Action and Reaction Between Moving Charges

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ONE of the fundamental axioms of Newtonian dynamics asserts that the forces of action and reaction between two particles are equal in magnitude and opposite in direction. Its two most important consequences are (i) the law of conservation of the *linear* momentum of a group of particles subject to no external forces, and (ii) the conclusion that the forces between the particles play no part in determining the motion of the center of mass of a group of particles. The law of action and reaction is not sufficient, however, to prove the law of conservation of the *angular* momentum of a group of particles subject to no external torques; we must assume in addition that the forces between two particles are along the line joining them.

The laws of conservation of linear and of angular momentum are so well verified by experiments in mechanics that we cannot doubt the validity of the axioms on which they rest, at least insofar as *time-average* forces are concerned. Yet we believe that the atoms of which matter is constructed are composed of rapidly moving charged particles, and the electromagnetic forces between moving charged particles satisfy, in general, neither the condition that the forces between two particles are along the line joining them nor the condition that these forces are equal in magnitude and oppositely directed. It is the object of this paper to discuss the forces and torques exerted by one charged particle on another with special reference to the law of action and reaction. We shall employ Gaussian units (electrostatic units for electric quantities, including current, and electromagnetic units for magnetic quantities). Then the power to which $1/c$ appears as a factor (c is the ratio of the electromagnetic to the electrostatic unit of charge), determines the order of magnitude of a quantity. Thus terms in a series that do not contain $1/c$ will be designated as of zero order, those in $1/c$ as of the first order of smallness, and those in $(1/c)^2$ as of the second order of smallness. In our analysis we shall retain all terms through the second order, but none of higher order.

TWO CURRENT ELEMENTS

The magnetic field intensity $\mathbf{H}$ caused by a current element—an element of length $\Delta\boldsymbol{\lambda}$ of a linear circuit carrying a current i—at a field point P at a vector distance $\mathbf{r}$ from the current element, is generally specified by the formula

$$\mathbf{H} = i\Delta\boldsymbol{\lambda}\times\mathbf{r}/cr^3. \tag{1}$$

While this expression is often referred to as Ampère's law, it is not the expression originally proposed by Ampère. In fact, the original Ampère's law,[1] while like Eq. (1) in that it yields correct results for *constant* currents in *closed* circuits, does not specify correctly even the first-order magnetic field of a current element. This first-order field is given correctly by Eq. (1). However, we know today that Eq. (1) is merely the first term of an infinite series[2] in powers of $1/c$. Nevertheless, this term alone yields an exceedingly accurate value of the field, owing to the fact that the next term in the series happens to have the coefficient zero, and therefore the error in Eq. (1) is of the order of $(1/c)^3$. Since we are interested here only in terms of the second and lower orders, we can employ Eq. (1) without modification.

A current element is usually thought of as a charge e moving along the circuit so as to constitute a current, associated with a charge $-e$ at rest, so that the total charge is zero. For the present, we shall dispense with the stationary charge $-e$ and confine our attention to a single charge e moving with velocity $\mathbf{v}$. Then, if ρ_λ is the charge per unit length of the circuit, $e=\rho_\lambda\Delta\lambda$ and $i=\rho_\lambda v$. Hence $e\mathbf{v}=i\Delta\boldsymbol{\lambda}$, and we may write Eq. (1) in the form

$$\mathbf{H} = e\mathbf{v}\times\mathbf{r}/cr^3. \tag{2}$$

To calculate the force exerted by one moving charge on another we also need the formula for the electric field intensity. The electric field of a moving charge differs from the simple inverse-square field of a charge at rest, the intensity at the field point P being specified by

$$\mathbf{E} = \frac{e\mathbf{r}}{r^3} - \frac{e}{2c^2}\left\{\frac{\mathbf{f}}{r} + \left(\frac{\mathbf{f}\cdot\mathbf{r}}{r^3} - \frac{v^2}{r^3} + 3\frac{\overline{\mathbf{v}\cdot\mathbf{r}}^2}{r^5}\right)\mathbf{r}\right\} \tag{3}$$

[1] See Maxwell, *Electricity and magnetism*, Part IV, chap. II, for a detailed discussion of the original Ampère's law.

[2] Page and Adams, *Electrodynamics*, p. 175.

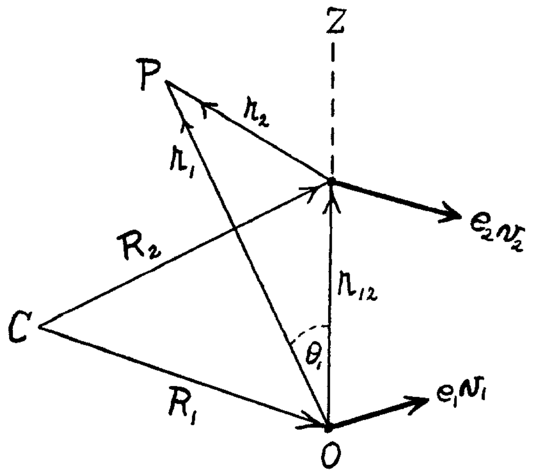

FIG. 1. Two moving charges.

through second-order terms.[3] Here $\mathbf{f}$ is the acceleration of the charge.

Now we are ready to calculate the forces between two moving charges, $e_1\mathbf{v}_1$ and $e_2\mathbf{v}_2$ (Fig. 1). Let $\mathbf{r}_{12}$ be the vector distance of e_2 from e_1, and $\mathbf{r}_{21}[=-\mathbf{r}_{12}]$ that of e_1 from e_2. First we calculate the magnetic forces $\mathbf{F}_1^H$ on the current element $e_1\mathbf{v}_1$ and $\mathbf{F}_2^H$ on $e_2\mathbf{v}_2$. Using Eq. (2) and the familiar formula,

$$\mathbf{F}^H=\frac{1}{c}e\mathbf{v}\times\mathbf{H},$$

for the force on a current element $e\mathbf{v}$ in a magnetic field $\mathbf{H}$, we find

$$\mathbf{F}_1^H=\frac{e_1\mathbf{v}_1}{c}\times\left\{\frac{e_2\mathbf{v}_2\times\mathbf{r}_{21}}{cr^3}\right\}=\frac{e_1e_2}{c^2}\frac{\mathbf{v}_1\times(\mathbf{v}_2\times\mathbf{r}_{21})}{r^3},$$

$$\mathbf{F}_2^H=\frac{e_2\mathbf{v}_2}{c}\times\left\{\frac{e_1\mathbf{v}_1\times\mathbf{r}_{12}}{cr^3}\right\}=\frac{e_1e_2}{c^2}\frac{\mathbf{v}_2\times(\mathbf{v}_1\times\mathbf{r}_{12})}{r^3},$$

where $r[=|\mathbf{r}_{12}|=|\mathbf{r}_{21}|]$ is the distance between the charges.

If the magnetic forces obeyed the law of action and reaction, $\mathbf{F}_1^H+\mathbf{F}_2^H$ would vanish. But, instead,

$$\mathbf{F}_1^H+\mathbf{F}_2^H=\frac{e_1e_2}{c^2r^3}(\mathbf{v}_2\times\mathbf{v}_1)\times\mathbf{r}_{12}, \quad (4)$$

which vanishes only when (*a*) $\mathbf{v}_2$ is parallel to $\mathbf{v}_1$, or (*b*) $\mathbf{v}_1$ and $\mathbf{v}_2$ are both perpendicular to $\mathbf{r}_{12}$. So, except in these special cases, the magnetic forces between two moving charges do not constitute an equal and opposite pair satisfying the law of action and reaction, but have a vector resultant different from zero.

Next we calculate the electric forces $\mathbf{F}_1^E$ on $e_1\mathbf{v}_1$ and $\mathbf{F}_2^E$ on $e_2\mathbf{v}_2$ from Eq. (3) and the formula $\mathbf{F}^E=e\mathbf{E}$. The zero-order terms form an equal and opposite pair satisfying the law of action and reaction, but the second-order terms do not. In fact, we find for the resultant electric force,

$$\mathbf{F}_1^E+\mathbf{F}_2^E=\frac{e_1e_2}{2c^2}\left[-\frac{\mathbf{f}_1+\mathbf{f}_2}{r}+\left\{-(\mathbf{f}_1+\mathbf{f}_2)\cdot\mathbf{r}_{12}+v_1^2-v_2^2-3\frac{\overline{\mathbf{v}_1\cdot\mathbf{r}_{12}}^2}{r^2}+3\frac{\overline{\mathbf{v}_2\cdot\mathbf{r}_{12}}^2}{r^2}\right\}\frac{\mathbf{r}_{12}}{r^3}\right], \quad (5)$$

which does not vanish in general, although it is zero in the special case $\mathbf{f}_2=-\mathbf{f}_1$, $\mathbf{v}_2=\pm\mathbf{v}_1$.

Evidently the total resultant force through second-order terms, obtained by adding Eqs. (4) and (5), does not vanish in general. Therefore the law of action and reaction does not hold for two moving charges. This conclusion, however, should occasion no surprise, for it is a well-known consequence of the electromagnetic equations that the so-called mechanical momentum of a system of charged particles subject to no external forces does not remain constant in time. On the contrary, the conservation law applies only to the *sum* of the mechanical and the electromagnetic momentums. This statement applies both to linear and to angular momentum. Thus a photon emitted from the sun subtracts from the mechanical momentum of the latter an amount equal to its own electromagnetic momentum.

It is, therefore, of interest to calculate the electromagnetic linear momentum of the field of the two moving charges under consideration, in order to show that its time-rate of increase, when added to the sum of Eqs. (4) and (5), gives zero resultant.

In Gaussian units the electromagnetic linear momentum per unit volume is given[4] by $(1/4\pi c)\mathbf{E}\times\mathbf{H}$. Consider a field point P (Fig. 1) at a vector distance $\mathbf{r}_1$ from $e_1\mathbf{v}_1$ and $\mathbf{r}_2$ from $e_2\mathbf{v}_2$. Let $\mathbf{E}_1$, $\mathbf{H}_1$ be the field intensities caused by the first current element, and $\mathbf{E}_2$, $\mathbf{H}_2$ those caused by the second. Then the electromagnetic linear momentum per unit volume is

$$\frac{1}{4\pi c}\{\mathbf{E}_1\times\mathbf{H}_1+\mathbf{E}_1\times\mathbf{H}_2+\mathbf{E}_2\times\mathbf{H}_1+\mathbf{E}_2\times\mathbf{H}_2\}.$$

[3] Reference 2, p. 175.

[4] Reference 2, p. 272.

Evidently the two terms in which the subscripts are the same remain constant if the velocities of the two charges do not change, and are responsible for the electromagnetic masses of the individual charges in accelerated motion. As we are interested here only in the interaction of the one current element with the other, we need retain only the mutual terms, and calculate the portion of the electromagnetic linear momentum per unit volume represented by

$$\mathbf{g}_l=\frac{1}{4\pi c}\{\mathbf{E}_1\times\mathbf{H}_2+\mathbf{E}_2\times\mathbf{H}_1\}. \quad (6)$$

Since Eq. (2) for $\mathbf{H}$ is of the first order, we need retain only the first term in Eq. (3) for $\mathbf{E}$ in order to obtain $\mathbf{g}_l$ through second-order terms. Thus

$$\mathbf{g}_l=\frac{e_1e_2}{4\pi c^2r_1^3r_2^3}\{\mathbf{r}_1\times(\mathbf{v}_2\times\mathbf{r}_2)+\mathbf{r}_2\times(\mathbf{v}_1\times\mathbf{r}_1)\}. \quad (7)$$

To obtain the total mutual electromagnetic linear momentum $\mathbf{G}_l$ we must integrate this expression over all space. To do this easily we expand the triple vector products and introduce spherical coordinates r_1, θ_1, ϕ with O as origin and OZ as polar axis. From the trigonometric relation $r_2^2=r^2-2rr_1\cos\theta_1+r_1^2$ we find,[5] for constant r_1, that $r_2dr_2=r_1r\sin\theta_1d\theta_1$. Hence the volume element $d\tau=r_1^2\sin\theta_1dr_1d\theta_1d\phi$ may be written $d\tau=(r_1r_2/r)dr_1dr_2d\phi$ in terms of r_1, r_2, ϕ as variables. For $r_1<r$ the limits of r_2 are $r-r_1$ and $r+r_1$, whereas for $r_1>r$, r_2 goes from r_1-r to r_1+r. The integration, which involves no difficulties, yields the result

$$\mathbf{G}_l=\frac{e_1e_2}{2c^2}\left\{\left(\frac{\mathbf{v}_1}{r}+\frac{\mathbf{v}_1\cdot\mathbf{r}_{12}\mathbf{r}_{12}}{r^3}\right)+\left(\frac{\mathbf{v}_2}{r}+\frac{\mathbf{v}_2\cdot\mathbf{r}_{12}\mathbf{r}_{12}}{r^3}\right)\right\}. \quad (8)$$

The reader can show easily that the time derivative of $\mathbf{G}_l$, when added to the sum of Eqs. (4) and (5), gives zero resultant, thus verifying the statement that the sum of the mechanical and electromagnetic linear momentums remains constant in time. The fact that the electromagnetic momentum changes when $\mathbf{r}_{12}$ is not constant (that is, when $\mathbf{v}_2$ is not equal to $\mathbf{v}_1$) *requires* that the forces of action and reaction between the two moving charges should *not* be equal in magnitude and oppositely directed. Otherwise the total momentum could *not* remain constant in time.

Next we calculate the torque or moment about an arbitrary point C (Fig. 1) of the force $\mathbf{F}_1^H+\mathbf{F}_1^E$ on $e_1\mathbf{v}_1$ and of the force $\mathbf{F}_2^H+\mathbf{F}_2^E$ on $e_2\mathbf{v}_2$. Denoting the position vectors of the two charges relative to C by $\mathbf{R}_1$ and $\mathbf{R}_2$, we have for the sum of the torques of the magnetic forces,

$$\mathbf{R}_1\times\mathbf{F}_1^H+\mathbf{R}_2\times\mathbf{F}_2^H=\frac{e_1e_2}{c^2r^3}[-\mathbf{R}_1\times\{\mathbf{v}_1\times(\mathbf{v}_2\times\mathbf{r}_{12})\}+\mathbf{R}_2\times\{\mathbf{v}_2\times(\mathbf{v}_1\times\mathbf{r}_{12})\}], \quad (9)$$

and for the sum of the torques of the electric forces,

$$\mathbf{R}_1\times\mathbf{F}_1^E+\mathbf{R}_2\times\mathbf{F}_2^E=\frac{e_1e_2}{2c^2}\left[\mathbf{R}_1\times\left\{-\frac{\mathbf{f}_1}{r}+\left(-\mathbf{f}_1\cdot\mathbf{r}_{12}+v_1^2-3\frac{\overline{\mathbf{v}_1\cdot\mathbf{r}_{12}}^2}{r^2}\right)\frac{\mathbf{r}_{12}}{r^3}\right\}+\mathbf{R}_2\times\left\{-\frac{\mathbf{f}_2}{r}+\left(-\mathbf{f}_2\cdot\mathbf{r}_{12}-v_2^2+3\frac{\overline{\mathbf{v}_2\cdot\mathbf{r}_{12}}^2}{r^2}\right)\frac{\mathbf{r}_{12}}{r^3}\right\}\right]. \quad (10)$$

Evidently the resultant torque, obtained by adding Eqs. (9) and (10), does not vanish in general, indicating that the mechanical angular momentum of the pair of moving charges does not remain constant in time. We conclude that this lack of balance is due to the presence of a changing electromagnetic angular momentum, which we proceed to calculate.

Relative to the point C the position vector of the field point P is $\mathbf{R}_1+\mathbf{r}_1=\mathbf{R}_2+\mathbf{r}_2$. Hence the mutual electromagnetic angular momentum per unit volume is

$$\mathbf{g}_a=(\mathbf{R}_1+\mathbf{r}_1)\times\mathbf{g}_l.$$

Integrating over all space, we find for the total mutual electromagnetic angular momentum about C,

$$\mathbf{G}_a=\mathbf{R}_1\times\mathbf{G}_l+\frac{e_1e_2}{2c^2}\frac{\mathbf{r}_{12}\times\mathbf{v}_1}{r},$$

[5] Of course r is not a variable for this integration.

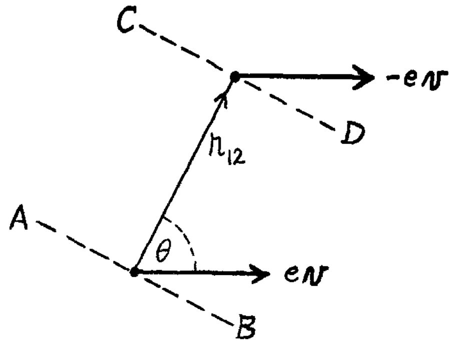

FIG. 2. The Trouton-Noble experiment.

which can be put in the more symmetrical form,

$$\mathbf{G}_a=\frac{e_1e_2}{2c^2}\left\{\mathbf{R}_2\times\left(\frac{\mathbf{v}_1}{r}+\frac{\mathbf{v}_1\cdot\mathbf{r}_{12}\mathbf{r}_{12}}{r^3}\right)+\mathbf{R}_1\times\left(\frac{\mathbf{v}_2}{r}+\frac{\mathbf{v}_2\cdot\mathbf{r}_{12}\mathbf{r}_{12}}{r^3}\right)\right\}. \quad (11)$$

Comparing with Eq. (8) we see that the portion of the linear momentum involving the velocity $\mathbf{v}_1$ of the first particle is to be considered as located at the second particle, and *vice versa*. The time derivative of $\mathbf{G}_a$, added to the torques of the forces on the two current elements specified by the sum of Eqs. (9) and (10), gives zero resultant, in accord with the law of conservation of the *sum* of the mechanical and electromagnetic angular momentums. Neither one of these two types of angular momentum alone is constant in time in the general case, even though no external torques are operating.

It is worthy of note that both $\mathbf{G}_l$ and $\mathbf{G}_a$ are of the second order. This is because $\mathbf{H}$ contains no zero-order term. Therefore the law of action and reaction fails to hold only by terms of the second or higher orders.

Finally, we must extend our analysis to cover the case of two conventional current elements, in each of which there is associated with the charge e moving with velocity $\mathbf{v}$ an equal and opposite stationary charge $-e$. Since the net charge of each current element is zero, the electric force on it vanishes. Therefore the entire resultant force is given by Eq. (4).

The zero-order electric field of each current element is zero at the instant when the moving charge coincides with the stationary charge of opposite sign. At this instant the mutual electromagnetic linear momentum $\mathbf{G}_l$ vanishes. However, $\mathbf{G}_l$ does not remain zero as these charges separate, and therefore the time derivative of $\mathbf{G}_l$ does not vanish, in general. For let us designate the coordinates of the two current elements by (x_1, y_1, z_1), (x_2, y_2, z_2), respectively, so that

$$\mathbf{r}_{12}=\mathbf{i}(x_2-x_1)+\mathbf{j}(y_2-y_1)+\mathbf{k}(z_2-z_1)$$

and

$$r^2=(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2.$$

Now, when we differentiate the $\mathbf{r}_{12}$ and r appearing in Eq. (8), where both e_1 and e_2 are *moving* charges, we must differentiate both sets of coordinates. But in the case of the complete current elements which we are now discussing, the expression for the mutual electromagnetic linear momentum contains, in addition to the right-hand member of Eq. (8), the term

$$-\frac{e_1e_2}{2c^2}\left(\frac{\mathbf{v}_1}{r}+\frac{\mathbf{v}_1\cdot\mathbf{r}_{12}\mathbf{r}_{12}}{r^3}\right),$$

in which e_1 is a *moving* charge and $-e_2$ a *stationary* charge, and the term

$$-\frac{e_1e_2}{2c^2}\left(\frac{\mathbf{v}_2}{r}+\frac{\mathbf{v}_2\cdot\mathbf{r}_{12}\mathbf{r}_{12}}{r^3}\right),$$

in which e_2 is *moving* and $-e_1$ is *stationary*. In differentiating the first of these terms we must hold x_2, y_2, z_2 constant, and in differentiating the second, x_1, y_1, z_1. The final result is the same as if, in differentiating Eq. (8), we differentiated only the x_2, y_2, z_2 contained implicitly in $\mathbf{r}_{12}$ and r inside the first parentheses, and only the x_1, y_1, z_1 inside the second parentheses. This gives, for the time-rate of increase of $\mathbf{G}_l$, the negative of Eq. (4), confirming the conservation of the *sum* of the mechanical and the electromagnetic linear momentums.

APPLICATIONS

An example of the theory developed, which is of great historical interest, is illustrated in Fig. 2. Here we have two equal and opposite charges moving with the same constant velocity $\mathbf{v}$ in a direction *not* at right angles to the vector distance $\mathbf{r}_{12}$ between them. Evidently $\mathbf{G}_l$ is constant in time, and therefore the forces between the current elements are equal and opposite. This is not true, however, of the torques, for

$$\frac{d\mathbf{G}_a}{dt}=-\frac{e^2}{c^2}\frac{\mathbf{v}\cdot\mathbf{r}_{12}\mathbf{v}\times\mathbf{r}_{12}}{r^3},$$

and therefore there exists an equal and opposite torque on the moving charges, to wit,

$$\mathfrak{L} = \frac{e^2v^2}{c^2r}\cos\theta\sin\theta \qquad (12)$$

in the sense of increasing θ.

The arrangement in Fig. 2 is effectively that used in the Trouton-Noble[6] experiment, which excited much interest during the last generation, the charges being spread over the positively charged plate AB and the negatively charged plate CD of a parallel plate condenser. Trouton and Noble concluded from Eq. (12) that a charged condenser carried along by the earth in its motion around the sun would turn until its plates are parallel to the direction of motion, a phenomenon which is obviously at variance with the relativity principle. The fallacy in this reasoning lies in the neglect of the torque of the forces exerted on the condenser plates by the insulating separators necessary to keep the plates from approaching each other under their mutual electric attraction. The relativity principle, completely confirmed by the null result of the experiment, demands that these forces show the same aberration as the electromagnetic forces.

Next consider two charged particles, moving around their common center of mass under their mutual forces, such as the electron and the proton constituting a hydrogen atom. We shall suppose that the velocities attained are small enough so that we can neglect the variation of mass with velocity. The zero-order force is the inverse-square Coulomb force of attraction (or repulsion), and under such a force the two particles would revolve about the center of mass with constant angular momentum. We shall show that, when account is taken of second-order terms, the mechanical angular momentum does not remain constant, but, in the case of an (approximately) elliptical orbit, oscillates between a minimum at perihelion and a maximum at aphelion.

Take the center of mass as origin, and denote the position vectors of the two particles, of mass and charge m_1, e_1 and m_2, e_2, respectively, by $\mathbf{r}_1$ and $\mathbf{r}_2$. Let r be the distance between the two particles, and θ the angle in the plane of the orbit which the line joining them makes with some fixed line. From Eq. (11) we calculate the electromagnetic angular momentum about the

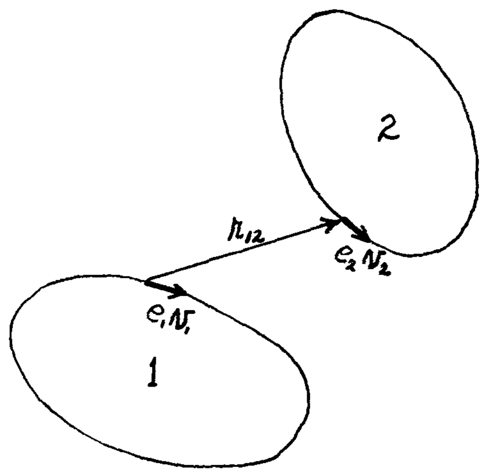

FIG. 3. Two complete circuits.

center of mass, finding

$$\mathbf{G}_a = \frac{e_1e_2}{2c^2}\left\{\frac{\mathbf{r}_1\times\mathbf{v}_2+\mathbf{r}_2\times\mathbf{v}_1}{r}\right\}.$$

But $|\mathbf{r}_1\times\mathbf{v}_2| = |\mathbf{r}_2\times\mathbf{v}_1| = -r_1r_2\dot{\theta}$, and, if μ is the reduced mass, $m_1r_1 = m_2r_2 = \mu r$. Hence

$$G_a = -\frac{e_1e_2}{c^2}\frac{r_1r_2}{r}\dot{\theta} = -\frac{e_1e_2\mu^2}{c^2m_1m_2}r\dot{\theta} \qquad (13)$$

in the sense of increasing θ. Now, so far as the zero-order motion is concerned, $r^2\dot{\theta} = h$ (a positive constant), and

$$G_a = -\frac{e_1e_2\mu^2}{c^2m_1m_2}\frac{h}{r}. \qquad (14)$$

If the charges are of opposite sign, the zero-order force is an attraction, and G_a is positive, attaining its maximum value when the particles are nearest together. Therefore, since the sum of the mechanical and the electromagnetic angular momentums must remain constant, the mechanical angular momentum is a minimum when the particles are closest together. In the case of an elliptical orbit, the mechanical angular momentum oscillates between a smallest value at perihelion and a largest value at aphelion. On the other hand, when the charges are of the same sign, so as to make the zero-order force a repulsion, G_a is negative, and therefore the electromagnetic angular momentum is an (algebraic) minimum and the mechanical angular momentum a maximum at the position of nearest approach. Since the zero-order mechanical angular mo-

[6] Proc. Roy. Soc. **72**, 132 (1903).

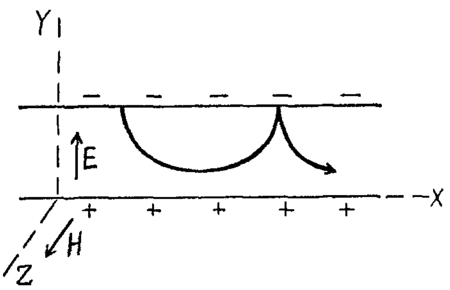

FIG. 4. Motion of electrons in perpendicular fields.

mentum is μh in either case, Eq. (14) may be regarded as expressing the electromagnetic angular momentum in terms of the zero-order mechanical angular momentum and the separation r of the particles. If we pass over from the two-body problem to the corresponding one-body problem by making one of the masses infinite, G_a vanishes, as expected.

As our next example we shall calculate the mutual electromagnetic linear momentum of two complete circuits *1* and *2*, Fig. 3, which we shall designate by the subscripts 1 and 2. Here we are concerned, not with the mutual momentum of two elements of the *same* circuit, but only with that of pairs of current elements such that one element of each pair is part of one circuit and the other is part of the other circuit.

At first we shall consider the case where the electric field of the moving charge is not neutralized by that of an equal stationary charge of opposite sign. Replacing e_1 and e_2 in Eq. (8) by de_1 and de_2, and using the relation $\mathbf{v}de = i d\boldsymbol{\lambda}$, we obtain, after integrating Eq. (8) over circuit *2*,

$$d\mathbf{G}_l = \frac{1}{2c^2}\left\{ de_1\mathbf{v}_1 \oint_2 \frac{de_2}{r} + de_1\mathbf{v}_1\cdot \oint_2 \frac{\mathbf{r}_{12}\mathbf{r}_{12}de_2}{r^3} + de_1 i_2 \oint_2 \frac{d\boldsymbol{\lambda}_2}{r} + de_1 i_2 \oint_2 \frac{\mathbf{r}_{12}\cdot d\boldsymbol{\lambda}_2\mathbf{r}_{12}}{r^3}\right\}.$$

Now, for the integration around circuit *2*, $d\boldsymbol{\lambda}_2 = d\mathbf{r}_{12}$. Hence, using the identity

$$d\left(\frac{\mathbf{r}_{12}}{r}\right) = \frac{d\mathbf{r}_{12}}{r} - \frac{\mathbf{r}_{12}\cdot d\mathbf{r}_{12}\mathbf{r}_{12}}{r^3},$$

we have

$$\oint_2 \frac{\mathbf{r}_{12}\cdot d\boldsymbol{\lambda}_2\mathbf{r}_{12}}{r^3} = \oint \frac{d\boldsymbol{\lambda}_2}{r},$$

which enables us to combine the last two integrals in the expression for $d\mathbf{G}_l$. Integrating next over circuit *1*, and proceeding in the same manner with the first two integrals, we obtain finally for the electromagnetic linear momentum of the two circuits,

$$\mathbf{G}_l = \frac{i_1}{c^2}\oint_1\oint_2 \frac{de_2 d\boldsymbol{\lambda}_1}{r} + \frac{i_2}{c^2}\oint_2\oint_1 \frac{de_1 d\boldsymbol{\lambda}_2}{r}. \qquad (15)$$

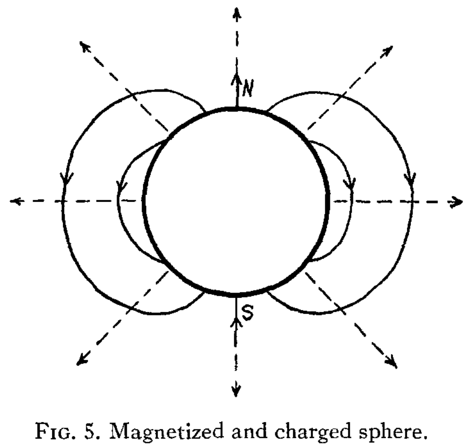

FIG. 5. Magnetized and charged sphere.

So long as the currents are constant, $\mathbf{G}_l$ is constant in time, and therefore the forces of action and reaction between the two circuits are equal in magnitude and opposite in direction.

Finally, if the moving charge in each circuit is associated with an equal charge of the opposite sign at rest, as in the conventional current circuit, the static electric field is everywhere zero, and consequently both $\mathbf{G}_l$ and its time derivative vanish. Therefore the law of action and reaction is valid for two such circuits carrying constant currents.

In closing we shall mention two other examples, the details of which will be found in our *Electrodynamics*. Consider a charged parallel plate condenser (Fig. 4), with plates perpendicular to the Y-axis, placed in a uniform magnetic field parallel to the Z-axis. Let electrons be liberated, say photoelectrically, from the upper negatively charged plate. Under the action of the combined fields they will describe cycloidal paths as indicated, and at the bottom of these paths will have acquired linear momentum in the X-direction. It can be shown easily that the mechanical linear momentum so acquired is exactly accounted for by the loss of electromagnetic linear momentum in the X-direction due to the weakening of the electric and magnetic fields by the displacement and motion of the electrons.

Finally, consider a sphere (Fig. 5) which is uniformly magnetized in the direction *SN* and at the same time carries a uniformly distributed positive electric charge on its surface. In Fig. 5 the lines of magnetic force are indicated by full lines, and the lines of electric force by broken lines. Evidently the electromagnetic field possesses angular momentum about the *SN*-axis in the positive sense. Now, suppose a negative ion, initially at rest, approaches the sphere under the electric attraction exerted by the latter. Owing to the deflecting force of the magnetic field, it will acquire angular momentum about the *SN*-axis in the positive sense, which it may communicate to the sphere upon impact. It can be shown without difficulty that the mechanical angular momentum so acquired is equal to the electromagnetic angular momentum lost as a result of the weakening of the field of the sphere by the attachment of the negative ion.

A Demonstration Laboratory for Advanced Dynamics

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THE Lagrangian treatment of analytical dynamics has long been a powerful tool in the hands of the physicist. Owing to its many advantages in treating applied problems involving not only mechanical systems but electrical, electromechanical and others, the Lagrangian method is fast becoming an everyday tool in the hands of the engineer and applied scientist. It seems certain that in due time, perhaps very shortly, it will constitute an important part of the working equipment of every well-trained engineer. Thus the question as to how best to present this subject to advanced engineers and graduate students of physics presents a pedagogic problem that deserves careful consideration.

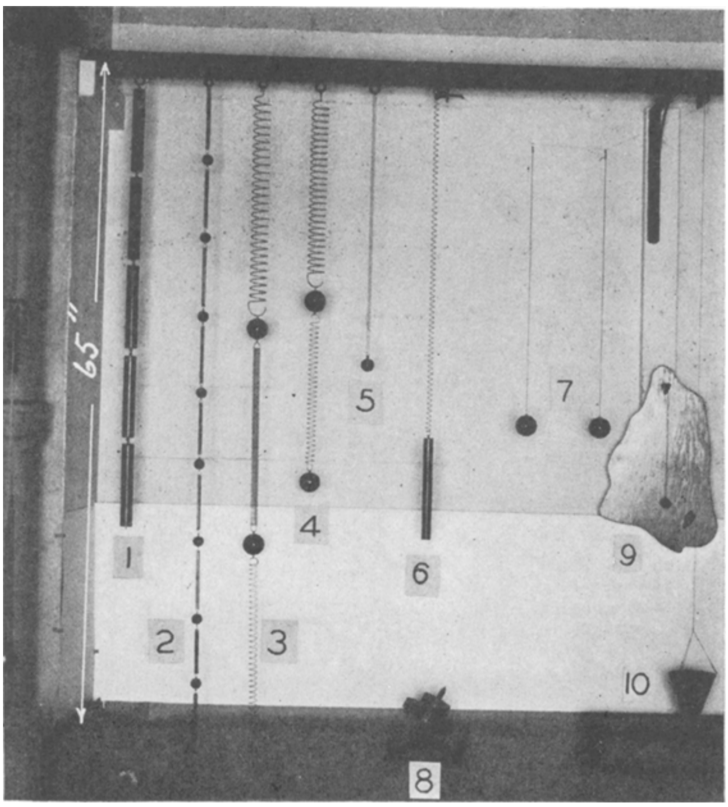

FIG. 1.

Now analytical dynamics can be treated from either of two quite different points of view:

(1) It can be regarded as an abstract mathematical subject in which generalized coordinates are just so many variable "parameters;" degrees of freedom and degrees of constraint merely connote certain mathematical relations; a nonholonomic system is one of those mathematical abortions in which occur certain differential relations that cannot be integrated; the Lagrangian function and generalized forces are but convenient mathematical definitions; and so on. Or

(2) it can be presented from a physical point of view in which care is taken to give the terms and mathematical relations employed simple and easy-to-recognize physi-