AMERICAN JOURNAL OF PHYSICS VOLUME 37, NUMBER 6 JUNE 1969

Examples of Momentum Distributions in the Electromagnetic Field and in Matter*

W. H. FURRY
Department of Physics, Harvard University, Cambridge, Massachusetts 02138
(Received 7 October 1968)

Momentum conservation and the validity of the center-of-mass law are examined for systems made up of electrostatic charges and magnets, in terms of the requirements of special relativity theory. The approach used is that of a quasimicroscopic electromagnetic theory, in which the interaction of the field with material bodies is described by using models for these bodies which involve charge and current densities. It is shown that if small "electromagnetic mass" terms are neglected, both conservation of momentum and the center-of-mass law hold in all cases. In some cases a "hidden momentum" contained in stationary matter plays an important role, as pointed out recently by Shockley and James [W. Shockley and R. P. James, Phys. Rev. Lett. 18, 876 (1967).] In such cases the center-of-mass law can fail in nonrelativistic theory. This is illustrated by the discussion of a special model. Another case in which a "hidden momentum" required by relativity theory is important is the explanation of the null result of the Trouton–Noble experiment. This is discussed in Sec. V.

INTRODUCTION

Some recent papers[1–3] in this Journal have emphasized the importance of accepting the Poynting vector as a description of the space distribution of energy flux and momentum density in the electromagnetic field. As these authors have remarked, the deprecatory attitude toward the Poynting vector traditionally taken in textbooks is not only unfortunately misleading, but actually entirely incorrect. The example of angular momentum in a static field given by Pugh and Pugh[2] is particularly conclusive evidence for this.

In the present paper we shall discuss this subject from the point of view of the requirements of the special theory of relativity, and consider examples in which there are distributions of momentum not only in the electromagnetic field, but also in material bodies. The importance of the possibility that material momentum may be present in a stationary body has recently been emphasized in discussions of the case of a magnet in the presence of a point charge.[1,4,5] This type of material momentum is also of importance in the explanation of the historic Trouton–Noble experiment,[6] and we use a simplified version of the electromechanical system of that experiment as an example in the last section of this paper.

In the following section we shall discuss the relation of the Poynting vector to the free-field, energy–momentum tensor, of which it is a part, and fix the general basis of our discussions of examples. In Sec. III we shall consider the calculation of the angular momentum and the linear momentum as integrals containing the Poynting vector for the case of a magnet in the presence of a point charge. The angular momentum has a unique value, but to get a definite value for the linear momentum we have to choose a model for the structure of the magnet. In Sec. IV we complete the discussion of the magnet-plus-charge example, for a realistic Ampere-current model of the magnet, by calculating the material momentum (called "hidden momentum" by Shockley and James[7]) contained in the stationary magnet. A number of cases are considered and a general theorem on the validity of the center-of-mass law in problems of this sort is proved. In the last section we discuss a simplified model of the Trouton–Noble experiment in terms of the "hidden momentum" of the material system.

II. POYNTING VECTOR AND ENERGY–MOMENTUM TENSOR

The idea that the Poynting vector is not to be taken seriously as a detailed distribution of energy

* Supported in part by the Air Force Office of Scientific Research.

1 R. H. Romer, Amer. J. Phys. **34,** 772 (1966); **35,** 445 (1967).

2 E. M. Pugh and G. E. Pugh, Amer. J. Phys. **35,** 153 (1967).

3 M. G. Calkin, Amer. J. Phys. **34,** 921 (1966).

4 O. Costa de Beauregard, Phys. Lett. **24A,** 177 (1967).

5 S. Coleman and J. H. Van Vleck, Phys. Rev. **171,** 1370 (1968).

6 F. T. Trouton and H. R. Noble, Phil. Trans. Roy. Soc. London A**202,** 165 (1903).

7 W. Shockley and R. P. James, Phys. Rev. Lett. **18,** 876 (1967).

flow and momentum density is a relic from the time when the energy–balance theorem for the electromagnetic field was taken as an isolated result. This single theorem indeed remains valid if one changes the Poynting vector by adding to it any vector whose divergence is zero. Such a change is clearly forbidden, however, by the theorems for momentum and angular momentum. The example given by Pugh and Pugh[2] conclusively vindicates the Poynting vector for the notorious case of a charged magnet for which generations of textbook writers declared that there could obviously be no actual energy flux.

The relations between the theorems for energy, momentum, and angular momentum can be summarized best in the relativistic notation of the energy–momentum tensor. We here consider special relativity only, and give only a summary of the formulas. Discussion and proofs are given in standard books on relativity, for example that by Møller.[8]

In the notation $x_4=ict$, which is most convenient for dealing with special relativity only, the Poynting vector S can be written

$$S_\mu=-icT_{4\mu}\ (\mu=1,2,3). \tag{1}$$

The space components of the electromagnetic energy–momentum tensor $T_{\mu\nu}$, with $\mu, \nu=1, 2, 3$, are the components of Maxwell's electromagnetic stress tensor. The momentum density $\mathbf{g}$ and the energy density W make up the remaining components of the tensor T_{kl} $(k, l=1, 2, 3, 4)$:

$$g_\mu=-(i/c)T_{\mu4}\ (\mu=1,2,3), \tag{2}$$

$$W=-T_{44}. \tag{3}$$

If a field whose energy–momentum tensor is T_{kl} interacts with another system (e.g., if the electromagnetic field interacts with matter), the force per unit volume f exerted by the field on the other system and the rate per unit time and volume, icf_4, at which the field does work on the other system are given by

$$f_k=-(\partial/\partial x_l)T_{kl}, \tag{4}$$

the negative divergence of the energy–momentum tensor of the field. (Latin letters take values 1, 2, 3, 4; Greek letters, values 1, 2, 3.) If T_{kl} is a closed system, not in interaction with anything else, the equation

$$(\partial/\partial x_l)T_{kl}=0, \tag{5}$$

expresses the conservation laws for energy and momentum. The total momentum $\mathbf{P}$ and total energy $-icP_4$ of the closed system are given by

$$P_k=-(i/c)\int T_{k4}d^3x, \qquad d^3x=dx_1dx_2dx_3, \tag{6}$$

and are constant in time,

$$(d/dx_4)P_k=0, \tag{7}$$

as a consequence of Eq. (5). For a closed system the quantities P_k are the components of a four-vector.

For any closed system the conservation of angular momentum has the consequence[9] that

$$T_{kl}=T_{lk}. \tag{8}$$

The energy–momentum tensor of a closed system must be symmetric. Since the free-space electromagnetic field is a closed system, it follows that the momentum density, Eq. (2), is equal to the energy–flux vector [Eq. (1)] times a factor c^{-2}:

$$\mathbf{g}=c^{-2}\mathbf{S}. \tag{9}$$

The symmetry condition, Eq. (8), is not only a necessary but also a sufficient condition for the conservation of angular momentum. The three space components M_{23}, M_{31}, M_{12} of the antisymmetric tensor,

$$M_{ik}=-(i/c)\int(x_iT_{k4}-x_kT_{i4})d^3x, \tag{10}$$

make up the angular-momentum axial three-vector,

$$\mathbf{M}=\int\mathbf{x}\times\mathbf{g}d^3x, \tag{11}$$

and it follows[10] from Eqs. (5) and (8) that

$$(d/dx_4)M_{ik}=0. \tag{12}$$

Equation (12) shows not only that $\mathbf{M}$ is constant, but also $(k=4)$ that,

$$\int\mathbf{x}(W/c^2)d^3x=t\int\mathbf{g}d^3x+\text{const}. \tag{13}$$

[8] C. Møller, *The Theory of Relativity* (Oxford University Press, London, 1952).

[9] See Ref. 8, pp. 164 ff.

[10] See Ref. 8, pp. 168 ff.

The mass of the system being $\int (W/c^2) d^3x$, this equation means that

$$\mathbf{P} = \int \mathbf{g} d^3x = \left[\int (W/c^2) d^3x \right] \cdot (d\mathbf{x}_c/dt), \quad (14)$$

where

$$\mathbf{x}_c = \left[\int (W/c^2) d^3x \right]^{-1} \int \mathbf{x} (W/c^2) d^3x, \quad (15)$$

is the position vector of the center of mass. This *law of the center of mass* always holds for a closed system in special relativity theory. The conservation laws for energy and momentum, Eq. (7), follow simply from the divergence condition, Eq. (5). To prove the law in Eq. (14) we must also have the symmetry condition, Eq. (8).

In nonrelativistic mechanics the symmetry of the three-dimensional stress tensor follows from the conservation of angular momentum and is equivalent to it. There is, however, no necessary connection between the stress tensor and the energy flux and momentum density, and therefore no necessary relation between these latter quantities. Accordingly, the law of the center of mass does not necessarily hold in nonrelativistic mechanics,[11] though conservation of momentum does always hold. We shall see in Sec. IV an example of a nonrelativistic system in which the law of the center of mass does not hold.

The examples we shall discuss involve the electromagnetic field and macroscopic bodies. Two general types of theory can be used for such systems. One is a macroscopic phenomenological electrodynamics in which the relations between the electromagnetic field and matter are dealt with by means of constitutive relations, hysteresis curves, boundary conditions, ponderomotive force laws, and so on. The other is a theory like the Lorentz electron theory, in which the electromagnetic field is always treated with the equations of vacuum electrodynamics, except that sources of the fields are included in the field equations. The electromagnetic properties of the matter are dealt with in terms of models made with these sources—charge and current densities for realistic models, and on occasion, mythical densities of magnetic pole strength for hypothetical discussion.

Phenomenological electrodynamics is useful and necessary for many practical problems. It is however, complicated, especially in relativistic formulations. From the earliest years of relativity theory there were two rival formulations, given by Minkowski and by Abraham.[12] In Minkowski's phenomenological theory the field energy–momentum tensor is not symmetric.

The formulation following the general approach of the electron theory is much simpler, and appeals to the physicist's wish to think in terms of a model of "what happens inside" bodies. This is the basis we shall use for the discussion of examples.

An essential point is that in this formulation the energy–momentum tensor $T_{kl}^{(f)}$ of the electromagnetic field is symmetric, as in the case of an isolated field. Since the total tensor $T_{kl}^{(f)} + T_{kl}^{(m)}$ must be symmetric, the matter energy–momentum tensor $T_{kl}^{(m)}$ is symmetric. The symmetry equations (8) and (9) hold for the field and matter separately [superscript (f) or (m)] though the conservation laws, Eqs. (5), (7), (12)–(14), hold only for the total system. A particularly important fact is that in the matter, as in the field, a flow of energy always has a momentum associated with it. We have for the field,

$$\mathbf{g}^{(f)} = c^{-2} \mathbf{S}^{(f)}, \quad (16)$$

and also for the matter,

$$\mathbf{g}^{(m)} = c^{-2} \mathbf{S}^{(m)}. \quad (17)$$

III. FIELD-ANGULAR MOMENTUM AND LINEAR MOMENTUM FOR A MAGNET AND AN EXTERNAL CHARGE

We consider a small magnet of magnetic moment $\mathbf{m}$ located at the origin, and a charge q at position $\mathbf{x} = \mathbf{a}$. The electric field of the charge is

$$\mathbf{E} = q(\mathbf{x} - \mathbf{a}) \mid \mathbf{x} - \mathbf{a} \mid^{-3} = -q \nabla \mid \mathbf{x} - \mathbf{a} \mid^{-1}, \quad (18)$$

and at points well outside the magnet ($\mathbf{r} \gg$ dimensions of magnet) its magnetic field is

$$\mathbf{B} = -\mathbf{m} r^{-3} + 3(\mathbf{m} \cdot \mathbf{x}) \mathbf{x} r^{-5}. \quad (19)$$

The field–momentum density in this region is

$$\mathbf{g}^{(f)} = c^{-2} \mathbf{S}^{(f)} = (4\pi c)^{-1} \mathbf{E} \times \mathbf{B}, \quad (20)$$

[11] Another feature of relativity theory which is absent in nonrelativistic mechanics is the equivalence of mass and energy, which is clearly visible in Eqs. (14), (15) as the inertia of energy.

[12] See Ref. 8, pp. 196 ff.

and the density of angular momentum is

$$\mathbf{x}\times\mathbf{g}^{(f)}=(4\pi c)^{-1}\mathbf{x}\times(\mathbf{E}\times\mathbf{B}). \quad (21)$$

The point $\mathbf{x}=\mathbf{a}$ gives no trouble in integrating these expressions to get the total momentum and total angular momentum of the field, because the volume element in spherical coordinates centered at this point has a factor $|\mathbf{x}-\mathbf{a}|^2$ which cancels the factor $|\mathbf{x}-\mathbf{a}|^{-2}$ in $\mathbf{E}$. At the origin, however, there is an r^{-3} singularity in $\mathbf{B}$. The extra factor r in Eq. (21) makes the integral for the angular momentum converge when we simply integrate over all space. If we assume the magnet is extremely small (dimensions $\ll a$), we can get the correct value of the angular momentum without taking the structure of the magnet into account. We cannot calculate the momentum by using Eqs. (19) and (20) thoughout all space, because the integral does not converge at the origin. To get the correct value we must replace Eq. (19) by some other expression for small values of r, so as to take into account the structure of the magnet.

We formulate the integrations in a spherical coordinate system with the polar axis, $\theta=0$, through the point $\mathbf{x}=\mathbf{a}$. In terms of the unit vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ for corresponding rectangular coordinates we have

$$\begin{aligned}&\mathbf{a}=\mathbf{k}a,\\&\mathbf{m}=m(\mathbf{i}\sin\theta_0+\mathbf{k}\cos\theta_0),\\&\mathbf{x}=r(\mathbf{i}\sin\theta\cos\varphi+\mathbf{j}\sin\theta\sin\varphi+\mathbf{k}\cos\theta).\end{aligned} \quad (22)$$

We use the expansion

$$\begin{aligned}|\mathbf{x}-\mathbf{a}|^{-1}&=\sum_{l=0}^{\infty}r^la^{-l-1}P_l(\cos\theta), \qquad r<a\\&=\sum_{l=0}^{\infty}a^lr^{-l-1}P_l(\cos\theta), \qquad r>a,\end{aligned} \quad (23)$$

from which we have

$$\begin{aligned}\nabla|\mathbf{x}-\mathbf{a}|^{-1}&=\sum_{l=0}^{\infty}lr^{l-2}a^{-l-1}\mathbf{x}P_l(\cos\theta)\\&\qquad-r^{l-1}a^{-l-1}\boldsymbol{\theta}_1P_l'(\cos\theta)\sin\theta,\\&=\sum_{l=0}^{\infty}-(l+1)a^lr^{-l-3}\mathbf{x}P_l(\cos\theta)\\&\qquad-a^lr^{-l-2}\boldsymbol{\theta}_1P_l'(\cos\theta)\sin\theta.\end{aligned} \quad (24)$$

Here $\boldsymbol{\theta}_1$ is the unit vector in the direction of increasing θ:

$$\boldsymbol{\theta}_1=\mathbf{i}\cos\theta\cos\varphi+\mathbf{j}\cos\theta\sin\varphi-\mathbf{k}\sin\theta. \quad (25)$$

We shall use these formulas in a calculation of the field momentum $\mathbf{P}^{(f)}=\int\mathbf{g}^{(f)}d^3x$. We first give simply the result for the field–angular momentum:

$$\begin{aligned}\mathbf{M}^{(f)}&=\int\mathbf{x}\times\mathbf{g}^{(f)}d^3x=(4\pi q/a)(\mathbf{m}-m\cos\theta_0\cdot\mathbf{k}),\\&=4\pi q[a^{-1}\mathbf{m}-a^{-3}(\mathbf{m}\cdot\mathbf{a})\mathbf{a}];\end{aligned} \quad (26)$$

here the last expression contains no reference to a particular coordinate system. This can be calculated by using Eqs. (18), (19), (21) and (22)–(25) in the way illustrated below in our calculation of $\mathbf{P}^{(f)}$; the integral converges everywhere, and we simply use Eq. (19) everywhere. Since the magnet is assumed very small, the contribution from the region where Eq. (19) does not hold is negligible.

Having verified that the structure of the magnet does not affect the value of $\mathbf{M}^{(f)}$, we can also obtain this value, Eq. (26), by using a result originally due to Thomson[13] and quoted more recently by Saha,[14] Wilson,[15] and others. The system of an electric charge q and a magnetic pole μ has angular momentum of magnitude $q\mu/c$ and direction along the line from q to μ. If we regard the magnet as consisting of two poles $\pm\mu$ at points $\mathbf{x}=\mathbf{m}/2\mu$ and calculate $\mathbf{M}^{(f)}$ by vector addition, the result, Eq. (26), is obtained in the limit $\mu\to\infty$, $m/\mu\to 0$.

We now proceed to the calculation of $\mathbf{P}^{(f)}=\int\mathbf{g}^{(f)}d^3x$. In calculating $\mathbf{E}\times\mathbf{B}$ from Eqs. (18), (19), and (24) we encounter vector products which can be evaluated with Eqs. (22) and (25):

$$\begin{aligned}\mathbf{m}\cdot\mathbf{x}&=mr(\cos\theta_0\cos\theta+\sin\theta_0\sin\theta\cos\varphi),\\\mathbf{m}\times\mathbf{x}&=mr[-\mathbf{i}\cos\theta_0\sin\theta\sin\varphi+\mathbf{j}(\cos\theta_0\sin\theta\cos\varphi\\&\qquad-\sin\theta_0\cos\theta)+\mathbf{k}\sin\theta_0\sin\theta\sin\varphi],\end{aligned}$$

[13] J. J. Thomson, *Electricity and Matter*, Silliman Lectures (Yale University Press, New Haven, Conn., 1904); *Elements of the Mathematical Theory of Electricity and Magnetism* (Cambridge University Press, Cambridge, England, 1904), 3rd and later eds. The details of the integration, which Thompson did not give in these publications, are readily supplied if we use expressions (24).

[14] M. N. Saha, Ind. J. Phys. **10**, 145 (1936); Phys. Rev. **75**, 1968 (1949).

[15] H. A. Wilson, Phys. Rev. **75**, 309 (1949).

$$\mathbf{m}\times\boldsymbol{\theta}_1=m[-\mathbf{i}\cos\theta_0\cos\theta\sin\varphi+\mathbf{j}(\cos\theta_0\cos\theta\cos\varphi+\sin\theta_0\sin\theta)+\mathbf{k}\sin\theta_0\cos\theta\sin\varphi]. \quad (27)$$

$$\mathbf{x}\times\mathbf{x}=0$$

$$\mathbf{x}\times\boldsymbol{\theta}_1(=r\boldsymbol{\varphi}_1)=r(-\mathbf{i}\sin\varphi+\mathbf{j}\cos\varphi).$$

We can now integrate over φ:

$$\int_0^{2\pi} d\varphi\cdot\mathbf{m}\times\mathbf{x}=-2\pi mr\mathbf{j}\sin\theta_0\cos\theta,$$

$$\int_0^{2\pi} d\varphi\cdot\mathbf{m}\times\boldsymbol{\theta}_1=2\pi m\mathbf{j}\sin\theta_0\sin\theta,$$

$$\int_0^{2\pi} d\varphi(\mathbf{m}\cdot\mathbf{x})(\mathbf{x}\times\boldsymbol{\theta}_1)=\pi mr^2\mathbf{j}\sin\theta_0\sin\theta. \quad (28)$$

We are then concerned with the following integrals over θ:

$$\int_0^{\pi}\sin\theta d\theta\cdot\cos\theta P_l(\cos\theta)=\int_{-1}^{1}dx\cdot xP_l(x)=\tfrac{2}{3}\delta_{l1},$$

$$\int_0^{\pi}\sin\theta d\theta\cdot\sin^2\theta P_l'(\cos\theta)=\int_{-1}^{1}dx(1-x^2)P_l'(x),$$

$$=\int_{-1}^{1}dx\cdot 2xP_l(x)=\tfrac{4}{3}\delta_{l1}. \quad (29)$$

When the results, Eqs. (28) and (29), are used along with Eqs. (18)–(20) and (24), we get, on collecting the terms,

$$\int_{r>r_0} d^3x\cdot\mathbf{g}^{(f)}=-(qm/c)\mathbf{j}\sin\theta_0\times\left(\int_{r_0}^{a}a^{-2}r^{-1}\cdot 0\cdot dr+\int_a^{\infty}ar^{-4}dr\right),$$

$$=-(qm/3a^2c)\mathbf{j}\sin\theta_0,$$

$$=(q/3a^3c)(\mathbf{m}\times\mathbf{a}), \quad (30)$$

where r_0 is any distance $<a$ at which Eq. (19) still describes the magnetic field. The factor 0 in the first term in curly brackets comes from the integration over θ and φ. The factor r^{-1} indicates that the original integral actually diverges in the limit $r_0\to 0$, and we cannot get a correct result for the total momentum by simply setting $r_0=0$.

We can write Eq. (30) in the form

$$\mathbf{P}^{(f)}=\int d^3x\cdot\mathbf{g}^{(f)},$$

$$=(q/3a^3c)(\mathbf{m}\times\mathbf{a})+\int_{r<r_0}d^3x\mathbf{g}^{(f)}, \quad (31)$$

where for $r<r_0$ we must take the structure of the magnet into account and use an expression other than Eq. (19) for **B**.

The simplest model of the magnet, for this purpose, is a uniformly magnetized sphere of radius r_0. The magnetization (moment per unit volume) is

$$\mathbf{I}=3\mathbf{m}r_0^{-3}/4\pi. \quad (32)$$

Equation (19) holds exactly outside the sphere.

If we adopt a "realistic" Ampere-current model for this magnet, the current flows in the surface $r=r_0$, with surface density

$$\mathbf{K}=(3/4\pi r_0^4)(\mathbf{m}\times\mathbf{x}), \qquad r=r_0. \quad (33)$$

The field inside the magnet has the constant value

$$\mathbf{B}=2\mathbf{m}r_0^{-3}, \qquad r<r_0. \quad (34)$$

The normal components of the vectors, Eqs. (19) and (34), are equal at the sphere's surface, $r=r_0$, and the difference of the tangential components—the "surface curl"—is

$$[(\mathbf{B}_<-\mathbf{B}_>)\times(\mathbf{x}/r)]_{r=r_0}=4\pi\mathbf{K}. \quad (35)$$

When we now use Eqs. (20), (18), (34), and the first line of Eq. (24) to evaluate the last term in Eq. (31), we find that the integrals over angles are those already done in Eq. (28) and (29). The result is

$$\int_{r<r_0}d^3x\mathbf{g}^{(f)}=-(2qm/3a^2c)\mathbf{j}\sin\theta_0,$$

$$=(2q/3a^3c)(\mathbf{m}\times\mathbf{a}), \quad (36)$$

and from Eq. (31) we have as the value of the total field momentum

$$\mathbf{P}^{(f)}=\int d^3x\mathbf{g}^{(f)}=(q/a^3c)(\mathbf{m}\times\mathbf{a}), \quad (37)$$

for the model of Ampere currents on the surface of a sphere.

Let us look at the case of a "mythical" magnetic-pole model of the spherical magnet. The surface density of magnetic pole strength in such a model would be

$$\sigma = (3/4\pi r_0^3)(\mathbf{m}\cdot\mathbf{x}), \qquad r = r_0, \tag{38}$$

and the field inside the sphere would be

$$\mathbf{B} = -\mathbf{m}r_0^{-3}, \qquad r < r_0. \tag{39}$$

The tangential components $(\mathbf{B}\times\mathbf{x})$ of the vectors, Eqs. (19) and (39), are equal, and the difference of the normal components—the "surface divergence"—is

$$[(\mathbf{B}_< - \mathbf{B}_>)\cdot\mathbf{x}/r]_{r=r_0} = 4\pi\sigma. \tag{40}$$

When we compare Eqs. (34) and (39), we see that since $\mathbf{g}^{(f)}$ is a linear function of $\mathbf{B}$, the value of the field momentum inside the sphere is just $(-\frac{1}{2})$ times the value of Eq. (36), and the total field momentum, Eq. (31), is now

$$\mathbf{P}^{(f)} = 0 \text{ for magnetic-pole model.} \tag{41}$$

It can now be noted that this result, Eq. (41), holds for any model which uses mythical magnetic poles and no ampere currents. For the case of a point charge and a single point pole the lines of $\mathbf{E}\times\mathbf{B}$ are circles, with the line joining charge and pole as axis of symmetry. The volume integral of $\mathbf{g}^{(f)}$ is obviously zero. This is also true if there are any number of poles and any number of charges, since in such a case

$$\mathbf{E}\times\mathbf{B} = \sum_i \mathbf{E}_i \sum_k \mathbf{B}_k = \sum_{ik} (\mathbf{E}_i\times\mathbf{B}_k), \tag{42}$$

and the integrals for the individual terms are all zero. When we regard continuous distributions as limits of distributions of point charges and point poles, we see that:

For any distribution of electrostatic charge and a stationary magnet or magnets composed of mythical poles with no ampere currents, the electromagnetic field momentum is zero, as stated in Eq. (41). The magnet or magnets can be of any size, shape, and distribution of magnetization.

We can now obtain simple expressions for the field momentum for the case of the realistic model in which we describe the magnet or magnets in terms of ampere currents. First, we note that for the magnetic-pole model the magnetic field which our electron-theory approach gives for the interior of a magnet is the field $\mathbf{H}$ of simple macroscopic theory, and for the ampere-current model it is the field $\mathbf{B}$ of macroscopic theory. In Gaussian units $\mathbf{H}$ and $\mathbf{B}$ are equal outside magnets, and inside magnets the relation is

$$\mathbf{B} = \mathbf{H} + 4\pi\mathbf{I} \text{ (macroscopic theory)}, \tag{43}$$

where $\mathbf{I}$ is the intensity of magnetization, or magnetic moment per unit volume,

$$\int \mathbf{I}d^3x = \sum_k \mathbf{m}_k, \tag{44}$$

the sum being over all magnets in the system. Since $\mathbf{I}=0$ outside magnets, Eq. (43) can be taken to hold everywhere.

We now have

$$\mathbf{P}^{(f)}(\text{A currents}) - \mathbf{P}^{(f)}(\text{mythical poles}) = (4\pi c)^{-1}\int d^3x[\mathbf{E}\times(\mathbf{B}-\mathbf{H})], \tag{45}$$

where $\mathbf{B}$ and $\mathbf{H}$ of the right number are those of simple macroscopic theory. Using Eqs. (41) and (43), we have

$$\mathbf{P}^{(f)} = c^{-1}\int d^3x\mathbf{E}\times\mathbf{I} \quad \text{for A-current model.} \tag{46}$$

This can be put in an interesting form if we use $\mathbf{E} = -\nabla V$ and perform an integration by parts:

$$\mathbf{P}^{(f)} = c^{-1}\int d^3x\mathbf{I}\times\nabla V,$$

$$= c^{-1}\int d^3x V\nabla\times\mathbf{I}. \tag{47}$$

In the absence of conduction currents, the macroscropic vector $\mathbf{H}$ in a stationary system satisfies the equation

$$\nabla\times\mathbf{H} = 0 \text{ (macroscopic)}, \tag{48}$$

and the vector $\mathbf{B}$ satisfies

$$\nabla\times\mathbf{B} = (4\pi/c)\mathbf{J}, \tag{49}$$

where $\mathbf{J}$ is the ampere current density. Using Eqs. (43), (48), and (49), we get from Eq. (47)

$$\mathbf{P}^{(f)} = c^{-2}\int d^3x V\mathbf{J}. \tag{50}$$

Equations (46)–(48), (50) express the field momentum $\mathbf{P}^{(f)}$ of a stationary system in terms of integrals over the region or regions occupied by magnetic material. $\mathbf{P}^{(f)}$ can also be expressed in terms of quantities in the electrostatic charge distribution. If this consists of a point charge q at the point $\mathbf{x}=\mathbf{a}$, we have

$$V(\mathbf{x})=q\,|\,\mathbf{x}-\mathbf{a}\,|^{-1}, \tag{51}$$

and when this is substituted in Eq. (50) we have

$$\mathbf{P}^{(f)}=(q/c)\mathbf{A}(\mathbf{a}), \tag{52}$$

where $\mathbf{A}(\mathbf{a})$ is the vector potential of the magnet or magnets at the position $\mathbf{a}$ of the charge

$$\mathbf{A}(\mathbf{x})=c^{-1}\int d^3x'\mathbf{J}(\mathbf{x}')\,|\,\mathbf{x}-\mathbf{x}'\,|^{-1}. \tag{53}$$

If the electrostatic field is produced by a charge distribution $\rho(\mathbf{x})$ instead of a single point charge, we have

$$\mathbf{P}^{(f)}=c^{-1}\int d^3x\mathbf{A}\rho. \tag{54}$$

This result was given earlier, in a slightly different form, by Calkin.[3]

IV. EFFECTS OF CHANGES IN CHARGE–MAGNET SYSTEM, MOMENTUM IN STATIONARY MATTER

A. Demagnetization of a Magnet

If the magnetic moment of a magnet is changed, the change of magnetic field causes a transient electric field which will accelerate any nearby charges. We consider the case of a stationary system of a magnet and an external charge, and calculate the motions given to the charge and the magnet when the magnetic moment is reduced to zero from its initial value $\mathbf{m}$.

The electric field produced at the point $\mathbf{a}$ by the change of the magnetic field is

$$\mathbf{E}(\mathbf{a})=-c^{-1}\partial\mathbf{A}(\mathbf{a})/\partial t. \tag{55}$$

If the change occurs quickly, so that the charge q does not move from the point $\mathbf{a}$ until afterwards, the momentum given to the charge is

$$\mathbf{p}_q=\int\mathbf{E}'(\mathbf{a})q\,dt=-(q/c)[\mathbf{A}(\mathbf{a},t')-\mathbf{A}(\mathbf{a},0)],$$
$$=(q/c)\mathbf{A}(\mathbf{a},0), \tag{56}$$

since $\mathbf{A}(\mathbf{a},t')=0$ after the demagnetization. We note that for the case of an ampere-current model this momentum given to the charge is the field momentum $\mathbf{P}^{(f)}$, Eq. (52).

1. *Fictitious-Pole Model of the Magnet*

For this unimportant model we consider only the case of a very small magnet (size $\ll a$). The vector potential can be taken axially symmetric around the line of the moment $\mathbf{m}$, and it is easily verified that the form

$$\mathbf{A}=(\mathbf{m}\times\mathbf{x})/r^3, \tag{57}$$

gives the field $\mathbf{B}$ of Eq. (19), when we calculate $\mathbf{B}=\nabla\times\mathbf{A}$. Then

$$\mathbf{p}_q=(q/ca^3)(\mathbf{m}\times\mathbf{a}), \tag{58}$$

for the general case of a very small magnet located at the origin.

In a theory using magnetic poles, a pole moving in an electric field must experience a force

$$-(\mu/c)(\mathbf{v}\times\mathbf{E}), \tag{59}$$

analogous to the Lorentz force

$$(q/c)(\mathbf{v}\times\mathbf{B}), \tag{60}$$

on a charge moving in a magnetic field. For the motion of poles which changes the magnetic moment from the value m to zero, we must have

$$\sum_i\int\mu_i\mathbf{v}_i\,dt=-\mathbf{m}, \tag{61}$$

and the electric field at the position of the magnet—the origin—is $E=-q\mathbf{a}/a^3$. The time integral of the sum of the forces, Eq. (59), on the poles is found to be

$$\mathbf{p}_m=-(q/ca^3)(\mathbf{m}\times\mathbf{a})=-\mathbf{p}_q. \tag{62}$$

In this case both the charge and the magnet receive momentum from the electromagnetic field. The total momentum they receive is zero and the initial and final values of $\mathbf{P}^{(f)}$ are zero [Eq. (41)].

2. *Ampere-Current Model of a Conducting (or Shielded) Magnet*

This is the most realistic model. The magnetization is described in terms of ampere currents, and the magnetic material has sufficient conductivity so that the electric field strength is zero in its interior. Our discussion of this model also covers the cases of a magnet, conducting or nonconducting, surrounded by a metal shield, and of a solenoid.

The calculation of the momentum given to the charge q has already been given in Eqs. (55) and (56). We have, using Eq. (53),

$$\mathbf{p}_q = (q/c)\mathbf{A}(\mathbf{a}, 0),$$

$$= (q/c^2)\int d^3x\mathbf{J}(\mathbf{x})\mid \mathbf{a}-\mathbf{x}\mid^{-1},$$

$$= c^{-2}\int d^3x V_q(\mathbf{x})\mathbf{J}(\mathbf{x}), \qquad (63)$$

where $V_q(\mathbf{x})$ is the potential which would be produced at the point $\mathbf{x}$ by the charge q if there were no conducting surface—of magnet, metal shield, or solenoid wire—carrying induced charge.

The induced charge q' distributed over the conducting surface is also acted on by the transient field $\mathbf{E}'$ of Eq. (55). The momentum given to this charge, and thus to the magnet, solenoid, or shield, is

$$\mathbf{p}_{q'} = c^{-1}\int dS\sigma(\mathbf{x}')\mathbf{A}(\mathbf{x}', 0),$$

$$= c^{-2}\int dS\sigma(\mathbf{x}')\int d^3x\mathbf{J}(\mathbf{x})\mid \mathbf{x}-\mathbf{x}'\mid^{-1}, \qquad (64)$$

where $\sigma(\mathbf{x})dS$ is an element of induced surface charge. Then

$$\mathbf{p}_{q'} = c^{-2}\int d^3x V_{q'}(\mathbf{x})\mathbf{J}(\mathbf{x}), \qquad (65)$$

where $V_{q'}(\mathbf{x})$ is the electrostatic potential produced at x by the induced charge. Since the total potential inside the screening surface is constant, we have

$$V_{q'}(\mathbf{x}) = V(\mathbf{x}) - V_q(\mathbf{x}) = \text{const} - V_q(\mathbf{x}). \qquad (66)$$

The result is

$$\mathbf{p}_{q'} = -\mathbf{p}_q. \qquad (67)$$

The point charge and the magnet (or solenoid, or shield) receive equal and opposite momenta.[16] As in the preceding case, both of these momenta are received from the field. The field momentum is zero throughout, since V in Eq. (50) is constant.

3. *Ampere-Current Model of a Nonconducting Magnet*

The conductivity of the magnet is so small that the presence of the charge q does not cause any appreciable induced charge on its surface during the time of the experiment. A conceptual experiment showing that such a system is possible in principle is described by Shockley and James.[7] In their model the current is produced by a symmetrical pair of counterrotating disks with oppositely charged rims, and the "demagnetization" is accomplished by having a frictional force bring the disks to rest.

The discussion of the force on the charge q is the same as in the other two cases, and gives the result in Eq. (63). In this case the potential V_q is the whole electrostatic potential V, and from Eq. (63) and (50) we have

$$\mathbf{p}_q = \mathbf{P}_{initial}^{(f)}. \qquad (68)$$

The charge q receives the momentum that was originally in the electromagnetic field. There is no action of the field on the currents in the magnet, except a self-inductance effect tending to maintain them, which is overcome by the processes causing demagnetization.

If we were to consider only the electromagnetic field and its actions on charges and currents, there would seem to be a paradox. In a complete treatment we must consider also the distribution of energy and momentum in the material of the magnet. Shockley and James[7] pointed out that the "hidden" material momentum in the magnet is of the correct magnitude and sign to provide the magnet with a "recoil," after demagnetization, with momentum $-\mathbf{p}_q$.

Let us calculate a component of the momentum

---

[16] Calkin (Ref. 3) gives a correct and useful discussion of the forces on external charges, but overlooks the existence of shielding charges. This leads him to assert on the first page of his paper that when the flux through a long solenoid is changed, there is a force on a nearby charge, but no corresponding force on the solenoid.

P corresponding to a momentum density **g**:

$$P_1=\int g_1 d^3x,$$

$$=\left(x_1\int g_1 dx_2 dx_3\right)_{-\infty}^{\infty}-\int x_1(\partial g_1/\partial x_1)d^3x, \quad (69)$$

by an integration by parts. The first term vanishes if **g** falls off more rapidly than r^{-3} for $r\to\infty$. We then have

$$P_1=-\int x_1[\nabla\cdot\mathbf{g}-(\partial g_2/\partial x_2)-(\partial g_3/\partial x_3)]d^3x. \quad (70)$$

The second term is seen to vanish when we perform the integration over x_2, and the third term likewise vanishes. We then have the result

$$\mathbf{P}=-\int d^3x\mathbf{x}\nabla\cdot\mathbf{g}. \quad (71)$$

The total energy–flux vector, $\mathbf{S}=\mathbf{S}^{(f)}+\mathbf{S}^{(m)}$, has zero divergence in our initial stationary system. Then

$$\nabla\cdot\mathbf{S}^{(m)}=-\nabla\cdot\mathbf{S}^{(f)}. \quad (72)$$

Since the momentum densities $\mathbf{g}^{(m)}$ and $\mathbf{g}^{(f)}$ are proportional to the corresponding energy fluxes, we have, from Eqs. (16), (17), (71), and (72),

$$\mathbf{P}^{(m)}=-\mathbf{P}_{initial}{}^{(f)}. \quad (73)$$

This momentum is initially present in the magnet as momentum density associated with the energy flow through the matter of the magnet from regions where $\mathbf{E}\cdot\mathbf{J}>0$ to regions where $\mathbf{E}\cdot\mathbf{J}<0$. After the demagnetization this same momentum is still present in the magnet, in total amount $\mathbf{P}^{(m)}$ and with density equal to the product of the mass density and the velocity of the magnet. Accordingly, the product of the mass of the magnet and its recoil velocity is $\mathbf{P}^{(m)}=-\mathbf{P}^{(f)}(\text{initial})=-\mathbf{p}_q$. The center-of-mass law holds, and there is no paradox.

4. *Discussion of a Special Model*[17]

This general result, that the center-of-mass law holds when a nonconducting magnet is demagnetized in the presence of a charge, is based directly on the use of relativity theory, which gives us the relation, Eq. (17), between $\mathbf{g}^{(m)}$ and $\mathbf{S}^{(m)}$. This relativistic relation does not depend on the structure of the magnet. Nonrelativistic theory provides no such general formula for $\mathbf{g}^{(m)}$. We shall now consider a special model of the structure of the magnet, which allows us to make detailed calculations of $\mathbf{P}^{(m)}$ in both nonrelativistic theory (Newtonian mechanics) and relativistic theory.

We construct the magnet with two smooth tubes which confine the moving charged particles to circular paths of radius b in the xy plane, with center at the origin. One tube contains particles of charge $+e$ moving counterclockwise, and the other tube, closely adjacent to it, contains particles of charge $-e$ moving clockwise. The point charge q is at the point $\mathbf{a}=-\mathbf{i}a$, and $a\gg b$, so that the field **E** can be taken to have a constant value over the paths of the particles

$$\mathbf{E}=\mathbf{i}q/a^2, \qquad r=b, \quad (74)$$

and the electrostatic potential is given in good approximation by

$$V=q/a-(bq/a^2)\cos\varphi. \quad (75)$$

The kinetic energies of particles of charges $\pm e$ are then

$$T_\pm=T_0\pm e(bq/a^2)\cos\varphi. \quad (76)$$

Writing $n_\pm(\varphi)$ for the numbers of particles of charges $\pm e$ per unit path length, $v_\pm(\varphi)$ for their velocities in the $+\varphi$ direction ($v_-<0$), and $m_\pm'$ for their masses, we have

$$P_y{}^{(m)}=\int m_+'v_+(\varphi)\cos\varphi\cdot n_+(\varphi)bd\varphi$$

$$+\int m_-'v_-(\varphi)\cos\varphi\cdot n_-(\varphi)bd\varphi. \quad (77)$$

Now $n_+(\varphi)v_+(\varphi)$ is the number of particles of charge $+e$ passing a point φ per unit time. Accordingly, it is independent of φ and can be written simply (n_+v_+). Similarly, $n_-(\varphi)v_-(\varphi)=$

[17] A model of this general type has been discussed, with a different approach and different emphasis, by Paul Renfield Jr. and Hermann A. Haus in *Electrodynamics of Moving Media* (Technology Press, Cambridge, Massachusetts, 1967), pp. 214–216.

const $= (n_- v_-)$. Then

$$P_y^{(m)} = (n_+ v_+) b \int m_+' \cos\varphi d\varphi + (n_- v_-) b \int m_-' \cos\varphi d\varphi. \quad (78)$$

The expression for $P_x^{(m)}$ is obtained by replacing $\cos\varphi$ with $-\sin\varphi$.

In nonrelativistic theory m_+' and m_-' are constants, and we find that

$$\mathbf{P}^{(m)} = 0 \text{ (special model, Newtonian mechanics)}. \quad (79)$$

There is indeed conservation of momentum. Initially all of the momentum is in the field, and finally the charge q has received all of this momentum. The center-of-mass law is violated, however. The magnet does not contain any "hidden" momentum initially, and remains at rest finally while the charge q is set in motion.

In relativistic theory the masses $m_\pm'$ depend on the speeds of the particles. To the rest mass of a particle we must add the mass of its kinetic energy:

$$m_\pm = m_{0\pm} + c^{-2} T_0 \pm (e/c^2)(bq/a^2) \cos\varphi. \quad (80)$$

Then

$$P_y^{(m)} = (b^2 q/a^2 c^2)[e(v_+ n_+) - e(v_- n_-)] \int \cos^2\varphi d\varphi,$$

$$= (\pi b^2 q/a^2 c^2)[e(v_+ n_+) - e(v_- n_-)]. \quad (81)$$

Since $\int \cos\varphi \sin\varphi d\varphi = 0$, we again have $P_x^{(m)} = 0$. The expression in brackets is the current i around the circle, and $\pi b^2 (i/c)$ is the magnetic moment m:

$$P_y^{(m)} = (mq/a^2 c), \qquad P_x^{(m)} = P_z^{(x)} = 0. \quad (82)$$

In our present case we have $\mathbf{m} = m\mathbf{k}$, $\mathbf{a} = -a\mathbf{i}$, so that

$$\mathbf{P}^{(m)} = -(q/ca^3)(\mathbf{m} \times \mathbf{a}). \quad (83)$$

Comparison with Eqs. (58), (68), and (73) shows that this result of detailed calculation for a special model checks with our general formulas for the relativistic theory.

B. Slow Motion of the Point Charge

We consider cases in which a quasistationary treatment is valid. There is no appreciable radiation, the relative position of charge and magnet does not change very much during a period of any internal motions in the magnet, and no corrections of order $(v/c)^2$ are needed in the expressions for fields.

1. *Fictitious-Pole Model of the Magnet*

We consider first a charge q at point $\mathbf{a}$, moving with velocity $\mathbf{v}$, and a single pole μ at point $\mathbf{x}$. The field of the pole at a is

$$\mathbf{B}(\mathbf{a}) = \mu(\mathbf{a} - \mathbf{x}) \, |\mathbf{a} - \mathbf{x}|^{-3}, \quad (84)$$

and the force on the moving charge is

$$\mathbf{F}_q = (q/c)(\mathbf{v} \times \mathbf{B}) = (\mu q/c)[\mathbf{v} \times (\mathbf{a} - \mathbf{x})] \, |\mathbf{a} - \mathbf{x}|^{-3}. \quad (85)$$

The magnetic field produced at the point $\mathbf{x}$ by the moving charge is

$$\mathbf{B}'(\mathbf{x}) = [(q\mathbf{v}/c) \times (\mathbf{x} - \mathbf{a})] \, |\mathbf{x} - \mathbf{a}|^3. \quad (86)$$

The force on the pole is

$$\mathbf{F}_\mu = \mu \mathbf{B}'(\mathbf{x}) = -\mathbf{F}_q. \quad (87)$$

For a magnet composed of poles μ_i, we simply sum both expressions and get

$$\mathbf{F}_m = -\mathbf{F}_q. \quad (88)$$

2. *Ampere-Current Model of a Conducting (or Shielded) Magnet*

The force on the moving point charge is

$$\mathbf{F}_q = (q/c)(\mathbf{v} \times B),$$

$$= (q/c^2) \int \mathbf{v} \times [\mathbf{J}(\mathbf{x}) \times (\mathbf{a} - \mathbf{x})] \, |\mathbf{a} - \mathbf{x}|^{-3} d^3x. \quad (89)$$

The moving point charge produces the magnetic field, Eq. (86), at the point $\mathbf{x}$, and this magnetic field exerts a force on any ampere current at this point. The force on the magnet produced in this way is

$$\mathbf{F}_{mq} = (q/c^2) \int \mathbf{J}(\mathbf{x}) \times [\mathbf{v} \times (\mathbf{x} - \mathbf{a})] \, |\mathbf{x} - \mathbf{a}|^{-3} d^3x. \quad (90)$$

The motion of the point charge q causes a redistribution of the shielding charge. When we consider this charge as composed of portions q_i

($i=1, \cdots N$) which move with velocities $\mathbf{v}_i$ to effect the redistribution, we find that the force on the shield is, in analogy with Eq. (89),

$$\mathbf{F}_S=\sum_{i=1}^{N}(q_i/c^2)\times\int\mathbf{v}_i\times[\mathbf{J}(\mathbf{x})\times(\mathbf{x}_i-\mathbf{x})]\,|\,\mathbf{x}_i-\mathbf{x}\,|^{-3}d^3x. \quad (91)$$

The motions of the charges q_i produce magnetic fields $\mathbf{B}_i'(\mathbf{x})$ which exert forces on the ampere currents. The force on the magnet produced in this way is, in analogy with Eq. (90),

$$F_{mS}=\sum_{i=1}^{N}(q_i/c^2)\times\int\mathbf{J}(\mathbf{x})\times[\mathbf{v}_i\times(\mathbf{x}-\mathbf{x}_i)]\,|\,\mathbf{x}-\mathbf{x}_i\,|^{-3}. \quad (92)$$

We now consider the sum of all these forces,

$$\sum\mathbf{F}=\mathbf{F}_q+\mathbf{F}_S+\mathbf{F}_{mq}+\mathbf{F}_{mS}. \quad (93)$$

Writing for convenience $q_0=q$, $x_0=\mathbf{a}$, $v_0=\mathbf{v}$, we have

$$\begin{aligned}\sum\mathbf{F}&=\sum_{i=0}^{N}(q_i/c^2)\int\{\mathbf{v}_i\times[\mathbf{J}(\mathbf{x})\times(\mathbf{x}_i-\mathbf{x})]\\&\quad-\mathbf{J}(\mathbf{x})\times[\mathbf{v}_i\times(\mathbf{x}_i-\mathbf{x})]\}\cdot|\,\mathbf{x}_i-\mathbf{x}\,|^{-3}d^3x,\\&=\sum_{i=0}^{N}(q_i/c^2)\int\{\mathbf{J}(\mathbf{x})[\mathbf{v}_i\cdot(\mathbf{x}_i-\mathbf{x})]\\&\quad-\mathbf{v}_i[\mathbf{J}(\mathbf{x})\cdot(\mathbf{x}_i-\mathbf{x})]\}\,|\,\mathbf{x}_i-\mathbf{x}\,|^{-3}d^3x. \quad (94)\end{aligned}$$

Now

$$\begin{aligned}\int\mathbf{J}(\mathbf{x})\cdot(\mathbf{x}_i-\mathbf{x})\,|\,\mathbf{x}_i-\mathbf{x}\,|^{-3}d^3x&=\int\mathbf{J}(\mathbf{x})\cdot\nabla\,|\,\mathbf{x}_i-\mathbf{x}\,|^{-1}d^3x,\\&=\int|\,\mathbf{x}_i-\mathbf{x}\,|^{-1}\nabla\cdot\mathbf{J}(\mathbf{x})\,d^3x=0, \quad (95)\end{aligned}$$

so that the second term in the curly brackets in Eq. (94) contributes nothing to $\sum\mathbf{F}$. To deal with the first term, we note that the total electrostatic potential of the charge q and the shielding charge is

$$V(\mathbf{x})=\sum_{i=0}^{N}q_i\,|\,\mathbf{x}_i-\mathbf{x}\,|^{-1}. \quad (96)$$

This quantity is constant throughout the region where $J(x)\neq0$, and its time derivative

$$(d/dt)V(\mathbf{x})=-\sum_{i=0}^{N}q_i\mathbf{v}_i\cdot(\mathbf{x}_i-\mathbf{x})\,|\,\mathbf{x}_i-\mathbf{x}\,|^{-3}, \quad (97)$$

is also constant in this region. Accordingly $\sum\mathbf{F}$ is a constant multiple of $\int\mathbf{J}(\mathbf{x})d^3x$, and therefore is zero.[18]

Our final result,

$$\mathbf{F}_q+\mathbf{F}_S+\mathbf{F}_{mq}+\mathbf{F}_{mS}=0, \quad (98)$$

shows that we will find that "action and reaction are equal and opposite" if we consider the point charge on one hand, and the magnet with rigidly attached shield on the other. If we consider three bodies—charge, magnet, and shield not rigidly attached—we do not find a pairwise balancing of action and reaction, but we do find that both conservation of momentum and the center-of-mass law are valid.[19]

3. *Ampere-Current Model of a Nonconducting Magnet*

The force on the point charge is given by Eq. (89). The entire force on the magnet is now given by Eq. (90). According to Eqs. (73) and (50), the momentum contained in the matter of of the magnet is (the prime emphasizes that this is "hidden" momentum)

$$\begin{aligned}\mathbf{P}'^{(m)}=-\mathbf{P}^{(f)}&=-c^{-2}\int d^3xV\mathbf{J}\\&=-(q/c^2)\int d^3x\mathbf{J}(\mathbf{x})\,|\,\mathbf{x}-\mathbf{a}\,|^{-1}. \quad (99)\end{aligned}$$

Its time dependence comes from $d\mathbf{a}/dt=\mathbf{v}$, and we

[18] We have already used the fact $\int\mathbf{J}(x)d^3x=0$, which is intuitively obvious for a finite stationary system, in concluding Eq. (67) from Eqs. (65) and (66). The formal proof uses the mathematics of Eqs. (69)–(71): $\int\mathbf{J}d^3x=-\int\mathbf{x}\nabla\cdot\mathbf{J}d^3x$.

[19] G. T. Trammell in Phys. Rev. **134B**, 1183 (1964), overlooks the shielding effect—the terms $\mathbf{F}_S+\mathbf{F}_{mS}$—and is led to assert violation of conservation of momentum in relative motions of a charge and a long solenoid.

have

$$-(d/dt)\mathbf{P}'^{(m)}=-(q/c^2)\times\int[\mathbf{v}\cdot(\mathbf{a}-\mathbf{x})]\mathbf{J}(\mathbf{x})\,|\mathbf{a}-\mathbf{x}|^{-3}d^3x. \quad (100)$$

Combining the three expressions we have

$$\mathbf{F}_q+\mathbf{F}_m-(d/dt)\mathbf{P}'^{(m)}$$

$$=(q/c^2)\int\{\mathbf{v}\times[\mathbf{J}\times(\mathbf{a}-\mathbf{x})]+\mathbf{J}\times[\mathbf{v}\times(\mathbf{x}-\mathbf{a})]-[\mathbf{v}\cdot(\mathbf{a}-\mathbf{x})]\mathbf{J}\}\,|\mathbf{a}-\mathbf{x}|^{-3}d^3x,$$

$$=(q/c^2)\mathbf{v}\int[\mathbf{J}\cdot(\mathbf{x}-\mathbf{a})]\,|\mathbf{x}-\mathbf{a}|^{-3}d^3x. \quad (101)$$

This last integral is the one evaluated in Eq. (95). Therefore we have

$$\mathbf{F}_q+\mathbf{F}_m-(d/dt)\mathbf{P}'^{(m)}=0. \quad (102)$$

This is the form of the law of action and reaction for the present case. $\mathbf{F}_q$ is the rate at which the field gives momentum to the charge. The total rate of change of the "overt" momentum, macroscopic mass-times-velocity, of the magnet is the sum of $\mathbf{F}_m$, the force of the field on the magnet, and $-(d/dt)\mathbf{P}'^{(m)}$, the rate at which momentum contained in the magnet ceases to be "hidden" and becomes "overt."

C. A General Conclusion

All of the cases we have presented in detail can be regarded as consequences of a general result, which we now establish. The argument used has much in common with that used for Eq. (73), but is formulated so that it is not restricted to stationary systems, and so that it refers only to the "hidden" part of the material momentum. (In a strictly stationary system this is of course the entire material momentum.)

We use primed symbols $\mathbf{S}'^{(m)}$, $\mathbf{g}'^{(m)}$, $\mathbf{P}'^{(m)}$ to refer to the parts of the energy flux, momentum density, and total momentum which are "hidden" in the sense that they are associated with microscopic processes of energy flow through matter. The "overt" quantities $\mathbf{S}^{(m)}-\mathbf{S}'^{(m)}$, $\mathbf{g}^{(m)}-\mathbf{g}'^{(m)}$, $\mathbf{P}^{(m)}-\mathbf{P}'^{(m)}$ can be simply expressed in terms of the mass density and the macroscopic velocity. For example,

$$\mathbf{P}^{(m)}-\mathbf{P}'^{(m)}=\sum_i M_i\mathbf{v}_i, \quad (103)$$

where the M_i are the masses of all material bodies (including our "external point charge," if any) and $\mathbf{v}_i$ are their velocities.

In any stationary or slowly moving magnet or set of magnets composed of ampere currents we have

$$\nabla\cdot\mathbf{S}'^{(m)}=\mathbf{E}\cdot\mathbf{J}. \quad (104)$$

Also by Eq. (71),

$$\mathbf{P}'^{(m)}=-\int\mathbf{x}\nabla\cdot\mathbf{g}'^{(m)}d^3x,$$

$$=-c^{-2}\int\mathbf{x}\nabla\cdot\mathbf{S}'^{(m)}d^3x. \quad (105)$$

Using Eq. (104) and integrating by parts, we get

$$\mathbf{P}'^{(m)}=-c^{-2}\int\mathbf{x}(\mathbf{E}\cdot\mathbf{J})d^3x,$$

$$=c^{-2}\int\mathbf{x}(\mathbf{J}\cdot\nabla V)d^3x,$$

$$=-c^{-2}\int V(\nabla\cdot\mathbf{J})\mathbf{x}d^3x, \quad (106)$$

where in the last expression the differential operator acts on $\mathbf{x}$ as well as on $\mathbf{J}$, so that the expression does not vanish. We then find

$$\mathbf{P}'^{(m)}=-c^{-2}\int V\mathbf{J}d^3x. \quad (107)$$

When we consider moving bodies we have two contributions to $\mathbf{P}^{(f)}$. One, which we may call $\mathbf{P}''^{(f)}$, has not been mentioned before in this paper. It comes from the vector product of $\mathbf{E}$ and the magnetic field of the slow motions of the charges, and can be represented in terms of "electromagnetic masses" of the charges, plus interference terms of the same general nature. This term $\mathbf{P}''^{(f)}$ is always extremely small compared to all the other momenta in a problem involving magnets.

The other part of $\mathbf{P}^{(f)}$, which we call $\mathbf{P}'^{(f)}$, comes from the vector product of $\mathbf{E}$ and the magnetic field of magnets. It is the quantity already calculated in Eq. (50). From Eqs. (50) and (107) we have

$$\mathbf{P}'^{(m)}+\mathbf{P}'^{(f)}=0. \quad (108)$$

By the general law of conservation of momentum [Eq. (7)] we have

$$(d\mathbf{P}/dt) = (d/dt)[\mathbf{P}^{(m)}+\mathbf{P}^{(f)}] = \sum_i \mathbf{F}_{i\ \text{ext}}, \quad (109)$$

where the $\mathbf{F}_{i\ \text{ext}}$ are any external forces applied to the bodies.

Using Eq. (103), and neglecting the very small quantity $\mathbf{P}''^{(f)}$, which is very much smaller than any of the terms we keep, we get

$$(d/dt)[\sum_i M_i\mathbf{v}_i+\mathbf{P}'^{(m)}+\mathbf{P}'^{(f)}] = \sum \mathbf{F}_{\text{ext}}, \quad (110)$$

and therefore by Eq. (108)

$$(d/dt)\sum_i M_i v_i = 0. \quad (111)$$

This result assures that *in any problem involving changes of magnetic moment and/or slow motions of magnets and charge distributions, the center-of-mass law will hold,* to an accuracy limited only by the neglect of "electromagnetic mass" terms. We are thus assured that there will be no paradoxes, and that conservation of momentum always holds for the macroscopic or "overt" momenta of bodies. In all two-body cases this assures that action and reaction are equal and opposite, but in cases with three or more bodies this concept can fail.

V. THE TROUTON–NOBLE EXPERIMENT: "HIDDEN" MOMENTUM IN A STRESSED MOVING BODY

One of the early experiments aimed at observing an absolute motion of the Earth was done by Trouton and Noble.[6] They tried to detect a predicted slight torque on a charged condenser. The relativistic explanation of the null result found in the experiment is given in a book by Pauli.[20] It depends on the presence of "hidden momentum" in the material structure of the condenser.

We shall briefly consider a simplified model used by Pauli to show the principle of the relativistic treatment of this experiment. Since our purpose is only to give further emphasis to the importance of the hidden momentum contained in the material structure, we consider a case of very slow motion and make our calculations only to the lowest order in v/c in which the torque in question appears.

In the rest frame of reference K of the apparatus we have a charge $+q$ at the point $(x, y, z) = (a, b, 0)$, and a charge $-q$ at the point $(-a, -b, 0)$. These charges are separated by a thin rigid rod of length $2l = 2(a^2+b^2)^{1/2}$, which holds them apart, against the forces of their electrostatic attraction. In the rest frame K the system has zero momentum and zero angular momentum, and there is no torque on the material structure.

We now consider the system in another frame of reference K', in which the rod and the charges are moving with the constant velocity iv, $v \ll c$. In the frame K' the motion of the charges produces a magnetic field. The field produced at the position of the charge $+q$ by the motion of the charge $-q$ is

$$\mathbf{B}_1 = -(q/8l^3c)[\mathbf{i}v \times (2a\mathbf{i}+2b\mathbf{j})],$$
$$= -(qvb/4l^3c)\mathbf{k}. \quad (112)$$

Because of its motion in this field the force on the charge $+q$ is

$$\mathbf{F}_1 = (q/c)\,iv \times \mathbf{B}_1 = (q^2b/4l^3)(v/c)^2\mathbf{j}. \quad (113)$$

There is also a force $-\mathbf{F}_1$ on the charge $-q$ owing to its motion in the magnetic field produced by the motion of the charge $+q$. The two forces form a couple whose torque is

$$\mathbf{N} = (2a\mathbf{i}+2b\mathbf{j}) \times \mathbf{F}_1 = (q^2ab/2l^3)(v/c)^2\mathbf{k}. \quad (114)$$

This couple must produce a change of the angular momentum $\mathbf{M}'$ of the system in the frame of reference K'. Because it is obvious that in the rest frame K the rod and charges are not caused to rotate, the relativity principle requires that the system is not set into rotation in the frame K'. Relativity theory provides an explanation of the increase of $\mathbf{M}'$ without rotation of the rod: it is an increase of the "hidden angular momentum" associated with hidden momentum in the rod.

There must indeed be hidden momentum in the rod, because there is a flow of energy along the rod. The electric field produced by the charge $-q$ at the position of the charge $+q$ is

$$\mathbf{E}_1 = -(q/8l^3)(2a\mathbf{i}+2b\mathbf{j}). \quad (115)$$

The upper end of the rod pushes the charge $+q$ along with velocity $\mathbf{i}v$ against the attraction of the other charge, so that the upper end of the

[20] W. Pauli, *Theory of Relativity* (Pergamon Press, Inc., New York, 1958), pp. 127–130.

rod does work (puts energy into the electromagnetic field) at the rate

$$-\int_{\text{top of rod}} \nabla' \cdot \mathbf{S}^{(m)} d^3x' = q(-\mathbf{E}_1 \cdot \mathbf{i}v) = (q^2av/4l^3). \quad (116)$$

At the lower end of the rod the force of the electric field on the charge has a positive component in the direction of the motion, and this end of the rod does negative work (receives energy from the field):

$$-\int_{\text{bottom of rod}} \nabla' \cdot \mathbf{S}^{(m)} d^3x' = -(q^2av/4l^3). \quad (117)$$

Using Eqs. (116) and (117) in the general formula, Eq. (105), we find that the hidden momentum in the rod is

$$\mathbf{P}'^{(m)} = (q^2av/4l^3c^2)(2a\mathbf{i}+2b\mathbf{j}). \quad (118)$$

We may conveniently calculate the angular momentum relative to the origin of the reference frame K'. If we take the origins of K' and K coincident at time $t'=t=0$, the position vector in K' for the center of the rod is $\mathbf{i}vt'$, and the angular momentum corresponding to the hidden momentum is

$$\mathbf{M}' = \mathbf{i}vt' \times \mathbf{P}'^{(m)} = (q^2ab/2l^3)(v/c^2)t'\mathbf{k}. \quad (119)$$

The torque $\mathbf{N}$ found in Eq. (114) is seen to be indeed equal to $d\mathbf{M}'/dt'$.

The electromagnetic field of course also has momentum in this case, with density c^{-2} times the Poynting vector. Since there are no magnets, the momentum $\mathbf{P}^{(f)}$ is entirely of the kind we called $\mathbf{P}''^{(f)}$ in our discussion of Sec. IV, Part C. If we use subscripts + and − to distinguish the fields produced by the charges $+q$ and $-q$, for the field momentum density we have

$$\mathbf{g}^f = c^{-2}\mathbf{S}^{(f)} = (4\pi c)^{-1} \times (\mathbf{E}_+ \times \mathbf{B}_+ + \mathbf{E}_- \times \mathbf{B}_- + \mathbf{E}_+ \times \mathbf{B}_- + \mathbf{E}_- \times \mathbf{B}_+). \quad (120)$$

The first two terms give the momenta corresponding to the "electromagnetic masses" of the two charges. The two interference terms contain a part corresponding to a joint "electromagnetic mass" owing to the proximity of the charges, but they also contain a part which gives a momentum opposite to $\mathbf{P}'^{(m)}$. Indeed, the energy theorem tells us that

$$\nabla' \cdot \mathbf{S}^{(f)} = -(\partial/\partial t')(E^2+B^2)/8\pi - \nabla' \cdot \mathbf{S}^{(m)}. \quad (121)$$

When used in Eq. (71) the first term gives the electromagnetic-mass momentum, which is proportional to the velocity $\mathbf{i}v$ and occasions no angular momentum. The second term gives momentum $-\mathbf{P}'^{(m)}$ and angular momentum $-\mathbf{M}'$. The complete closed system of matter and field has zero angular momentum around the origin of K'. The need to discuss a torque, a momentum not parallel to the direction of motion, and a changing angular momentum arises only when we consider the material system alone, which is not a closed system.

Discussion of a Special Model

The presence of hidden momentum in a material system subjected to stress by the action of external forces can be illustrated with the particularly simple example of an ideal gas. We consider a gas which has zero total momentum in the rest system K,

$$\mathbf{P} = \sum nm\mathbf{u} = 0. \quad (122)$$

Here the summation is over sets of gas molecules having (practically) the same velocity $\mathbf{u}$, and m is the mass of a molecule having velocity $\mathbf{u}$—in relativity theory

$$m = m_0(1-u^2/c^2)^{-1/2}, \quad (123)$$

where m_0 is the rest mass of a molecule.

In nonrelativistic mechanics the components of a molecule's velocity $\mathbf{u}'$ in the moving coordinate system K' are

$$u_x' = u_x+v, \qquad u_y' = u_y, \qquad u_z' = u_z. \quad (124)$$

The only nonvanishing component of the momentum P' in the system K' is

$$P_x' = \sum nm(u_x+v) = (\sum nm)v = Mv. \quad (125)$$

This result holds both for the case of gas freely expanding in space, and for the case of a confined gas, since Eq. (122) holds for both these cases. In nonrelativistic mechanics the momentum of the gas is the product of the total mass and the average velocity.

In relativity theory the velocity u' is given by[21]

$$u_x' = (u_x+v)(1+vu_x/c^2)^{-1},$$
$$u_y' = u_y(1-v^2/c^2)^{1/2}(1+vu_x/c^2)^{-1},$$
$$u_z' = u_z(1-v^2/c^2)^{1/2}(1+vu_x/c^2)^{-1}, \quad (126)$$

and we have

$$1-u'^2/c^2 = (1-v^2/c^2)(1-u^2/c^2)(1+vu_x/c^2)^{-2}. \quad (127)$$

When we calculate the total momentum P', in the reference system K', for the case of a gas freely expanding in space,

$$P' = \sum nm_0\mathbf{u}'(1-u'^2/c^2)^{-1/2}, \quad (128)$$

by using Eqs. (122) and (123), we find that the only nonvanishing component is

$$P_x' = (1-v^2/c^2)^{-1/2}\sum nm_0(1-u^2/c^2)^{-1/2}(u_x+v),$$
$$= (1-v^2/c^2)^{-1/2}[\sum nm_0(1-u^2/c^2)^{-1/2}]v,$$
$$= M_0v(1-v^2/c^2)^{-1/2}, \quad (129)$$

where

$$M_0 = \sum nm_0(1-u^2/c^2)^{-1/2}, \quad (130)$$

is the rest mass of the gas—its total mass in the rest system K. The total energy in the reference system K' is

$$U' = \sum nm_0c^2(1-u'^2/c^2)^{-1/2},$$
$$= c^2(1-v^2/c^2)^{-1/2}$$
$$\times \sum nm_0(1-u^2/c^2)^{-1/2}(1+vu_x/c^2),$$
$$= c^2(1-v^2/c^2)^{-1/2}\sum nm_0(1-u^2/c^2)^{-1/2},$$
$$= M_0c^2(1-v^2/c^2)^{-1/2}. \quad (131)$$

The relation between energy and momentum is

$$\mathbf{P}' = M\mathbf{v} = (U'/c^2)\mathbf{v}, \quad (132)$$

as it must be. The freely expanding gas is a closed system, interacting with nothing else, and its energy and momentum form a four-vector $P_k = (P, iU/c)$ (Sec. II).

Let us now consider the case of a gas confined in a container which is at rest in the reference frame K. In this case the numbers of molecules in the various sets are not the same in the two reference systems. For a gas in equilibrium the collisions among the molecules themselves do not produce changes in the numbers in the various sets. In the rest reference frame K this is also true of collisions with the walls of the container. In particular, if the motion of the particles in the x direction is limited on the right and on the left by plane walls, while there are collisions with the left-hand wall which change u_x from the value $-|u_x|$ to the value $+|u_x|$, there are corresponding simultaneous collisions with the right-hand wall which make the opposite change in the velocity of an equal number of molecules. In the reference frame K', however, these particular collisions at the left-hand and right-hand walls are not simultaneous. There is, of course, a correspondence between collisions with the two walls in K' which keeps the numbers n', as observed in K', constant in time. But because the correspondence is not the same as in K, the numbers n' are not equal to the numbers n.

Consider two particular sets of molecules with velocities $(+|u_x|, u_y, u_z)$ and $(-|u_x|, u_y, u_z)$ in the reference frame K. In K each set has n molecules. Let us call the respective numbers of molecules in these sets as observed in K' n_+' and n_-'. The x components of their velocities in K' are

$$u_{x+}' = (|u_x|+v)(1+v|u_x|/c^2)^{-1},$$
$$u_{x-}' = (-|u_x|+v)(1-v|u_x|/c^2)^{-1}. \quad (133)$$

Since the gas remains uniform both in total density and in velocity distribution throughout the container, there is no net transfer of molecules of these two groups through a plane perpendicular to the x axis and fixed in K. When we write the condition for this in the frame K' we have

$$(n_+'/l')(u_{x+}'-v)+(n_-'/l')(u_{x-}'-v) = 0, \quad (134)$$

where l' is the length of the container as measured in K'. Substituting the values, Eq. (133), and dividing by $(|u_x|/l')(1-v^2/c^2)$, we have

$$n_+'(1+v|u_x|/c^2)^{-1} - n_-'(1-v|u_x|/c^2)^{-1} = 0. \quad (135)$$

From this equation and

$$n_+'+n_-' = 2n, \quad (136)$$

we find the values

$$n_+' = n(1+v|u_x|/c^2), \qquad n_-' = n(1-v|u_x|/c^2). \quad (137)$$

[21] Reference 8, pp. 51 ff, Eqs. (45), (56).

We can drop the subscripts and write simply

$$n'=n(1+vu_x/c^2), \qquad (138)$$

for all the sets of molecules.

The total momentum $\mathbf{P}'$ still has P_x' as its only nonvanishing component, since it is clear by symmetry that sums containing u_xu_y or u_xu_z vanish. We have

$$P_x'=\sum n'm_0u_x'(1-u'^2/c^2)^{-1/2}, \qquad (139)$$

and on substituting from Eqs. (126), (127) and (138) and using Eqs. (122) and (130) we get

$$P_x'=[M_0+c^{-2}\sum nm_0u_x^2(1-u^2/c^2)^{-1/2}]\times v(1-v^2/c^2)^{-1/2}. \qquad (140)$$

From elementary kinetic theory we know that

$$\sum nm_0u_x^2(1-u^2/c^2)^{-1/2}=\sum nu_x\cdot mu_x=pV, \qquad (141)$$

the product of pressure and volume in the rest system K. The factor V appears because the n's are total numbers, not number densities. Then

$$P_x'=(M_0+pV/c^2)v(1-v^2/c^2)^{-1/2}. \qquad (142)$$

Similarly, replacing n with n' in the calculation of Eq. (131), we find for the total energy of the confined gas in the reference frame K' the result

$$U'=(M_0+pVv^2/c^2)c^2(1-v^2/c^2)^{-1/2} \qquad (143)$$

The relation, in Eq. (132), between energy and momentum does not hold for a confined gas. Its energy and momentum do not form a four-vector, because it is not a closed system, being subjected to external forces by the walls of the container.[22]

We can now calculate the hidden momentum. Since we are already using a prime to refer to the reference frame K', we distinguish the hidden momentum with a subscript h:

$$P_{xh}=P_x'-Mv=P_x'-(U'/c^2)v,$$
$$=(pV/c^2)\cdot v\cdot[1-(v^2/c^2)]^{1/2}. \qquad (144)$$

Let us now write $V=Al$, where A is the area of the vessel in a section perpendicular to the x axis, and l is its length in the reference frame K. Then $l'=l(1-v^2/c^2)^{1/2}$ is the length in K'. We now have

$$P_{xh}'=pA\cdot(v/c^2)\cdot l'. \qquad (145)$$

The pressure p' in the reference frame K' can be found by calculating the transfer of momentum per unit area and time across a surface perpendicular to the x axis and moving with the vessel. The number density of the molecules in a group being $n'/l'A$, we have

$$p'=\sum (n'/l'A)(u_x'-v)\cdot mu_x', \qquad (146)$$

and when we substitute the values of n' and u_x' we find[22]

$$p'=\sum (1-v^2/c^2)^{1/2}(n/l'A)mu_x^2$$
$$=\sum (n/lA)mu_x^2=p. \qquad (147)$$

Accordingly we can replace the factor pA in Eq. (144) by $p'A=F'$, the force the gas exerts on an end wall, as measured in the frame K' (this force has the same value as in K). Then

$$P_{xh}'=(F'v/c^2)\cdot l'. \qquad (148)$$

Energy leaves the gas at the right-hand end wall at the rate $F'v$ and reenters it at the left-hand wall.[23] This energy flow corresponds to a momentum density, as in our general formulas for hidden momentum. For a rigid container, the return energy flow and the momentum $-P_{xh}'$ are in the longitudinal stress in the side walls. If we cancel this stress by putting charges $\pm q$ on the ends so that their attraction provides the force F', the return energy flow and the momentum $-P_{xh}'$ are in the electromagnetic field.

[22] Cf. Møller, Ref. 8, p. 183.

[23] This flow of energy can be verified directly in the kinetic model by a calculation of energy transfer analogous to the calculation of momentum transfer in Eqs. (146) and (147).