Cullwick's Paradox:
Charged Particle on the Axis of a Toroidal Magnet

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1 Problem

In an induction linac [1] a toroidal magnet carries a time-dependent current $I(t)$ such that the induced electric field can transfer energy from the magnet to charged particles that move along the axis of the toroid.

Discuss the force and momentum balance in an idealized induction linac consisting of a single magnet whose form is a torus of major radius a and minor radius $b \ll a$, and a single electron of charge e that moves along the symmetry axis of the toroid. The current I is the total current crossing any major circle on the surface of the torus.
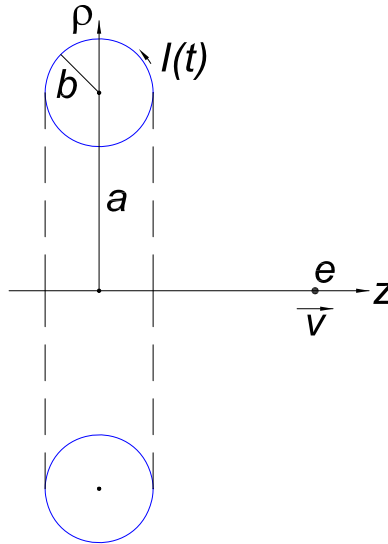
While actual induction linacs contain high-permeability ferrites inside the toroid, whose windings are made from shielded or unshielded conductors, it suffices here to consider a nonconducting toroid (without ferrites) whose currents are due to electric charges fixed on the rims of rotating disks. Neighboring disks have opposite charges and rotate in opposite senses so that the net electric charge (and the net mechanical angular momentum) of the toroid is zero. This configuration of a nonconducting toroid has no azimuthal current, in contrast to a single-layer helical winding on the toroid which includes, in effect, a single azimuthal current loop.

You may assume that (unlike the case of an induction linac) the velocity v of the moving charge is small compared to c, the speed of light, and that the time variation of the current in the toroid is slow enough that radiation and retarded effects can be ignored.

Provide an analysis in the rest frame of the moving charge as well as in the lab frame, *i.e.*, the rest frame of the toroid.

Cullwick [2, 3] has noted that this example is paradoxical because no force is exerted on the moving charge when the current is constant in the toroid, but the moving charge exerts a nonzero force on the toroid.

2   Solution

The force $\mathbf{F}_e$ on the electric charge e due to the toroid causes a time rate of change of the mechanical momentum $\mathbf{P}_e$ of the electron according to

$$\mathbf{F}_e = \frac{d\mathbf{P}_e}{dt}, \tag{1}$$

and likewise the force $\mathbf{F}_T$ on the toroid changes the mechanical momentum $\mathbf{P}_T$ of the latter according to

$$\mathbf{F}_T = \frac{d\mathbf{P}_T}{dt}. \tag{2}$$

The paradox (which dates back to Ampère) is that the magnetic interaction of a moving charge and a current (as well as the magnetic interaction of two moving charges) does not in general obey Newton's third law, $\mathbf{F}_e \neq -\mathbf{F}_T$, so that the total mechanical momentum of the system, $\mathbf{P}_{\mathrm{mech}} = \mathbf{P}_e + \mathbf{P}_T$, is not constant in time, in apparent violation of Newton's first law for an isolated system.

The resolution of such paradoxes is that electromechanical systems in general possess an additional momentum, $\mathbf{P}_{\mathrm{EM}}$, associated with the interaction of the charges and currents with the electromagnetic field such that the total momentum of an isolated system, $\mathbf{P}_e + \mathbf{P}_T + \mathbf{P}_{\mathrm{EM}}$ in the present example, is constant in time.

A further subtlety is that the sum $\mathbf{P}_{\mathrm{mech}} + \mathbf{P}_{\mathrm{EM}}$, while constant, may appear to have a nonzero value for an isolated system at rest. However, a "hidden" mechanical momentum $\mathbf{P}_h$ can be identified that restores the total momentum of a system at rest to zero.

2.1   Analysis in the Lab Frame

2.1.1   The Electromagnetic Momentum

For systems in which effects of radiation and of retardation can be ignored, the electromagnetic momentum can be calculated in various equivalent ways [4] (in Gaussian units),

$$\mathbf{P}_{\mathrm{EM}} = \int \frac{\varrho \mathbf{A}}{c} \, d\mathrm{Vol} = \int \frac{\mathbf{E} \times \mathbf{B}}{4\pi c} \, d\mathrm{Vol} = \int \frac{\Phi \mathbf{J}}{c^2} \, d\mathrm{Vol}, \tag{3}$$

where ϱ is the electric charge density, $\mathbf{A}$ is the magnetic vector potential (in the Coulomb gauge where $\boldsymbol{\nabla} \cdot \mathbf{A} = 0$), $\mathbf{E}$ is the electric field, Φ is the electric (scalar) potential, and $\mathbf{J}$ is the electric current density. The first form is due to Faraday [5] and Maxwell [6], the second form is due to Poynting [7] and Abraham [8], and the third form was introduced by Furry [9].

To calculate the electromagnetic momentum using the first form of eq. (3), we need the vector potential $\mathbf{A}_T$ of the toroid at the position of the charge e, but we do not need the

vector potential of the charge since the toroid is assumed to be electrically neutral. The vector potential of the toroid obeys

$$\boldsymbol{\nabla} \times \mathbf{A}_{\mathrm{T}} = \mathbf{B}_{\mathrm{T}} = B_{\mathrm{T}}\,\hat{\boldsymbol{\phi}}, \tag{4}$$

where the magnetic field is $B_{\mathrm{T}} = 2I/ac$ inside the toroid and zero outside, and $\hat{\boldsymbol{\phi}}$ is a unit vector in the azimuthal direction in a cylindrical coordinate system (ρ, ϕ, z). The toroid is centered on the origin with the z-axis as its axis, as shown in the figure below (with radius b exaggerated for clarity).
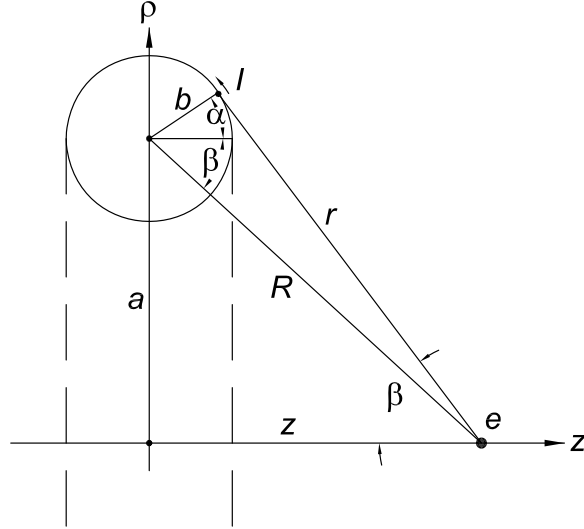
Cullwick notes [3] that the relation (4) has the same form as Maxwell's equation for the magnetic field due to a conducting wire that forms a (solid) torus of the same dimensions as the (hollow) toroidal magnet when the wire carries azimuthal current density $\mathbf{J} = J\,\hat{\boldsymbol{\phi}}$,

$$\boldsymbol{\nabla} \times \mathbf{B}_{\mathrm{loop}} = \frac{4\pi}{c}\mathbf{J} = \frac{4\pi}{c}J\,\hat{\boldsymbol{\phi}}. \tag{5}$$

From the Biot-Savart law we know that the magnetic field along the axis of the current loop is, for $b \ll a$,

$$\mathbf{B}_{\mathrm{loop}}(0,0,z) \approx \frac{2\pi}{c}\frac{\pi b^2 J a^2}{(z^2 + a^2)^{3/2}}\,\hat{\mathbf{z}}. \tag{6}$$

Comparing eqs. (4) and (5), we see that on replacing $4\pi J$ in eq. (6) by $2I/a$ we obtain the vector potential on the axis of the toroid when $b \ll a$,

$$\mathbf{A}_{\mathrm{T}}(0,0,z) \approx \frac{\pi b^2 I}{c}\frac{a}{(z^2 + a^2)^{3/2}}\,\hat{\mathbf{z}}. \tag{7}$$

Hence, the electromagnetic momentum of the system when charge e is at position z on the axis of the toroid is

$$\mathbf{P}_{\mathrm{EM}} = \frac{e\mathbf{A}_{\mathrm{T}}(0,0,z)}{c} = \frac{\pi b^2 I e}{c^2}\frac{a}{(z^2 + a^2)^{3/2}}\,\hat{\mathbf{z}}, \tag{8}$$

which is independent of the velocity of the charge.

3

To calculate the electromagnetic momentum using the second form of eq. (3), we note that the electric field at the toroid due to charge e has magnitude $E_e = e/(z^2 + a^2)$ on average, and that the z-component of $\mathbf{E}_e \times \mathbf{B}_{\mathrm{T}}$ (which is the only one remaining after the integral over the toroid volume) is $E_e B_{\mathrm{T}} a/\sqrt{z^2 + a^2}$. Hence,[1]

$$\mathbf{P}_{\mathrm{EM}} = \int \frac{\mathbf{E}_e \times \mathbf{B}_{\mathrm{T}}}{4\pi c}\, d\mathrm{Vol} \approx \frac{e}{z^2 + a^2} \frac{2I}{ac} \frac{a}{\sqrt{z^2 + a^2}} \frac{2\pi a \pi b^2}{4\pi c}\, \hat{\mathbf{z}} = \frac{\pi b^2 I e}{c^2} \frac{a}{(z^2 + a^2)^{3/2}}\, \hat{\mathbf{z}}. \quad (9)$$

For completeness, we calculate the electromagnetic momentum using the third form of eq. (3). We must keep the first correction to the spatial dependence of the electric potential Φ_e of charge e over the toroid. Referring to the figure above, we see that $r = \sqrt{R^2 - 2bR\cos(\alpha + \beta) + b^2} \approx R[1 - \frac{b}{R}\cos(\alpha + \beta)]$, $\sin\beta = (a + b\sin\alpha)/r \approx a/R$, and $R = \sqrt{z^2 + a^2}$. Only the z-component of the integral survives, so noting that $J_z\, d\mathrm{Vol} \to -Ib\sin\alpha\, d\alpha$, we find

$$
\begin{aligned}
\mathbf{P}_{\mathrm{EM}} &= \int \frac{\Phi_e \mathbf{J}}{c^2}\, d\mathrm{Vol} = -\int_0^{2\pi} \frac{eI}{c^2 r} b\sin\alpha\, d\alpha\, \hat{\mathbf{z}} \\
&\approx -\frac{eIb}{c^2 R} \int_0^{2\pi} \sin\alpha\, d\alpha \left(1 + \frac{b}{R}(\cos\alpha\cos\beta - \sin\alpha\sin\beta)\right) \hat{\mathbf{z}} \\
&= \frac{\pi b^2 I e}{c^2} \frac{a}{(z^2 + a^2)^{3/2}}\, \hat{\mathbf{z}}.
\end{aligned}
\quad (10)
$$

2.1.2 The Force on the Electric Charge

The force $\mathbf{F}_e$ on the electric charge is due to the electric field $\mathbf{E}_{\mathrm{T}}$ induced when the current in the toroid changes. This field is conveniently calculated as the time derivative of the vector potential (7). Thus,

$$\mathbf{F}_e = e\mathbf{E}_{\mathrm{T}} = -\frac{e}{c}\frac{\partial \mathbf{A}_{\mathrm{T}}}{\partial t} = -\frac{\pi b^2 \dot{I} e}{c^2} \frac{a}{(z^2 + a^2)^{3/2}}\, \hat{\mathbf{z}}, \quad (11)$$

where $\dot{I} = dI/dt$, independent of the velocity of the charge. This force is nonzero only when the current I in the toroid is changing.

2.1.3 The Force on the Toroid

The magnetic field $\mathbf{B}_e$ at a distance $\mathbf{r}$ from the moving charge e is given by

$$\mathbf{B}_e = e\frac{\mathbf{v}}{c} \times \frac{\hat{\mathbf{r}}}{r^2} = \frac{ev\rho}{cr^3}\, \hat{\boldsymbol{\phi}}, \quad (12)$$

where $\mathbf{v}$ is its velocity, and ρ is distance from the observation point from the z-axis. This magnetic field acts on the current I in the toroid to exert a force on the latter given by

$$\mathbf{F}_T = \oint \frac{I}{c}\, d\mathbf{l} \times \mathbf{B}_e = \frac{evI}{c^2} \oint dl_\rho \frac{\rho}{r^3}\, \hat{\mathbf{z}} \quad (13)$$

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[1]The "self momentum" of charge e associated with the cross product $\mathbf{E}_e \times \mathbf{B}_e$ is, as usual, assumed to be part of the mechanical momentum of the charge.

Referring to the figure above, we see that $dl_\rho = b \cos\alpha\, d\alpha$, $\rho = a + b \sin\alpha \approx a$, $r = \sqrt{R^2 - 2bR\cos(\alpha+\beta) + b^2} \approx R[1 - \frac{b}{R}\cos(\alpha+\beta)]$, $\cos\beta = (z - b\cos\alpha)/r \approx z/R$, and $R = \sqrt{z^2 + a^2}$. Then,

$$\begin{aligned}
\mathbf{F}_T &\approx \frac{evI}{c^2} \int_0^{2\pi} b\cos\alpha\, d\alpha \, \frac{a}{R^3}\left(1 + 3\frac{b}{R}(\cos\alpha\cos\beta - \sin\alpha\sin\beta)\right)\hat{\mathbf{z}} \\
&= \frac{3evI\pi b^2}{c^2}\frac{az}{(z^2+a^2)^{5/2}}\hat{\mathbf{z}} = -\frac{ev}{c}\frac{\partial \mathbf{A}_T}{\partial z},
\end{aligned} \tag{14}$$

recalling eq. (7). This force is nonzero whenever the velocity v of the charge and the current I in the toroid are nonzero.

2.1.4 Momentum Balance in the Lab Frame

The sum of the electromagnetic forces on the system is

$$\mathbf{F}_T + \mathbf{F}_e = -\frac{e}{c}\frac{\partial \mathbf{A}_T}{\partial t} - \frac{ev}{c}\frac{\partial \mathbf{A}_T}{\partial z} = -\frac{e}{c}\frac{d\mathbf{A}_T}{dt}, \tag{15}$$

where d/dt is the convective derivative according to an observer on the charge e. The total force is nonzero when the charge is moving and/or the current in the toroid is changing, in apparent violation of Newton's third law.

Consistency with Newton's laws is restored if we recall eq. (8) for the electromagnetic momentum of the system, so that we can write

$$\mathbf{F}_T + \mathbf{F}_e = -\frac{d\mathbf{P}_{\mathrm{EM}}}{dt} = -\frac{\partial \mathbf{P}_{\mathrm{EM}}}{\partial t} - v\frac{\partial \mathbf{P}_{\mathrm{EM}}}{\partial z}, \tag{16}$$

noting that the electromagnetic momentum varies both with the current in the toroid and with the position z of charge e. Then, using eqs. (1) and (2) we see that the total momentum of the system is constant in time,

$$\frac{d\mathbf{P}_T}{dt} + \frac{d\mathbf{P}_e}{dt} + \frac{d\mathbf{P}_{\mathrm{EM}}}{dt} = \frac{d\mathbf{P}_{\mathrm{total}}}{dt} = 0. \tag{17}$$

2.1.5 "Hidden" Mechanical Momentum

While eq. (17) is a satisfactory representation of overall momentum balance, another aspect of momentum in this example remains paradoxical. Namely, that if the velocity of charge e is zero and the current in the toroid is constant, then the mechanical momenta $\mathbf{P}_e$ and $\mathbf{P}_T$ are zero, yet the electromagnetic momentum $\mathbf{P}_{\mathrm{EM}}$ of eq. (8) is nonzero. If the total momentum of an isolated system at rest is to be zero, in accordance with usual expectations, there must be an additional, "hidden" momentum in the system that is equal and opposite to the $\mathbf{P}_{\mathrm{EM}}$.

The question of whether the electromagnetic momentum (3) itself corresponds to a kind of "hidden" mechanical momentum was considered by Maxwell in secs. 552 and 590 of [10], who felt that the issue could not be settled at that time. Cullwick appears to have concluded that the electromagnetic momentum associated with currents actually is the mechanical

momentum of the moving charges that comprise the currents. See. chap. 18 of [3]. However, this view does not ensure that the total momentum is zero for an isolated system at rest.

Rather, we argue the charge e is brought sufficient slowly from "infinity" to rest near the toroid, then the force (14) is negligible and negligible work is done on the toroid during this process. The toroid remains at rest so long as the velocity of the charge e is negligible. The total energy of a charge q of rest mass M that participates in the current I of the toroid remains Mc^2. When that charge is in the electric potential Φ of charge e, its electrical potential energy is $q\Phi$, so the effective mass M_{eff} of charge q is must be lower than M, such that $\Delta M_{\text{eff}} = M_{\text{eff}} - M = -q\Phi/c^2$ according to Einstein's relation for the equivalence of mass and energy. Following initial discussion of this effect by Shockley [11] and by Coleman and Van Vleck [12], a useful expression for the "hidden" mechanical momentum $\mathbf{P}_h$ was given by Furry [9],

$$\mathbf{P}_h = -\int \frac{\Phi \mathbf{J}}{c^2}\, d\text{Vol}. \tag{18}$$

who noted that for a single charge q, $\mathbf{J}\, d\text{Vol} \leftrightarrow q\mathbf{v}$, so that the "hidden" mechanical momentum associated with ΔM_{eff} is the $d\mathbf{P}_h = -(q\Phi/c^2)\mathbf{v} \leftrightarrow -(\Phi \mathbf{J}/c^2)\, d\text{Vol}$. Comparing with eq. (3) we see that

$$\mathbf{P}_h = -\mathbf{P}_{\text{EM}}, \tag{19}$$

(and not $+\mathbf{P}_{\text{EM}}$ as argued by Cullwick [3]), so that the total momentum is indeed zero.[2]

2.2 Analysis in the Rest Frame of the Moving Charge

The transformation from the lab frame to the rest frame of charge e requires a boost by the small velocity v, and so we expect the forces to be the same in both frames. However, in the rest frame of the charge e that charge creates no magnetic field, so it appears that the force on the toroid is zero in this frame, and hence Galilean invariance may be violated.

The resolution of this aspect of Cullwick's paradox is to be found in the relativistic transformation of charge and current density, which form a 4-vector, $(c\varrho, \mathbf{J})$. We consider only the case that the velocity v of charge e is small compared to the speed of light, so that $\gamma = 1/\sqrt{1 - v^2/c^2} \approx 1$.

In the lab frame there is no charge density ϱ associated in the toroid, but in the rest frame of charge e, whose lab velocity is $\mathbf{v}$, the toroid has a nonzero charge density ϱ' given by

$$\varrho' = \gamma \left(\varrho - \frac{\mathbf{J} \cdot \mathbf{v}}{c^2} \right) \approx -\frac{\mathbf{J} \cdot \mathbf{v}}{c^2} = -\frac{vJ_z}{c^2}, \tag{20}$$

where the $'$ indicates quantities measured in the rest frame of the charge. The lab-frame current consists of positive and negative charge densities that are equal and opposite but which have different velocities. On transforming to a moving frame, the positive and negative charge densities are no longer the same, and a net charge density (20) is observed. See sec. 86 of [16] for further discussion, including the example of a moving ring of current.

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[2]The present example, in which the mechanical momentum of conduction currents includes a "hidden" component, contrasts with others, [13, 14, 15] in which part or all of an isolated electromechanical system has tiny net velocity, such that the associated mechanical momentum is effectively "hidden".

Since the current density $\mathbf{J}$ resides on the surface of the toroid, the volume charge density (20) can be re-expressed as a surface charge density σ' given by

$$\sigma' \approx \frac{vI}{2\pi ac^2}\sin\alpha, \tag{21}$$

where α the angle shown in the figure above. The charge distribution (21) on the toroid is positive for radial distances ρ greater than a and negative for $\rho < a$, so that total charge on the toroid is zero in the rest frame (as well as in the lab frame).

Charge e exerts an electrostatic force on the charge distribution (21) on the toroid, and, of course, charge e experiences an equal and opposite electrostatic force from the toroid (in addition to the force if the current is changing).

For low-velocity transformations, the current density is unchanged since $\varrho = 0$ in the lab frame,

$$\mathbf{J}' = \gamma\left(\mathbf{J}_\parallel - \varrho\mathbf{v}\right) + \mathbf{J}_\perp \approx \mathbf{J}. \tag{22}$$

We now calculate the electromagnetic momentum $\mathbf{P}'_{\rm EM}$, and the forces $\mathbf{F}'_{\rm T}$ on the toroid and $\mathbf{F}'_e$ on the charge e in the rest frame of charge e.

2.2.1 The Electromagnetic Momentum

It is simplest to use the first form of eq. (3) to evaluate the electromagnetic momentum in the rest frame. The only vector potential in this frame is that due to the current $\mathbf{J}' \approx \mathbf{J}$ in the toroid. Hence, the rest frame vector potential $\mathbf{A}'$ obeys

$$\mathbf{A}' = \mathbf{A}'_{\rm T} \approx \mathbf{A}_{\rm T}. \tag{23}$$

The rest-frame electromagnetic momentum is therefore,[3]

$$\mathbf{P}'_{\rm EM} = \int \frac{\varrho'\mathbf{A}'}{c}\,d{\rm Vol}' \approx \frac{e\mathbf{A}_{\rm T}}{c} = \mathbf{P}_{\rm EM}, \tag{24}$$

the same as in the lab frame. This result illustrates how electromagnetic momentum that is tied to charges and currents does not transform like the space part of an energy-momentum 4-vector. See, for example, sec. 12.10 of [17] for additional comments.

We can, however, relate the electromagnetic momentum to the charge/current-density 4-vector, $(\Phi, \mathbf{A})$. In the lab frame the electric potential $\Phi_{\rm T}$ of the toroid vanishes, so the transformation of the toroid's lab-frame 4-vector $(\Phi_{\rm T} = 0, \mathbf{A}_{\rm T})$ to the rest frame of charge e gives

$$\Phi'_{\rm T} = \gamma\left(\Phi_{\rm T} - \frac{\mathbf{v}\cdot\mathbf{A}_{\rm T}}{c}\right) \approx -\frac{v}{c}A_{{\rm T},z}, \qquad \mathbf{A}'_{\rm T} = \gamma\left(\mathbf{A}_{{\rm T},\parallel} - \Phi_{\rm T}\frac{\mathbf{v}}{c}\right) + \mathbf{A}_{{\rm T},\perp} \approx \mathbf{A}_{\rm T}. \tag{25}$$

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[3]We do not include the term $\sigma'\mathbf{A}'_{\rm T}/c$ in eq. (24) as this is suppressed a factor of c^2.

2.2.2 The Force on the Toroid

The electric field $\mathbf{E}'_e \approx \mathbf{E}_e = e\hat{\mathbf{r}}/r^2$ of charge e exerts a force $\mathbf{F}'_T$ on the charge distribution σ' on the toroid in the charge's rest frame given by[4]

$$
\begin{aligned}
\mathbf{F}'_T &= \int \sigma' \mathbf{E}'_e \, d\text{Area}' = \int \sigma' \frac{e\hat{\mathbf{r}}}{r^2} \, d\text{Area}' \\
&\approx \int_0^{2\pi} \frac{vI \sin\alpha}{2\pi a c^2} \frac{e(\sin\beta \, \hat{\boldsymbol{\rho}} - \cos\beta \hat{\mathbf{z}})}{R^2} \left(1 + 2\frac{b}{R}\cos(\alpha + \beta)\right) 2\pi ab \, d\alpha.
\end{aligned}
\tag{26}
$$

A subtlety compared to the calculations in sec. 2.1 is that the factor $\cos\beta$ in the expression for $\hat{\mathbf{r}}$ in eq. (26) must be expanded to the next order of accuracy. Indeed,

$$
\cos\beta = \frac{z - b\cos\alpha}{r} \approx \frac{z}{r} \approx \frac{z}{R}\left(1 + \frac{b}{R}\cos(\alpha + \beta)\right).
\tag{27}
$$

Thus,

$$
\begin{aligned}
\mathbf{F}'_T &\approx \frac{vIb}{c^2 R^3} \int_0^{2\pi} \sin\alpha \, e(a\,\hat{\boldsymbol{\rho}} - z\,\hat{\mathbf{z}})\left(1 + 3\frac{b}{R}(\cos\alpha\cos\beta - \sin\alpha\sin\beta)\right) d\alpha \\
&\approx \frac{3evI\pi b^2}{c^2} \frac{az}{(z^2 + a^2)^{5/2}}\hat{\mathbf{z}} = -\frac{ev}{c}\frac{\partial \mathbf{A}_T}{\partial z} = \mathbf{F}_T.
\end{aligned}
\tag{28}
$$

2.2.3 The Force on the Electric Charge

The force $\mathbf{F}'_e$ on the electric charge in its rest frame is due to the electric field $\mathbf{E}'_{T,\text{ind}}$ induced when the vector potential of the toroid changes at the position of the charge e, and also due to the electric field $\mathbf{E}'_{T,\sigma'}$ of the charge distribution (20) on the toroid. The vector potential $\mathbf{A}'_T$ at the charge e is the same in the rest frame as in the lab frame, but in the rest frame $\mathbf{A}'_T$ changes due to the velocity $\mathbf{v}_T = -\mathbf{v}$ of the toroid, as well as due to changes in the current I. Hence, the force due to the changing vector potential of the toroid is

$$
\mathbf{F}'_{e,\text{ind}} = e\mathbf{E}'_{T,\text{ind}} = -\frac{e}{c}\frac{\partial \mathbf{A}'_T}{\partial t} - \frac{e}{c}(\mathbf{v}_T \cdot \boldsymbol{\nabla}'_T)\mathbf{A}'_T = -\frac{e}{c}\frac{\partial \mathbf{A}_T}{\partial t} - \frac{ev}{c}\frac{\partial \mathbf{A}_T}{\partial z},
\tag{29}
$$

noting that $\boldsymbol{\nabla}'_T = -\boldsymbol{\nabla}'_e (= -\boldsymbol{\nabla}_e)$ since the former refers to the coordinates of the (center of the) toroid while the latter refers to the coordinates of the charge e. The electrostatic force on charge e is equal and opposite to the electrostatic force (26) on the toroid,

$$
\mathbf{F}'_{e,\sigma'} = \int e\mathbf{E}'_{T,\sigma'} \, d\text{Area}' = \int e\frac{-\sigma'\hat{\mathbf{r}}}{r^2} \, d\text{Area}' = -\mathbf{F}'_T = \frac{ev}{c}\frac{\partial \mathbf{A}_T}{\partial z} = -e\boldsymbol{\nabla}'_e \Phi'_T = -e\boldsymbol{\nabla}_e \Phi'_T(0,0,z),
\tag{30}
$$

where the last form refers to the electric potential (25) of the toroid in the rest frame of charge e.

The total force on charge e in its rest frame is

$$
\mathbf{F}'_e = -\frac{e}{c}\frac{\partial \mathbf{A}_T}{\partial t} = \mathbf{F}_e.
\tag{31}
$$

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[4]A briefer argument works backwards from the end of eq. (30).

2.2.4   Momentum Balance in the Rest Frame of Charge e

Once it is recognized that, in the rest frame of charge e, the moving toroid appears to have a nonzero surface charge distribution, we find that the forces on the charge and on the toroid are the same as in the lab frame. Also, the electromagnetic momentum is the same in both frames (which shows that $\mathbf{P}_{\mathrm{EM}}$ does not behave exactly like an ordinary momentum in all respects). Hence, the details of momentum balance are the same in both frames.

The sum of the forces on the charge e and on the toroid in the rest frame is

$$\mathbf{F}'_e + \mathbf{F}'_{\mathrm{T}} = -\frac{e}{c}\frac{\partial \mathbf{A}_{\mathrm{T}}}{\partial t} = -\frac{e}{c}\frac{d \mathbf{A}'_{\mathrm{T}}}{dt} = -\frac{d\mathbf{P}'_{\mathrm{EM}}}{dt}, \tag{32}$$

since the partial and total time derivatives of the vector potential at charge e are the same in the charge's rest frame. Relating the forces to the corresponding time rates of change of momentum, we have

$$\frac{d\mathbf{P}'_e}{dt} + \frac{d\mathbf{P}'_{\mathrm{T}}}{dt} + \frac{d\mathbf{P}'_{\mathrm{EM}}}{dt} = \frac{d\mathbf{P}'_{\mathrm{total}}}{dt} = 0. \tag{33}$$

As expected, the total momentum is constant in the rest frame of the charge e.

2.2.5   "Hidden" Mechanical Momentum in the Rest Frame of Charge e

According to the prescription of Furry [9], the "hidden" mechanical momentum in the rest frame of charge e can be calculated as[5]

$$\mathbf{P}'_h = -\int \frac{\Phi'_e \mathbf{J}'}{c^2}\, d\mathrm{Vol}' = -\mathbf{P}'_{\mathrm{EM}}. \tag{34}$$

We have seen in eq. (22) that $\mathbf{J}' \approx \mathbf{J}$. Similarly, the electric potential of charge e in its rest frame is that same as that in the lab frame when $v \ll c$, so that $\Phi'_e \approx \Phi_e$. Hence,

$$\mathbf{P}'_h \approx \mathbf{P}_h. \tag{35}$$

Thus, "hidden" mechanical momentum does not transform between moving frames like an ordinary mechanical momentum. The "hidden" momentum, as does the electromagnetic momentum, transforms like the charge-current 4-vector rather than like an energy-momentum 4-vector. Hence, both of these concepts must be treated with care in problems involving transformations between moving frames. See [18] for additional commentary.

2.3   Energy Flow in an Induction Linac

In the lab frame the charge e is accelerated by the electric field $\mathbf{E}_{\mathrm{T}}$ that exists when the current in the toroid is changing. The power P absorbed by the charge is

$$P = \mathbf{F}_e \cdot \mathbf{v} = ev E_{\mathrm{T}}(0,0,z) = -\frac{ev}{c}\frac{\partial}{\partial t} A_{\mathrm{T}}(0,0,z) = -\frac{ev\dot{I}\pi b^2}{c}\frac{a}{(z^2+a^2)^{3/2}}. \tag{36}$$

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[5]We neglect the contribution from $\Phi'_{\mathrm{T}}\mathbf{J}'/c^2$ in eq. (34) as this is suppressed by two additional powers of c.

The flow of power from the toroid to the charge is described by the Poynting vector, or more precisely, by the interaction part of the Poynting vector,

$$\mathbf{S}_{\text{int}} = \frac{c}{4\pi}\mathbf{E}_{\text{T}} \times \mathbf{B}_e + \frac{c}{4\pi}\mathbf{E}_e \times \mathbf{B}_{\text{T}}. \tag{37}$$

It would be nice to have a plot of the field lines of the Poynting vector (37), which would show them emanating from the toroid and converging on the charge e. Lacking such a plot, we content ourselves with verification that the total Poynting flux across a small surface surrounding charge e, and also across the surface of the toroid, is equal to the power P of eq. (36).

We first consider a small cylindrical surface of radius ρ and length $2l \gg \rho$ centered on the charge e. The electric field due to the toroid is essentially uniform over this surface, so $\mathbf{E}_{\text{T}} \approx E_{\text{T}}(0,0,z)$, and the magnetic field $\mathbf{B}_e$ of charge e is given by eq. (12). Outside the toroid the magnetic field $\mathbf{B}_{\text{T}}$ vanishes, so only the first term in eq. (37) contributes there. We can neglect the Poynting flux on the ends of the small cylinder since $\rho \ll l$. Hence, the inward Poynting flux over the surface of this cylinder is

$$
\begin{aligned}
-\oint \mathbf{S}_{\text{int}} \cdot d\mathbf{Area} &= -\frac{c}{4\pi}\oint \mathbf{E}_{\text{T}} \times \mathbf{B}_e \cdot d\mathbf{Area} \approx -\frac{c}{4\pi}\int_{-l}^{l} E_{\text{T}}\,\hat{\mathbf{z}} \times \frac{ev\rho}{cr^3}\,\hat{\boldsymbol{\phi}} \cdot 2\pi\rho\,dz\,\hat{\boldsymbol{\rho}} \\
&\approx \frac{evE_{\text{T}}\rho^2}{2}\int_{-l}^{l}\frac{dz}{(z^2+\rho^2)^{3/2}} = evE_{\text{T}}\frac{l}{\sqrt{l^2+\rho^2}} \to evE_{\text{T}} = P, \tag{38}
\end{aligned}
$$

in the limit that the radius ρ of the cylinder goes to zero.

To evaluate the outward Poynting flux from the surface of the toroid we consider a toroidal surface just outside the actual toroid, so that the second term of eq. (37) can again be neglected. The electric field $\mathbf{E}_{\text{T}}$ due to the changing current $\dot{I}$ flows in loops of radius b just outside the toroid. From Faraday's law, the magnitude of the induced electric field at the surface of the toroid is

$$E_{\text{T}} = -\frac{b\dot{B}_\phi}{2c} = -\frac{b\dot{I}}{ac}, \tag{39}$$

recalling that $B_\phi = 2I/ac$ inside the toroid. The magnetic field of charge e at the toroid has magnitude $B_e = eva/c(z^2+a^2)^{3/2}$. The cross product $\mathbf{E}_{\text{T}} \times \mathbf{B}_e$ is directed along the outward normal to the surface of the toroid. Hence, the total outward Poynting flux from the toroid is

$$
\begin{aligned}
\oint \mathbf{S}_{\text{int}} \cdot d\mathbf{Area} &= \frac{c}{4\pi}E_{\text{T}}B_e\,\text{Area} = -\frac{c}{4\pi}\frac{b\dot{I}}{ac}\frac{eva}{c(z^2+a^2)^{3/2}}\,2\pi a\,2\pi b \\
&= -\frac{ev\dot{I}\pi b^2}{c}\frac{a}{(z^2+a^2)^{3/2}} = P. \tag{40}
\end{aligned}
$$

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