"Hidden" Momentum in a Sound Wave

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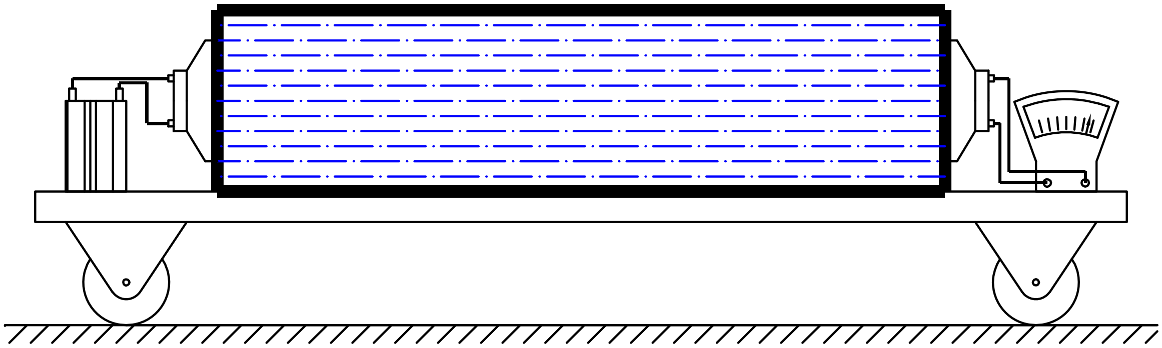
1 Problem

Discuss the momentum of a sound wave. In particular, consider the example sketched below in which a sound wave propagates through a volume of gas at pressure P. The energy of the sound wave is provided by a battery. The entire apparatus is mounted on a horizontal platform that can roll without friction in the direction of the sound wave.



2 Solution

The gas molecules appear to have zero average velocity in the rest frame of the pressure vessel, so it appears that they have zero average momentum in this frame.

Of course, the gas molecules exert a force $F = AP$ on a portion of area A of the wall of the pressure vessel, and since force equals the time rate of change of momentum, we can say that the gas can transmit momentum without possessing it [1].

When a sound wave is present, energy is transferred even when there is no net transfer of particles, as in the rest frame of the pressure vessel in the present example. If the rate of energy flow is dE/dt, then the (vector) energy flux is

$$\mathbf{S}_{\rm sound} = \frac{1}{A}\frac{dE}{dt}\,\hat{\mathbf{v}}_{\rm sound} = \frac{u_{\rm sound}}{A}\,\mathbf{v}_{\rm sound} \tag{1}$$

where A is the cross sectional area of the pressure vessel, $u_{\rm sound}$ is the density of energy in the sound wave, and $\mathbf{v}_{\rm sound}$ is the velocity of sound with respect to the pressure vessel.

Following the insight of Einstein [2] that energy and mass are related by $E = mc^2$, where c is the speed of light in vacuum, the energy density $u_{\rm sound}$ is associated with a mass density $u_{\rm sound}/c^2$, and hence there is a momentum density in the sound wave given by

$$\mathbf{p}_{\rm sound} = \frac{u_{\rm sound}}{Ac^2}\,\mathbf{v}_{\rm sound} = \frac{\mathbf{S}_{\rm sound}}{c^2}\,. \tag{2}$$

This momentum density is very small, due to the factor c^2 in the denominator of eq. (2); it is a relativistic effect.

It appears that the fluid-dynamics community considers that the laws of relativity are not relevant to this branch of physics, and that the momentum density (2) can and should be ignored.[1] For example, a delightful article by McIntyre implies that it is a myth that sound waves possess momentum, even though he acknowledges the content of eqs. (1)-(2) in a footnote [3]. An interesting article by Stone [4] expands on the theme of McIntyre that translational invariance of of a wave with respect to its supporting medium leads to conservation of a quantity that is better called pseudomomentum than Newtonian ($m\mathbf{v}$) momentum. But Stone argues (in sec. IV of [4]) that there is neither momentum nor pseudomomentum in the present example.[2]

In electromechanical systems where "mechanical" momentum of order $1/c^2$ is present in association with electromagnetic momentum, the "mechanical" momentum has come to be called "hidden" momentum [5, 6, 7, 8, 10, 9, 11, 12, 13]. It seems appropriate to this author to call the tiny momentum of eq. (2) a "hidden" momentum.

In any case, energy/mass is transferred from the battery to the absorber/meter at the right end of the apparatus, such that the center of mass of the system moves to the right with respect to the pressure vessel.

As the system is mounted on frictionless rollers, the system must roll to the left to leave the position of the center of mass unchanged in the lab frame. During this rolling, the system has Newtonian momentum

$$\mathbf{P} = -\mathbf{p}_{\rm sound} V = -\frac{\mathbf{S}_{\rm sound} V}{c^2} = -\frac{V}{Ac^2}\frac{dE}{dt}\,\hat{\mathbf{v}}_{\rm sound} = -\frac{L}{c^2}\frac{dE}{dt}\,\hat{\mathbf{v}}_{\rm sound}, \tag{3}$$

with respect to the lab frame, where L is the length of the pressure vessel and V is its volume. This tiny momentum could also be called a "hidden" momentum. Then we could summarize by saying that the system possesses two equal and opposite "hidden" momenta, one of which is a Newtonian momentum and the other is a pseudomomentum.

As the gas molecules move to the right during the wave motion, they carry greater kinetic energy on average than when they move to the left, in the frame of the pressure vessel. That is, the average velocity of right-moving gas molecules is greater than that of left-moving molecules in this frame.[3] So, in a microscopic view we can say that the gas possesses net Newtonian momentum with respect to the pressure vessel, which momentum points to the right. In the lab frame, the entire system moves to the left, so the gas molecules also move to

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[1] It is underappreciated that ordinary kinetic energy is a tiny relativistic correction to the total energy of a moving mass: $E_{\rm total} = mc^2/\sqrt{1 - v^2/c^2} \approx mc^2(1 + v^2/2c^2) = mc^2 + mv^2/2 = E_{\rm rest\ mass} + E_{\rm kinetic}$. Thus, all discussions of kinetic energy in classical mechanics concern tiny relativistic corrections of order $1/c^2$, in which, however, the factor $1/c^2$ is not explicitly displayed.

[2] In a private communication with the author, Stone acknowledges the existence of the momentum described by eq. (2) as well as its association with "hidden" momentum.

[3] Linear longitudinal waves imply zero average velocity for the molecules, so we infer that nonlinear effects must be considered to obtain a full understanding of the momentum density of a sound wave. See, for example, sec. IV of [4]. For sound waves not confined between walls, there can also be a small bulk motion of the medium due to the wave, known as the Stokes drift [14].

the left on average in this frame. Since only a small fraction of the total mass of the system is in the gas molecules, the left-pointing momentum of the gas molecules in the lab frame is less than their right-pointing momentum, and the net momentum of the gas molecules in the lab frame points to the right, even though they are moving to the left on average in this frame.

3 Acknowledgment

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4 References

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