Hidden momentum and the electromagnetic mass of a charge and current carrying body

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The contribution of the electromagnetic self-field to the energy and momentum of a charge and/or current carrying body is considered within classical physics and discussed in simple terms using nontrivial exactly solvable examples. The Lorentz-transformation properties of the energy and momentum of the body can be maintained by taking into account the relativistic effects of the mechanical stresses due the sources of the self-field, or by separating the energy and momentum of a moving self-field from those of a nonelectromagnetic origin in a relativistically covariant way. In either procedure, a proper account must be taken of the hidden mechanical momentum that, in general, is contained in a stationary body that carries both charge and current, and which is equal and opposite to the momentum of the static electromagnetic self-field. Hidden momentum and the momentum of a static electromagnetic field are indispensable concepts of direct physical significance in classical electrodynamics.

I. INTRODUCTION

Great efforts to construct a purely electromagnetic theory of the electron were made by Abraham and Lorentz at the beginning of this century,[1] and the classical electron theory associated with their names was developed almost simultaneously with Einstein's theory of special relativity. As it happened, the electron theory of Lorentz gave exactly the same prediction as special relativity for the variation of electron mass with velocity.[2] However, in the classical electron theory, the problem of the electromagnetic mass was beset from the start with difficulties, which the special theory of relativity helped to put in sharper relief.

When an electron, or for that matter any charged body, moves, it carries along its Coulomb field. In the case of a uniform motion, we have to do with a system of the charged body and its field in a joint motion. This self-field is, of course, no longer an electrostatic Coulomb field when measured in a frame in which the charged body moves, as there is now also the magnetic field created by the motion of the charge and, in addition, the electric Coulomb field undergoes a relativistic distortion. A moving self-field contains a linear momentum as well as energy, and these must belong to the joint system of the charged body plus its self-field. In the general case of an accelerated motion, the retarded electromagnetic field produced by a charged particle, called the Liénard–Wiechert field,[3] is the sum of a part that depends only on the velocity of the particle, and a part that depends on the acceleration. The velocity field falls off as $1/r^2$ with the distance r from the particle and is exactly the self-field of a particle moving with a constant velocity, while the acceleration field is that of the electromagnetic radiation produced by the accelerated motion of the particle and varies as $1/r$. The radiation field is detached from the charged body as the radiation propagates away from it with the speed of light, whereas the velocity field remains attached to the body and participates in its motion. It is, obviously, the velocity field, or self-field, that is connected with the electromagnetic mass and momentum of the body, while the detached acceleration field is responsible for the energy and momentum emitted in electromagnetic waves by the charged body. The velocity self-field cannot be separated from the entity we call a charged particle, and so the speculation arose at the turn of the last century that the entire observed mass of the electron was due to its electromagnetic self-field.

The first difficulty the electromagnetic theory of the electron faced can be seen to follow from the simple fact that the energy U_0 of the electrostatic self-field of a charged particle, and hence also its electromagnetic rest mass $m=U_0/c^2$, are inversely proportional to the radius a of the particle. Thus when the limit $a\to 0$ is taken, the mass becomes infinitely large, and the appealing concept of the electron as a structureless elementary particle of no spatial extension is not possible, at least not in any straightforward fashion. So, it seemed necessary to endow the electron with a finite size and internal structure, and the electron was conceived of as an extended body with a radius of the order of "the classical electron radius" $a_c=e^2/4\pi\epsilon_0 m_e c^2=2.8$ fm,[4] where e and m_e are the observed charge and rest mass of the electron, respectively. But once the electron has a structure, the hope of a purely electromagnetic description must be abandoned, as the question of the stability of the extended, deformable charge distribution in the particle arises immediately, and forces of nonelectromagnetic origin must be introduced to hold the electron together against electrostatic repulsion of the charges within it. Moreover, it was eventually established experimentally that the electron must be a particle of much smaller size than the classical electron radius a_c, with the strange consequence that the purely electromagnetic mass of the electron is greater than its total, observed mass.

But it was the lack of relativistic covariance that, with the advent of special relativity, became a most serious problem for the classical electron theory. This problem, and its solutions, have a long history,[5] replete with debates and controversies that still keep flaring up now and then.[6] The problem arises immediately when the energy and momentum of the electromagnetic field of a uniformly moving charged body are calculated using the standard formulae of electrodynamics, and the calculated quantities are confronted with the requirements of special relativity. Modeling the electron as, in its rest frame, a spherical shell of radius a, uniformly charged on its surface,[7] one gets for the electromagnetic energy U and momentum $\mathbf{P}$

$$U=\frac{\epsilon_0}{2}\int(\mathbf{E}^2+c^2\mathbf{B}^2)d^3r=\gamma\frac{e^2}{8\pi\epsilon_0 a}\left(1+\frac{1}{3}\beta^2\right), \quad (1)$$

$$\mathbf{P}=\epsilon_0\int\mathbf{E}\times\mathbf{B}d^3r=\frac{4}{3}\gamma\frac{e^2}{8\pi\epsilon_0 ac^2}\mathbf{v}, \quad (2)$$

where $\gamma=(1-\beta^2)^{-1/2}$ and $\beta=|\mathbf{v}|/c$, with $\mathbf{v}$ the velocity of the electron. The fields $\mathbf{E}$ and $\mathbf{B}=\mathbf{v}\times\mathbf{E}/c^2$ are the exact relativistic fields of the uniformly moving electron, which is appropriately Lorentz-contracted. The integrals are most easily carried out by performing a Lorentz transformation of the coordinates and fields into the rest frame of the electron, where there is only a spherically symmetric electrostatic field.[8] When $\mathbf{v}=0$, $U=U_0=e^2/8\pi\epsilon_0 a$, and $\mathbf{P}=0$. According to the theory of relativity,

$$m_e=\frac{1}{c^2}U_0=\frac{e^2}{8\pi\epsilon_0 ac^2} \quad (3)$$

is then the rest mass, which must be a relativistic invariant. Equations (1) and (2) can now be written in terms of the electromagnetic rest mass m_e as

$$U=\gamma m_e c^2(1+\tfrac{1}{3}\beta^2), \quad (4)$$

$$\mathbf{P}=\tfrac{4}{3}\gamma m_e\mathbf{v}, \quad (5)$$

and it is obvious immediately that these quantities do not transform properly as an energy-momentum four-vector and that the relativistic relation $U^2=\mathbf{P}^2c^2+m_e^2c^4$ between the energy and momentum of a particle is violated. Leaving aside the factor $1+\frac{1}{3}\beta^2$ in the expression for the energy, differing from unity only in second order in β, the famous factor $\frac{4}{3}$ appearing in the formula for the electromagnetic momentum was most puzzling. It seemed to indicate that the attached self-field is not carried along by the electron in a relativistically covariant way, and that would be a very serious flaw in a theory based on relativistically covariant electrodynamics.

Facing the problem of the stability of the electron led very early on to a way out of the difficulty with the wrong relation between the electromagnetic energy and momentum of the electron. Poincaré[9] postulated cohesive forces of nonelectromagnetic origin to render the electron stable, and it can be shown that when the total forces, which result in a divergenceless total energy-momentum four-tensor $T_{\rm tot}^{\mu\nu}$ of the electron,

$$\frac{\partial T_{\rm tot}^{\mu\nu}}{\partial x^\mu}=0, \quad (6)$$

are considered, the total energy-momentum $P_{\rm tot}^\mu=(U_{\rm tot}/c,\mathbf{P}_{\rm tot})$ of the electron, defined as

$$P_{\rm tot}^\mu=\frac{1}{c}\int T_{\rm tot}^{\mu 0}d^3r, \quad (7)$$

is a relativistically covariant four-vector, as it must be for a closed system.[10] Accordingly, the experimentally observed mass of the electron then consists of two parts, an electromagnetic part and a nonelectromagnetic, "mechanical" contribution. However, the two contributions cannot be split and dealt with separately in a relativistically covariant way, as only the resulting total has the correct invariance properties. Here, one encounters a first example of mass renormalization, albeit of a noncovariant kind. Poincaré's solution is a satisfactory one as far as the theory of the electron, or the mass of any extended charged body is concerned, but it should be possible to formulate the purely electrodynamic aspects of a theory in a covariant way.

In the late 1940s, major progress occurred in *quantum* electrodynamics (QED). By casting the perturbation scheme of QED in a relativistically covariant form, it became possible to eliminate the infinities of QED by absorbing them into the experimentally observed mass and charge of the electron. This procedure is called renormalization, and when relativistic covariance is required, it can be done in a well-defined and unambiguous way. And what is most important, the predictions of renormalized QED agree incredibly well with experiment.[11] A comparison with classical electrodynamics then seemed to imply a peculiar situation. On the one hand, it was possible in QED to reduce the problem of the self-energy and stability of the electron to accepting simply the electron mass as a parameter to be determined by experiment and given to the theory from outside, but, on the other hand, it did not seem possible to carry out such a procedure covariantly in the classical theory. The solution to this problem was found in 1960 by Rohrlich,[12] and it was then realized soon that the same solution had appeared in the literature more than once before, to be unnoticed or forgotten.[13]

The fault with the definitions that are used in Eqs. (1) and (2) for the energy and momentum of the self-field of a moving charge is simply that they are not relativistically covariant—they can be written as

$$U=\int T^{00}d^3r, \quad P_i=\frac{1}{c}\int T^{i0}d^3r, \quad (8)$$

where $T^{\mu\nu}$ is the energy-momentum four-tensor of the electromagnetic self-field,[14] which is not divergenceless. The integrations in Eqs. (1) and (2) extend over field points that are simultaneously at rest with respect to an observer who sees the charge in motion, whereas when the rest self-field energy U_0 is calculated, it refers to contributions from field points that are simultaneous in the rest frame. The self-field of the charge is an entity that moves together with the charge, and to calculate the energy of a moving self-field, one must refer to field points that move along with the particle. Seen from such a perspective,[15] the definitions (8) are correct only when they refer to the energy and momentum of a free, or detached, electromagnetic field, and for the self-field of a moving charge, they have to be modified. A manifestly covariant definition of the energy-momentum four-vector P^μ of an electromagnetic self-field is given by[16]

$$P^\mu=(U/c,\mathbf{P})=\frac{1}{c^2}\int v_\nu T^{\mu\nu}\gamma d^3r, \quad (9)$$

where $v^\nu=(\gamma c,\gamma\mathbf{v})$ is the four-vector of the velocity with which the charged particle, or whatever is the source of the self-field, is moving with respect to a given frame. Equation (9) defines a four-vector because the contraction $v_\nu T^{\mu\nu}$ is obviously a four-vector and the proper volume element γd^3r is an invariant. The time and space components of P^μ, written explicitly in terms of the components of $T^{\mu\nu}$ and v^ν, give

$$U=\epsilon_0\gamma^2\int\left[\frac{1}{2}(\mathbf{E}^2+c^2\mathbf{B}^2)-\mathbf{v}\cdot(\mathbf{E}\times\mathbf{B})\right]d^3r, \quad (10)$$

$$\mathbf{P}=\epsilon_0\gamma^2\int[\mathbf{E}\times\mathbf{B}+(\mathbf{v}\cdot\mathbf{E})\mathbf{E}/c^2+(\mathbf{v}\cdot\mathbf{B})\mathbf{B}$$
$$-\tfrac{1}{2}(\mathbf{E}^2+c^2\mathbf{B}^2)\mathbf{v}/c^2]d^3r. \quad (11)$$

In the rest frame of the charged particle, $\mathbf{v}=0$ and $\gamma=1$, and Eqs. (10) and (11) reduce to the usual definitions of electromagnetic energy and momentum used in Eq. (8). The covariant definitions (10) and (11) are thus guaranteed to replace Eqs. (4) and (5) by the relativistically correct values $U=\gamma m_e c^2$ and $\mathbf{P}=\gamma m_e\mathbf{v}$ for the electromagnetic energy and momentum of a moving electron. Both procedures, the traditional one that takes explicit account of the mechanical stresses in the body and the use of covariant definitions of the electromagnetic energy and momentum, are valid solutions to the problem of the electromagnetic mass of a macroscopic body in classical physics insofar as they both result in the body's total energy, momentum and rest mass having the correct Lorentz transformation properties;[17] however, these solutions entail different physical perspectives, with the latter procedure being a classical analog of the mass renormalization in quantum electrodynamics.

As outlined above, the problem of the electromagnetic mass of a moving classical body arose in the context of the classical electron theory. It is for this reason that only the problem of the electromagnetic mass of purely electrostatic systems, i.e., systems that do not generate any magnetic field in their rest frames, has been given close attention in the literature so far. Extending the attention to systems that have magnetic fields in their rest frames seems to complicate the picture, however. In particular, a stationary body that carries a stationary electric current distribution, as well as a stationary charge distribution, generates static magnetic and electric self-fields that, in general, contribute a nonzero electromagnetic momentum, apart from an electromagnetic energy, to the *stationary* body. This apparently paradoxical result appears to reinforce the often expressed reluctance to ascribe physical significance, or "reality," to a nonzero momentum of a static electromagnetic field.[18] However, a stationary configuration of mass, charge and current may contain a nonzero mechanical momentum, called hidden momentum, which is not associated with the motion of the center of mass of the system.[19] Hidden momentum arises as a relativistic effect from the motion of the current carriers, be they charged particles or a charged incompressible fluid, in the body that supports the currents, and it is equal and opposite to the momentum of the static electromagnetic field in the system.[20] Thus the contribution of the electromagnetic momentum to the momentum of the stationary body is compensated by the hidden mechanical momentum. Indeed, it can be shown easily on most general grounds that the total linear momentum of any finite stationary distribution of matter must vanish.[21] Therefore, already on the strength of this general requirement, one has to ascribe as much of physical reality to the momentum of a static electromagnetic field as one is prepared to give to the hidden mechanical momentum of a stationary body. The existence of hidden momentum is, however, an inescapable consequence of relativistic mechanics. Moreover, the presence of hidden momentum in a current-carrying body has to be taken into account when evaluating the force on the body due to an external electromagnetic field,[22] and such effects are,[23] in principle, testable experimentally.

In this paper, a classical body that may carry electric currents as well as charge is considered. Paying due attention to hidden mechanical momentum, the contribution of the body's electromagnetic self-field to its energy and momentum is evaluated both in the traditional approach, which takes explicitly into account the relativistic effects of the mechanical stresses due to the sources of the self-field, and in the covariant formalism of the electromagnetic energy momentum. In Sec. II, traditional-approach calculations are carried out of the electromagnetic energy and momentum of a moving configuration of charge and/or current distributions, and of the stresses produced by the sources of the electromagnetic self-field, to show how the Lorentz-transformation properties of the total energy and momentum of the body are maintained. In Sec. III, the covariant definition of the electromagnetic energy momentum is applied to the same problem, thus separating the energy and momentum of the moving self-field of the body from the corresponding nonelectromagnetic quantities in a relativistically covariant way. Conclusions are drawn in Sec. IV.

II. ENERGY, MOMENTUM, AND STRESSES IN A SYSTEM OF CHARGE AND CURRENT

As a nontrivial exactly solvable example, we use the system of two concentric spherical shells: one of radius a with a surface charge distribution such that its electric field $\mathbf{E}$ is uniform at distances from the center $r<a$ and equal to the field of an electric dipole $\mathbf{p}$ for $r>a$, and the other of radius b with a surface current distribution such that its magnetic field $\mathbf{B}$ is uniform for $r<b$ and equal to the field of a magnetic dipole $\mathbf{m}$ for $r>b$.[24] The fields are thus

$$\mathbf{E}(\mathbf{r})=\begin{cases}-\dfrac{\mathbf{p}}{4\pi\epsilon_0 a^3}, & r<a\\ \dfrac{1}{4\pi\epsilon_0 r^3}[3(\mathbf{p}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}}-\mathbf{p}], & r>a\end{cases}, \quad (12)$$

$$\mathbf{B}(\mathbf{r})=\begin{cases}\dfrac{\mu_0}{4\pi b^3}2\mathbf{m}, & r<b\\ \dfrac{\mu_0}{4\pi r^3}[3(\mathbf{m}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}}-\mathbf{m}], & r>b\end{cases}. \quad (13)$$

The charge and current distributions ρ and $\mathbf{J}$, respectively, that give rise to these fields are given explicitly by

$$\rho(\mathbf{r})=\frac{3\,\delta(r-a)}{4\pi a^3}\,\mathbf{p}\cdot\hat{\mathbf{r}}, \quad (14)$$

$$\mathbf{J}(\mathbf{r})=\frac{3\,\delta(r-b)}{4\pi b^3}\,\mathbf{m}\times\hat{\mathbf{r}}. \quad (15)$$

All these specifications refer, of course, to the rest frame, i.e., the frame where the two shells are at rest.

A. Rest-frame quantities

Let us first calculate the energy and momentum of the electromagnetic field of this system in the rest frame. The rest-frame electromagnetic energy U_0, calculated as in Eq. (1), is given by the sum of electrostatic and magnetostatic energies. Choosing the direction of $\mathbf{p}$ as the polar axis, one gets for the electrostatic energy $U_0^{(e)}$

$$U_0^{(e)}=\frac{\epsilon_0}{2}\int \mathbf{E}^2d^3r=\frac{\mathbf{p}^2}{32\pi^2\epsilon_0}\left[4\pi\int_0^a\frac{1}{a^6}r^2dr\right.$$
$$\left.+2\pi\int_a^\infty\frac{1}{r^6}r^2dr\int_0^\pi(3\cos^2\theta+1)\sin\theta\, d\theta\right]$$
$$=\frac{\mathbf{p}^2}{8\pi\epsilon_0a^3}. \quad (16)$$

A similar calculation of the magnetostatic energy $U_0^{(m)}$ results in the value

$$U_0^{(m)}=\frac{1}{2\mu_0}\int \mathbf{B}^2d^3r=\frac{\mu_0\mathbf{m}^2}{4\pi b^3}, \quad (17)$$

and the total rest-frame electromagnetic energy is thus

$$U_0=U_0^{(e)}+U_0^{(m)}=\frac{\mu_0}{8\pi}\left(\frac{\mathbf{p}^2c^2}{a^3}+\frac{2\mathbf{m}^2}{b^3}\right). \quad (18)$$

The rest-frame electromagnetic momentum $\mathbf{P}_0$ can be calculated directly, using Eq. (2) with the fields $\mathbf{E}$ and $\mathbf{B}$ as given by Eqs. (12) and (13), but a more economical route is to use here the formula

$$\mathbf{P}_0=\epsilon_0\int \mathbf{E}\times\mathbf{B}d^3r=\frac{1}{c^2}\int \Phi\mathbf{J}d^3r, \quad (19)$$

which holds for an electrostatic potential Φ and current density $\mathbf{J}$ such that $\mathbf{E}=-\nabla\Phi$ and $\nabla\times\mathbf{B}=\mu_0\mathbf{J}$.[25] Using the electrostatic potential that gives the electric field of Eq. (12),

$$\Phi(\mathbf{r})=\frac{\mathbf{p}\cdot\mathbf{r}}{4\pi\epsilon_0g(r,a)}, \quad (20)$$

where $g(r,a)=a^3$ for $r\leqslant a$ and $g(r,a)=r^3$ for $r>a$, together with the current density $\mathbf{J}$ of Eq. (15), the rest-frame electromagnetic momentum is evaluated as

$$\mathbf{P}_0=\frac{1}{c^2}\int \Phi\mathbf{J}d^3r=\frac{3\mu_0}{16\pi^2b^3}\,\mathbf{m}\times\int\frac{\delta(r-b)}{g(r,a)}(\mathbf{p}\cdot\mathbf{r})\hat{\mathbf{r}}d^3r$$
$$=\frac{\mu_0}{4\pi d^3}\,\mathbf{m}\times\mathbf{p}, \quad (21)$$

where $d=\max(a,b)$. The momentum of the static field $(\mathbf{E},\mathbf{B})$ is thus nonzero, as long as the dipole moments $\mathbf{p}$ and $\mathbf{m}$ are not parallel. However, as discussed in Sec. I, in the shell that supports the current density $\mathbf{J}$ there must be hidden a mechanical momentum $\mathbf{P}_h$ that compensates the field momentum $\mathbf{P}_0$,

$$\mathbf{P}_h=-\mathbf{P}_0, \quad (22)$$

which guarantees that the total momentum $\mathbf{P}_0+\mathbf{P}_h$ of the stationary system vanishes.

The charges and currents on the surfaces of the shells experience forces due to the electric and magnetic fields in the system, resulting in mechanical stresses. These stresses, or forces acting per unit area, can be calculated using Maxwell's stress tensor

$$T_{ij}^{(M)}=\epsilon_0\left(E_iE_j-\frac{1}{2}\,\delta_{ij}\mathbf{E}^2\right)+\frac{1}{\mu_0}\left(B_iB_j-\frac{1}{2}\,\delta_{ij}\mathbf{B}^2\right). \quad (23)$$

The Cartesian components F_i of the force per unit area on a surface, specified by an outward unit normal with components n_j, are given by the contraction

$$F_i=T_{ij}^{(M)}n_j, \quad (24)$$

with only the electric-field (or the magnetic-field) terms in the stress tensor $T_{ij}^{(M)}$ when there are only charges (or currents) on the surface. To calculate the force acting per unit surface area of the charge-carrying shell, one must add the expressions (24) with Maxwell's tensors for the electric fields inside and just outside the shell, which are associated with opposite unit normals.[26] With the electric field (12) and using the direction of the moment $\mathbf{p}$ as the polar axis, one obtains, after straightforward algebra, the following values for the components of this force:

$$F_1^{(e)}=\frac{9U_0^{(e)}}{4\pi a^3}\cos^2\theta\sin\theta\cos\phi,$$
$$F_2^{(e)}=\frac{9U_0^{(e)}}{4\pi a^3}\cos^2\theta\sin\theta\sin\phi, \quad (25)$$
$$F_3^{(e)}=\frac{9U_0^{(e)}}{4\pi a^3}\left(\cos^2\theta-\frac{2}{3}\right)\cos\theta,$$

where $U_0^{(e)}$ is the electrostatic energy of Eq. (16). The volume-averaged tensor of mechanical stresses in the shell due to the electric self-field, $\bar{\sigma}_{ij}^{(e)}$, can be expressed as a surface integral evaluated over the surface S_e of the shell,[27]

$$\bar{\sigma}_{ij}^{(e)}=\frac{1}{V_e}\int_{V_e}\sigma_{ij}^{(e)}d^3r=\frac{1}{V_e}\oint_{S_e}F_i^{(e)}x_jda, \quad (26)$$

where V_e is the volume occupied by the mass of the shell. Evaluating the surface integral (26) with the force (25), only the diagonal terms of $\bar{\sigma}_{ij}^{(e)}$ survive, with the values

$$\bar{\sigma}_{11}^{(e)}=\bar{\sigma}_{22}^{(e)}=\frac{3}{5}\frac{U_0^{(e)}}{V_e},\quad \bar{\sigma}_{33}^{(e)}=-\frac{1}{5}\frac{U_0^{(e)}}{V_e}. \quad (27)$$

The calculation of the force acting per unit surface area of the current-carrying shell proceeds in the same way as that for the charge-carrying shell, using now the magnetic field (13). The result is a force per unit area with the components

$$F_1^{(m)}=\frac{9U_0^{(m)}}{8\pi b^3}(\cos^2\theta+\tfrac{1}{3})\sin\theta\cos\phi,$$
$$F_2^{(m)}=\frac{9U_0^{(m)}}{8\pi b^3}(\cos^2\theta+\tfrac{1}{3})\sin\theta\sin\phi, \quad (28)$$
$$F_3^{(m)}=-\frac{9U_0^{(m)}}{8\pi b^3}\sin^2\theta\cos\theta,$$

where the direction of the moment $\mathbf{m}$ is now used as the polar axis and $U_0^{(m)}$ is the magnetostatic energy (17). The tensor of the volume-averaged stresses in the current-carrying shell, $\bar{\sigma}_{ij}^{(m)}$, is then calculated according to a formula similar to that of Eq. (26) using the force (28), with the results

$$\bar{\sigma}_{11}^{(m)}=\bar{\sigma}_{22}^{(m)}=\frac{4}{5}\frac{U_0^{(m)}}{V_m},\quad \bar{\sigma}_{33}^{(m)}=-\frac{3}{5}\frac{U_0^{(m)}}{V_m}, \quad (29)$$

where V_m is the volume occupied by the mass of the current-carrying shell; the nondiagonal elements of $\bar{\sigma}_{ij}^{(m)}$ are zero.

B. Moving charge-carrying shell

Before considering the system of the two shells in uniform motion, let us calculate the electromagnetic energy and momentum of a simpler system of a moving single shell, in its rest frame spherical and charged as specified by Eq. (14), producing the electric field (12). To be specific, let us assume that the electric dipole moment **p** of the shell is directed along the z axis and that the shell is moving with a constant velocity **v** along the x axis.

The electromagnetic energy $U_{\rm elm}^{(e)}$ of the moving charge-carrying shell is to be calculated as in Eq. (1), and the simplest way of evaluating the integral is to Lorentz transform the integration variables and the fields to the rest frame of the shell. The Lorentz transformations of the electric field **E** and magnetic field **B** are given by[28]

$$E_1=E_1', \quad E_2=\gamma(E_2'+vB_3'), \quad E_3=\gamma(E_3'-vB_2') \tag{30}$$

$$B_1=B_1', \quad B_2=\gamma(B_2'-vE_3'/c^2), \quad B_3=\gamma(B_3'+vE_2'/c^2), \tag{31}$$

where the quantities associated with the rest frame are now denoted by a prime. With the rest-frame fields $\mathbf{E}'$ equal to **E** of Eq. (12) and $\mathbf{B}'=0$, the electromagnetic energy $U_{\rm elm}^{(e)}$ is thus evaluated as

$$\begin{aligned} U_{\rm elm}^{(e)}&=\frac{\epsilon_0}{2}\int(\mathbf{E}^2+c^2\mathbf{B}^2)d^3r \\ &=\frac{\epsilon_0}{2}\int[E_1'^2+\gamma^2(E_2'^2+E_3'^2) \\ &\quad+\gamma^2\beta^2(E_2'^2+E_3'^2)]\,\frac{d^3r'}{\gamma} \\ &=\frac{\epsilon_0}{2}\,\gamma\int[(1+\beta^2)\mathbf{E}'^2-2\beta^2E_1'^2]d^3r' \\ &=\gamma(1+\tfrac{3}{5}\beta^2)U_0^{(e)}, \end{aligned} \tag{32}$$

using Eq. (16), and evaluating the integral over $(\epsilon_0/2)E_1'^2$ as $\frac{1}{5}U_0^{(e)}$. The electromagnetic field momentum $\mathbf{P}_{\rm elm}^{(e)}$ is to be calculated as in Eq. (2). It is, obviously, directed along the velocity **v**, and again, as in Eq. (32), it can be evaluated most easily by Lorentz-transforming into the rest frame of the shell:

$$\begin{aligned} \mathbf{P}_{\rm elm}^{(e)}&=\epsilon_0\int\mathbf{E}\times\mathbf{B}d^3r \\ &=\epsilon_0\int(E_2B_3-E_3B_2)d^3r\,\frac{\mathbf{v}}{v} \\ &=\frac{\epsilon_0\gamma}{c^2}\int(\mathbf{E}'^2-E_1'^2)d^3r'\mathbf{v} \\ &=\frac{8}{5}\,\gamma\,\frac{1}{c^2}\,U_0^{(e)}\mathbf{v}. \end{aligned} \tag{33}$$

The electromagnetic energy (32) and momentum (33) of the moving shell thus differ from the values $\gamma U_0^{(e)}$ and $\gamma(U_0^{(e)}/c^2)\mathbf{v}$, respectively, that would be expected on the grounds of relativistic invariance. However, there are nonelectromagnetic, mechanical, contributions to the observed total energy and momentum of the shell, and these must be taken properly into account in this approach. The important point here is that, according to special relativity, the mechanical energy and momentum of a stressed body transform differently from those of an unstressed body. The energy density, momentum density and the stress tensor σ_{ij} of a macroscopic body form together a four-tensor,[29] and that has immediate consequences for the transformation properties of the energy and momentum of a stressed body. In the rest frame, the mechanical energy-momentum four-tensor $\bar{T}_e'^{\mu\nu}$ of the charge-carrying shell, averaged over its volume,[30] is given by

$$\bar{T}_e'^{\mu\nu}=\begin{pmatrix} \bar{\rho}_0^{(e)}c^2 & 0 & 0 & 0 \\ 0 & -\bar{\sigma}_{11}^{(e)} & 0 & 0 \\ 0 & 0 & -\bar{\sigma}_{22}^{(e)} & 0 \\ 0 & 0 & 0 & -\bar{\sigma}_{33}^{(e)} \end{pmatrix}, \tag{34}$$

where $\bar{\sigma}_{ii}^{(e)}$ are the stress-tensor elements (27) and $\bar{\rho}_0^{(e)}c^2$ is the volume-averaged rest-frame energy density, with $\bar{\rho}_0^{(e)}$ the volume-averaged rest-mass density. In general, the elements A^{00} and A^{0i} of a symmetric four-tensor $A^{\mu\nu}$ in a frame in which the rest frame moves with a velocity **v** along the x axis are given in terms of the tensor $A'^{\mu\nu}$ in the rest frame as follows:[31]

$$\begin{aligned} A^{00}&=\gamma^2(A'^{00}+2\beta A'^{01}+\beta^2A'^{11}), \\ A^{01}&=\gamma^2[(1+\beta^2)A'^{01}+\beta A'^{00}+\beta A'^{11}], \\ A^{02}&=\gamma(A'^{02}+\beta A'^{12}), \\ A^{03}&=\gamma(A'^{03}+\beta A'^{13}). \end{aligned} \tag{35}$$

Using the transformation equation for A^{00} with the elements (34) of the rest-frame tensor $\bar{T}_e'^{\mu\nu}$, the mechanical energy of the moving charge-carrying shell, $U_{\rm mec}^{(e)}$, is then

$$\begin{aligned} U_{\rm mec}^{(e)}&=\bar{T}_e^{00}\,\frac{V_e}{\gamma} \\ &=\gamma^2(\bar{\rho}_0^{(e)}c^2-\beta^2\bar{\sigma}_{11}^{(e)})\,\frac{V_e}{\gamma} \\ &=\gamma m_0^{(e)}c^2-\tfrac{3}{5}\gamma\beta^2U_0^{(e)}, \end{aligned} \tag{36}$$

where $m_0^{(e)}=\bar{\rho}_0^{(e)}V_e$ is the mechanical rest mass of the shell, and where Eq. (27) is used for the stress-tensor element $\bar{\sigma}_{11}^{(e)}$. The mechanical momentum of the charge-carrying shell, $\mathbf{P}_{\rm mec}^{(e)}$, is calculated similarly, using the transformation equation (35) with the rest-frame energy-momentum tensor (34):

$$\begin{aligned} \mathbf{P}_{\rm mec}^{(e)}&=\bar{T}_e^{01}\,\frac{V_e}{\gamma c}\,\frac{\mathbf{v}}{v} \\ &=\gamma^2\beta(\bar{\rho}_0^{(e)}c^2-\bar{\sigma}_{11}^{(e)})\,\frac{V_e}{\gamma c}\,\frac{\mathbf{v}}{v} \\ &=\gamma m_0^{(e)}\mathbf{v}-\frac{3}{5}\,\gamma\,\frac{1}{c^2}\,U_0^{(e)}\mathbf{v}. \end{aligned} \tag{37}$$

The total energy $U_{\rm tot}^{(e)}$ and momentum $\mathbf{P}_{\rm tot}^{(e)}$ of the moving charge-carrying shell are then, using Eqs. (32) and (33) and Eqs. (36) and (37),

$$U_{\rm tot}^{(e)}=U_{\rm elm}^{(e)}+U_{\rm mec}^{(e)}=\gamma m_{0\rm tot}^{(e)}c^2, \tag{38}$$

$$\mathbf{P}_{\rm tot}^{(e)}=\mathbf{P}_{\rm elm}^{(e)}+\mathbf{P}_{\rm mec}^{(e)}=\gamma m_{0\rm tot}^{(e)}\mathbf{v}, \tag{39}$$

where

$$m_{0\rm tot}^{(e)}=U_0^{(e)}/c^2+m_0^{(e)} \tag{40}$$

is the total rest mass of the charge-carrying shell. The total energy and momentum of the shell thus satisfy the relativistic relation $U_{\rm tot}^{(e)2}=\mathbf{P}_{\rm tot}^{(e)2}c^2+m_{\rm tot}^{(e)2}c^4$.

The motion along the x axis, i.e., in a direction perpendicular to the electric dipole moment $\mathbf{p}$ of the shell was considered above. To confirm that our results concerning the relativistic properties of the total energy and momentum of the shell have a more general validity, let us outline the calculations for the charge-carrying shell moving along the z axis, i.e., along the direction of the moment $\mathbf{p}$. It can be seen easily from Eq. (32) that the electromagnetic energy $U_{\rm elm}^{(e)}$ of the shell is then given by

$$U_{\rm elm}^{(e)}=\frac{\epsilon_0}{2}\gamma\int[(1+\beta^2)\mathbf{E}'^2-2\beta^2E_3'^2]d^3r'$$
$$=\gamma(1-\tfrac{1}{5}\beta^2)U_0^{(e)}, \tag{41}$$

after evaluating the integral over $(\epsilon_0/2)E_3'^2$ as $\frac{3}{5}U_0^{(e)}$. Similarly, Eq. (33) for the electromagnetic momentum $\mathbf{P}_{\rm elm}^{(e)}$ modifies now to

$$\mathbf{P}_{\rm elm}^{(e)}=\frac{\epsilon_0\gamma}{c^2}\int(\mathbf{E}'^2-E_3'^2)d^3r'\mathbf{v}=\frac{4}{5}\gamma\frac{1}{c^2}U_0^{(e)}\mathbf{v}. \tag{42}$$

The mechanical energy $U_{\rm mec}^{(e)}$ and momentum $\mathbf{P}_{\rm mec}^{(e)}$ of the charge-carrying shell moving along the z axis are obtained by modifying appropriately Eqs. (36) and (37):

$$U_{\rm mec}^{(e)}=\gamma^2(\bar\rho_0^{(e)}c^2-\beta^2\bar\sigma_{33}^{(e)})\frac{V_e}{\gamma}=\gamma m_0^{(e)}c^2+\frac{1}{5}\gamma\beta^2U_0^{(e)}, \tag{43}$$

$$\mathbf{P}_{\rm mec}^{(e)}=\gamma^2\beta(\bar\rho_0^{(e)}c^2-\bar\sigma_{33}^{(e)})\frac{V_e}{\gamma c}\frac{\mathbf{v}}{v}$$
$$=\gamma m_0^{(e)}\mathbf{v}+\frac{1}{5}\gamma\frac{1}{c^2}U_0^{(e)}\mathbf{v}. \tag{44}$$

Equations (41)–(44) show that the total energy $U_{\rm tot}^{(e)}$ and momentum $\mathbf{P}_{\rm tot}^{(e)}$ of the shell satisfy Eqs. (38) and (39), complying thus again with the relativistic relation between energy and momentum. It is now obvious that, under a general transformation from one inertial frame to another, the total energy and momentum of the charge-carrying shell will behave properly as an energy-momentum four-vector.

C. Moving current-carrying shell

We now consider a single shell, in its rest frame spherical and carrying currents (15), so that its magnetic field is given by Eq. (13). The magnetic dipole moment $\mathbf{m}$ of the shell is assumed to be directed along the z axis, while its velocity $\mathbf{v}$ is, say, along the x axis.

The calculations of the electromagnetic and mechanical energies and momenta of the current-carrying shell are the same in form as those for the charge-carrying shell in the preceding subsection. In analogy to Eqs. (32) and (33), the electromagnetic energy $U_{\rm elm}^{(m)}$ and momentum $\mathbf{P}_{\rm elm}^{(m)}$ of the current-carrying shell are calculated as

$$U_{\rm elm}^{(m)}=\frac{1}{2\mu_0}\gamma\int[(1+\beta^2)\mathbf{B}'^2-2\beta^2B_1'^2]d^3r'$$
$$=\gamma(1+\tfrac{4}{5}\beta^2)U_0^{(m)}, \tag{45}$$

$$\mathbf{P}_{\rm elm}^{(m)}=\frac{\gamma}{\mu_0c^2}\int(\mathbf{B}'^2-B_1'^2)d^3r'\mathbf{v}=\frac{9}{5}\gamma\frac{1}{c^2}U_0^{(m)}\mathbf{v}, \tag{46}$$

where $U_0^{(m)}$ is the magnetostatic energy (17), and the integral over $(1/2\mu_0)B_1'^2$ is evaluated as $\frac{1}{10}U_0^{(m)}$. The mechanical energy and momentum of the moving current-carrying shell, $U_{\rm mec}^{(m)}$ and $\mathbf{P}_{\rm mec}^{(m)}$, respectively, are calculated by transforming the rest-frame energy-momentum tensor of the shell, $\bar T_m'^{\mu\nu}$ [cf. Eq. (34)],

$$\bar T_m'^{\mu\nu}=\begin{pmatrix}\bar\rho_0^{(m)}c^2 & 0 & 0 & 0\\ 0 & -\bar\sigma_{11}^{(m)} & 0 & 0\\ 0 & 0 & -\bar\sigma_{22}^{(m)} & 0\\ 0 & 0 & 0 & -\bar\sigma_{33}^{(m)}\end{pmatrix}, \tag{47}$$

to the frame in which the shell is moving with the velocity $\mathbf{v}$. In analogy to Eqs. (36) and (37), these quantities are then given by

$$U_{\rm mec}^{(m)}=\bar T_m^{00}\frac{V_m}{\gamma}=\gamma^2(\bar\rho_0^{(m)}c^2-\beta^2\bar\sigma_{11}^{(m)})\frac{V_m}{\gamma}$$
$$=\gamma m_0^{(m)}c^2-\tfrac{4}{5}\gamma\beta^2U_0^{(m)}, \tag{48}$$

$$\mathbf{P}_{\rm mec}^{(m)}=\bar T_m^{01}\frac{V_m}{\gamma c}\frac{\mathbf{v}}{v}=\gamma^2\beta(\bar\rho_0^{(m)}c^2-\bar\sigma_{11}^{(m)})\frac{V_m}{\gamma c}\frac{\mathbf{v}}{v}$$
$$=\gamma m_0^{(m)}\mathbf{v}-\frac{4}{5}\gamma\frac{1}{c^2}U_0^{(m)}\mathbf{v}, \tag{49}$$

with $m_0^{(m)}=\bar\rho_0^{(m)}V_m$ the mechanical rest mass of the shell, and with the value of Eq. (29) for the stress-tensor element $\bar\sigma_{11}^{(m)}$. Using Eqs. (45) and (46) and Eqs. (48) and (49), the total energy $U_{\rm tot}^{(m)}$ and momentum $\mathbf{P}_{\rm tot}^{(m)}$ of the moving current-carrying shell are

$$U_{\rm tot}^{(m)}=U_{\rm elm}^{(m)}+U_{\rm mec}^{(m)}=\gamma m_{0\,\rm tot}^{(m)}c^2, \tag{50}$$

$$\mathbf{P}_{\rm tot}^{(m)}=\mathbf{P}_{\rm elm}^{(m)}+\mathbf{P}_{\rm mec}^{(m)}=\gamma m_{0\,\rm tot}^{(m)}\mathbf{v}, \tag{51}$$

with

$$m_{0\,\rm tot}^{(m)}=U_0^{(m)}/c^2+m_0^{(m)} \tag{52}$$

its total rest mass; the relativistic relation between energy and momentum, $U_{\rm tot}^{(m)2}=\mathbf{P}_{\rm tot}^{(m)2}c^2+m_{0\,\rm tot}^{(m)2}c^4$, thus ensues.

It remains to be confirmed that these transformation properties of the total energy and momentum are not peculiar to only the shell's motion along the x axis, i.e., along a direction that is perpendicular to the magnetic moment $\mathbf{m}$ of the shell. For this purpose, let us assume that the shell is moving in the direction of its magnetic moment $\mathbf{m}$, i.e., along the z axis. Equations (45) and (46) for the electromagnetic energy $U_{\rm elm}^{(m)}$ and momentum $\mathbf{P}_{\rm elm}^{(m)}$ then modify to

$$U_{\rm elm}^{(m)}=\frac{1}{2\mu_0}\,\gamma\int[(1+\beta^2){\bf B}'^2-2\beta^2B_3'^2]d^3r'$$

$$=\gamma(1-\tfrac{3}{5}\beta^2)U_0^{(m)}, \qquad (53)$$

$$\mathbf{P}_{\rm elm}^{(m)}=\frac{\gamma}{\mu_0c^2}\int(\mathbf{B}'^2-B_3'^2)d^3r'\mathbf{v}=\frac{2}{5}\,\gamma\,\frac{1}{c^2}\,U_0^{(m)}\mathbf{v}, \quad (54)$$

where the integral over $(1/2\mu_0)B_3'^2$ is evaluated as $\frac{4}{5}U_0^{(m)}$. The mechanical energy $U_{\rm mec}^{(m)}$ and momentum $\mathbf{P}_{\rm mec}^{(m)}$ are now given as [cf. Eqs. (48) and (49)]:

$$U_{\rm mec}^{(m)}=\gamma^2(\bar{\rho}_0^{(m)}c^2-\beta^2\bar{\sigma}_{33}^{(m)})\,\frac{V_m}{\gamma}$$

$$=\gamma m_0^{(m)}c^2+\tfrac{3}{5}\gamma\beta^2U_0^{(m)}, \qquad (55)$$

$$\mathbf{P}_{\rm mec}^{(m)}=\gamma^2\beta(\bar{\rho}_0^{(m)}c^2-\bar{\sigma}_{33}^{(m)})\,\frac{V_m}{\gamma c}\,\frac{\mathbf{v}}{v}$$

$$=\gamma m_0^{(m)}\mathbf{v}+\frac{3}{5}\,\gamma\,\frac{1}{c^2}\,U_0^{(m)}\mathbf{v}, \qquad (56)$$

with the value of Eq. (29) for the stress-tensor element $\bar{\sigma}_{33}^{(m)}$. Equations (53) and (54) and Eqs. (55) and (56) lead again to the relativistically correct Eqs. (50) and (51) for the total energy $U_{\rm tot}^{(m)}$ and momentum $P_{\rm tot}^{(m)}$ of the moving current-carrying shell, confirming the generality of our results.

D. Moving system of charge and current

Let us now consider the full system of the two shells, one carrying charge and the other carrying current, in uniform motion with a velocity $\mathbf{v}$. To simplify the calculations, we assume that the electric dipole moment $\mathbf{p}$ and the magnetic dipole moment $\mathbf{m}$ of the shells are perpendicular to each other, with, say, $\mathbf{m}$ along the x axis and $\mathbf{p}$ along the y axis.

First, we consider motion along the z axis, i.e., along the direction of the rest-frame electromagnetic field momentum $\mathbf{P}_0$ [see Eq. (21)]. The electromagnetic energy $U_{\rm elm}$ of the system is calculated by Lorentz-transforming the fields and variables into the rest-frame of the system using Eqs. (30) and (31), modified for motion along the z axis, as

$$U_{\rm elm}=\frac{\epsilon_0}{2}\int(\mathbf{E}^2+c^2\mathbf{B}^2)d^3r$$

$$=\frac{\epsilon_0\gamma}{2}\int[(1+\beta^2)\mathbf{E}'^2-2\beta^2E_3'^2]d^3r'+\frac{\gamma}{2\mu_0}$$

$$\times\int[(1+\beta^2)\mathbf{B}'^2-2\beta^2B_3'^2]d^3r'+2\gamma v\epsilon_0$$

$$\times\int(E_1'B_2'-E_2'B_1')d^3r'$$

$$=\gamma(1+\tfrac{3}{5}\beta^2)U_0^{(e)}+\gamma(1+\tfrac{4}{5}\beta^2)U_0^{(m)}+2\gamma vP_0, \qquad (57)$$

where $P_0=|\mathbf{P}_0|$ is the rest-frame field momentum, Eqs. (19) and (21). Here we used Eqs. (32) and (45), with due regard for the fact that the z axis is now perpendicular to the moments $\mathbf{p}$ and $\mathbf{m}$, to evaluate the terms proportional to $U_0^{(e)}$ and $U_0^{(m)}$. Thus the electromagnetic energy (57) is the sum of the energies $U_{\rm elm}^{(e)}$ and $U_{\rm elm}^{(m)}$ of the single shells [see Eqs. (32) and (45)], plus a term that is proportional to the rest-frame field momentum P_0. The electromagnetic momentum $\mathbf{P}_{\rm elm}$ is calculated similarly:

$$\mathbf{P}_{\rm elm}=\epsilon_0\int\mathbf{E}\times\mathbf{B}d^3r$$

$$=\frac{\epsilon_0\gamma}{c^2}\int(\mathbf{E}'^2-E_3'^2)d^3r'\mathbf{v}$$

$$+\frac{\gamma}{\mu_0c^2}\int(\mathbf{B}'^2-B_3'^2)d^3r'\mathbf{v}$$

$$+\gamma(1+\beta^2)\epsilon_0\int(E_1'B_2'-E_2'B_1')d^3r'\,\frac{\mathbf{v}}{v}$$

$$=\gamma[(1/c^2)(\tfrac{8}{5}U_0^{(e)}+\tfrac{9}{5}U_0^{(m)})+(1+\beta^2)P_0/v]\mathbf{v}. \quad (58)$$

Equations (33) and (46) were used here to evaluate the terms proportional to $U_0^{(e)}$ and $U_0^{(m)}$.

To calculate the mechanical energy $U_{\rm mec}$ and momentum $\mathbf{P}_{\rm mec}$ of the moving system of two shells, the volume-averaged mechanical energy-momentum tensor of the system in its rest frame, $\bar{T}'^{\mu\nu}_{e+m}$, is required. Due to the presence of the hidden mechanical momentum $\mathbf{P}_h=-\mathbf{P}_0$, the tensor $\bar{T}'^{\mu\nu}_{e+m}$ is not just a volume-weighted sum of the tensors $\bar{T}'^{\mu\nu}_e$ and $\bar{T}'^{\mu\nu}_m$ of the single shells, Eqs. (34) and (47), respectively, but is given by

$$\bar{T}'^{\mu\nu}_{e+m}=\frac{1}{V_{e+m}}\,[V_e\bar{T}'^{\mu\nu}_e+V_m\bar{T}'^{\mu\nu}_m-P_0c(\delta_0^\mu\delta_3^\nu+\delta_3^\mu\delta_0^\nu)], \qquad (59)$$

where $V_{e+m}=V_e+V_m$ is the total rest volume of the system; the components $\bar{T}'^{03}_{e+m}=\bar{T}'^{30}_{e+m}$ of this tensor equal c times the volume average $-P_0/V_{e+m}$ of the hidden mechanical momentum of the system $-\mathbf{P}_0=-P_0\mathbf{v}/v$ ($\mathbf{v}$ is assumed above to be along the z axis).[32] The nonzero momentum elements in the *rest-frame* tensor (59) turn out to be crucial for guaranteeing the correct transformation properties of the total energy and mass of the system. Transforming the rest-frame tensor $\bar{T}'^{\mu\nu}_{e+m}$ of Eq. (59) to the frame in which the system of two shells moves with the velocity $\mathbf{v}$ by using Eq. (35), modified appropriately for a motion along the z axis, one obtains the mechanical energy $U_{\rm mec}$ and momentum $\mathbf{P}_{\rm mec}$ of the system as

$$U_{\rm mec}=\bar{T}^{00}_{e+m}V_{e+m}/\gamma$$

$$=\gamma^2(\bar{T}'^{00}_{e+m}+2\beta\bar{T}'^{03}_{e+m}+\beta^2\bar{T}'^{33}_{e+m})V_{e+m}/\gamma$$

$$=\gamma[(V_e\bar{\rho}_0^{(e)}+V_m\bar{\rho}_0^{(m)})c^2-2\beta P_0c$$

$$-\beta^2(V_e\bar{\sigma}_{33}^{(e)}+V_m\bar{\sigma}_{33}^{(m)})]$$

$$=\gamma(m_0^{(e)}+m_0^{(m)})c^2-2\gamma\beta P_0c$$

$$-\gamma\beta^2(\tfrac{3}{5}U_0^{(e)}+\tfrac{4}{5}U_0^{(m)}), \qquad (60)$$

$$\mathbf{P}_{\rm mec}=\bar{T}^{03}_{e+m}\,\frac{V_{e+m}}{\gamma c}\,\frac{\mathbf{v}}{v}.$$

$$=\gamma^2[(1+\beta^2)\bar{T}'^{03}_{e+m}+\beta\bar{T}'^{00}_{e+m}+\beta\bar{T}'^{33}_{e+m}]\frac{V_{e+m}}{\gamma c}\,\frac{\mathbf{v}}{v}$$

$$=\gamma[-(1+\beta^2)P_0+\beta c(V_e\bar{\rho}_0^{(e)}+V_m\bar{\rho}_0^{(m)})$$

$$-(\beta/c)(V_e\bar{\sigma}_{33}^{(e)}+V_m\bar{\sigma}_{33}^{(m)})]\,\frac{\mathbf{v}}{v}$$

$$=\gamma[-(1+\beta^2)P_0/v+m_0^{(e)}$$

$$+m_0^{(m)}-(1/c^2)(\tfrac{3}{5}U_0^{(e)}+\tfrac{4}{5}U_0^{(m)})]\mathbf{v}, \qquad (61)$$

where the values used for the stress-tensor elements $\bar{\sigma}_{33}^{(e)}$ and $\bar{\sigma}_{33}^{(m)}$ are those of $\bar{\sigma}_{11}^{(e)}=\bar{\sigma}_{22}^{(e)}$ and $\bar{\sigma}_{11}^{(m)}=\bar{\sigma}_{22}^{(m)}$ in Eqs. (27) and (29), respectively, as a motion in a direction perpendicular to both the moments **p** and **m** of the shells is considered. It now follows immediately from Eqs. (57) and (58) and Eqs. (60) and (61) that the total energy $U_{\rm tot}$ and momentum $\mathbf{P}_{\rm tot}$ of the system satisfy the relativistically correct relations

$$U_{\rm tot}=U_{\rm elm}+U_{\rm mec}=\gamma m_{0\,{\rm tot}}c^2, \qquad (62)$$

$$\mathbf{P}_{\rm tot}=\mathbf{P}_{\rm elm}+\mathbf{P}_{\rm mec}=\gamma m_{0\,{\rm tot}}\mathbf{v}, \qquad (63)$$

where

$$m_{0\,{\rm tot}}=\frac{1}{c^2}\,(U_0^{(e)}+U_0^{(m)})+m_0^{(e)}+m_0^{(m)} \qquad (64)$$

is the total rest mass of the system of the two shells.

To check that this result is of a more general validity, let us now assume that the system is moving in a direction perpendicular to the rest-frame field momentum $\mathbf{P}_0$, say along the direction of the magnetic moment **m** of the current-carrying shell, i.e., along the x axis. The electromagnetic energy $U_{\rm elm}$ is then given by [cf. Eq. (57)]

$$U_{\rm elm}=\frac{\epsilon_0}{2}\int(\mathbf{E}^2+c^2\mathbf{B}^2)d^3r$$

$$=\frac{\epsilon_0\gamma}{2}\int[(1+\beta^2)\mathbf{E}'^2-2\beta^2E_1'^2]d^3r'$$

$$+\frac{\gamma}{2\mu_0}\int[(1+\beta^2)\mathbf{B}'^2-2\beta^2B_1'^2]d^3r'$$

$$+2\gamma v\epsilon_0\int(E_2'B_3'-E_3'B_2')d^3r'$$

$$=\gamma(1+\tfrac{3}{5}\beta^2)U_0^{(e)}+\gamma(1-\tfrac{3}{5}\beta^2)U_0^{(m)}, \qquad (65)$$

where Eq. (32) is used to evaluate the term proportional to $U_0^{(e)}$, and Eq. (53), which is relevant for a motion along the direction of the moment **m**, is used for the term proportional to $U_0^{(m)}$; the term mixed in the electric and magnetic fields is proportional to the x component of the rest-frame field momentum, which vanishes. The electromagnetic momentum $\mathbf{P}_{\rm elm}$ has now a component along the direction of the velocity **v**, which is taken as the x axis direction, as well as a component along the z axis, which is the direction of the rest-frame electromagnetic momentum $\mathbf{P}_0$; the former arises from the motion of the system, while the latter turns out to be unaffected by the motion and equals $\mathbf{P}_0$. Indeed, the electromagnetic momentum $\mathbf{P}_{\rm elm}$ is calculated as

$$\mathbf{P}_{\rm elm}=\epsilon_0\int\mathbf{E}\times\mathbf{B}d^3r$$

$$=\left[\frac{\epsilon_0\gamma}{c^2}\int(\mathbf{E}'^2-E_1'^2)d^3r'\right.$$

$$+\frac{\gamma}{\mu_0c^2}\int(\mathbf{B}'^2-B_1'^2)d^3r'$$

$$\left.+\gamma(1+\beta^2)\epsilon_0\int(E_2'B_3'-E_3'B_2')d^3r'\,\frac{1}{v}\right]\mathbf{v}$$

$$+\epsilon_0\int(E_1'B_2'-E_2'B_1')d^3r'\,\frac{\mathbf{P}_0}{P_0}$$

$$=\gamma(\tfrac{8}{5}U_0^{(e)}/c^2+\tfrac{2}{5}U_0^{(m)}/c^2)\mathbf{v}+\mathbf{P}_0. \qquad (66)$$

Here, the term along the direction of **v** is obtained similarly to Eq. (58), using Eqs. (33) and (54) to evaluate the terms proportional to $U_0^{(e)}$ and $U_0^{(m)}$, respectively, (the term mixed in the electric and magnetic fields vanishes, being proportional to the x component of $\mathbf{P}_0$); the term along the direction of $\mathbf{P}_0$ is the only nonzero term arising from the z component of the vector product $\mathbf{E}\times\mathbf{B}$.

Transforming the mechanical energy-momentum tensor $\bar{T}'^{\mu\nu}_{e+m}$ of Eq. (59) from the rest frame using Eqs. (35), the mechanical energy $U_{\rm mec}$ and momentum $\mathbf{P}_{\rm mec}$ are now [cf., Eqs. (60) and (61)]

$$U_{\rm mec}=\bar{T}^{00}_{e+m}V_{e+m}/\gamma$$

$$=\gamma^2(\bar{T}'^{00}_{e+m}+2\beta\bar{T}'^{01}_{e+m}+\beta^2\bar{T}'^{11}_{e+m})V_{e+m}/\gamma$$

$$=\gamma[(V_e\bar{\rho}_0^{(e)}+V_m\bar{\rho}_0^{(m)})c^2$$

$$-\beta^2(V_e\bar{\sigma}_{11}^{(e)}+V_m\bar{\sigma}_{11}^{(m)})]$$

$$=\gamma(m_0^{(e)}+m_0^{(m)})c^2-\gamma\beta^2(\tfrac{1}{5}U_0^{(e)}-\tfrac{3}{5}U_0^{(m)}), \qquad (67)$$

$$\mathbf{P}_{\rm mec}=\left(\bar{T}^{01}_{e+m}\,\frac{\mathbf{v}}{v}+\bar{T}^{03}_{e+m}\,\frac{\mathbf{P}_0}{P_0}\right)\frac{V_{e+m}}{\gamma c}$$

$$=\left\{\gamma^2[(1+\beta^2)\bar{T}'^{01}_{e+m}+\beta\bar{T}'^{00}_{e+m}+\beta\bar{T}'^{11}_{e+m}]\frac{\mathbf{v}}{v}\right.$$

$$\left.+\gamma(\bar{T}'^{03}_{e+m}+\beta\bar{T}'^{13}_{e+m})\,\frac{\mathbf{P}_0}{P_0}\right\}\frac{V_{e+m}}{\gamma c}$$

$$=\gamma[\beta c(V_e\bar{\rho}_0^{(e)}+V_m\bar{\rho}_0^{(m)})-(\beta/c)$$

$$\times(V_e\bar{\sigma}_{11}^{(e)}+V_m\bar{\sigma}_{11}^{(m)})]\,\frac{\mathbf{v}}{v}-\mathbf{P}_0$$

$$=\gamma[m_0^{(e)}+m_0^{(m)}-(1/c^2)$$

$$\times(\tfrac{1}{5}U_0^{(e)}-\tfrac{3}{5}U_0^{(m)})]\mathbf{v}-\mathbf{P}_0, \qquad (68)$$

where the fact that $\bar{T}'^{01}_{e+m}=\bar{T}'^{13}_{e+m}=0$ is utilized [see Eq. (59)], and the values used for the stress-tensor elements $\bar{\sigma}_{11}^{(e)}$ and $\bar{\sigma}_{11}^{(m)}$ are those of $\bar{\sigma}_{11}^{(e)}$ and $\bar{\sigma}_{33}^{(m)}$ in Eqs. (27) and (29), respectively. Using now Eqs. (65) and (66) and Eqs. (67) and (68), the relativistically correct Eqs. (62) and (63) are obtained again for the total energy $U_{\rm tot}$ and momentum $\mathbf{P}_{\rm tot}$, with the invariant total rest mass $m_{0\,{\rm tot}}$ of Eq. (64). Clearly, the requirement of special relativity that the relation

$$U_{\rm tot}^2=\mathbf{P}_{\rm tot}^2c^2+m_{0\,\rm tot}^2c^4 \tag{69}$$

holds for the observed energy $U_{\rm tot}$ and momentum $\mathbf{P}_{\rm tot}$ of a body with an observed rest mass $m_{0\,\rm tot}$ will be satisfied in any uniform motion of the system of the two shells.

III. ELECTROMAGNETIC ENERGY AND MOMENTUM OF A MOVING SYSTEM OF CHARGE AND CURRENT—COVARIANT FORMULATION

Having investigated in some detail the contribution of the electromagnetic self-field of a moving system of charge and current to the system's observed energy and momentum in the traditional approach, we now turn to the alternative way of dealing with the problem by applying the relativistically covariant definitions of electromagnetic energy and momentum, Eqs. (10) and (11), to the system considered. After the calculations of Sec. II, it is not necessary to start again with the simpler problem of single shells, and we proceed directly to the system of the two shells in uniform motion. To simplify the calculations, we assume once more that the electric dipole moment $\mathbf{p}$ of the charge-carrying shell is perpendicular to the magnetic dipole $\mathbf{m}$ of the current-carrying shell, with x and y axes along $\mathbf{m}$ and $\mathbf{p}$, respectively. The rest-frame electromagnetic momentum $\mathbf{P}_0$ of the system is then along the z axis [see Eq. (21)].

First, let us assume that the system is moving along the direction of the rest-frame electromagnetic momentum $\mathbf{P}_0$, i.e., along the z axis, with a velocity $\mathbf{v}$. The electromagnetic energy $U_{\rm elm}$ of the system, defined covariantly as in Eq. (10), is then

$$\begin{aligned}U_{\rm elm}&=\epsilon_0\gamma^2\int\left[\frac{1}{2}(\mathbf{E}^2+c^2\mathbf{B}^2)-\mathbf{v}\cdot(\mathbf{E}\times\mathbf{B})\right]d^3r\\&=\gamma^3[(1+\tfrac{3}{5}\beta^2)U_0^{(e)}+(1+\tfrac{4}{5}\beta^2)U_0^{(m)}\\&\quad+2vP_0]-\gamma^3[(1/c^2)(\tfrac{8}{5}U_0^{(e)}+\tfrac{9}{5}U_0^{(m)})\\&\quad+(1+\beta^2)P_0/v]v^2\\&=\gamma(U_0^{(e)}+U_0^{(m)}+vP_0)=\gamma(U_0+\mathbf{v}\cdot\mathbf{P}_0).\end{aligned} \tag{70}$$

Here, $U_0=U_0^{(e)}+U_0^{(m)}$ is the rest-frame electromagnetic energy of the system, and Eqs. (57) and (58) were used to evaluate the two terms of the integral. The electromagnetic momentum $\mathbf{P}_{\rm elm}$ of the system, defined covariantly as in Eq. (11), is

$$\begin{aligned}\mathbf{P}_{\rm elm}&=\epsilon_0\gamma^2\int[\mathbf{E}\times\mathbf{B}+(\mathbf{v}\cdot\mathbf{E})\mathbf{E}/c^2+(\mathbf{v}\cdot\mathbf{B})\mathbf{B}\\&\quad-\tfrac{1}{2}(\mathbf{E}^2+c^2\mathbf{B}^2)\mathbf{v}/c^2]d^3r\\&=\gamma^3[(1/c^2)(\tfrac{8}{5}U_0^{(e)}+\tfrac{9}{5}U_0^{(m)})+(1+\beta^2)P_0/v]\mathbf{v}\\&\quad+\gamma\,\tfrac{2}{5}U_0^{(e)}\mathbf{v}/c^2+\gamma\,\tfrac{1}{5}U_0^{(m)}\mathbf{v}/c^2-\gamma^3[(1+\tfrac{3}{5}\beta^2)U_0^{(e)}\\&\quad+(1+\tfrac{4}{5}\beta^2)U_0^{(m)}+2vP_0]\mathbf{v}/c^2\\&=\gamma(\mathbf{P}_0+U_0\mathbf{v}/c^2).\end{aligned} \tag{71}$$

Here the first and last terms of the integral are again evaluated using Eqs. (58) and (57), while the second and third terms are, after transforming to the rest frame, proportional to integrals over $\epsilon_0E_3'^2$ and $B_3'^2/\mu_0$, which are evaluated as $\frac{2}{5}U_0^{(e)}$ and $\frac{1}{5}U_0^{(m)}$, respectively. Equations (70) and (71) give the relativistically correct expressions for the electromagnetic energy and momentum of the moving system in terms of the rest-frame electromagnetic energy U_0 and momentum $\mathbf{P}_0$, as the Lorentz transformations of a four-vector $A^\mu=(A^0,\mathbf{A})$ to its rest-frame components $(A'^0,\mathbf{A}')$ are[33]

$$A^0=\gamma(A'^0+\mathbf{v}\cdot\mathbf{A}'/c),\quad \mathbf{A}_\parallel=\gamma(\mathbf{A}'_\parallel+A'^0\mathbf{v}/c),\quad \mathbf{A}_\perp=\mathbf{A}'_\perp, \tag{72}$$

where the vector components parallel and perpendicular to $\mathbf{v}$ are denoted by $\parallel$ and $\perp$, respectively.

The mechanical energy $U_{\rm mec}$ and momentum $\mathbf{P}_{\rm mec}$ of the moving system are now *postulated* to transform, independently of the mechanical stresses in the system, also as an energy-momentum four-vector:

$$U_{\rm mec}=\gamma(m_0c^2+\mathbf{v}\cdot\mathbf{P}_h), \tag{73}$$

$$\mathbf{P}_{\rm mec}=\gamma(\mathbf{P}_h+m_0c^2\mathbf{v}/c^2), \tag{74}$$

where $m_0=m_0^{(e)}+m_0^{(m)}$ is the mechanical rest mass, and m_0c^2 and $\mathbf{P}_h$ are the rest-frame mechanical energy and momentum, respectively, of the system. The rest-frame mechanical momentum $\mathbf{P}_h$ is, of course, the hidden momentum that is equal and opposite to the rest-frame electromagnetic momentum $\mathbf{P}_0$, $\mathbf{P}_h=-\mathbf{P}_0$, and so Eqs. (70) and (71) and Eqs. (73) and (74) give for the observed total energy $U_{\rm tot}$ and momentum $\mathbf{P}_{\rm tot}$ of the system

$$U_{\rm tot}=U_{\rm elm}+U_{\rm mec}=\gamma(U_0+m_0c^2)=\gamma m_{0\,\rm tot}c^2, \tag{75}$$

$$\mathbf{P}_{\rm tot}=\mathbf{P}_{\rm elm}+\mathbf{P}_{\rm mec}=\gamma(U_0/c^2+m_0)\mathbf{v}=\gamma m_{0\,\rm tot}\mathbf{v}, \tag{76}$$

where $m_{0\,\rm tot}$ is the total rest mass (64). Thus we arrive at the same relativistically correct results for the total energy $U_{\rm tot}$ and momentum $\mathbf{P}_{\rm tot}$ of the system as those of Eqs. (62) and (63) in Sec. II.

To confirm that these transformation properties hold generally, we now consider the system of the two shells in motion along the x axis, i.e., in a direction perpendicular to the rest-frame electromagnetic momentum $\mathbf{P}_0$. The electromagnetic energy $U_{\rm elm}$ is now

$$\begin{aligned}U_{\rm elm}&=\epsilon_0\gamma^2\int\left[\frac{1}{2}(\mathbf{E}^2+c^2\mathbf{B}^2)-\mathbf{v}\cdot(\mathbf{E}\times\mathbf{B})\right]d^3r\\&=\gamma^3[(1+\tfrac{3}{5}\beta^2)U_0^{(e)}+(1-\tfrac{3}{5}\beta^2)U_0^{(m)}]\\&\quad-\gamma^3(\tfrac{8}{5}U_0^{(e)}/c^2+\tfrac{2}{5}U_0^{(m)}/c^2)v^2\\&=\gamma(U_0^{(e)}+U_0^{(m)})=\gamma U_0.\end{aligned} \tag{77}$$

To evaluate the two terms of the integral, Eqs. (65) and (66) were used here. The electromagnetic momentum $\mathbf{P}_{\rm elm}$ can no longer be directed exclusively along the velocity $\mathbf{v}$ as it must contain a component in the direction of the rest-frame momentum $\mathbf{P}_0$. Indeed,

$$\begin{aligned}\mathbf{P}_{\rm elm}&=\epsilon_0\gamma^2\int[\mathbf{E}\times\mathbf{B}+(\mathbf{v}\cdot\mathbf{E})\mathbf{E}/c^2+(\mathbf{v}\cdot\mathbf{B})\mathbf{B}\\&\quad-\tfrac{1}{2}(\mathbf{E}^2+c^2\mathbf{B}^2)\mathbf{v}/c^2]d^3r\\&=\{\gamma^3(\tfrac{8}{5}U_0^{(e)}+\tfrac{2}{5}U_0^{(m)})+\gamma\,\tfrac{2}{5}U_0^{(e)}+\gamma\,\tfrac{8}{5}U_0^{(m)}\\&\quad-\gamma^3[(1+\tfrac{3}{5}\beta^2)U_0^{(e)}+(1-\tfrac{3}{5}\beta^2)U_0^{(m)}]\}\mathbf{v}/c^2\\&\quad+\gamma^2(1-\beta^2)\mathbf{P}_0\\&=\gamma(U_0^{(e)}/c^2+U_0^{(m)}/c^2)\mathbf{v}+\mathbf{P}_0\\&=\gamma(U_0/c^2)\mathbf{v}+\mathbf{P}_0.\end{aligned}\tag{78}$$

Here, the first and fourth terms of the integral are again evaluated using Eqs. (65) and (66); the second and third terms contribute to both the components along $\mathbf{v}$ and $\mathbf{P}_0$: in the direction of $\mathbf{v}$, they lead to integrals over $\epsilon_0 E_1'^2$ and $B_1'^2/\mu_0$, which are evaluated as $\frac{2}{5}U_0^{(e)}$ and $\frac{8}{5}U_0^{(m)}$, respectively, and, in the direction of $\mathbf{P}_0$, they contribute with $-\gamma^2\beta^2\mathbf{P}_0$. The electromagnetic energy (77) and momentum (78) thus transform together as a four-vector, i.e., according to Eq. (72) with $A'^0=U_0/c$, $\mathbf{A}'_\parallel=0$ and $\mathbf{A}'_\perp=\mathbf{P}_0$.

When the velocity $\mathbf{v}$ of the system is perpendicular to the mechanical rest-frame momentum $\mathbf{P}_h$, the mechanical energy $U_{\rm mech}$ and momentum $\mathbf{P}_{\rm mech}$, postulated to form an energy-momentum four-vector, transform in accordance with Eq. (72) as

$$U_{\rm mec}=\gamma m_0c^2,\tag{79}$$

$$\mathbf{P}_{\rm mec}=\gamma m_0\mathbf{v}+\mathbf{P}_h.\tag{80}$$

It follows from Eqs. (77) and (78) and Eqs. (79) and (80), and the fact that $\mathbf{P}_0=-\mathbf{P}_h$, that the total energy $U_{\rm tot}=U_{\rm elm}+U_{\rm mec}$ and momentum $\mathbf{P}_{\rm tot}=\mathbf{P}_{\rm elm}+\mathbf{P}_{\rm mec}$ of the system have the relativistically correct values of Eqs. (75) and (76). The observed total rest mass, $m_{0\,\rm tot}=U_0/c^2+m_0$, of the system of the two shells is thus a relativistic invariant. It should be noted that this result is not guaranteed automatically by the covariant definitions of the electromagnetic energy and momentum unless the existence of the rest-frame hidden mechanical momentum is recognized and accounted for properly.

A point concerning the procedure of using the covariant definitions (10) and (11) of the electromagnetic energy and momentum requires clarification. The definitions single out a frame of reference, "the rest frame," relative to which the velocity $\mathbf{v}$ is defined. Jackson[34] holds explicitly, and Konopinski[15] implies that such a frame can be chosen arbitrarily, so that it is a matter of convenience what is regarded as the rest frame. However, it can be seen easily that the only rest frame that can be chosen consistently with the theorem[21] that guarantees zero total momentum for a system of stationary distribution of matter, charge, and current is the frame where matter, charge, and current are distributed in a stationary manner, say frame A—unless one abandons the standard definition of mechanical momentum in frame A. Indeed, if another frame is adopted, say frame B moving with a velocity $\mathbf{v}$ relative to frame A, as the "rest" frame for the covariant definitions of electromagnetic energy and momentum, then the electromagnetic momentum obtained by the Lorentz transformation from frame B to frame A is equal and opposite to the mechanical momentum in frame A only when the latter quantity is defined as the Lorentz transform of the standard mechanical energy and momentum in frame B, treated formally as the components of a four-vector.

IV. CONCLUSIONS

The two different ways of treating the problem of the contribution of the electromagnetic self-field to the energy and momentum of a moving body, one that was called here traditional, in which an explicit account is taken of the mechanical stresses in the body, and the other, in which relativistically covariant definitions of the electromagnetic quantities are employed, were illustrated with an example, worked out in detail, of a body that may carry electric currents as well as charge. Such a body contains, in general, a hidden mechanical momentum, which is not associated with the motion of its center of mass. The existence of hidden momentum turned out to be crucial in both approaches for establishing the correct Lorentz-transformation behavior of the observed energy, momentum and rest mass of the body.

Hidden momentum is equal and opposite to the momentum of the static electromagnetic self-field of the body in the rest frame where its center of mass is at rest and where the total momentum, electromagnetic plus mechanical, is guaranteed to vanish on the grounds of a general theorem. Thus the physical significance, or reality, of the momentum of a static electromagnetic field, so often questioned if not rejected outright, is tied up inexorably with that of hidden momentum. Hidden momentum is a direct consequence of relativistic mechanics, with implications that are, in principle, testable experimentally for the force that a current-carrying body experiences in an external electromagnetic field. Moreover, as shown in the present paper, hidden momentum is necessary for maintaining the correct relativistic properties of the observed total energy, momentum, and rest mass of a charge and current carrying body. In classical electrodynamics, hidden momentum, and with it the momentum of a static electromagnetic field, are indispensable concepts of direct physical significance.

[1]A. Pais, "The early history of the theory of the electron: 1897–1947," in *Aspects of Quantum Theory*, edited by A. Salam and E. P. Wigner (Cambridge U. P., Cambridge, 1972), pp. 79–93.

[2]Lorentz's formula for the momentum of the electron was $(1-\mathbf{v}^2/c^2)^{-1/2}m\mathbf{v}$, see Pais, Ref. 1. An interesting episode in the early history of special relativity concerning experimental tests of the differing Abraham and Lorentz–Einstein predictions for the variation of the electron mass with velocity is described in great detail by J. T. Cushing, "Electromagnetic mass, relativity, and the Kaufmann experiments," Am. J. Phys. **49**(12), 1113–1149 (1981).

[3]See, for example, J. D. Jackson, *Classical Electrodynamics* (Wiley, New York, 1975), 2nd ed., Sec. 14.1.

[4]The exact value of the radius depends on the details of the postulated distribution of charge within the electron. With the most often used distribution of a uniform surface charge density of radius a, the energy of the electric field $\mathbf{E}$ produced by the distribution is $(\epsilon_0/2)\int\mathbf{E}^2d^3r=2\pi\epsilon_0\int_a^\infty(e/4\pi\epsilon_0r^2)^2r^2dr=e^2/8\pi\epsilon_0a$, and hence the radius $a=a_c/2$ for the energy to equal m_ec^2.

[5]See Pais, Ref. 1.

[6]A relatively recent exchange is between T. H. Boyer, "Classical model of the electron and the definition of electromagnetic field momentum," Phys. Rev. D **25**, 3246–3250 (1982); and F. Rohrlich, "Comment on preceding paper by T. H. Boyer," *ibid.*, 3251–3255 (1982).

[7]Because of its simplicity, this particular charge distribution is very popular in classical models of the electron; using another spherically symmetric distribution with a well-defined surface would only change the details of the expression for the rest mass m_e in terms of the radius a [see our Eq. (3)], without affecting any of the points under the discussion.

[8]Similar calculations, using the device of Lorentz transforming into the rest frame, will be carried out in more detail in Sec. II of this paper. A simple nonrelativistic calculation displaying the factor of 4/3 discrepancy has been given recently by P. Moylan, "An elementary account of the factor of 4/3 in the electromagnetic mass," Am. J. Phys. **63**(9), 818–820 (1995).

[9]The original papers of H. Poincaré, Co. R. Acad. Sci. Paris **140**, 1504 (1905), and "Sur la dynamique de l'électron," Rend. Circ. Mat. Palermo **21**, 129 (1906), are not easily accessible; however, there is an English, "modernized" presentation of the latter paper by H. M. Schwartz, "Poincaré's Rendiconti paper on relativity. Part I," Am. J. Phys. **39**(11), 1287–1294 (1971); "Part II," *ibid.* **40**(6), 862–872 (1972); and "Part III," *ibid.* **40**(9), 1282–1287 (1972); an early discussion of his ideas is by H. A. Lorentz, *The Theory of Electrons* (Dover, New York, 1952), 2nd ed., pp. 213–215; see also Jackson, Ref. 3, pp. 792–793.

[10]See, for example, L. D. Landau and E. M. Lifshitz, *The Classical Theory of Fields* (Pergamon, Oxford, 1975), 4th ed., Secs. 33 and 34.

[11]An authoritative, detailed history of QED has appeared recently: S. S. Schweber, *QED and the Men Who Made It* (Princeton U.P., Princeton, NJ, 1994).

[12]F. Rohrlich, "Self-energy and stability of the classical electron," Am. J. Phys. **28**, 639–643 (1960); "Electromagnetic momentum, energy, and mass," *ibid.* **38**(11), 1310–1316 (1970); *Classical Charged Particles* (Addison-Wesley, Reading, MA, 1965 and 1990).

[13]E. Fermi, "Über einen Widerspruch zwischen der elektrodynamischen und der relativistischen Theorie der elektromagnetische Masse," Z. Phys. **24**, 340–346 (1922); B. Kwal, "Les expressions de l'energie et l'impulsion du champ électromagnetique propre de l'électron en mouvement," J. Phys. Radium **10**, 103–104 (1949).

[14]See, for example, Jackson, Ref. 3, pp. 604–605, where, however, the symbol $\Theta^{\mu\nu}$ is used for the symmetric energy momentum, or stress tensor (our $T^{\mu\nu}$).

[15]E. J. Konopinski, *Electromagnetic Fields and Relativistic Particles* (McGraw-Hill, New York, 1981), pp. 432–436.

[16]Rohrlich, Ref. 12; see also Jackson, Ref. 3, Sec. 17.5, and Konopinski, Ref. 15.

[17]D. J. Griffiths and R. E. Owen, "Mass renormalization in classical electrodynamics," Am. J. Phys. **51**(12), 1120–1126 (1983). These authors also consider, and analyze in detail on the example of a "dumbbell" configuration of charge, a dynamic mechanism for the generation of electromagnetic mass in the self-force on an accelerating extended charged body, i.e., in the force that arises from the retarded electromagnetic field of the body itself. In the classical electron theory, the inclusion of the self-force led to the Abraham–Lorentz equation of motion, which includes the radiative reaction on a charged particle, see Jackson, Ref. 3, Sec. 17.3.

[18]This viewpoint has been expressed again very recently by K. McDonald, "Answer to Question #26," Am. J. Phys. **64**(1), 15–16 (1996), in an answer to R. H. Romer, "Question #26. Electromagnetic field momentum," *ibid.* **63**(9), 777–779 (1995), who posed the question whether the usual definition of electromagnetic field momentum, which may yield a nonzero momentum for a static field, is appropriate in the case of a field "bound" to its sources, i.e., a self-field. Romer, *loc. cit.* gives several references to the literature on the electromagnetic momentum of static fields.

[19]W. Shockley and R. P. James, " 'Try simplest cases' discovery of 'hidden momentum' forces on 'magnetic currents,' " Phys. Rev. Lett. **18**, 876–879 (1967); H. A. Haus and P. Penfield, "Force on a current loop," Phys. Lett. **26A**, 412–413 (1968).

[20]L. Vaidman, "Torque and force on a magnetic dipole," Am. J. Phys. **58**(10), 978–983 (1990), and references therein; V. Hnizdo, "Hidden momentum of a relativistic fluid carrying current in an external electric field," Am. J. Phys. **65**(1), 92–94 (1997).

[21]S. Coleman and J. H. Van Vleck, "Origin of 'hidden momentum forces' on magnets," Phys. Rev. **171**, 1370–1375 (1968); M. G. Calkin, "Linear momentum of the source of a static electromagnetic field," Am. J. Phys. **39**(5), 513–516 (1971); Y. Aharanov, P. Pearle, and L. Vaidman, "Comment on 'Proposed Aharanov–Casher effect: Another example of Aharanov–Bohm effect arising from a classical lag,' " Phys. Rev. A **37**, 4052–4055 (1988); Vaidman, Ref. 20; E. Comay, "Exposing 'hidden momentum,' " Am. J. Phys. **64**, 1028–1034 (1996).

[22]Vaidman, Ref. 20; V. Hnizdo, "Comment on 'Torque and force on a magnetic dipole,' " Am. J. Phys. **60**(3), 279–280 (1992).

[23]One consequence of hidden momentum is that the forces acting between a current-carrying body and a moving charged particle satisfy Newton's third law in the nonrelativistic limit of slow motion, see W. H. Furry, "Examples of momentum distributions in the electromagnetic field and in matter," Am. J. Phys. **37**(6), 621–636 (1969), and V. Hnizdo, "Conservation of linear and angular momentum and the interaction of a moving charge with a magnetic dipole," *ibid.* **60**(3), 242–246 (1992).

[24]Romer, Ref. 18. The shells are assumed to be massive, i.e., not massless, of small but macroscopic thickness, and, if needed, with thin struts that would keep them fixed to the common center, independently from each other (the struts of the larger shell would pass through holes in the smaller shell); the charges/currents are on the outer surfaces of the shells and occupy layers of negligible thickness. The electric field is that of a charge distribution that would be induced on the surface of a conducting sphere by an external uniform electric field, but there is no external field here and the charges are assumed to be fastened on the shell; similarly, the currents are assumed to be "fastened" on the second shell, where the macroscopic charge density vanishes. The shells must be made, therefore, of a nonconducting material.

[25]See, for example, Calkin, Ref. 21.

[26]Another way to calculate this quantity is as the force on a surface element of charge due to the arithmetic mean of the electric fields inside and outside the surface of the shell.

[27]L. D. Landau and E. M. Lifshitz, *Theory of Elasticity* (Pergamon, Oxford, 1986), 3rd ed., Sec. 2.

[28]See, for example, Jackson, Ref. 3, p. 552.

[29]Landau and Lifshitz, Ref. 10, Sec. 35.

[30]A volume-averaged energy-momentum tensor in the rest frame, $\bar{T}'^{\mu\nu}$, is defined by $\bar{T}'^{\mu\nu}=(1/V)\int T'^{\mu\nu}(\mathbf{r}')d^3r'$, where $T'^{\mu\nu}(\mathbf{r}')$ is the energy-momentum tensor at a point $\mathbf{r}'$ in the rest frame, where the system is stationary, and V is the rest volume of the system. The volume-averaged energy-momentum tensor $\bar{T}^{\mu\nu}$ in a frame where the system moves as a whole with a nonzero constant velocity can be obtained simply by Lorentz transforming the rest-frame volume-averaged tensor $\bar{T}'^{\alpha\beta}$: $\bar{T}^{\mu\nu}=(\gamma/V)\int T^{\mu\nu}(\mathbf{r})d^3r=(\gamma/V)\int L^{\mu\nu}_{\alpha\beta}T'^{\alpha\beta}(d^3r'/\gamma)=L^{\mu\nu}_{\alpha\beta}\bar{T}'^{\alpha\beta}$, with $L^{\mu\nu}_{\alpha\beta}$ the matrix of the Lorentz transformation.

[31]Landau and Lifshitz, Ref. 10, Sec. 6, Problem 1.

[32]While the pressure, or stresses, induced by the electric field of the charge-carrying shell in the current-carrying medium on the surface of the current-carrying shell may be essential for the creation of the hidden momentum in the latter shell (see Vaidman, or Hnizdo, Ref. 20), these stresses do not contribute to the *volume-averaged* stress components $\bar{T}'^{ij}_{e+m}$, as the charge density, and hence also the electric-field force, are assumed to vanish on the surface of the current-carrying shell and so will make no contribution to the volume-averaged stress tensor in a surface integral like that of Eq. (26).

[33]See, for example, Jackson, Ref. 3, p. 518.

[34]Jackson, Ref. 3, pp. 791–796.

THE OTHER CHICKEN GOT AWAY

A story is told of an investigation in which chickens were subjected to a certain treatment. It was then reported that $33\frac{1}{3}$ per cent of the chickens recovered, $33\frac{1}{3}$ per cent died, and no conclusion could be drawn from the other $33\ \frac{1}{3}$ per cent because that one ran away!

E. Bright Wilson, Jr., *An introduction to Scientific Research* (McGraw-Hill, New York, 1952), p. 46.