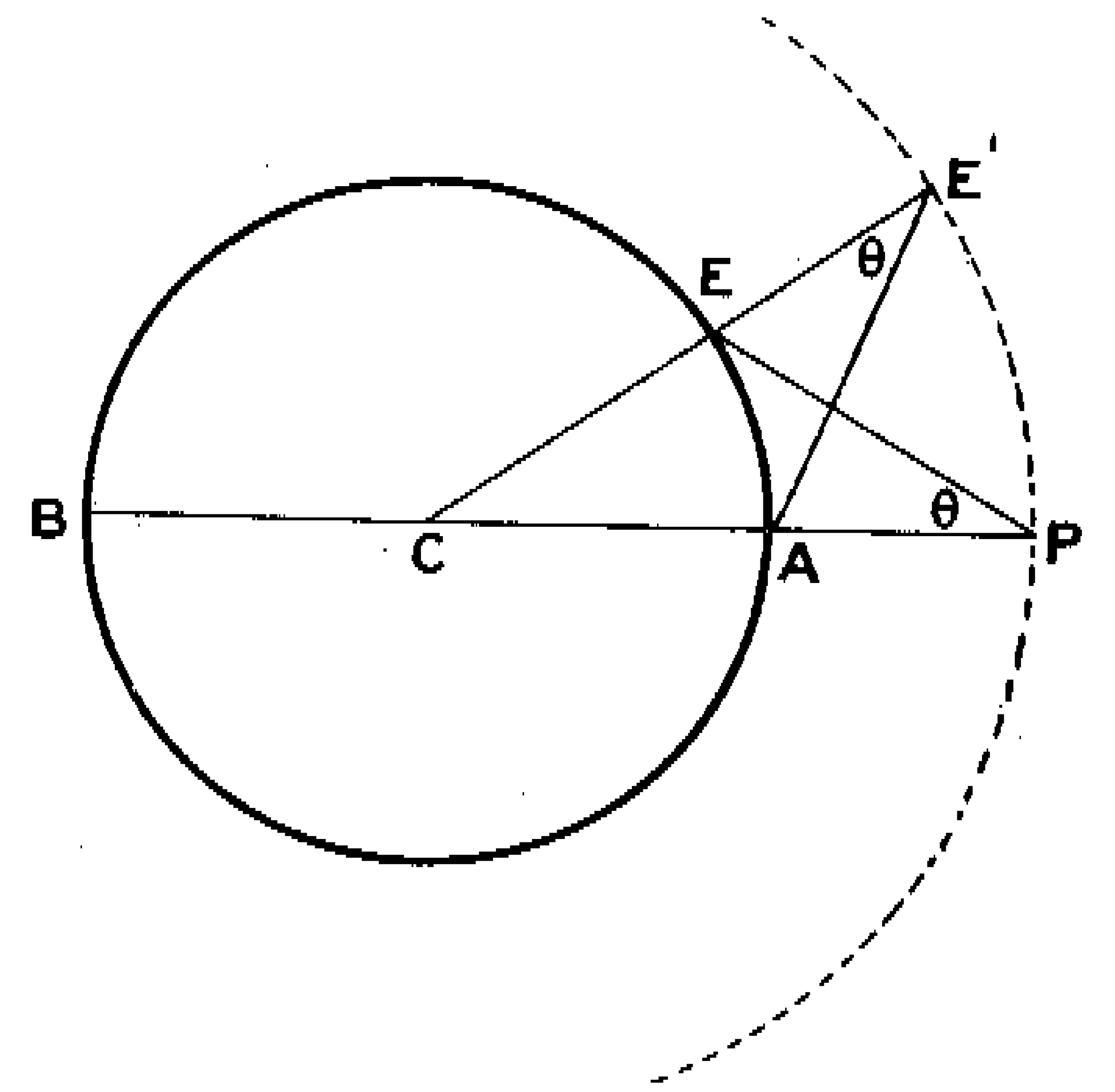
XLIII. *On a reciprocal relation between the Electrostatic Fields of certain Distributions of Electricity and the Magnetic Fields of corresponding Uniform Currents.* By Professor A. GRAY, *F.R.S.**

1. *A uniform circular linear distribution of electricity and a uniform circular current.*

THE heavy circle in fig. 1 represents the circular electrical distribution of line density ρ and radius a. P is a point external to the circle and its plane, at a distance a' from its centre, and A is the intersection of CP with the circle.

Fig. 1.



Consider an element of the circle at E of length ds. Project this element radially to E′ on the concentric circle (of radius a') described through P, and denote by ds' the length of the projection. Then $ds'/ds=a'/a$. Denote EP by r, $\angle$CPE by θ; we have then also AE′$=r$, and $\angle$CE′A$=\theta$.

The repulsion of the charge on ds exerted on a unit charge at P, that is the electric field intensity at P, is

$$F_e=\rho\frac{\cos\theta\, ds}{r^2}=\rho\frac{a}{a'}\frac{\cos\theta\, ds'}{r^2}\quad .\ .\ .\ .\ .\quad (1)$$

Now the second of these forms is, by the diagram, and

* Communicated by the Author.

according to the so-called law of Laplace, the *magnetic field intensity produced at* A by the element ds' of a circular conductor coincident with the concentric circle drawn through P, and carrying a current of strength $\gamma=\rho a/a'$. Thus the whole magnetic field intensity F_m at A, due to a current γ in the concentric circle through P, is given by

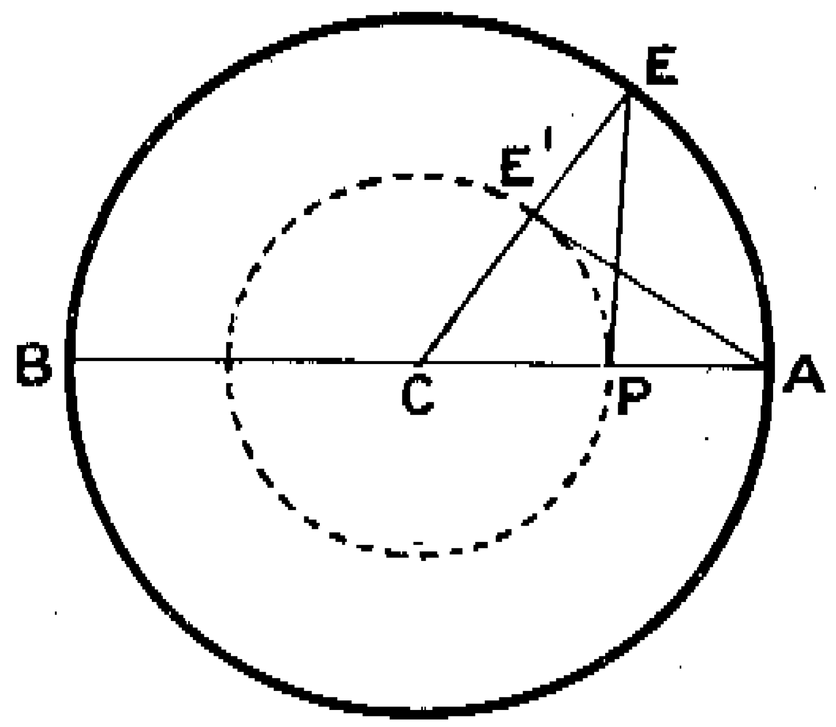
$$F_m=\gamma\int\frac{\cos\theta}{r^2}ds'=\rho\int\frac{\cos\theta}{r^2}ds=F_e,\quad .\ .\quad (2)$$

where F_e is the electric (or gravitational) field intensity at P due to the circle of radius a and uniform line density ρ.

Of course F_e is directed radially outward, while F_m is normal to the plane of the circles.

The same result holds *mutatis mutandis* when the point P is within the given circle (fig. 2).

Fig. 2.



2. *The mutual inductance of two concentric circles is proportional to the electric (or gravitational) field intensity produced by a uniformly charged disk, the edge of which coincides with one circle, at a point on the circumference of the other.*

Imagine the charged disk divided into an infinite number of concentric narrow circular strips, and consider the outward repulsion of each of these on unit charge at P. These repulsions combine into a radially outward force at P equal to their sum. But each element of repulsion is equal to the product of the area of the strip producing it and the magnetic field intensity produced at any point of that strip by a fixed current flowing round the other circle. This conclusion seems very remarkable: I am not aware that it has been noticed before.

One direct mode of calculating the mutual inductance of two coplanar concentric circles, would therefore seem to be as follows. Imagine the circle AEB (fig. 1) replaced by a uniformly charged disk and calculate the repulsion of the disk on a unit charge placed at P. This will be most easily done by dividing the disk into narrow strips all at right angles to the diameter through A. Then the repulsion exerted on unit charge at P, by one of these strips at distance x from P, of breadth dx, and having D, F for its extremities, is equal to the repulsion of the incomplete circular strip of radius x, breadth dx, and intercepted between the lines PD and PF.

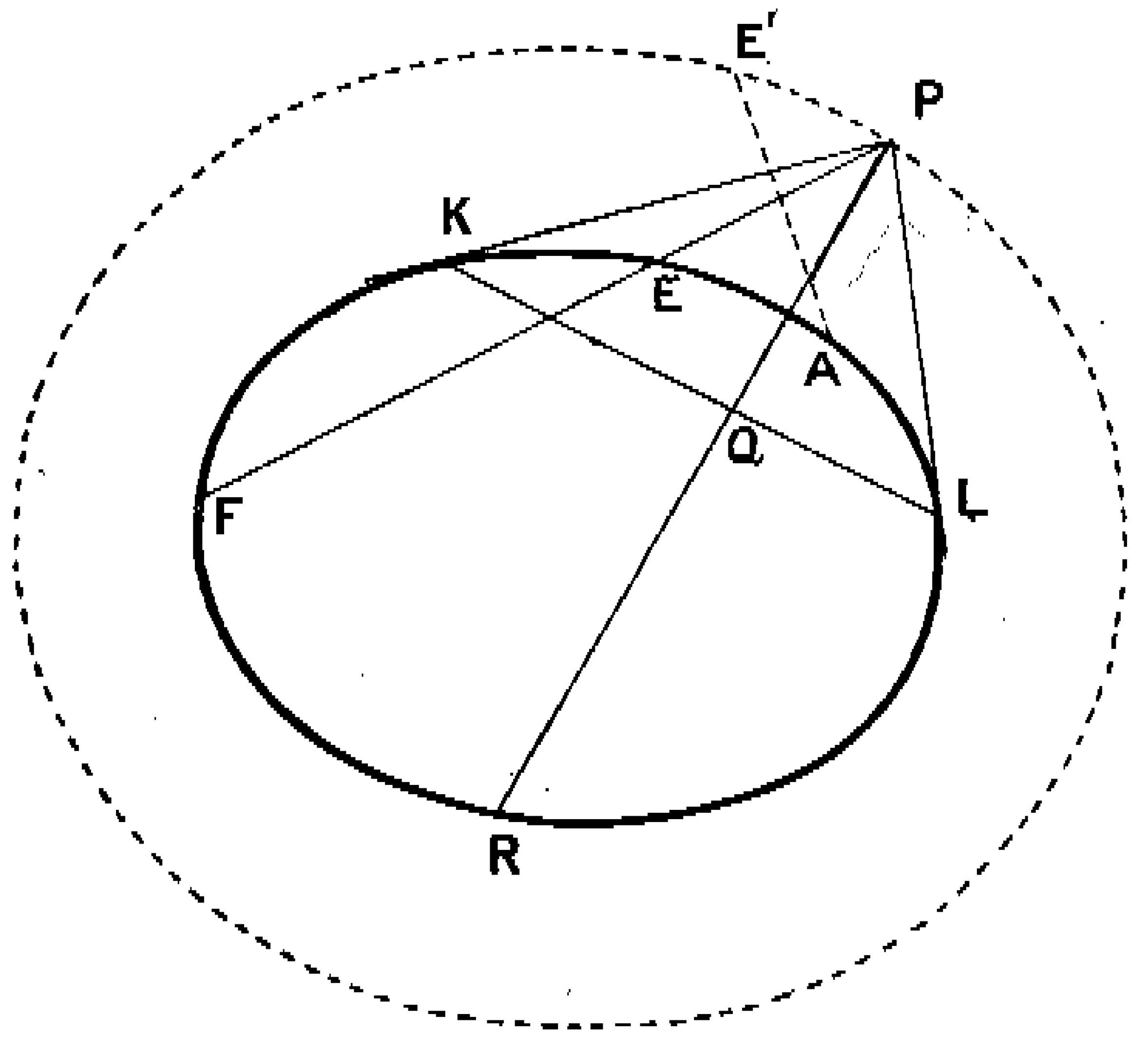
3. *A distribution of electricity on a plane conductor bounded by two close, similar, and similarly situated ellipses, and a uniform current in the confocal ellipse through the point considered.*

To generalize the theorem of § 1 consider the space between the ellipses of which the equations are

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\kappa, \quad \frac{x^2}{a^2}+\frac{y^2}{b^2}=\kappa-d\kappa. \quad . \quad . \quad . \quad (3)$$

One of these is shown by the heavy curve in fig. 3. We take an element of elliptic arc at E of length ds, and suppose

Fig. 3.



that over the area of this length, which lies at E between the two curves, electricity is distributed with uniform surface density σ. If the length of the perpendicular from the

centre on the tangent at E is p, the area is $\frac{1}{2}p\,ds\,d\kappa/\kappa$, and so the charge is $\frac{1}{2}\sigma p\,ds\,d\kappa/\kappa$.

Now through P let an ellipse confocal with the given ellipse LAK be described. A point A on the latter curve corresponds to P on the confocal, and a point E′ on the confocal corresponds to E on the given ellipse. Join E to P and A to E′. These lines have the same length, r. Let p' be the length of the perpendicular from the centre on the tangent at E′, and θ denote the angle between the line AE′ and that perpendicular. Let also p_0', θ_0 be the corresponding quantities for the point P and the line EP. [Care of course is to be taken that the lines are reckoned in the directions indicated by the letters, and that the perpendiculars are regarded as drawn both inward or both outward, so that there is no ambiguity as to the signs of the cosines.] In a former paper (Phil. Mag. April 1907) I have proved for two confocal ellipsoids the geometrical theorem (not, apparently, previously known)

$$p'\sec\theta=p_0'\sec\theta_0, \quad . \quad . \quad . \quad . \quad . \quad (4)$$

where p', θ and p_0', θ_0 refer to pairs of corresponding points on the ellipsoids, and have applied it to the complete and instantaneous evaluation of the integral for the force produced at an internal or external point by an elliptic homœoid, and hence to the solution of the problem of the attraction of a solid ellipsoid of uniform or of varying density.

The geometrical theorem as stated above asserts that the product $p'\sec\theta$ is invariant over the confocal ellipsoid: exactly the same theorem holds of course for the given ellipsoid. Moreover, since an ellipsoid is one of its own confocals, the theorem holds also for any two points P, Q on a given ellipsoid, the perpendiculars from the centre on the tangent planes at P, Q, and the chord joining these points. The theorem holds also for *any* confocal surfaces of the second degree.

In the present case let us take on the confocal ellipse through P, the element of arc ds', the points on which *correspond* to the points on ds; then since the equation of the confocal is

$$\frac{x^2}{a^2+u}+\frac{y^2}{b^2+u}=\kappa, \quad . \quad . \quad . \quad . \quad . \quad (5)$$

we have, as can easily be proved,

$$p\,ds=\frac{ab}{\{(a^2+u)(b^2+u)\}^{\frac{1}{2}}}p'\,ds'. \quad . \quad . \quad . \quad (6)$$

The electric field intensity at P, resolved along the normal to the confocal, is

$$F_e=\tfrac{1}{2}\sigma\frac{d\kappa}{\kappa}\int p\cos\theta_0\frac{ds}{r^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

where the integration is taken round the given elliptic strip. By the previous equation this can be written

$$F_e=\tfrac{1}{2}\sigma\frac{d\kappa}{\kappa}\frac{ab}{\{(a^2+u)(b^2+u)\}^{\frac{1}{2}}}\int p'\cos\theta_0\frac{ds'}{r^2}, \quad . \quad (8)$$

where the integration is taken round the confocal ellipse. But by the geometrical theorem stated above, the value of the component field intensity thus found becomes

$$F_e=\tfrac{1}{2}\sigma\frac{d\kappa}{\kappa}\frac{abp_0'}{\{(a^2+u)(b^2+u)\}^{\frac{1}{2}}}\int\cos\theta\frac{ds'}{r^2}. \quad . \quad . \quad (9)$$

or, if we write

$$\gamma=\tfrac{1}{2}\sigma\frac{d\kappa}{\kappa}\frac{abp_0'}{\{(a^2+u)(b^2+u)\}^{\frac{1}{2}}}, \quad . \quad . \quad . \quad (10)$$

$$F_e=\int\gamma\cos\theta\frac{ds'}{r^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

This is evidently the magnetic field intensity F_m produced at A by a current of strength γ flowing round the confocal ellipse. It is of course in the direction at right angles to F_e, that is perpendicular to the plane of the ellipse.

This relation between the normal component of the electric field intensity at P of the charged elliptic strip and the magnetic field intensity at the corresponding point A due to a current in the confocal ellipse, is curious and appears to be new. There is not, so far as I can see, any direct practical application of the theorem which can be made with advantage.

I may here recall that in the paper of April 1907, referred to above, the expression

$$F=\tfrac{1}{2}\rho\frac{d\kappa}{\kappa}\frac{abc}{\{(a^2+u)(b^2+u)(c^2+u)\}^{\frac{1}{2}}}p_0\int\frac{\cos\theta'}{r^2}dS' \quad (12)$$

[where p_0 is the perpendicular let fall from the centre on the tangent plane to the confocal at the point P, and θ' is the angle between the perpendicular from the centre to the tangent plane at the element dS', at E′, of the confocal surface, and the line AE′ (see fig. 3), and the integral is taken over the confocal] was found, by the geometrical theorem (4) quoted above, for the attraction of a homœoid at the external point P. It was remarked that this value of F at P is, to a constant factor, equal to the potential

produced at any point internal to itself by a uniform magnetic shell coinciding with the confocal surface. The strength of this shell, though constant over the surface, is proportional to the length of the perpendicular from the centre on the tangent plane to the confocal at P, and therefore varies with the position of P on the surface.

The value of the integral $\int\cos\theta'\,dS'/r^2$ is of course 4π and so F the force at P due to the given elliptic homœoid, or its equivalent the magnetic potential within the confocal is obtained at once *.

4. *Evaluation of the field intensities for the case of* § 1.

The values of the electric and magnetic field intensities specified in § 1 in terms of elliptic integrals are of course well known, (see for example a paper by Dr. Alexander Russell, Phil. Mag. April 1870). But a process of integration, differing somewhat from the usual one, is of some interest and seems possibly capable of some extensions, which may be given in a subsequent note.

The direct process of evaluation of the electric field intensity F_e from

$$F_e=2\rho\frac{a}{a'}\int_0^{\pi}\frac{\cos\theta\,ds'}{r^2} \quad . \quad . \quad . \quad . \quad . \quad (13)$$

in elliptic integrals is as follows. By fig. 1 we see at once that $\cos\theta=(a'-a\cos\phi)/r$ so that we get

$$F_e=2\rho\frac{a}{a'}\int_0^{\pi}\frac{a'-a\cos\phi}{r^3}ds'. \quad . \quad . \quad . \quad (14)$$

This leads to

$$F_e=2\rho\frac{a}{a'}\left\{\frac{1}{a'-a}E(k)+\frac{1}{a'+a}K(k)\right\}, \quad . \quad . \quad (15)$$

if $k=2\sqrt{aa'}/(a+a')$, and we use the relation

$$\int_0^{\frac{\pi}{2}}\frac{d\psi}{(1-k^2\sin^2\psi)^{\frac{3}{2}}}=\frac{E}{1-k^2}, \quad . \quad . \quad . \quad (16)$$

which can be verified by direct integration. This is an elliptic integral of the third kind with parameter $-k^2$.

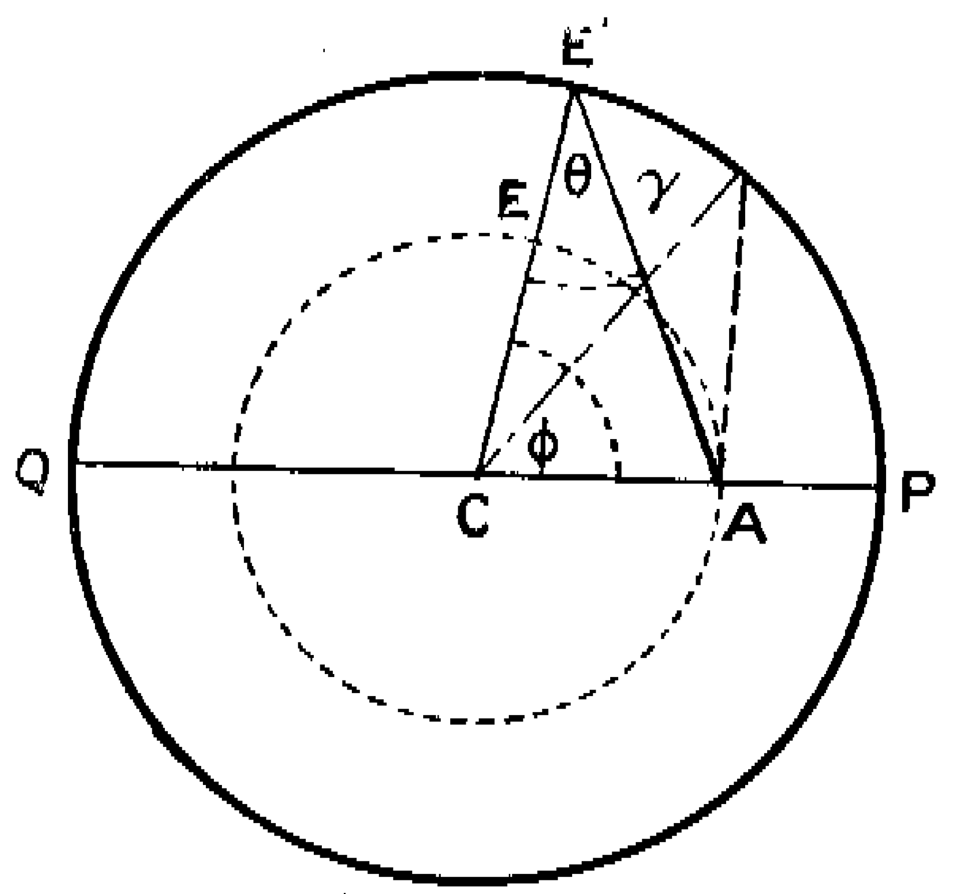
* In the particular case in which the confocals are concentric spherical surfaces, the attraction of a shell coincident with the inner sphere reduces at once to $4\pi\sigma a^2/a'^2$, since $\int\cos\theta\,dS'/r^2=4\pi$.

Similarly for the attraction at an internal point, the point A is external to the sphere radius a', and the solid angle is zero. Hence the internal attraction is zero.

But we may also, since by fig. 4 $r(d\phi+d\theta)/ds'=\cos\theta$, write

$$\int_0^\pi \frac{\cos\theta\, ds'}{r^2}=\int_0^\pi \frac{d\phi}{r}+\int\frac{d\theta}{r}. \quad . \quad . \quad . \quad (17)$$

Fig. 4.



The first integral is obtained at once. It comes out

$$\int_0^\pi \frac{d\phi}{r}=\frac{2}{a+a'}\mathrm{K}(k), \quad . \quad . \quad . \quad . \quad (18)$$

and it is to be observed that multiplied by $2\rho' a'$ it is the potential at A due to a uniform distribution of electricity (or gravitating matter) of line density ρ' on the outer circle.

For the second integral $\int d\theta/r$ we take

$$a^2=a'^2+r^2-2a'r\cos\theta,$$

that is

$$r=a'\cos\theta\pm(a^2-a'^2\sin^2\theta)^{\frac{1}{2}}. \quad . \quad . \quad . \quad (19)$$

Thus the largest value of θ is $\sin^{-1}(a/a')$, and is shown by the dotted lines in fig. 4, where a right angle at A is intended to be indicated. As successive positions of E and E' are taken on the semicircles, AEB and PE'Q, θ varies from zero to $\sin^{-1}(a/a')$ and then falls off to zero.

For the first part of this range of variation we have to take the lower sign in the last equation, and for the second part the upper sign. Thus we get for the first part, putting

θ_1 for $\sin^{-1}(a/a')$,

$$\int\frac{d\theta}{r}=\int_0^{\theta_1}\frac{d\theta}{a'\cos\theta-(a^2-a'^2\sin^2\theta)^{\frac{1}{2}}}$$

$$=\frac{1}{a'^2-a^2}\int_0^{\theta_1}d\theta\{a'\cos\theta+(a^2-a'^2\sin^2\theta)^{\frac{1}{2}}\},$$

or again, since $a'\sin\theta_1=a$,

$$\int\frac{d\theta}{r}=\frac{1}{a'^2-a^2}\left\{-a+a'\frac{a^2}{a'^2}\int_0^{\frac{1}{2}\pi}\frac{\cos^2\chi\, d\chi}{\left(1-\frac{a^2}{a'^2}\sin^2\chi\right)^{\frac{3}{2}}}\right\}, \quad (20)$$

where $\chi=\sin^{-1}\{(a'/a)\sin\theta\}$.

The second part of the integral $\int d\theta/r$ for the semicircle is found similarly to be

$$\int\frac{d\theta}{r}=\frac{1}{a'^2-a^2}\left\{a+a'\frac{a^2}{a'^2}\int_0^{\frac{1}{2}\pi}\frac{\cos^2\chi\, d\chi}{\left(1-\frac{a^2}{a'^2}\sin^2\chi\right)^{\frac{3}{2}}}\right\}. \quad (21)$$

Thus the total value of $\int d\theta/r$ for the circle is

$$\frac{4a'k_1^2}{a'^2-a^2}\int_0^{\frac{1}{2}\pi}\frac{\cos^2\chi\, d\chi}{(1-k_1^2\sin^2\chi)^{\frac{3}{2}}}$$

$$=\frac{4a'}{a'^2-a^2}\{k_1^2\mathrm{K}(k_1)+\mathrm{E}(k_1)-\mathrm{K}(k_1)\}, \quad (22)$$

where (with $a'>a$) $k_1^2=a^2/a'^2$. Also

$$k=2\sqrt{aa'}/(a+a')=2\sqrt{k_1}/(1+k_1).$$

Now k and k_1 are the moduli connected by Landen's transformation, and so the magnetic field intensity at A due to a current γ in the circle of radius a' is, as stated in § 1 above,

$$\mathrm{F}_m=\frac{4a'\gamma}{a'^2-a^2}\mathrm{E}(k_1)=2\gamma\left\{\frac{\mathrm{E}(k)}{a'-a}+\frac{\mathrm{K}(k)}{a'+a}\right\}. \quad . \quad (23)$$

A comparison of the reduced form of (14), from which (15) was obtained by assuming (16), with (23) establishes (16), and so evaluates this simple form of the elliptic integral of the third kind.

5. *Field intensities not in the plane of the circular distributions.*

Returning to fig. 1 we may regard the points E' and P as in a plane parallel to that of the circle AEB, and distant b from it, so that the two circles are now coaxial. The two distances EP and AE′, and the two angles (CPE and CE′A) marked θ are equal. This angle is not, however, any longer the complement of the angle between AE′ and the tangent to the circle at E′, drawn back towards P. We may calculate the components of the electric field intensity at P, denoting by X the component parallel to CA, and by Z the component parallel to the axis of the system.

If as before a, a' be the radii of the circles, and r be EP, so that now $r^2=a^2+a'^2+b^2-2aa'\cos\phi$, the components due to $ad\phi$ at E are, as we see at once,

$$dX_e=\rho\frac{a\,d\phi}{r^2}\frac{a'-a\cos\phi}{r}, \quad dZ_e=\rho b\frac{a\,d\phi}{r^3}.$$

Thus integrating round the circle AEB we get

$$X_e=2\rho a\int_0^\pi\frac{a'-a\cos\phi}{r^3}d\phi, \quad Z_e=2\rho ba\int_0^\pi\frac{d\phi}{r^3}. \qquad (24)$$

To pass from these to the components of magnetic field intensity at A due to a current of strength $\rho a/a'$ in the circle PE′, we have only to interchange X and Z, and take note of the directions of the components. As a little consideration will show, we have to replace X_e, which is parallel to CP, by a component along PC, and then suppose that turned normally out from the paper to give Z_m. It will be seen that the "vertical" component Z_e gives a component X_m parallel to EC, and that we have

$$X_m=-2\gamma a'b\int_0^\pi\frac{\cos\phi\,d\phi}{r^3}, \quad Z_m=2\gamma a'\int\frac{(a'-a\cos\phi)d\phi}{r^3}. \quad (25)$$

These components can be at once expressed in elliptic integrals. The component Z_m is the more important as it enables the mutual inductance of the two circles to be found by integration over the circle AEB. The mutual inductance between the coaxial circles could be found by calculating the total X_m at P for each of a series of narrow concentric rings into which the circle of radius a, say, is divided, multiplying each by the area of the ring to which it belongs and calculating the sum of products thus obtained.

These reciprocal theorems of coaxial circles have not so far as I know been stated explicitly before; but Sir George Greenhill has pointed out to me that in a paper in the 'American Journal of Mathematics,' vol. xxxix. p. 439 (1917), he has given certain general reciprocal relations from which they may be deduced.

XLIV. *Notes on the "Break" of a Magneto or Induction-Coil.* *By* NORMAN CAMPBELL, *Sc.D.**

NOTE.—The work described in these notes was carried out at the National Physical Laboratory under the direction of the Advisory Committee for Aeronautics. The results have been communicated in a confidential report to the Internal Combustion Engine Sub-Committee of that Committee, who have now given their consent to the publication of any portions which appear of pure scientific interest.

(1) *Introduction.*

IT is well known that if the greatest possible efficiency is to be obtained from a magneto or induction-coil, that is, the greatest possible ratio of the maximum secondary potential to the primary current broken, it is necessary to avoid sparking between the terminals at which the primary current is broken. No elaborate theory is necessary to explain the loss of efficiency due to sparking. In the first place the spark involves the dissipation of some of the electromagnetic energy originally present in the primary current which would otherwise have been available for conversion into electrostatic energy of the secondary. In the second place the passage of the spark, even if it involved no loss of energy, would prolong the time which elapses between the first decrease of the primary current, as the contact opens, and its total cessation; very general considerations will show that, if this time is so prolonged as to become an appreciable fraction of the period of the oscillations excited, a loss of efficiency will usually follow.

It is always possible to suppress the spark at the primary break by inserting a condenser of sufficient capacity in parallel with the separating terminals. Until an adequate theory of the induction-coil was developed, chiefly by the work of Prof. Taylor Jones and his collaborators †, it seems to have been believed that the suppression of the spark was the only useful function of the primary condenser; and that if the suppression could be achieved by any other means, such as an increase in the speed of separation of the terminals, the addition of capacity to the primary circuit would be unnecessary. This view was, of course, largely based on the classical experiments of Lord Rayleigh ‡, who showed

* Communicated by the Author.

† Phil. Mag. Jan. 1909, p. 28; Nov. 1911, p. 706; April 1914, p. 580; Jan. 1915, p. 1; Aug. 1915, p. 224; April 1917, p. 322.

‡ Phil. Mag. ii. p. 581 (1901).