Momentum in a DC Circuit

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1 Problem

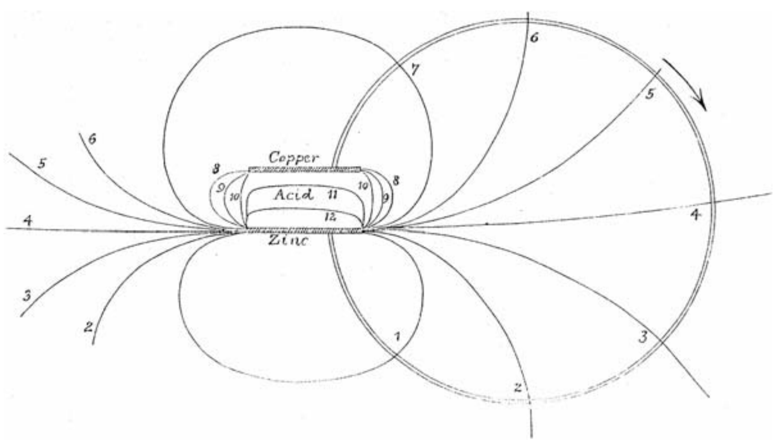
Discuss the electromagnetic and mechanical momentum in a DC current loop of resistance R that is powered by a battery of voltage V. The loop is at rest in the laboratory.

2 Solution

This problem was inspired by an extensive e-dialogue with Vladimir Hnizdo. For a related example of a DC circuit with a different physical configuration, see [1].

Because the loop is at rest in the laboratory, its total momentum should be zero.

However, a DC current loop is a dynamical system in that its electrical resistance R causes dissipation of power at the rate $V^2/R = VI = I^2R$, where the DC current I is, of course, $I = V/R$. This power flows from the battery to the loop in a manner first well described by Poynting. As shown in the figure below by Poynting [2], the power does not flow down the wire of the loop, but rather it flows through the air/vacuum and enters the wire at right angles to its surface.



Abraham noted that Poynting's vector, when divided by c^2 in Gaussian units, where c is the speed of light, represents the volume density of electromagnetic momentum [3]. The total electromagnetic momentum associated with the circuit is then given in terms of the electric field $\mathbf{E}$ and the magnetic field $\mathbf{B}$ by

$$\mathbf{P}_{\rm EM} = \int \frac{\mathbf{E} \times \mathbf{B}}{4\pi c}\, d\text{Vol}. \tag{1}$$

The above figure suggests that the DC circuit has a nonzero electromagnetic momentum that points to the right.

Then, for the circuit to have zero total momentum, it must also possess a "hidden" mechanical momentum that points to the left.

In sec. 2.1 we deduce the electromagnetic momentum of the circuit from various points of view, and in sec. 2.2 we discuss the "hidden" mechanical momentum.

2.1 Electromagnetic Momentum

A direct evaluation of the electromagnetic momentum of the circuit according to eq. (1) is cumbersome, so we begin with an indirect calculation in sec. 2.1.1. A lengthier, direct calculation is presented in sec. 2.1.2. An alternative calculation based on the concept of canonical electromagnetic momentum is given in sec. 2.1.3.

2.1.1 Indirect Calculation of the Electromagnetic Momentum

It is convenient to express the electromagnetic momentum (1) in terms of an integral of the electric potential Φ and the current density $\mathbf{J}$, following Furry [4]. See also [5]. For this we note that in a static situation the electric and magnetic fields obey $\mathbf{E} = -\boldsymbol{\nabla}\Phi$ and $\boldsymbol{\nabla} \times \mathbf{B} = (4\pi/c)\,\mathbf{J}$. Then,

$$\begin{aligned}
\mathbf{P}_{\rm EM} &= \int \frac{\mathbf{E} \times \mathbf{B}}{4\pi c}\, d\mathrm{Vol} = -\int \frac{\boldsymbol{\nabla}\Phi \times \mathbf{B}}{4\pi c}\, d\mathrm{Vol} = \int \frac{\Phi\boldsymbol{\nabla} \times \mathbf{B}}{4\pi c}\, d\mathrm{Vol} - \int \frac{\boldsymbol{\nabla} \times \Phi\mathbf{B}}{4\pi c}\, d\mathrm{Vol} \\
&= \int \frac{\Phi\mathbf{J}}{c^2}\, d\mathrm{Vol} - \oint \frac{d\mathbf{Area} \times \Phi\mathbf{B}}{4\pi c} \\
&= \int \frac{\Phi\mathbf{J}}{c^2}\, d\mathrm{Vol},
\end{aligned} \tag{2}$$

whenever the charges and currents are contained within a finite volume.

In the present problem we suppose the circuit forms a circle of radius a in the x-y plane, centered on the origin. The battery of voltage V is located at $(x, y, z) = (-a, 0, 0)$ and is oriented so that the current vector at position $(a, \phi, 0)$ in a cylindrical coordinate system is

$$\mathbf{I} = -\frac{V}{R}\,\hat{\boldsymbol{\phi}} = \frac{V}{R}(\hat{\mathbf{x}}\,\sin\phi - \hat{\mathbf{y}}\,\cos\phi). \tag{3}$$

We suppose that the resistance R is uniformly distributed over the circumference of the circuit, so that the electric potential along the circuit is

$$\Phi(a, \phi, 0) = \frac{V\phi}{2\pi} \qquad (-\pi < \phi < \pi). \tag{4}$$

The electric potential varies from $-V/2$ to $V/2$ within the battery, so the potential is actually continuous in the vicinity of $\phi = \pm\pi$. Using eqs. (3) and (4) in (2), and expressing $\mathbf{J}\, d\mathrm{Vol}$ as $a\mathbf{I}\, d\phi$, we find the electromagnetic momentum of the circuit to be

$$\mathbf{P}_{\rm EM} = \frac{aV^2}{2\pi c^2 R}\int_{-\pi}^{\pi} d\phi\,(\hat{\mathbf{x}}\,\sin\phi - \hat{\mathbf{y}}\,\cos\phi) = \frac{aV^2}{c^2 R}\,\hat{\mathbf{x}} = \frac{aVI}{c^2}\,\hat{\mathbf{x}} = \frac{aI^2 R}{c^2}\,\hat{\mathbf{x}}, \tag{5}$$

which points to the right as anticipated above.

2.1.2 Direct Calculation of the Electromagnetic Momentum

A direct evaluation of the electromagnetic momentum (1) is difficult in that a calculation of the magnetic field of a current loop leads to elliptic integrals. However, analytic calculations become tractable in a two-dimensional approximation of a current loop by a current cylinder [6].

We now suppose that R is the resistance per unit length along the cylinder, so that I is the current per unit length that flows around its circumference due to the battery of voltage V that lies along the line $\phi = -\pi$. The magnetic field $\mathbf{E}$ is that of an infinite solenoid,

$$\mathbf{B} = \begin{cases} -\frac{4\pi I}{c}\hat{\mathbf{z}} & (r < a), \\ 0 & (r > a). \end{cases} \tag{6}$$

The electric field $\mathbf{E}$ can be deduced from a calculation of the potential $\Phi(r, \phi, z)$. This potential now has the value $V\phi/2\pi$ for all z on the cylinder $r = a$. As this is an odd function of the angle ϕ, the potential can be represented by the expansion

$$\Phi(r, \phi, z) = \begin{cases} \sum_{n=1}^{\infty} A_n \left(\frac{r}{a}\right)^n \sin n\phi & (r < a), \\ \sum_{n=1}^{\infty} A_n \left(\frac{a}{r}\right)^n \sin n\phi & (r > a). \end{cases} \tag{7}$$

The Fourier coefficients A_n are given by

$$A_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{V\phi}{2\pi} \sin n\phi \; d\phi = -(-1)^n \frac{V}{n\pi}. \tag{8}$$

The radial component of the electric field is

$$\begin{aligned} E_r = -\frac{\partial \Phi}{\partial r} &= \frac{V}{\pi a} \begin{cases} -\sum_{n=1}^{\infty} \left(\frac{-r}{a}\right)^{n-1} \sin n\phi & (r < a), \\ \sum_{n=1}^{\infty} \left(\frac{-a}{r}\right)^{n+1} \sin n\phi & (r > a), \end{cases} \\ &= \frac{aV}{\pi} \begin{cases} -\frac{\sin\phi}{r^2 + 2ar\cos\phi + a^2} & (r < a), \\ \frac{\sin\phi}{r^2 + 2ar\cos\phi + a^2} & (r > a). \end{cases} \end{aligned} \tag{9}$$

using Dwight 417.4. Similarly, the azimuthal electric field is

$$\begin{aligned} E_\phi &= -\frac{1}{r}\frac{\partial \Phi}{\partial \phi} = -\frac{V}{\pi a} \begin{cases} \sum_{n=1}^{\infty} \left(\frac{-r}{a}\right)^{n-1} \cos n\phi & (r < a), \\ \sum_{n=1}^{\infty} \left(\frac{-a}{r}\right)^{n+1} \cos n\phi & (r > a), \end{cases} \\ &= -\frac{V}{\pi} \begin{cases} \frac{r + a\cos\phi}{r^2 + 2ar\cos\phi + a^2} & (r < a), \\ \frac{a}{r}\frac{a + r\cos\phi}{r^2 + 2ar\cos\phi + a^2} & (r < a), \end{cases} \end{aligned} \tag{10}$$

using Dwight 417.3.

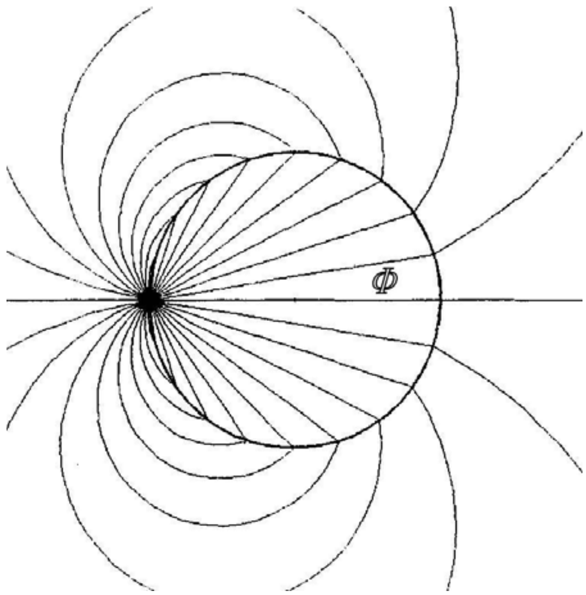
We digress briefly to discuss the equipotentials and field lines. For this we note that

$$\int \frac{a\sin\phi}{r^2 + 2ar\cos\phi + a^2}\, dr = \tan^{-1}\frac{r\sin\phi}{a + r\cos\phi} = -\tan^{-1}\frac{a\sin\phi}{r + a\cos\phi}\,, \tag{11}$$

so that the potential can be obtained by integrating eq. (9). Hence,

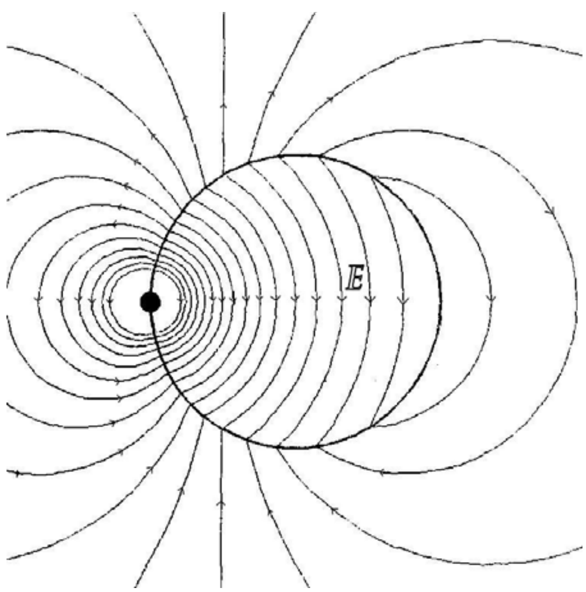
$$\Phi(r,\phi,z) = \frac{V}{\pi}\begin{cases} \tan^{-1}\frac{r\sin\phi}{a+r\cos\phi} = \theta & (r<a), \\ \tan^{-1}\frac{a\sin\phi}{r+a\cos\phi} & (r>a), \end{cases} \tag{12}$$

which is continuous at $r = a$, where θ is the azimuthal angle with respect to the x-axis measured from the location of the battery. When $r = a$, angle θ equals $\phi/2$ so that eq. (12) agrees with eq. (4). The equipotentials are shown in the figure below, from [6].

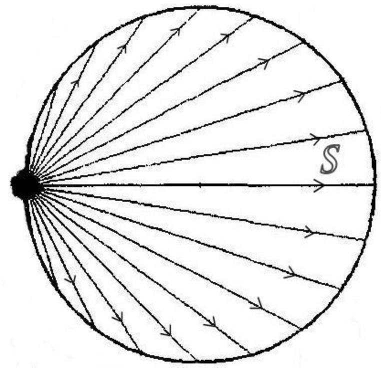


Inside the circuit the equipotentials are straight lines that emanate from the battery. Outside the circuit the equipotentials are circles that pass through the battery.

The corresponding electric field lines are shown below.



The Poynting vector, $\mathbf{S} = (c/4\pi)\mathbf{E}\times\mathbf{B}$, is nonzero only inside the circuit, where $\mathbf{S}$ is perpendicular to $\mathbf{E}$ and hence parallel to the equipotentials. That is, the equipotential lines inside the circuit on the upper figure on p. 4 also represent the flow of energy from the battery to the resistive circuit, as shown below as well.



A subtlety is that the electric field inside the conductor at $r = a$ is purely azimuthal, as needed to drive the current in the resistive medium. This is consistent with eq. (9) in that the average of the radial electric field at $r = a^-$ and $r = a^+$ is zero. In contrast, the azimuthal electric field (10) is continuous at $r = a$. Hence, the flow of energy into the resistor at $r = a$ is described by $\mathbf{S} = cE_\phi \hat{\boldsymbol{\phi}} \times \mathbf{B}/4\pi = VI\,\hat{\mathbf{r}}/2\pi a$, which is the power dissipated per unit length around the circumference of the resistive surface.

To evaluate the electromagnetic momentum (1), we first express the electric field for $r < a$ in rectangular coordinates,

$$\begin{aligned}\mathbf{E}(r < a) &= (E_r \cos\phi - E_\phi \sin\phi)\,\hat{\mathbf{x}} + (E_r \sin\phi + E_\phi \cos\phi)\,\hat{\mathbf{y}} \\ &= \frac{V}{\pi}\frac{r\sin\phi\,\hat{\mathbf{x}} - (a + r\cos\phi)\,\hat{\mathbf{y}}}{r^2 + 2ar\cos\phi + a^2} = E_x\,\hat{\mathbf{x}} + E_y\,\hat{\mathbf{y}}. \end{aligned} \tag{13}$$

We combine eqs. (1), (6) and (13) to calculate the electromagnetic momentum per unit length along z of the cylindrical circuit as

$$\begin{aligned}\mathbf{P}_{\rm EM} &= \int \frac{(E_x\,\hat{\mathbf{x}} + E_y\,\hat{\mathbf{y}}) \times B\hat{\mathbf{z}}}{4\pi c}\,d\mathrm{Vol} = \int \frac{E_y B\,\hat{\mathbf{x}} - E_x B\,\hat{\mathbf{y}}}{4\pi c}\,d\mathrm{Vol} \\ &= -\frac{VI}{\pi c^2}\int_0^a r\,dr \int_{-\pi}^{\pi} d\phi\,\frac{(a + r\cos\phi)\,\hat{\mathbf{x}} + r\sin\phi\,\hat{\mathbf{y}}}{r^2 + 2ar\cos\phi + a^2} = \frac{2VI}{ac^2}\int_0^a r\,dr\,\hat{\mathbf{x}} \\ &= \frac{aVI}{c^2}\,\hat{\mathbf{x}}, \end{aligned} \tag{14}$$

using Dwight 859.122. This is in agreement with the indirect calculation (5) of sec. 2.1.1.

2.1.3 The Canonical Electromagnetic Momentum

The electromagnetic momentum can also be calculated using the concept of the canonical momentum of a charge that interacts with a magnetic field [7], which notion dates back to Faraday [8] and Maxwell [9]. Namely, the electromagnetic part of the momentum associated with a charge distribution ϱ that is immersed in a vector potential $\mathbf{A}$ (in the Coulomb gauge, strictly speaking) is given by

$$\mathbf{P}_{\rm EM} = \int \frac{\varrho\mathbf{A}}{c}\,d\mathrm{Vol}. \tag{15}$$

For a resistive circuit to contain current I, there must be a longitudinal electric field inside the wire, and a nonzero surface charge density is needed to shape this electric field. In the case of a wire of radius $b \ll a$, the surface charge density λ per unit length along the wire is approximately[1]

$$\lambda \approx \frac{V\phi}{\ln(b/a)} \tag{16}$$

The vector potential at the surface of the wire is approximately

$$\mathbf{A} \approx \frac{I\ln(b/a)}{c}\hat{\mathbf{l}}. \tag{17}$$

The x component of the electromagnetic momentum (15) is then

$$P_{\mathrm{EM},x} = \int \frac{\lambda A_x}{c}\,dl \approx \int \frac{V\phi}{\ln(b/a)}\frac{I\ln(b/a)}{c}\sin\phi\frac{a\,d\phi}{c} \approx \frac{aVI}{c^2}\,, \tag{18}$$

in agreement with eq. (5).

For the example of sec. 2.1.2 of a cylindrical circuit, the surface charge distribution σ per unit length in z can be related to the radial component of the electric field (9),

$$\sigma = \frac{E_r(r = a^+) - E_r(r = a^-)}{4\pi} = \frac{V\sin\phi}{4\pi^2 a(1+\cos\phi)}\,. \tag{19}$$

The vector potential associated with the magnetic field (6) is purely azimuthal, and its value at radius $r = a$ follows from use of the relation $\mathbf{B} = \boldsymbol{\nabla}\times\mathbf{A}$ and Stokes' theorem,

$$A_\phi(r = a) = -\frac{Ba}{2} = -\frac{2\pi Ia}{c}\,. \tag{20}$$

The x component of the electromagnetic momentum per unit length along z is then,

$$\begin{aligned} P_{\mathrm{EM},x} &= \int_{-\pi}^{\pi}\frac{\sigma A_x}{c}\,a\,d\phi = -\int_{-\pi}^{\pi}\frac{\sigma A_\phi\sin\phi}{c}\,a\,d\phi = \frac{aVI}{2\pi c^2}\int_{-\pi}^{\pi}\frac{\sin^2\phi}{1+\cos\phi}\,d\phi \\ &= \frac{aVI}{2\pi c^2}\int_{-\pi}^{\pi}(1-\cos\phi)\,d\phi = \frac{aVI}{c^2}\,, \end{aligned} \tag{21}$$

as found previously by other methods.

2.2 "Hidden" Mechanical Momentum

In sec. 2.1 we showed by several methods that there is a nonzero electromagnetic momentum associated with a DC circuit that is at rest in the laboratory. If this momentum is related to the kind of momentum familiar in mechanical systems, then we expect the total momentum of a system at rest to be zero. Hence, consistency between field and mechanical momenta requires that the DC circuit at rest contain nonzero mechanical momentum that is equal and opposite to the electromagnetic momentum (5).

[1] Compare with an "exact" calculation of the surface charge on the inner conductor of a coaxial cable [1].

A DC circuit does contain moving charge carriers, but it would appear that the total momentum vector associated with this motion is zero for steady currents that flow in closed loops. The challenge, then, is to identify a kind of "hidden" mechanical momentum in the DC circuit.

The difficulty in reconciling Newton's third law of mechanics with the physics of moving electric charges was appreciated by Ampère, who concluded that isolated moving charges do not exist, and that all electric currents must be steady [10, 11]. It is generally considered that the introduction of the Poynting vector [2] provides the desired consistency between electromagnetism and mechanics for both time-dependent currents and moving, isolated charges. However, the present example shows that the details of such consistency are subtle.

Concerns of the sort raised by the present example can be traced at least as far back as 1952 to commentary by Cullwick [12, 13]. The physical character of "hidden" mechanical momentum (and that name) were first enunciated in 1967 by Shockley and James [14], and further clarified by Coleman and Van Vleck [15]. See also [4, 5, 16, 17].

An important clue is the factor of c^2 that appears in the denominator of eq. (5) which alerts us to the possibility of small relativistic corrections. As was discussed in an example of DC currents in a coaxial cable [1], energy flows from the battery to the resistive conductor. According to Einstein [18], a reduction ΔU in the energy of the battery is associated with a reduction $\Delta U/c^2$ in the mass of the battery, and a corresponding gain in mass of the resistor. Hence, the center of mass of the circuit moves in the $+x$ direction (in the rest frame of the circuit) as time increases. If the circuit is subject to no external forces in the lab frame, its center of mass must remain fixed. Therefore, the circuit must be moving as a whole (assuming that it does not deform) in the $-x$ direction with a very small velocity. The mechanical momentum associated with this small velocity is equal and opposite to the electromagnetic momentum (14) and (21). This mechanical momentum is not hidden in principle, but in practice it is very difficult to detect.[2]

3 References

[1] K.T. McDonald, *"Hidden" Momentum in a Coaxial Cable* (Mar. 28, 2002),
http://puhep1.princeton.edu/~mcdonald/examples/hidden.pdf

[2] J.H. Poynting, *On the Transfer of Energy in the Electromagnetic Field*, Phil. Trans. Roy. Soc. London **175**, 343 (1884),
http://puhep1.princeton.edu/~mcdonald/examples/EM/poynting_ptrsl_175_343_84.pdf

[3] M. Abraham, *Prinzipien der Dynamik des Elektrons*, Ann. Phys. **10**, 105 (1903),
http://puhep1.princeton.edu/~mcdonald/examples/EM/abraham_ap_10_105_03.pdf

[4] W.H. Furry, *Examples of Momentum Distributions in the Electromagnetic Field and in Matter*, Am. J. Phys. **37**, 621 (1969),
http://puhep1.princeton.edu/~mcdonald/examples/EM/furry_ajp_37_621_69.pdf

[2] Other examples involving "hidden" momentum are given in [19, 20, 21, 22].

[5] M.G. Calkin, *Linear Momentum of the Source of a Static Electromagnetic Field*, Am. J. Phys. **39**, 513 (1971),
http://puhep1.princeton.edu/~mcdonald/examples/EM/calkin_ajp_39_513_71.pdf

[6] M. Heald, *Electric fields and charges in elementary circuits*, Am. J. Phys. **52**, 522 (1984),
http://puhep1.princeton.edu/~mcdonald/examples/EM/heald_ajp_52_522_84.pdf

[7] L.D. Landau and E.M. Lifshitz, *The Classical Theory of Fields*, 4th ed. (Pergamon Press, Oxford, 1975), sec. 16.

[8] Part I, sec. IX of M. Faraday, *Experimental Researches in Electricity* (Dover Publications, New York, 2004; reprint of the 1839 edition).

[9] Secs. 22-24 and 57 of J.C. Maxwell, *A Dynamical Theory of the Electromagnetic Field*, Phil. Trans. Roy. Soc. London **155**, 459 (1865),
http://puhep1.princeton.edu/~mcdonald/examples/EM/maxwell_ptrsl_155_459_65.pdf

[10] A.M. Ampère, *Mémoire sur la détermination de la formule qui représente l'action mutuelle de deux portions infiniment petites de conducteurs voltaïques*, Ann. Chem. Phys. **20**, 398, 422 (1822).

[11] The most extensive discussion in English of Ampère's attitudes on the relation between magnetism and mechanics is given in R.A.R. Tricker, *Early Electrodynamics, the First Law of Circulation* (Pergamon, Oxford, 1965).
Another historical survey is by O. Darrigol, *Electrodynamics from Ampère to Einstein* (Oxford U.P., 2000).
See also sec. IIA of J.D. Jackson and L.B. Okun, *Historical roots of gauge invariance*, Rev. Mod. Phys. **73**, 663 (2001),
http://puhep1.princeton.edu/~mcdonald/examples/EM/jackson_rmp_73_663_01.pdf

[12] E.G. Cullwick, Nature **170**, 425 (1952),
http://puhep1.princeton.edu/~mcdonald/examples/EM/cullwick_nature_170_425_52.pdf

[13] E.G. Cullwick, *Electromagnetism and Relativity* (Longmans, Green & Co., London, 1959), pp. 232-238,
http://puhep1.princeton.edu/~mcdonald/examples/EM/cullwick_EM_relativity_59.pdf

[14] W. Shockley and R.P. James, *"Try Simplest Cases" Discovery of "Hidden Momentum" Forces on Magnetic Currents*, Phys. Rev. Lett. **18**, 876 (1967),
http://puhep1.princeton.edu/~mcdonald/examples/EM/shockley_prl_18_876_67.pdf

[15] S. Coleman and J.H. Van Vleck, *Origin of "Hidden Momentum" Forces on Magnets*, Phys. Rev. **171**, 1370 (1968),
http://puhep1.princeton.edu/~mcdonald/examples/EM/coleman_pr_171_1370_68.pdf

[16] L. Vaidman, *Torque and force on a magnetic dipole*, Am. J. Phys. **58**, 978 (1990),
http://puhep1.princeton.edu/~mcdonald/examples/EM/vaidman_ajp_58_978_90.pdf

[17] V. Hnizdo, *Hidden momentum and the electromagnetic mass of a charge and current carrying body*, Am. J. Phys. **65**, 55 (1997),
http://puhep1.princeton.edu/~mcdonald/examples/EM/hnizdo_ajp_65_55_97.pdf

[18] A. Einstein, *Ist die Trägheit eines Krpers von seinem Energieinhalt abhängig?*, Ann. Phys. **18**, 639 (1905),
http://www.physik.uni-augsburg.de/annalen/history/papers/1905_18_639-641.pdf
Translation: *Does the Inertia of a Body Depend upon its Energy-Content?*,
http://www.fourmilab.ch/etexts/einstein/E_mc2/www/

[19] K.T. McDonald, *Onoochin's Paradox* (Jan. 1, 2006),
http://puhep1.princeton.edu/~mcdonald/examples/onoochin.pdf

[20] K.T. McDonald, *Cullwick's Paradox: Charged Particle on the Axis of a Toroidal Magnet* (June 4, 2006), http://puhep1.princeton.edu/~mcdonald/examples/cullwick.pdf

[21] K.T. McDonald, *Energy, Momentum and Stress in a Belt Drive* (Oct. 20, 2007),
http://puhep1.princeton.edu/~mcdonald/examples/belt_drive.pdf

[22] K.T. McDonald, *"Hidden" Momentum in a Sound Wave* (Oct. 31, 2007),
http://puhep1.princeton.edu/~mcdonald/examples/hidden_sound.pdf