides the principal disturbance, which determines the size of the drops, there be another of twice the period, it is clear that the alternate drops break away under different conditions and therefore with different velocities. Complete periodicity is only attained after the passage of a *pair* of drops; and thus the odd series of drops pursues one path, and the even series another. All I propose at present is to bring forward a few facts connected with the influence of electricity, which are not mentioned in my former communication. To it, however, I must refer the reader for further explanations. The literature of the subject is given very fully in Plateau's second volume.

When the jet is projected upwards at a moderate obliquity, the sheaf is (as Savart describes it) confined to a vertical plane. Under these circumstances, there are few or no collisions, as the drops have room to clear one another, and moderate electrical influence is without effect. At a higher obliquity the drops begin to be scattered out of the vertical plane, which is a sign that collisions are taking place. Moderate electrical influence will now reduce the scattering again to

* "Proc. Roy. Soc," March 13, 1879. On the Influence of Electricity on Colliding Water Drops.

† Tyndall has shown that a pinhole gas burner may also be used with advantage.

the vertical plane, by causing the coalescence of drops which come into contact. When the projection is nearly vertical, the whole scattering is due to collisions, and is destroyed by electricity. If the resolution into drops is regularised by vibrations of suitable frequency, the principal drops follow the same path, and unless the projection is nearly vertical, there are no collisions, as explained in my former paper. It sometimes happens that the spherules are projected laterally in a distinct stream, making a considerable angle with the main stream. This is the result of collisions between the spherules and the principal drops. I believe that the former are often reflected backwards and forwards several times, until at last they escape laterally. Occasionally the principal drops themselves collide in a regular manner, and ultimately escape in a double stream. In all cases the behaviour under electrical influence is a criterion of the occurrence of collisions. The principal phenomena are easily observed directly, with the aid of instantaneous illumination.

Appendix I.

The subject of this appendix is the mathematical investigation of the motion of frictionless fluid under the action of capillary force, the configuration of the fluid differing infinitely little from that of equilibrium in the form of an infinite circular cylinder.

Taking the axis of the cylinder as axis of z, and polar co-ordinate r,θ in the perpendicular plane, we may express the form of the surface at any time t by the equation

$$r=a_0+f(\theta,z) \quad . \quad . \quad . \quad . \quad . \quad . \quad (11),$$

in which f (θ,z) is always a small quantity. By Fouriers' theorem, the arbitrary function f may be expanded in a series of terms of the type $\alpha_n \cos n\theta \cos kz$; and, as we shall see in the course of the investigation, each of these terms may be considered independently of the others. The summation extends to all positive values of k, and to all positive integral values of n, zero included.

During the motion the quantity a_0 does not remain absolutely constant, and must be determined by the condition that the inclosed *volume* is invariable. Now for the surface.

$$r=a_0+\alpha_n \cos n\theta \cos kz \quad . \quad . \quad . \quad . \quad . \quad (12),$$

we find

$$\text{volume}=\tfrac{1}{2}\iint r^2 d\theta dz=\int(\pi a_0^2+\tfrac{1}{2}\pi\alpha_n^2\cos^2 kz)dz=z(\pi a_0^2+\tfrac{1}{4}\pi\alpha_n^2);$$

so that, if a denote the radius of the section of the undisturbed cylinder,

$$\pi a^2=\pi a_0^2+\tfrac{1}{4}\pi\alpha_n^2,$$

whence approximately

$$a_0=a\left(1-\frac{\alpha_n^2}{8a^2}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13).$$

For the case $n=0$, (13) is replaced by

$$a_0=a\left(1-\frac{\alpha_0^2}{4a^2}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14).$$

We have now to calculate the area of the surface of (22), on which the potential energy of displacement depends. We have

$$\text{Surface}=\iint\left\{1+\left(\frac{dr}{dz}\right)^2+\left(\frac{dr}{rd\theta}\right)^2\right\}^{\frac{1}{2}} r\,d\theta\,dz$$

$$=\iint\left\{1+\frac{1}{2}\left(\frac{dr}{dz}\right)^2+\frac{1}{2r^2}\left(\frac{dr}{d\theta}\right)^2\right\} r\,d\theta\,dz$$

$$=\iint\{1+\tfrac{1}{2}k^2\alpha_n^2\cos^2 n\theta\sin^2 kz+\tfrac{1}{2}n^2\alpha_n^2a^{-2}\sin^2 n\theta\cos^2 kz\}r\,d\theta\,dz$$

$$=z\{2\pi a_0+\tfrac{1}{4}\pi k^2\alpha_n^2a+\tfrac{1}{4}\pi n^2\alpha_n^2a^{-1}\}\ ;$$

so that, if σ denote the surface corresponding on the average to the unit of length,

$$\sigma=2\pi a+\tfrac{1}{4}\pi a^{-1}(k^2a^2+n^2-1)\alpha_n^2 \quad . \quad . \quad . \quad (15),$$

the value of a_0 being substituted from (13).

The potential energy P, estimated per unit length, is therefore expressed by

$$\text{P}=\tfrac{1}{4}\pi a^{-1}\text{T}(k^2a^2+n^2-1)\alpha_n^2 \quad . \quad . \quad . \quad . \quad (16),$$

T being the superficial tension.

For the case $n=0$, (16) is replaced by

$$\text{P}=\tfrac{1}{2}\pi a^{-1}\text{T}(k^2a^2-1)\alpha_0^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (17).$$

From (16) it appears that, when n is unity or any greater integer, the value of P is positive, showing that, for all displacements of these kinds, the original equilibrium is stable. For the case of displacements symmetrical about the axis, we see from (17) that the equilibrium is stable or unstable according as ka is greater or less than unity, *i.e.*, according as the wave-length $(2\pi k^{-1})$ is less or greater than the circumference of the cylinder.

If the expression for r in (12) involve a number of terms with various values of n and k, the corresponding expression for P is found by simple addition of the expressions relating to the component terms, and contains only the squares (and not the products) of the quantities α.

The velocity potential (ϕ) of the motion of the fluid satisfies the equation

$$\frac{d^2\phi}{dr^2}+\frac{1}{r}\frac{d\phi}{dr}+\frac{1}{r^2}\frac{d^2\phi}{d\theta^2}+\frac{d^2\phi}{dz^2}=0\,;$$

or, if in order to correspond with (12) we assume that the variable part is proportional to $\cos n\theta \cos kz$,

$$\frac{d^2\phi}{dr^2}+\frac{1}{r}\frac{d\phi}{dr}-\left(\frac{n^2}{r^2}+k^2\right)\phi=0 \quad . \quad . \quad . \quad . \quad (18).$$

The solution of (18) under the condition that there is no introduction or abstraction of fluid along the axis of symmetry is—

$$\phi=\beta_n \mathrm{J}_n(ikr)\cos n\theta \cos kz \quad . \quad . \quad . \quad . \quad (19),$$

in which $i=\sqrt{(-1)}$, and J_n is the symbol of the Bessel's function of the nth order, so that

$$\mathrm{J}_n(ikr)=\frac{(kr)^n}{2^n\Gamma(n+1)}\left\{1+\frac{k^2r^2}{2\,.\,2n+2}+\frac{k^4r^4}{2\,.\,4\,.\,2n+2\,.\,2n+4}\right.$$
$$\left.+\frac{k^6r^6}{2\,.\,4\,.\,6\,.\,2n+2\,.\,2n+4\,.\,2n+6}+\quad . \quad . \quad \right\} \quad . \quad . \quad (20).$$

The constant β_n is to be found from the condition that the radial velocity when $r=a$ coincides with that implied in (12). Thus

$$ik\,\beta_n \mathrm{J}_n'(ika)=\frac{d\alpha_n}{dt} \quad . \quad . \quad . \quad . \quad . \quad (21).$$

The kinetic energy of the motion is, by Green's theorem,

$$\tfrac{1}{2}\rho\iint\left[\phi\frac{d\phi}{dr}\right]_{r=a} a\,d\theta\,dz=\tfrac{1}{4}\pi\rho z\,.\,ika\,.\,\mathrm{J}_n(ika)\,\mathrm{J}_n'(ika)\,.\,\beta_n^2\,;$$

so that, by (21), if K denote the kinetic energy per unit length,

$$\mathrm{K}=\tfrac{1}{4}\pi\rho a^2\frac{\mathrm{J}_n(ika)}{ika\,.\,\mathrm{J}_n'(ika)}\left(\frac{d\alpha_n}{dt}\right)^2 \quad . \quad . \quad . \quad (22).$$

When $n=0$, we must take, instead of (22),

$$\mathrm{K}=\tfrac{1}{2}\pi\rho a^2\frac{\mathrm{J}_0(ika)}{ika\,.\,\mathrm{J}_0'(ika)}\left(\frac{d\alpha_0}{dt}\right)^2 \quad . \quad . \quad . \quad (23).$$

The most general value of K is to be found by simple summation, with respect to n and k, from the particular values expressed in (22) and (23). Since the expressions for P and K involve only the squares, and not the products, of the quantities α, $\frac{d\alpha}{dt}$, it follows that the motions represented by (12) take place in perfect independence of one another.

For the free motion we get by Lagrange's method from (16), (22),

$$\frac{d^2\alpha_n}{dt^2}+\frac{T}{\rho a^3}\frac{ika\,.\,J_n'(ika)}{J_n(ika)}(n^2+k^2a^2-1)\alpha_n=0 \quad . \quad . \quad (24),$$

which applies without change to the case $n=0$. Thus, if $\alpha_n \propto \cos(pt-\epsilon)$,

$$p^2=\frac{T}{\rho a^3}\frac{ika\,J_n'(ika)}{J_n(ika)}(n^2+k^2a^2-1) \quad . \quad . \quad . \quad (25),$$

giving the frequency of vibration in the cases of stability. If $n=0$, and $ka<1$, the solution changes its form. If we suppose that $\alpha_n \propto e^{\pm qt}$.

$$q^2=\frac{T}{\rho a^3}\frac{ika\,J'_0(ika)}{J_0(ika)}(1-k^2a^2) \quad . \quad . \quad . \quad (26).$$

From this the table in the text was calculated.

When n is greater than unity, the values of p^2 in (25) are usually in practical cases nearly the same as if ka were zero, or the motion took place in two dimensions. We may therefore advantageously introduce into (25) the supposition that ka is small. In this way we get

$$p^2=n(n^2-1+k^2a^2)\frac{T}{\rho a^3}\left[1+\frac{k^2a^2}{n\,.\,2n+2}+-\right] \quad . \quad (27),$$

or, if ka be neglected altogether,

$$p^2=(n^3-n)\frac{T}{\rho a^3} \quad . \quad . \quad . \quad . \quad . \quad . \quad (28),$$

which agree with the formulæ used in the text. When $n=1$, there is no force of restitution for the case of a displacement in two dimensions.

Combining in the usual way two stationary vibrations, whose phases differ by a quarter of a period, we find as the expression of a progressive wave,

$$r=a_0+\gamma_n\cos n\theta\cos kz\cos pt+\gamma_n\cos n\theta\sin kz\sin pt$$

$$=a_0+\gamma_n\cos n\theta\cos(pt-kz) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (29).$$

For the application to a jet the progressive wave must be reduced to steady motion by the superposition of a common velocity (v) equal and opposite to that of the wave's propagation. The solution then becomes

$$r=a_0+\gamma_n\cos n\theta\cos kz \quad . \quad . \quad . \quad . \quad (30),$$

in which γ_n is an absolute constant. The corresponding velocity-potential is

$$\phi=-vz+\frac{p\,\gamma_n J_n(ikr)}{ikJ_n'(ika)}\sin kz\cos n\theta \quad . \quad . \quad . \quad (31).$$

It is instructive to verify these results by the formulæ applicable to steady motion. The resultant velocity q at any point is approximately equal to $\frac{d\phi}{dz}$; and

$$\frac{d\phi}{dz}=-v+\frac{p\,k\,\gamma_n J_n(ikr)}{ikJ_n'(ika)}\cos kz\cos n\theta.$$

At the surface we have approximately $r=a$, and

$$\tfrac{1}{2}q^2=\tfrac{1}{2}v^2-p\,k\,v\frac{\gamma_n J_n(ika)}{ik\,J'(ika)}\cos kz\cos n\theta.$$

Thus by the hydrodynamical equation of pressure, with use of (25), since $v=pk^{-1}$,

$$\text{Pressure}=\text{const.}+\gamma_n a^{-2}\text{T}\,(n^2-1+k^2a^2)\cos kz\cos n\theta \quad . \quad (32).$$

The pressure due to superficial tension is $\text{T}\,(\text{R}_1^{-1}+\text{R}_2^{-1})$, if R_1, R_2, are the radii of curvature in planes parallel and perpendicular to the axis; and from (30)

$$-\text{R}_2^{-1}=\frac{d^2r}{dz^2}=-k^2\gamma_n\cos n\theta\cos kz$$

$$\text{R}_1^{-1}=r^{-1}+\frac{d^2r^{-1}}{d\theta^2}=a^{-1}+\gamma_n a^{-2}(n^2-1)\cos n\theta\cos kz\,;$$

so that

$$\text{Pressure}=\text{const.}+\gamma_n a^{-2}(n^2-1+k^2a^2)\cos n\theta\cos kz.$$

Thus the pressure due to velocity is exactly balanced by the capillary force, and the surface condition of equilibrium is satisfied.

Appendix II.

We will now investigate in the same manner the vibrations of a liquid mass about a *spherical* figure, confining ourselves for brevity to modes of vibrations symmetrical about an axis, which is sufficient for the application in the text. These modes require for their expression only Legendre's functions P_n; the more general problem, involving Laplace's functions, may be treated in the same way, and leads to the same results.

The radius r may be expanded at any time t in the series

$$r=a_0+a_1\text{P}_1(\mu)+\quad . \quad . \quad . \quad +a_n\text{P}_n(\mu)+\quad . \quad . \quad (33),$$

where a_1 a_2 are small quantities relatively to a_0, and μ (according to the usual notation) represents the cosine of the colatitude (θ).

For the volume included within the surface (33) we have

$$V=\tfrac{2}{3}\pi\int_{-1}^{+1}r^3d\mu$$

$$=\tfrac{2}{3}\pi\int_{-1}^{+1}[a_0^3+3a_0^2\{a_1P_1(\mu)+\ldots+a_nP_n(\mu)+\ldots\}$$

$$+3a_0\{a_1P_1(\mu)+\ldots+a_nP_n(\mu)+\ldots\}^2+\ldots]d\mu$$

$$=\tfrac{2}{3}\pi a_0^3\int_{-1}^{+1}[1+3a_0^{-2}\Sigma a_n^2P_n^2(\mu)]d\mu$$

$$=\tfrac{4}{3}\pi a_0^3[1+3a_0^{-2}\Sigma(2n+1)^{-1}a_n^2,$$

approximately. If a be the radius of the sphere of equilibrium,

$$a^3=a_0^3[1+3a^{-2}\Sigma(2n+1)^{-1}a_n^2] \quad . \quad . \quad . \quad (34).$$

We have now to calculate the area of the surface S.

$$S=2\pi\int r\sin\theta\,\surd\left\{r^2+\left(\frac{dr}{d\theta}\right)^2\right\}d\theta=2\pi\int\left\{r^2+\tfrac{1}{2}\left(\frac{dr}{d\theta}\right)^2\right\}\sin\theta d\theta.$$

For the first part

$$\int_{-1}^{+1}r^2d\mu=2a_0^2+2\Sigma(2n+1)^{-1}a_n^2.$$

For the second part

$$\tfrac{1}{2}\int\left(\frac{dr}{d\theta}\right)^2\sin\theta\,d\theta=\tfrac{1}{2}\int_{-1}^{+1}(1-\mu^2)\left[\Sigma a_n\frac{dP_n}{d\mu}\right]^2d\mu.$$

The value of the quantity on the right hand side may be found with the aid of the formula*

$$\int_{-1}^{+1}(1-\mu^2)\frac{dP_m}{d\mu}\frac{dP_n}{d\mu}\,d\mu=n(n+1)\int_{-1}^{+1}P_mP_nd\mu.$$

Thus

$$\tfrac{1}{2}\int_{-1}^{+1}\left(\frac{dr}{d\theta}\right)^2\sin\theta\,d\theta=\tfrac{1}{2}\int_{-1}^{+1}(1-\mu^2)\Sigma a_n^2\left(\frac{dP_n}{d\mu}\right)^2d\mu$$

$$=\tfrac{1}{2}\Sigma n(n+1)a_n^2\int_{-1}^{+1}P_n^2d\mu=\Sigma\, n(n+1)(2n+1)^{-1}a_n^2.$$

Accordingly $\quad S=4\pi a_0^2+2\pi\Sigma(2n+1)^{-1}(n^2+n+2)a_n^2$;

or, since by (34)

$$a_0^2=a^2-2\Sigma(2n+1)^{-1}a_n^2,$$

$$S=4\pi a^2+2\pi\Sigma(n-1)(n+2)(2n+1)^{-1}a_n^2 \quad . \quad . \quad (35).$$

* Todhunter's "Laplace's Functions," § 62.

If T be the cohesive tension, the potential energy is

$$P=2\pi T\,\Sigma(n-1)(n+2)(2n+1)^{-1}a_n^2 \quad . \quad . \quad (36).$$

We have now to calculate the kinetic energy of the motion. The velocity-potential ϕ may be expanded in the series

$$\phi=\beta_0+\beta_1 r\,P_1(\mu)+ \quad . \quad . \quad +\beta_n r^n P_n(\mu)+ \quad . \quad (37);$$

and thus for K we get

$$K=\tfrac{1}{2}\rho\iint\phi\frac{d\phi}{dr}dS=\tfrac{1}{2}\rho\,.\,2\pi a^2\int_{-1}^{+1}\phi\frac{d\phi}{dr}\,d\mu=\tfrac{1}{2}\rho\,.\,4\pi a^2\,.\,\Sigma\,(2n+1)^{-1}n\,a^{2n-1}\beta_n^2.$$

But by comparison of the value of $\frac{d\phi}{dr}$ from (37) with (33), we find

$$n\,a^{n-1}\beta_n=\frac{da_n}{dt}\,;$$

and thus

$$K=2\pi\rho a^3\Sigma(2n+1)^{-1}n^{-1}\left(\frac{da_n}{dt}\right)^2 \quad . \quad . \quad . \quad (38).$$

Since the products of the quantities a_n and $\frac{da_n}{dt}$ do not occur in the expressions for P and K, the motions represented by the various terms occur independently of one another. The equation for a_n is by Lagrange's method

$$\frac{d^2a_n}{dt^2}+n(n-1)(n+2)\frac{T}{\rho a^3}a_n=0 \quad . \quad . \quad (39);$$

so that, if $a_n \propto \cos\,(pt+\epsilon)$,

$$p^2=n(n-1)(n+2)\frac{T}{\rho a^3} \quad . \quad . \quad . \quad . \quad . \quad (40).$$

The periodic time τ given in the text (equation (10)) follows from (40) by putting $\tau=2\pi p^{-1}$, $n=2$, $V=\frac{4}{3}\,\pi a^3$.

To find the radius of the sphere of water which vibrates seconds, put $p=2\pi$, $T=81$, $\rho=1$, $n=2$. Thus $a=2{\cdot}54$ centims., or one inch almost exactly.

The Society adjourned over Ascension Day to Thursday, May 29.

FIG. 1.

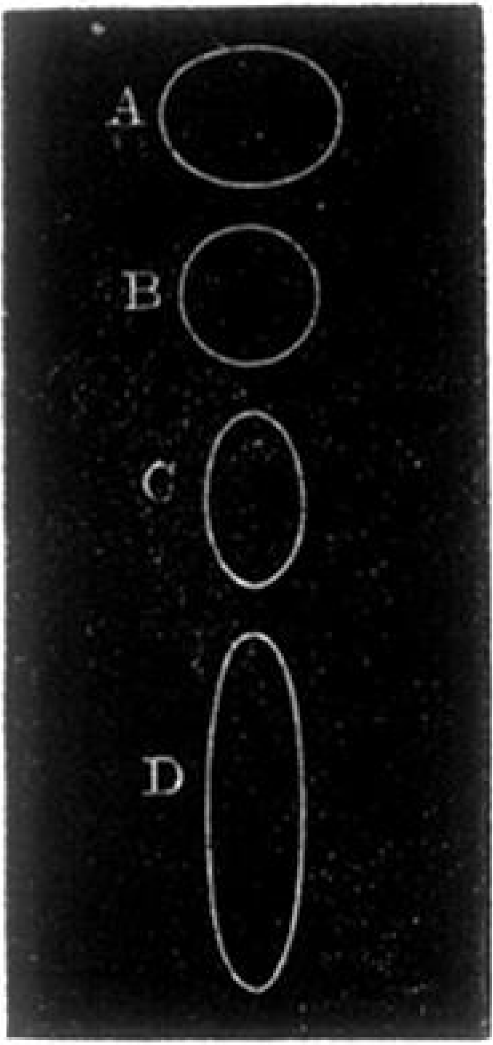


FIG. 2.