“Hidden” Momentum in a Coaxial Cable

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1 Problem

Calculate the electromagnetic momentum and identify the “hidden” mechanical momentum in a coaxial cable of length L, inner radius a, outer radius b, when a battery of voltage V is connected to one end and a load resistor R_0 is connected to the other. The current may be taken as uniformly distributed over the inner conductor, which has resistivity ρ. The outer conductor has negligible resistivity, and the current flows on it in a thin sheet at radius b. The battery has negligible internal resistance.

Deduce the charge per unit length on the outer surface of the inner conductor. Then, suppose the battery can be turned off in such a way that the current in the cable falls to zero with some time dependence $I(t)$. Calculate the impulse on the charge on the surface of the inner conductor due to the electric field induced by the transient current.

2 Solution

This problem is based on sec. 17 of [1], and on prob. 7.57, ex. 8.3 and ex. 12.12 of [2].

2.1 Electromagnetic Fields and Field Momentum

We denote the resistance per unit length along inner conductor as

$$R = \frac{\rho}{\pi a^2}\,. \tag{1}$$

Then, the total resistance of the cable plus load resistor is $R_0 + RL$. To have current I in the system, the battery must have voltage

$$V = I(R_0 + RL). \tag{2}$$

The current I causes a magnetic field that is readily calculated via Ampère’s law to be (in Gaussian units, and in a cylindrical coordinate system (r, ϕ, z) with the coaxial cable centered on the z axis),

$$\mathbf{B} = \frac{2I}{c}\hat{\phi}\begin{cases} \frac{r}{a^2} & (r < a), \\ \frac{1}{r} & (a < r < b), \\ 0 & (r > b). \end{cases} \tag{3}$$

Inside the wire the electric field is $\mathbf{E}(r < a) = IR\hat{\mathbf{z}}$, as needed to drive the current I against the resistivity ρ. Since the tangential component of the electric field is continuous across a boundary, there must be some electric field in the region $r > a$ as well. Indeed, a charge distribution $Q(z)$ is needed on the surface of the inner conductor to shape the interior electric field to be purely longitudinal.

An analysis of the electric field can be based on the convention that the electric potential $V(r,z)$ is equal to zero on the outer conductor, and is also zero on the plane $z = 0$ (which is not necessarily inside the wire of length L). That is, we suppose the cable extends from $z = -L - R_0/R$ (the position of the battery) to $z = -R_0/R$ (the position of the resistor), so that the electric potential for $r \le a$ can be written as

$$V(r \le a, z) = -IRz. \tag{4}$$

Thus, the potential of the inner conductor at the position of the load resistor is IR_0, and the potential at the position of the battery is $IR(L + R_0/R)$, *i.e.*, the battery voltage (2).

The capacitance per unit length between the inner and outer conductors of the coaxial cable is well known to be

$$C = \frac{1}{2\ln(b/a)}\,. \tag{5}$$

The charge $Q(z)$ per unit length on the inner conductor is therefore

$$Q(z) = CV(r = a, z) = -\frac{IRz}{2\ln(b/a)} = \frac{IRz}{2\ln(a/b)}\,, \tag{6}$$

assuming that $L \gg b$ so that $Q(z)$ is essentially constant over length $\Delta z \ll b$. Further, the potential in the region $a < r < b$ is essentially that for a long wire of charge density $Q(z)$, matched to the condition that $V(r = b) = 0$, namely

$$V(a < r < b, z) = -2Q(z)\ln(r/b) = -\frac{IRz\ln(r/b)}{\ln(a/b)}\,, \tag{7}$$

which also matches eq. (4) at $r = a$. The potential (7) can also be obtained by a separation-of-variables solution to Laplace's equation [1].

The electric field is obtained by taking the gradient of eq. (7), and we find

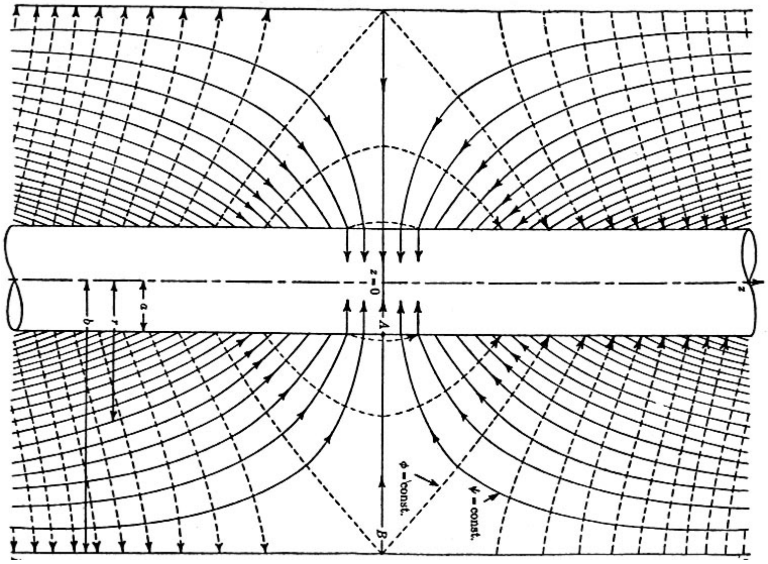
$$\mathbf{E} = IR\begin{cases} \hat{\mathbf{z}} & (r < a), \\ \frac{\ln(r/b)}{\ln(a/b)}\hat{\mathbf{z}} + \frac{z}{r\ln(a/b)}\hat{\mathbf{r}} & (a < r < b), \\ 0 & (r > b). \end{cases} \tag{8}$$

The electromagnetic momentum density is

$$\mathbf{p}_{\rm EM} = \frac{\mathbf{S}}{c^2} = \frac{\mathbf{E}\times\mathbf{B}}{4\pi c} = \frac{I^2R}{2\pi c^2}\begin{cases} -\frac{r}{a^2}\hat{\mathbf{r}} & (r < a), \\ -\frac{\ln(r/b)}{r\ln(a/b)}\hat{\mathbf{r}} + \frac{z}{r^2\ln(a/b)}\hat{\mathbf{z}} & (a < r < b), \\ 0 & (r > b). \end{cases} \tag{9}$$

The Poynting vector $\mathbf{S}$ quantifies the flow of energy from the battery in the region ($a < r < b, z = -L - R_0/R$) to the inner conductor and to the load resistor, where the energy is dissipated in Joule heating.

The figure on the next page (from [1]) shows lines of electric field and of Poynting flux in a coaxial cable that has no terminating resistor, but rather is symmetric about the origin and with power sources at both ends. The example considered here corresponds to, say, the left third of the figure, plus a terminating resistive plate; the power source is at the left of the figure.



The total electromagnetic momentum in the cable is

$$\begin{aligned} \mathbf{P}_{\rm EM} &= \int \mathbf{p}_{\rm EM}\, d{\rm Vol} = \frac{I^2 R\, \hat{\mathbf{z}}}{2\pi c^2 \ln(a/b)} \int_a^b 2\pi r\ dr \int_{-L-R_0/R}^{-R_0/R} dz \frac{z}{r^2} \\ &= \frac{I^2 RL(L + 2R_0/R)}{2c^2} \hat{\mathbf{z}}. \end{aligned} \tag{10}$$

2.2 "Hidden" Mechanical Momentum

Suppose the entire system of coaxial cable, battery and load resistor is isolated from the rest of the Universe and that the center of mass of the system is at rest. Then, we expect the total momentum of the system to be zero. While there is internal motion associated with the electrical current, we expect the net momentum of the current to be zero, since the steady current density $\mathbf{J}$ obeys

$$\int \mathbf{J}\, d{\rm Vol} = 0. \tag{11}$$

This implies that there is mechanical momentum "hidden" somewhere in the system such that the total mechanical momentum cancels the electromagnetic momentum (10).

Jon Thaler[1] (private communication, Aug. 26, 2007) reminds us that the present example is very close to that considered by Einstein in 1905 [3] from which he deduced that the emission of light of energy E lowers the mass of the emitting body according to

$$E = \Delta m c^2. \tag{12}$$

[1]This point was also recently made by Timothy Boyer (private communication, Sept. 21, 2007).

Here, the mass of the battery is reduced as the electromagnetic field carries energy away to the resistive inner conductor and the load resistor. As the latter absorb the energy their masses increase (ignoring possible thermal transport of the absorbed energy). Hence, the mass of the system at positive z is increasing with time, so the system as a whole must be moving in the negative z direction if the center of mass is to remain fixed.

The required motion of the system in the $-z$ direction can be traced to transient forces generated as the current increases from 0 to I [2].

Another confirmation of the result (10) can be found by supposing the current I drops to zero with time. The changing magnetic field induces a longitudinal electric field that pushes on the charge on the surface of the inner conductor, leading to a force on the wire. The force on the conduction electrons merely slows the decrease of the current, but does not cause a net force on the wire. By Faraday's law, the induced electric field at $r = a$ is[2]

$$E_{z,\text{induced}}(r = a) = -\frac{1}{c}\frac{d}{dt}\int_a^b B_\phi \; dr = -\frac{2}{c^2}\frac{dI}{dt}\ln(b/a), \tag{14}$$

noting that $E_{z,\text{induced}}(r = b) = 0$ since the outer (perfect) conductor can support no tangential electric field. The additional force on the surface charge is

$$F_{z,\text{induced}} = \int_{-L-R_0/R}^{-R_0/R} Q(z) E_{z,\text{induced}}(r = a) \; dz = -\frac{RL(L + 2R_0/R)}{2c^2}\frac{dI^2}{dt}, \tag{15}$$

using eq. (6). The momentum kick to the wire as the current rises from zero to I is therefore

$$\Delta\mathbf{P}_{\text{mech}} = \hat{\mathbf{z}}\int F_{z,\text{induced}} \; dt = -\frac{I^2 RL(L + 2R_0/R)}{2c^2}\hat{\mathbf{z}} = -\mathbf{P}_{\text{EM}}. \tag{16}$$

Thus, the back reaction to the process of emission of the electromagnetic energy into the coaxial cable results in a very small mechanical momentum of the cable as a whole in the direction opposite to the energy flow. The corresponding momentum is not, strictly speaking, "hidden", since the entire cable moves as a rigid body. However, this motion is effectively hidden by its tiny magnitude, which is proportional to $1/c^2$.

This result reinforces the interpretation of eq. (10) as field momentum stored in the system, that could be converted to back into mechanical momentum when the current drops to zero. Here, this conversion serves to cancel small but nonzero "hidden" mechanical momentum (16), returning the cable to zero velocity at zero current such that the total momentum is zero at all times.

Since the nonzero electromagnetic momentum of a coaxial cable at rest is always canceled by the "hidden" mechanical momentum, both of these entities can be safely neglected by the pragmatic physicist in this case. Electromagnetic momentum is of greater significance in dynamic phenomena, in which the mechanical momentum is "evident" rather than "hidden", and in which Newton's 3rd law for "evident" mechanical momentum is not satisfied unless the electromagnetic momentum is taken into account [4].

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[2] Alternatively, we can use Faraday's law in differential form,

$$(\boldsymbol{\nabla}\times\mathbf{E}_{\text{induced}})_\phi = \frac{\partial E_{r,\text{induced}}}{\partial z} - \frac{\partial E_{z,\text{induced}}}{\partial r} = -\frac{1}{c}\frac{\partial B_\phi}{\partial t} = -\frac{2\dot{I}}{c^2 r} \qquad (a < r < b). \tag{13}$$

There will be no radial component to the induced field, so eq. (13) integrates to the form (14) after enforcing the condition that $E_{z,\text{induced}}(r = b) = 0$.

3 Comments

The name "hidden" momentum has also been applied to examples in which a permanent magnet resides in a static electric field such that electromagnetic momentum

$$\mathbf{P}_{\text{EM}} = \int \frac{\mathbf{S}}{c^2} d\text{Vol} = \int \frac{\mathbf{E} \times \mathbf{B}}{4\pi c} d\text{Vol} \tag{17}$$

is nonzero [5, 6, 7, 8, 10, 9, 11, 12]. In such examples there is no flow of energy between a source and a sink, and hence no "hidden" motion of the system as a whole. The difficulty in locating the "hidden" mechanical momentum required to cancel the electromagnetic momentum in the isolated system is compounded by the inadequacy for classical models of permanent magnetism.

Suggestive arguments can be made by noting that in static systems the electromagnetic momentum (17) can also be calculated as [9, 10]

$$\mathbf{P}_{\text{EM}} = \int \frac{\mathbf{J}V}{c^2} d\text{Vol}, \tag{18}$$

where $\mathbf{J}$ is the current density and V is the electric potential.[3] If the total momentum is to be zero, then we must have a "hidden" mechanical momentum equal to

$$\mathbf{P}_{\text{hidden}} = -\int \frac{\mathbf{J}V}{c^2} d\text{Vol}. \tag{20}$$

A possible interpretation[4] of eq. (20) is that a charge e with velocity $\mathbf{v}$ that participates in current density $\mathbf{J}$ somehow has its relativistic mass γm reduced by amount eV/c^2, and hence its mechanical momentum is lower by amount $e\mathbf{v}V/c^2$. This interpretation would be justified if somehow the total energy $U = \gamma mc^2 + eV$ of the charge were independent of the strength of the potential V, which however is doubtful.

The physical interpretation of "hidden" mechanical momentum deserves further clarification, in the view of this author.

4 References

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[3] The electromagnetic potential in these conditions can also be calculated as

$$\mathbf{P}_{\text{EM}} = \int \frac{\rho \mathbf{A}}{c} d\text{Vol}, \tag{19}$$

where ρ is the electric charge density and $\mathbf{A}$ is the vector potential.

[4] This interpretation may have first been made by Penfield and Haus [7], although it became relatively well-known from its statement in footnote 9 of [8].

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